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ON Ω-CLOSED SETS AND Ωs-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. New classes of sets called Ω-closed sets and Ωs-closed sets are introduced and studied. Also, we introduce and study Ω-continuous functions and Ωs-continuous functions and prove pasting lemma for these functions. Moreover, we introduce classes of topological spaces called $\Omega - T_{\frac{1}{2}}$ and $\Omega - T_s$.

1. Introduction

Many concepts of topology have been generalized by considering the concept of semi-open sets due to Levine [10] instead of open sets. On the other hand, the study of generalized closed sets in a topological space was initiated by Levine [11] and the concept of $T_{\frac{1}{2}}$ spaces was introduced. In 1987, Bhattacharyya and Lahiri [4] introduced the class of semi-generalized closed sets and used them to obtain properties of semi-$T_{\frac{1}{2}}$ spaces. In 1990, Arya and Nour [1] defined the generalized semi-closed sets and studied some characterizations of s-normal spaces. The modified forms of generalized closed sets and generalized continuity were studied by Balachandran, Devi, Maki and Sundaram [3, 8]. This paper is a continuation of their works.

2. Preliminaries

Throughout this paper (X,τ) and (Y,σ) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X,τ), $\mathrm{Cl}\,(A)$, $\mathrm{Int}\,(A)$ and $D[A]$ [5, 6] denote the closure, the interior and the derived set of A, resp.

We recall some known definitions and properties needed in this paper.

Key words and phrases: Ω-closed, Ωs-closed, Ω-continuity, Ωs-continuity, $\Omega - T_{\frac{1}{2}}$ space and $\Omega - T_s$ space.

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Definition 2.1. Let (X,τ) be a topological space. A subset $A \subseteqq X$ is said to be

(1) semi-open [10] if $A \subseteqq \mathrm{Cl}\left(\mathrm{Int}\,(A)\right)$ and semi-closed if $\mathrm{Int}\left(\mathrm{Cl}\,(A)\right) \subseteqq A$,

(2) α-open [13] if $A \subseteqq \mathrm{Int}\left(\mathrm{Cl}\left(\mathrm{Int}\,(A)\right)\right)$,

(3) regular open if $A = \mathrm{Int}\left(\mathrm{Cl}\,(A)\right)$ and regular closed if $A = \mathrm{Cl}\left(\mathrm{Int}\,(A)\right)$.

Definition 2.2 [5]. Let (X,τ) be a topological space and $A, B \subseteqq X$. Then A is semi-closed if and only if $X - A$ is semi-open and the semi-closure of B, denoted by $\mathrm{sCl}\,(B)$, is the intersection of all semi-closed sets containing B.

Definition 2.3 [7]. Let (X,τ) be a topological space, $A \subseteqq X$ and $x \in X$. Then x is said to be a semi-limit point of A if and only if every semi-open set containing x contains a point of A different from x.

Definition 2.4 [7]. Let (X,τ) be a topological space and $A \subseteqq X$. The set of all semi-limit points of A is said to be the semi-derived set of A and is denoted by $D_s[A]$.

Definition 2.5 [5]. Let (X,τ) be a topological space and $A \subseteqq X$. The semi-interior of A, denoted by $\mathrm{sInt}\,(A)$, is the union of all semi-open subsets of A.

Definition 2.6. A subset $A \subseteqq X$ is said to be

(1) generalized closed (briefly g-closed) [11] if $\mathrm{Cl}\,(A) \subseteqq U$ whenever $A \subseteqq U$ and U is open,

(2) semi-generalized closed (briefly sg-closed) [4] if $\mathrm{sCl}\,(A) \subseteqq U$ whenever $A \subseteqq U$ and U is semi-open,

(3) generalized semi-closed (briefly gs-closed) [1] if $\mathrm{sCl}\,(A) \subseteqq U$ whenever $A \subseteqq U$ and U is open.

Definition 2.7. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(1) g-continuous [3] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V of (Y,σ),

(2) sg-continuous [13] if $f^{-1}(V)$ is sg-closed in (X,τ) for every closed set V of (Y,σ),

(3) gs-continuous [8] if $f^{-1}(V)$ is gs-closed in (X,τ) for every closed set V of (Y,σ),

(4) semi-continuous [10] if $f^{-1}(V)$ is semi-open in (X,τ) for every open set V of (Y,σ),

(5) contra-continuous [9] if $f^{-1}(V)$ is closed in (X,τ) for every open set V of (Y,σ),

(6) perfectly-continuous [2] if $f^{-1}(V)$ is both open and closed in (X,τ) for every open set V of (Y,σ).

3. Ω-closed sets and Ωs-closed sets

Definition 3.1. A subset A of (X,τ) is said to be Ω-closed in (X,τ) if $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,(U)$ whenever $A \subseteqq U$ and U is semi-open in (X,τ).

Definition 3.2. A subset A of (X,τ) is said to be Ωs-closed in (X,τ) if $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,\big(\mathrm{Cl}\,(U)\big)$ whenever $A \subseteqq U$ and U is semi-open in (X,τ).

Proposition 3.1. *If $D[E] \subseteqq D_s[E]$ for each subset E of a space (X,τ), then the union of two Ω-closed sets (resp. Ωs-closed sets) is Ω-closed (resp. Ωs-closed).*

Proof. Let A and B be Ω-closed (resp. Ωs-closed) subsets of X and let U be a semi-open set such that $A \cup B \subseteqq U$. Then, $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,(U)$ and $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,(U)$ (resp. $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,\big(\mathrm{Cl}\,(U)\big)$ and $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,\big(\mathrm{Cl}\,(U)\big)$). Since $D[A] \subseteqq D_s[A]$ and $D[B] \subseteqq D_s[B]$, then we have that $\mathrm{Cl}\,(A) = \mathrm{sCl}\,(A)$ and $\mathrm{Cl}\,(B) = \mathrm{sCl}\,(B)$. Therefore, $\mathrm{Cl}\,(A \cup B) = \mathrm{Cl}\,(A) \cup \mathrm{Cl}\,(B) = \mathrm{sCl}\,(A) \cup \mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,(U)$, i.e., $\mathrm{sCl}\,(A \cup B) \subseteqq \mathrm{Int}\,(U)$ (resp. $\mathrm{sCl}\,(A \cup B) \subseteqq \mathrm{Int}\,\big(\mathrm{Cl}\,(U)\big)$). Hence, $A \cup B$ is Ω-closed (resp. Ωs-closed).

Proposition 3.2. (1) *Every open and semi-closed subset of (X,τ) is Ω-closed.*

(2) *Every Ω-closed subset of (X,τ) is Ωs-closed and gs-closed.*

Proof. (1) Let A be an open and semi-closed subset of (X,τ) and $A \subseteqq U$, where U is a semi-open subset of X. Then, $\mathrm{sCl}\,(A) = A = \mathrm{Int}\,(A) \subseteqq \mathrm{Int}\,(U)$. Hence, A is Ω-closed.

(2) Let A be an Ω-closed subset of (X,τ) and $A \subseteqq U$, where U is a semi-open subset of X. Then, $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,(U) \subseteqq \mathrm{Cl}\,\big(\mathrm{Int}\,(U)\big)$. Hence, A is an Ωs-closed subset of (X,τ). To prove the second part let A be an Ω-closed subset of (X,τ) and $A \subseteqq U$, where U is an open subset of X. Then, $\mathrm{sCl}\,(A) \subseteqq \mathrm{Int}\,(U) \subseteqq U$. Hence, A is gs-closed.

REMARK 3.1. We have the following relationship between Ω-closed sets, Ωs-closed sets and related sets:

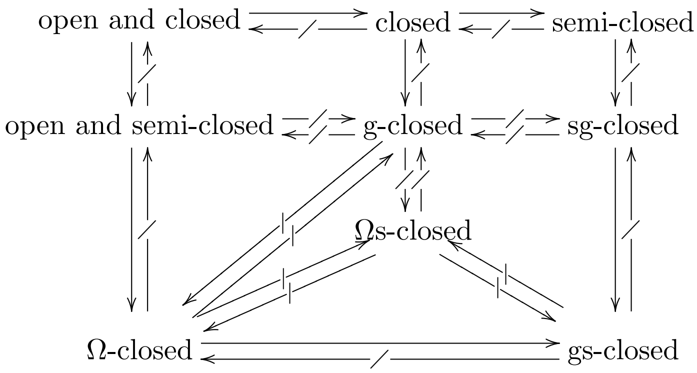

Fig. 1

REMARK 3.2. In Proposition 3.2, the converses are not necessarily true.
(1) Not every Ω-closed set is semi-closed as shown by Example 3.1.
(2) An Ωs-closed set need not be Ω-closed as shown by Example 3.2.
(3) A gs-closed set is not always Ω-closed as shown by Example 3.3.

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \big\{X, \phi, \{c, d\}\big\}$. The subset $\{a, b, d\}$ of X is Ω-closed but it is not semi-closed.

EXAMPLE 3.2. Let $X = \{a, b\}$ and $\tau = \big\{X, \phi, \{a\}\big\}$. The subset $\{a\}$ of X is Ωs-closed but it is neither Ω-closed nor gs-closed. Therefore, $\{a\}$ is not g-closed.

EXAMPLE 3.3. Let $X = \{a, b, c\}$ and $\tau = \big\{X, \phi, \{a\}, \{b\}, \{a, b\}\big\}$. The subset $\{c\}$ of X is g-closed and hence gs-closed but it is neither Ωs-closed nor Ω-closed.

REMARK 3.3. (1) Ω-closedness and g-closedness are independent by Examples 3.3 and 3.4 (below).
(2) Ωs-closedness and g-closedness are independent by Examples 3.2 and 3.3.
(3) Ωs-closedness and gs-closedness are independent by Examples 3.2 and 3.3.

EXAMPLE 3.4. Let $X = \{a, b, c\}$ and $\tau = \big\{X, \phi, \{a\}, \{a, b\}\big\}$. Then, the subset $\{b\}$ of (X, τ) is Ω-closed but it is not g-closed.

PROPOSITION 3.3. *If a subset A is Ω-closed in (X, τ), then* $\mathrm{sCl}\,(A) - A$ *does not contain a non empty semi-closed set.*

PROOF. Let F be a semi-closed set in (X,τ) such that $F \subseteqq \mathrm{sCl}(A) - A$. Then $F \subseteqq \mathrm{sCl}(A)$ and $A \subseteqq X - F$. Since A is Ω-closed, then $\mathrm{sCl}(A) \subseteqq \mathrm{Int}(X-F) = X - \mathrm{Cl}(F)$. Therefore, $F \subseteqq \mathrm{Cl}(F) \subseteqq X - \mathrm{sCl}(A)$. Hence, $F \subseteqq \big(X - \mathrm{sCl}(A)\big) \cap \mathrm{sCl}(A) = \phi$.

PROPOSITION 3.4. *If a subset A is Ωs-closed in (X,τ), then* $\mathrm{sCl}(A) - A$ *does not contain a non empty semi-open and semi-closed set.*

PROOF. Let F be semi-open and semi-closed in (X,τ) such that $F \subseteqq \mathrm{sCl}(A) - A$. Then we have that $A \subseteqq X - F$ and $\mathrm{sCl}(A) \subseteqq \mathrm{Int}\big(\mathrm{Cl}(X-F)\big) = X - \mathrm{Cl}\big(\mathrm{Int}(F)\big)$. Thus we obtain $F \subseteqq \mathrm{Cl}\big(\mathrm{Int}(F)\big) \subseteqq X - \mathrm{sCl}(A)$. Therefore, $F \subseteqq \big(X - \mathrm{sCl}(A)\big) \cap \mathrm{sCl}(A) = \phi$.

PROPOSITION 3.5. *If a subset A of (X,τ) is semi-open and Ω-closed, then it is semi-closed.*

PROOF. Since A is semi-open and Ω-closed, then $\mathrm{sCl}(A) \subseteqq \mathrm{Int}(A) \subseteqq A$. Hence, $\mathrm{sCl}(A) = A$ and A is semi-closed.

THEOREM 3.1. *A subset A of (X,τ) is regular open if and only if A is α-open and Ω-closed.*

PROOF. Suppose A is an α-open and Ω-closed set. Then A is semi-open and Ω-closed and by Proposition 3.5, A is semi-closed. So, $\mathrm{Int}\big(\mathrm{Cl}(A)\big) \subseteqq A$. Since A is α-open, then $A \subseteqq \mathrm{Int}\big(\mathrm{Cl}\big(\mathrm{Int}(A)\big)\big) \subseteqq \mathrm{Int}\big(\mathrm{Cl}(A)\big)$. Thus, $A = \mathrm{Int}\big(\mathrm{Cl}(A)\big)$ and A is regular open. Conversely, let A be regular open, then A is α-open. Since A is regular open, A is open and semi-closed. By Proposition 3.2, A is Ω-closed (see Fig. 1).

THEOREM 3.2. *An open set of (X,τ) is gs-closed if and only if it is Ω-closed.*

PROOF. Let A be an open and gs-closed set. Assume that $A \subseteqq U$, where U is a semi-open set in X. Thus $A = \mathrm{Int}(A) \subseteqq \mathrm{Int}(U)$. Since $\mathrm{Int}(U)$ is open in X and A is *gs*-closed, then $\mathrm{sCl}(A) \subseteqq \mathrm{Int}(U)$ and A is an Ω-closed set. Conversely, it is obvious that every Ω-closed set is *gs*-closed.

LEMMA 3.1 [14]. *Let $A \subset B \subset X$ where (X,τ) is a topological space and B is semi-open in the space X. Then A is semi-open in the space X if and only if A is semi-open in B.*

THEOREM 3.3. *Let B and Y be subsets of a space (X,τ) such that $B \subseteqq Y \subseteqq X$.*

(1) *If B is an Ω-closed set in the subspace Y and Y is an open and Ω-closed set in (X,τ), then B is an Ω-closed set in (X,τ).*

(2) *If Y is open and B is Ω-closed in (X,τ), then B is an Ω-closed set in the subspace Y.*

PROOF. (1) Let U be a semi-open set of (X,τ) such that $B \subseteqq U$. Since B is an Ω-closed set in the subspace Y, then we have $\mathrm{sCl}_Y(B) \subseteqq \mathrm{Int}_Y(U \cap Y)$. Then we obtain that $Y \cap \mathrm{sCl}\,(B) \subseteqq \mathrm{sCl}_Y(B) \subseteqq \mathrm{Int}_Y(U \cap Y) = \mathrm{Int}\,(U \cap Y)$. Hence $\mathrm{Int}\,(U \cap Y) \cup \{X - \mathrm{sCl}\,(B)\}$ is semi-open in (X,τ) and it contains Y. Since Y is Ω-closed in (X,τ), then we have

$$\mathrm{sCl}\,(B) \subseteqq \mathrm{sCl}\,(Y) \subseteqq \mathrm{Int}\left[\mathrm{Int}\,(U \cap Y) \cup \left\{X - \mathrm{sCl}\,(B)\right\}\right]$$

$$\subseteqq \mathrm{Int}\,(U) \cup \left\{X - \mathrm{sCl}\,(B)\right\}.$$

Thus $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,(U)$ and B is Ω-closed in (X,τ).

(2) Let $B \subseteqq U$, where U is semi-open in the subspace Y. Since Y is open in X, then by Lemma 3.1 U is semi-open in (X,τ). Since B is an Ω-closed set in (X,τ), then $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,(U)$. Since Y is open in X, then $\mathrm{Int}\,(U) = \mathrm{Int}_Y(U)$ and $\mathrm{sCl}_Y(B) = Y \cap \mathrm{sCl}\,(B) \subseteqq \mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\,(U) = \mathrm{Int}_Y(U)$. Hence, B is an Ω-closed set in the subspace Y.

COROLLARY 3.1. *Let $B \subseteqq Y \subseteqq X$ and Y be open and Ω-closed in (X,τ). Then, the following are equivalent:*

(1) *B is Ω-closed in (X,τ).*

(2) *B is an Ω-closed set in the subspace Y.*

THEOREM 3.4. *Let B and Y be subsets of a space (X,τ) such that $B \subseteqq Y \subseteqq X$.*

(1) *If B is an Ωs-closed set in the subspace Y and Y is an open and Ω-closed set in (X,τ), then B is an Ωs-closed set in (X,τ).*

(2) *If Y is open and B is Ωs-closed in (X,τ), then B is an Ωs-closed set in the subspace Y.*

PROOF. (1) Let U be a semi-open set of (X,τ) such that $B \subseteqq U$. Since B is an Ωs-closed set in the subspace Y, then $\mathrm{sCl}_Y(B) \subseteqq \mathrm{Int}_Y\left(\mathrm{Cl}_Y(U \cap Y)\right)$. So,

$$Y \cap \mathrm{sCl}\,(B) \subseteqq \mathrm{sCl}_Y(B) \subseteqq \mathrm{Int}_Y\left(\mathrm{Cl}_Y(U \cap Y)\right) = \mathrm{Int}\left(\mathrm{Cl}\,(U \cap Y)\right) \cap Y.$$

Hence $\mathrm{Int}\left(\mathrm{Cl}\,(U \cap Y)\right) \cup \left\{X - \mathrm{sCl}\,(B)\right\}$ is semi-open in (X,τ) and it contains Y. Since Y is an Ω-closed set in (X,τ), then we have that $\mathrm{sCl}\,(B) \subseteqq \mathrm{sCl}\,(Y) \subseteqq \mathrm{Int}\left[\mathrm{Int}\left(\mathrm{Cl}\,(U \cap Y)\right) \cup \left(\left\{X - \mathrm{sCl}\,(B)\right\}\right)\right] \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right) \cup \left\{X - \mathrm{sCl}\,(B)\right\}$. Thus, we obtain $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right)$ and B is Ωs-closed in (X,τ).

(2) Let $B \subseteqq U$, where U is semi-open in the subspace Y. Since Y is open in X, then by Lemma 3.1 U is semi-open in (X,τ). Since B is an Ωs-closed set in (X,τ), then $\mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right)$. Since Y is open in X, then $\mathrm{Int}\,(U) = \mathrm{Int}_Y(U)$ and $\mathrm{sCl}_Y(B) = Y \cap \mathrm{sCl}\,(B) \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right) \cap Y = \mathrm{Int}_Y\left(\mathrm{Cl}_Y(U)\right)$. Hence, B is an Ωs-closed set in the subspace Y.

4. Ω-continuity and Ωs-continuity

Let $f:(X,\tau)\to(Y,\sigma)$ be a function from a topological space (X,τ) into a topological space (Y,σ).

DEFINITION 4.1. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be Ω-continuous (resp. Ωs- continuous) if $f^{-1}(V)$ is Ω-closed (resp. Ωs-closed) in (X,τ) for every closed set V of (Y,σ).

DEFINITION 4.2. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be Ω-irresolute (resp. Ωs-irresolute) if $f^{-1}(V)$ is Ω-closed (resp. Ωs-closed) in (X,τ) for every Ω-closed (resp. Ωs-closed) set V of (Y,σ).

PROPOSITION 4.1. (1) *Every Ω-continuous function is gs-continuous.*
(2) *Every Ω-continuous function is Ωs-continuous.*
(3) *Every contra-continuous and semi-continuous function is Ω-continuous.*

REMARK 4.1. We have the following relationship between Ω-continuity and Ωs-continuity and other related generalized continuity.

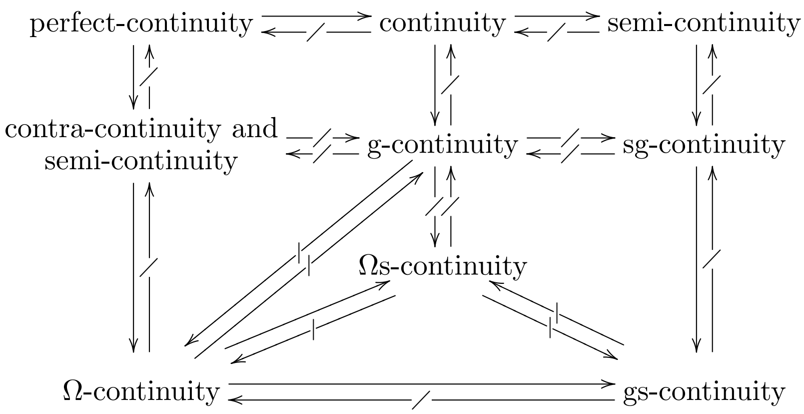

Fig. 2

REMARK 4.2. In Proposition 4.1, the converses are not necessarily true.
(1) Not every gs-continuous function is Ω-continuous as shown by Example 4.1.
(2) An Ωs-continuous function need not be Ω-continuous as shown by Example 4.2.
(3) An Ω-continuous function is not always semi-continuous as shown by Example 4.3.

EXAMPLE 4.1. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = f(b) = p$ and $f(c) = q$. Then f is continuous and hence g-continuous and gs-continuous. But it is not Ωs-continuous and hence not Ω-continuous. There exists a closed set $\{q\}$ of (Y, σ) such that $f^{-1}(\{q\})$ is not Ωs-closed in (X, τ).

EXAMPLE 4.2. Let $X = \{a, b\}$, $\tau = \{X, \phi, \{a\}\}$, $Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$. Define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = q$ and $f(b) = p$. Then f is an Ωs-continuous function which is not Ω-continuous. Also, it is neither gs-continuous nor g-continuous.

EXAMPLE 4.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$, $Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = f(c) = p$ and $f(b) = q$. Then f is Ω-continuous but it is not semi-continuous. There exists a closed set $\{q\}$ of (Y, σ) such that $f^{-1}(\{q\})$ is Ω-closed but not *semi*-closed in (X, τ).

EXAMPLE 4.4. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = f(c) = p$ and $f(b) = q$. Then f is Ω-continuous but not g-continuous. Because there exists a closed set $\{q\}$ of (Y, σ) such that $f^{-1}(\{q\})$ is Ω-closed but not g-closed in (X, τ).

REMARK 4.3. (1) Ω-continuity and g-continuity are independent by Examples 4.1 and 4.4.
(2) Ωs-continuity and g-continuity are independent by Examples 4.1 and 4.2.
(3) Ωs-continuity and gs-continuity are independent by Examples 4.1 and 4.2.

5. Applications

DEFINITION 5.1. A topological space (X, τ) is said to be
(1) $\Omega - T_{\frac{1}{2}}$ if every Ωs-closed set is semi-closed in (X, τ),
(2) $\Omega - T_s$ if every Ωs-closed set is closed in (X, τ),
(3) semi-$T_{\frac{1}{2}}$ [4] if every sg-closed set is semi-closed.

DEFINITION 5.2. A topological space (X, τ) is said to be semi-T_0 (resp. semi-T_1) [12] if for each pair of distinct points x, y of X, there exists a semi-open set U_x containing x but not y or (resp. and) a semi-open set U_y containing y but not x.

PROPOSITION 5.1. *Let* (X,τ) *be a topological space.*

(1) *For each* $x \in X, \{x\}$ *is semi-closed or its complement* $X-\{x\}$ *is* Ω*-closed in* (X,τ)*.*

(2) *For each* $x \in X, \{x\}$ *is open and semi-closed or its complement* $X-\{x\}$ *is* Ω*s-closed in* (X,τ)*.*

PROOF. (1) Suppose that $\{x\}$ is not semi-closed in (X,τ). Then $X-\{x\}$ is not semi-open and the only semi-open set containing $X-\{x\}$ is X. Therefore, $\mathrm{sCl}\left(X-\{x\}\right) \subseteqq \mathrm{Int}\,(X)=X$. So, $X-\{x\}$ is Ω-closed in (X,τ).

(2) Suppose that $\{x\}$ is not semi-closed in (X,τ). Then by (1), $X-\{x\}$ is Ω-closed in (X,τ) and then Ωs-closed. Suppose that $\{x\}$ is not open and let U be a semi-open set such that $X-\{x\} \subseteqq U$. If $U=X$, then

$$\mathrm{sCl}\left(X-\{x\}\right) \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right)=U.$$

If $U=X-\{x\}$, then we have that $\mathrm{Int}\left(\mathrm{Cl}\,(U)\right)=\mathrm{Int}\left(\mathrm{Cl}\left(X-\{x\}\right)\right)= \mathrm{Int}\,(X)=X$. Hence, $\mathrm{sCl}\left(X-\{x\}\right) \subseteqq \mathrm{Int}\left(\mathrm{Cl}\,(U)\right)$. Therefore, $X-\{x\}$ is Ωs-closed in (X,τ).

THEOREM 5.1. *For a topological space* (X,τ)*, the following are equivalent:*

(1) *Every* Ω*-closed set is semi-closed in* (X,τ)*;*

(2) *For each* $x \in X$*,* $\{x\}$ *is semi-open or semi-closed in* (X,τ)*;*

(3) (X,τ) *is semi-*$T_{\frac{1}{2}}$*.*

PROOF. (1) $\Longrightarrow$ (2). Suppose that for a point $x \in X$, $\{x\}$ is not semi-closed (X,τ). By Proposition 5.1 (1) $X-\{x\}$ is Ω-closed in (X,τ). By assumption, $X-\{x\}$ is semi-closed in (X,τ) and hence $\{x\}$ is semi-open. Therefore, each singleton is semi-open or semi-closed in (X,τ).

(2) $\Longrightarrow$ (1). Let A be an Ω-closed set in (X,τ). We want to prove that $\mathrm{sCl}\,(A)=A$. Suppose $x \in \mathrm{sCl}\,(A)$.

Case 1: $\{x\}$ is semi-open in (X,τ). Then $\{x\} \cap A \neq \phi$ which implies $x \in A$.

Case 2: $\{x\}$ is semi-closed in (X,τ) and $x \notin A$. Then $\mathrm{sCl}\,(A)-A$ contains a semi-closed set $\{x\}$ and this contradicts Proposition 3.3. Hence $x \in A$ and A is semi-closed. Therefore, Every Ω-closed set is semi-closed in (X,τ).

(2) $\Leftrightarrow$ (3). This is shown in [17, Theorem 4.8].

THEOREM 5.2. *For a topological space* (X,τ)*, the following properties hold:*

(1) *If* (X,τ) *is* $\Omega-T_s$*, then for each* $x \in X$ *the singleton* $\{x\}$ *is open or semi-closed;*

(2) (X,τ) *is* $\Omega - T_{\frac{1}{2}}$ *if and only if for each* $x \in X$, $\{x\}$ *is semi-open or semi-closed and open in* (X,τ);

(3) *If* (X,τ) *is* $\Omega - T_s$, *then it is* $\Omega - T_{\frac{1}{2}}$;

(4) *If* (X,τ) *is* $\Omega - T_{\frac{1}{2}}$, *then it is semi-*$T_{\frac{1}{2}}$.

PROOF. (1) Suppose that for some $x \in X$, $\{x\}$ is not semi-closed. By Proposition 5.1 $X - \{x\}$ is Ω-closed in (X,τ). Hence, $X - \{x\}$ is Ωs-closed in (X,τ). Since (X,τ) is $\Omega - T_s$, then $X - \{x\}$ is closed in (X,τ). Thus $\{x\}$ is open.

(2) *Necessity.* Suppose that a singleton $\{x\}$ is not semi-closed or open. By Proposition 5.1 $X - \{x\}$ is Ωs-closed in (X,τ). Using the assumption we have that $\{x\}$ is semi-open.

Sufficiency. It follows from the assumption that every subset is semi-open and semi-closed. Then (X,τ) is $\Omega - T_{\frac{1}{2}}$.

(3) It is straightforward from the definitions of $\Omega - T_s$ spaces and $\Omega - T_{\frac{1}{2}}$ spaces.

(4) It is obvious from (2) above and Theorem 5.1.

REMARK 5.1. We have the following implications. But a semi-$T_{\frac{1}{2}}$ space is not necessarily $\Omega - T_{\frac{1}{2}}$ as shown by Example 5.1 (below).

$$\begin{tikzcd}
\text{semi-}T_1 \arrow[r] & \text{semi-}T_{\frac{1}{2}} \arrow[r] & \text{semi-}T_0 \\
\Omega - T_s \arrow[r] & \Omega - T_{\frac{1}{2}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Fig. 3

EXAMPLE 5.1. Let $X = \{a,b,c\}$ and $\tau = \big\{X, \phi, \{a\}, \{a,b\}\big\}$. Then (X,τ) is a semi-$T_{\frac{1}{2}}$ space which is not $\Omega - T_{\frac{1}{2}}$ since $\{b\}$ is neither semi-open nor semi-closed.

THEOREM 5.3. *Let* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\upsilon)$ *be two functions.*

(1) $g \circ f$ *is* Ω*-continuous (resp.* Ω*s-continuous) if* g *is continuous and* f *is* Ω*-continuous (resp.* Ω*s-continuous).*

(2) $g \circ f$ *is* Ω*-irresolute (resp.* Ω*s-irresolute) if both* f *and* g *are* Ω*-irresolute (resp.* Ω*s-irresolute).*

(3) $g \circ f$ *is* Ω*-continuous (resp.* Ω*s-continuous) if* g *is* Ω*-continuous (resp.* Ω*s-continuous) and* f *is* Ω*-irresolute (resp.* Ω*s-irresolute).*

(4) *Let* (Y, σ) *be an* $\Omega - T_s$ *space. Then* $g \circ f$ *is continuous if* f *is continuous and* g *is* Ω*s-continuous.*

(5) *Let* f *be* Ω*s-continuous. Then* f *is continuous (resp. semi-continuous) if* (X, τ) *is* $\Omega - T_s$ *(resp.* $\Omega - T_{\frac{1}{2}}$*).*

PROOF. Obvious.

DEFINITION 5.3. A function $f : X \to Y$ is said to be semi-closed [15] (resp. pre-semi-closed [16]) if for each closed (resp. semi-closed) set F of X, $f(F)$ is semi-closed in Y.

THEOREM 5.4. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a function.*

(1) *Let* f *be an* Ω*s-irresolute and closed surjection. If* (X, τ) *is an* $\Omega - T_s$ *space, then* (Y, σ) *is also* $\Omega - T_s$*.*

(2) *Let* f *be an* Ω*s-irresolute and semi-closed surjection. If* (X, τ) *is an* $\Omega - T_s$ *space, then* (Y, σ) *is* $\Omega - T_{\frac{1}{2}}$*.*

(3) *Let* f *be an* Ω*s-irresolute and pre-semi-closed surjection. If* (X, τ) *is an* $\Omega - T_{\frac{1}{2}}$ *space, then* (Y, σ) *is* $\Omega - T_{\frac{1}{2}}$*.*

PROOF. Obvious.

6. Pasting lemma

Let $X = A \cup B$ and $f : A \to Y$ and $h : B \to Y$ be two functions. f and h are called compatible if $f(x) = h(x)$ for every $x \in A \cap B$. The combination $f \bigtriangledown h : (X, \tau) \to (Y, \sigma)$ is defined by $(f \bigtriangledown h)(x) = f(x)$ if $x \in A$ and $(f \bigtriangledown h)(x) = h(x)$ if $x \in B$.

THEOREM 6.1. *Suppose that* A *and* B *are both* Ω*-closed and open subsets of* (X, τ)*. Let* $f : (A, \tau_{/A}) \to (Y, \sigma)$ *and* $h : (B, \tau_{/B}) \to (Y, \sigma)$ *be compatible functions. Assume that* $D[E] \subsetneqq D_s[E]$ *for any* $E \subsetneqq X$*. If* f *and* h *are* Ω*-continuous (resp.* Ω*s-continuous), then the combination* $f \bigtriangledown h : (X, \tau) \to (Y, \sigma)$ *is also* Ω*-continuous (resp.* Ω*s-continuous).*

PROOF. Let F be a closed subset of (Y, σ). By definition $(f \bigtriangledown h)^{-1}(F) = f^{-1}(F) \cup h^{-1}(F)$. By the assumption $f^{-1}(F)$ is Ω-closed (Ωs-closed) in $(A, \tau_{/A})$ and $h^{-1}(F)$ is Ω-closed (Ωs-closed) in $(B, \tau_{/B})$. Since A and B are both Ω-closed and open subsets of (X, τ), by Theorem 3.3 (Theorem 3.4) $f^{-1}(F)$ and $h^{-1}(F)$ are both Ω-closed (resp. Ωs-closed) sets in (X, τ). Then by Proposition 3.1, $f^{-1}(F) \cup h^{-1}(F)$ is Ω-closed (Ωs-closed) in (X, τ). Hence, $f \bigtriangledown h$ is Ω-continuous (Ωs-continuous).

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