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Fuzzy γ-separation Axioms in Fuzzifying Topology

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Abstract:

In the present paper we introduce and study T_0^γ-, R_0^γ-, T_1^γ-, R_1^γ-, T_2^γ-, T_3^γ-, T_4^γ-strong T_3^γ- and strong T_4^γ-separation axioms in fuzzifying topology and give some of their characterizations as well as the relations of these axioms and other separation axioms in fuzzifying topology introduced by Shen, Fuzzy Sets and Systems, 57 (1993), 111-123.

Keywords and phrases:

Fuzzy logic, fuzzifying topology, Fuzzyifying γ-open sets, Fuzzifying separation axioms.

1. Introduction

In 1991, Ying [6] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from a completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. Andrijević [1] introduced the concepts of b-open sets and b-closed sets in general topology. In [2] Hanafy used the term γ-open sets instead of b-open sets and studied the concepts of γ-open sets and γ-continuity in fuzzy topology. In [4] the concepts of fuzzifying γ-open sets, fuzzifying γ-neighborhood structure of a point, fuzzifying γ-interior operation and fuzzifying γ-closure operation are introduced and studied. Shen [5] introduced T_0-, T_1-, T_2 (Hausdorff)-, T_3 (regularity)-, T_4 (normality)-separation axioms in fuzzifying topology. In the frame work of fuzzifying topology, the authors in [3] introduced the R_0-separation axioms and studied their relations with the T_1- and T_2-separation axioms, respectively. In the present paper, we introduce and study

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$T_0^\gamma-, R_0^\gamma-, T_1^\gamma-, R_1^\gamma-, T_2^\gamma$ (γ-Hausdorff)-, T_3^γ (γ-regularity)-, T_4^γ (γ-normality), strong T_3^γ-, strong T_4^γ-separation axioms in fuzzyifying topology. Also, we give some of their characterizations as well as the relations of these axioms and T_0-, R_0-, T_1-, R_1-, T_2 (Hausdorff)-, T_3 (regularity)-, T_4 (normality)-separation axioms in fuzzifying topology.

2. Preliminaries

First, we display the fuzzy logical and corresponding set theoretical notations [6-7] since we need them in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0,1]. We write $\models \varphi$ if $[\varphi]=1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) $[\alpha]=\alpha(\alpha\in[0,1])$; $[\varphi\wedge\psi]:=\min([\varphi],[\psi])$; $[\varphi\to\psi]:=\min(1,1-[\varphi]+[\psi])$.

(2) If $\tilde{A}\in\Im(X)$, where $\Im(X)$ is the family of all fuzzy sets of X, then $[x\in\tilde{A}]:=\tilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x\varphi(x)]:=\inf_{x\in X}[\varphi(x)]$. In addition the following derived formulae are given,

(1) $[\neg\varphi]:=[\varphi\to 0]:=1-[\varphi]$.

(2) $[\varphi\vee\psi]:=[\neg(\neg\varphi\wedge\neg\psi)]:=\max([\varphi],[\psi])$.

(3) $[\varphi\leftrightarrow\psi]:=[\varphi\to\psi]\wedge[\psi\to\varphi]$.

(4) $[\varphi\dot\wedge\psi]:=[\neg(\varphi\to\neg\psi)]:=\max(0,[\varphi]+[\psi]-1)$.

(5) $[\varphi\dot\vee\psi]:=[\neg(\neg\varphi\dot\wedge\neg\psi)]:=[\neg\varphi\to\psi]:=\min(1,[\varphi]+[\psi])$.

(6) $[\exists x\varphi(x)]:=[\neg\forall x\neg\varphi(x)]:=\sup_{x\in X}[\varphi(x)]$.

(7) If $\tilde{A},\tilde{B}\in\Im(X)$, then

$$[\tilde{A}\subseteq\tilde{B}]:=[\forall x(x\in\tilde{A}\to x\in\tilde{B})]:=\inf_{x\in X}\min(1,1-\tilde{A}(x)+\tilde{B}(x));$$

Second, we give some definitions and results in fuzzifying topology.

Definition 2.1 [6]. Let X be a universe of discourse, and let $\tau\in\Im(P(X))$, where $P(X)$ is the power set of X satisfying the following conditions:

(1) $\models X\in\tau$;

(2) for any $A,B\in P(X)$, $\models(A\in\tau)\wedge(B\in\tau)\to(A\cap B)\in\tau$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X)$, $\models \forall\lambda(\lambda \in \Lambda \to A_\lambda \in \tau) \to \bigcup_{\lambda\in\Lambda} A_\lambda \in \tau$.

Then τ is called a fuzzifying topology and (X,τ) is a fuzzifying topological space.

The family of all fuzzifying closed sets will be denoted by F_τ, or if there is no confusion by F, and defined as follows: $A \in F := (X - A) \in \tau$, where $X - A$ is the complement of A.

Definition 2.2 [6]. Let (X,τ) be a fuzzifying topological space.

(1) The fuzzifying neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows:

$$N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$$

(2) The interior of a set $A \in P(X)$ is denoted by $A^\circ \in \Im(X)$ and defined as follows: $A^\circ(x) = N_x(A)$.

(3) The closure of a set $A \in P(X)$ is denoted by $\overline{A} \in \Im(X)$ and defined as follows:

$$\overline{A}(x) = 1 - N_x(X - A).$$

(4) $\beta \in \Im(P(X))$ is a base of τ iff $\tau = \beta^{(\cup)}$ (Theorem 4.1 [6]), i.e.,

$$\tau(A) = \sup_{\bigcup_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \beta(B_\lambda).$$

(5) $\varphi \in \Im(P(X))$ is a subbase of τ if $\varphi^{(\cap)}$ is a base of τ, i.e.,

$$\tau(A) = \sup_{\bigcup_{\lambda\in\Lambda} D_\lambda = A} \inf_{\lambda\in\Lambda} \sup_{\bigcap_{\lambda\in I_\lambda} D_\lambda = D_\lambda} \inf_{\lambda\in I_\lambda} \varphi(D_\lambda).$$

Definition 2.3 [4]. (1) The family of fuzzifying γ-open sets, denoted by $\tau_\gamma \in (P(X))$, is defined as follows:

$$A \in \tau_\gamma := \forall x(x \in A \to x \in A^{-\circ} \cup A^{\circ-}),\ \text{i.e.,}\ \tau_\gamma(A) = \inf_{x\in A} \max(A^{-\circ}(x), A^{\circ-}(x)).$$

(2) The family of fuzzifying γ-closed sets, denoted by $F_\gamma \in \Im(P(X))$, is defined as follows:

$$A \in F_\gamma := (X - A) \in \tau_\gamma.$$

(3) Let $x \in X$. The fuzzifying γ-neighborhood system of x, denoted by $N_x^\gamma \in \Im(P(X))$, is defined as follows:

$$A \in N_x^\gamma := \exists B(x \in B \subseteq A \rightarrow B \in \tau_\gamma).$$

(4) The fuzzifying γ-closure of A is denoted and defined as follows:

$$Cl_\gamma(A)(x) = 1 - N_x^\gamma(X - A).$$

Theorem 2.1 [4]. *Let (X, τ) be a fuzzifying topological space. Then, we have* (1) $\models \tau \subset \tau_\gamma$; (2) $\models F \subseteq F_\gamma$.

Theorem 2.2 [4]. *The mapping $N^\gamma : X \rightarrow \Im^N(P(X))$, $x \mapsto N_x^\gamma$, where $\Im^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$, has the following properties:*

(1) $\models A \in N_x^\gamma \rightarrow x \in A$;

(2) $\models A \subseteq B \rightarrow (A \in N_x^\gamma \rightarrow B \in N_x^\gamma)$;

(3) $\models A \in N_x^\gamma \rightarrow \exists H(H \in N_x^\gamma \wedge H \subseteq A \wedge \forall y(y \in H \rightarrow H \in N_x^\gamma))$.

Theorem 2.3 [4].

$$\tau_\gamma(A) = \inf_{x \in A} N_x^\gamma(A).$$

Remark 2.1. For simplicity we use the following notations:

$$K(x,y) := \exists A((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A));$$

$$H(x,y) := \exists B \exists C((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C));$$

$$M(x,y) := \exists B \exists C(B \in N_x \wedge C \in N_y \wedge B \cap C = \phi);$$

$$V(x,D) := \exists A \exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B = \phi);$$

$$W(A,B) := \exists G \exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \phi).$$

Definition 2.4 [5]. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i \in \Im(\Omega)$, $i = 1, \cdots, 4$, and $R_i \in \Im(X)$, $i = 0,1$ are defined as follows, respectively

$$(X, \tau) \in T_0 := \forall x \forall y((x \in X \wedge y \in X \wedge x \neq y) \rightarrow K(x,y));$$

$$(X,\tau)\in T_1 := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to H(x,y)\big);$$

$$(X,\tau)\in T_2 := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to M(x,y)\big);$$

$$(X,\tau)\in T_3 := \forall x\forall D\big((x\in X\wedge D\in F\wedge x\notin D)\to V(x,D)\big);$$

$$(X,\tau)\in T_4 := \forall A\forall B\big((A\in F\wedge B\in F\wedge A\cap B=\phi)\to W(A,B)\big).$$

$$(X,\tau)\in R_0 := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to (K(x,y)\to H(x,y))\big);$$

$$(X,\tau)\in R_1 := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to (K(x,y)\to M(x,y))\big).$$

3. Fuzzifying γ-separation axioms

Remark 3.1. For simplicity we use the following notations:

$$K_\gamma(x,y) := \exists A\big((A\in N_x^\gamma\wedge y\notin A)\vee(A\in N_y^\gamma\wedge x\notin A)\big);$$

$$H_\gamma(x,y) := \exists B\exists C\big((B\in N_x^\gamma\wedge y\notin B)\wedge(C\in N_y^\gamma\wedge x\notin C)\big);$$

$$M_\gamma(x,y) := \exists B\exists C\big(B\in N_x^\gamma\wedge C\in N_y^\gamma\wedge B\cap C=\phi\big);$$

$$V_\gamma(x,D) := \exists A\exists B\big(A\in N_x^\gamma\wedge B\in \tau_\gamma\wedge D\subseteq B\wedge A\cap B=\phi\big);$$

$$W_\gamma(A,B) := \exists G\exists H\big(G\in\tau_\gamma\wedge H\in\tau_\gamma\wedge A\subseteq G\wedge B\subseteq H\wedge G\cap H=\phi\big).$$

Definition 3.1. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $\gamma-T_i$ (T_i^γ for short) $\in\Im(\Omega)$, $i=0,\cdots,4,$, γ-strong - T_i^γ ($T_i^{\gamma^s}$ for short $\in\Im(\Omega)$, $i=3,4$ and $\gamma-R_i$ (R_i^γ for short) $\in\Im(\Omega)$, $i=0,1$, are defined as follows, respectively

$$(X,\tau)\in T_0^\gamma := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to K_\gamma(x,y)\big);$$

$$(X,\tau)\in T_1^\gamma := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to H_\gamma(x,y)\big);$$

$$(X,\tau)\in T_2^\gamma := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to M_\gamma(x,y)\big);$$

$$(X,\tau)\in T_3^\gamma := \forall x\forall D\big((x\in X\wedge D\in F\wedge x\notin D)\to V_\gamma(x,D)\big);$$

$$(X,\tau)\in T_4^\gamma := \forall A\forall B\big((A\in F\wedge B\in F\wedge A\cap B=\phi)\to W_\gamma(A,B)\big).$$

$$(X,\tau)\in T_3^{\gamma^s} := \forall x\forall D\big((x\in X\wedge D\in F_\gamma\wedge x\notin D)\to V(x,D)\big);$$

$$(X,\tau)\in T_4^{\gamma^S} := \forall A\forall B\big((A\in F_\gamma \wedge B\in F_\gamma \wedge A\cap B=\phi)\to W(A,B)\big).$$

$$(X,\tau)\in R_0^{\gamma} := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to (K_\gamma(x,y)\to H_\gamma(x,y))\big);$$

$$(X,\tau)\in R_1^{\gamma} := \forall x\forall y\big((x\in X\wedge y\in X\wedge x\neq y)\to (K_\gamma(x,y)\to M_\gamma(x,y))\big).$$

Lemma 3.1. *For any fuzzifying topological space* (X,τ)

(1) $\models K(x,y)\to K_\gamma(x,y)$;

(2) $\models H(x,y)\to H_\gamma(x,y)$;

(3) $\models M(x,y)\to M_\gamma(x,y)$;

(4) $\models V(x,D)\to V_\gamma(x,D)$;

(5) $\models W(A,B)\to W_\gamma(A,B)$.

Proof. From Theorem 2.1 (1), $\models \tau\subseteq\tau_\gamma$ and so one can deduce that $N_x(A)\le N_x^\gamma(A)$ for any $A\in P(X)$, the proof is immediate.

Theorem 3.2. *For any fuzzifying topological space* (X,τ)

(1) $\models (X,\tau)\in T_i \to (X,\tau)\in T_i^\gamma$, *where* $i=0,\cdots,4$.

(2) $\models (X,\tau)\in T_i^{\gamma^S} \to (X,\tau)\in T_i$, *where* $i=3,4$.

(3) $\models (X,\tau)\in T_i^{\gamma^S} \to (X,\tau)\in T_i^\gamma$, *where* $i=3,4$.

Proof. (1) It is obtained from Lemma 3.1.

(2) It follows from Theorem 2.1 (2).

(3) It follows from (1) and (2).

Lemma 3.3. *For any fuzifying topological space* (X,τ)

(1) $\models M_\gamma(x,y)\to H_\gamma(x,y)$;

(2) $\models H_\gamma(x,y)\to K_\gamma(x,y)$;

(3) $\models M_\gamma(x,y)\to K_\gamma(x,y)$;

Proof. (1) If $N_x^\gamma(B)=0$ or $N_y^\gamma(C)=0$, then the result holds. Suppose that $N_x^\gamma(B)>0$ and $N_y^\gamma(C)>0$. By Theorem 2.2 (1) we have $[x\in B]=1$ and $[y\in C]=1$. So, $\{B,C\in P(X): B\cap C=\phi\}\subseteq\{B,C\in P(X): y\notin B\wedge x\notin C\}$. Thus

$$[M_\gamma(x,y)] = \sup_{B\cap C=\phi}\min\big(N_x^\gamma(B),N_y^\gamma(C)\big)\le \sup_{y\notin B,\, x\notin C}\min\big(N_x^\gamma(B),N_y^\gamma(C)\big)$$

$$=\left[H_\gamma(x,y)\right].$$

(2) We have that

$$\left[K_\gamma(x,y)\right]=\max\left(\sup_{y\notin A}N_x^\gamma(A),\sup_{y\notin A}N_y^\gamma(A)\right)\geq\sup_{y\notin A}N_x^\gamma(A)$$

$$\geq\sup_{y\notin A,x\notin B}\left(N_x^\gamma(A)\wedge N_y^\gamma(B)\right)=\left[H_\gamma(x,y)\right].$$

(3) It is obtained from (1) and (2).

Theorem 3.4. *For any fuzzifying topological space* (X,τ)

(1) $\models(X,\tau)\in T_1^\gamma\to(X,\tau)\in T_0^\gamma$;

(2) $\models(X,\tau)\in T_2^\gamma\to(X,\tau)\in T_1^\gamma$.

Proof. The proof of (1) and (2) are obtained from Lemma 3.3 (2) and (1), respectively.

Corollary 3.5. *For any fuzzifying topological space* (X,τ)

$$\models(X,\tau)\in T_2^\gamma\to(X,\tau)\in T_0^\gamma.$$

Proof. From Theorem 3.4 the proof is immediate.

Theorem 3.6. *For any fuzzifying topological space* (X,τ)

$$\models(X,\tau)\in T_0^\gamma\leftrightarrow\Big(\forall x\forall y\big(x\in X\wedge y\in X\wedge x\neq y\to\big(\neg\left(x\in Cl_\gamma(\{y\})\right)$$
$$\vee\neg\left(y\in Cl_\gamma(\{x\})\right)\big)\big)\Big)$$

Proof. Applying Theorem 2.2 (2) we have

$$\left[(X,\tau)\in T_0^\gamma\right]=\inf_{x\neq y}\max\left(\sup_{y\notin A}N_x^\gamma(A),\sup_{x\notin A}N_y^\gamma(A)\right)$$

$$=\inf_{x\neq y}\max\left(N_x^\gamma(X-\{y\}),N_y^\gamma(X-\{x\})\right)$$

$$=\inf_{x\neq y}\max\left(1-Cl_\gamma(\{y\})(x),1-Cl_\gamma(\{x\})(y)\right)$$

$$=\inf_{x\neq y}\left(\neg Cl_\gamma(\{y\})(x)\vee\neg Cl_\gamma(\{x\})(y)\right)$$

$$\left[\forall x\forall y\left(x\in X\wedge y\in X\wedge x\neq y\to\left(\neg\left(x\in Cl_\gamma(\{y\})\right)\right)\vee\neg\left(y\in Cl_\gamma(\{x\})\right)\right)\right].$$

Theorem 3.7. *Let (X,τ) be a fuzzifying topological space. Then*

$$\models (X,\tau)\in T_1^{\gamma} \leftrightarrow \forall x(\{x\}\in F_{\gamma})$$

Proof. For any $x_1,x_2,x_1\neq x_2$, we have from Theorems 2.3 and 2.2 (2) that

$$[\forall x(\{x\}\in F_{\gamma})]=\inf_{x\in X}F_{\gamma}(\{x\})=\inf_{x\in X}\tau_{\gamma}(X-\{x\})=\inf_{x\in X}\inf_{y\in X-\{x\}}N_y^{\gamma}(X-\{x\})$$

$$\leq \inf_{y\in X-\{x_2\}}N_y^{\gamma}(X-\{x_2\})\leq N_{x_1}^{\gamma}(X-\{x_2\})=\sup_{x_2\notin A}N_{x_1}^{\gamma}(A).$$

According to the same reason we can prove that

$$[\forall x(\{x\}\in F_{\gamma})]\leq \sup_{x_1\notin B}N_{x_2}^{\gamma}(A).$$

Therefore

$$[\forall x(\{x\}\in F_{\gamma})]\leq \inf_{x_1\neq x_2}\min\left(\sup_{x_2\notin A}N_{x_1}^{\gamma}(A),\sup_{x_1\notin B}N_{x_2}^{\gamma}(B)\right)$$

$$=\inf_{x_1\neq x_2}\sup_{x_2\notin A,x_1\notin B}\min(N_{x_1}^{\gamma}(A),N_{x_2}^{\gamma}(B))=[(X,\tau)\in T_1^{\gamma}].$$

On the other hand

$$[(X,\tau)\in T_1^{\gamma}]=\inf_{x_1\neq x_2}\min\left(\sup_{x_2\notin A}N_{x_1}^{\gamma}(A),\sup_{x_1\notin B}N_{x_2}^{\gamma}(B)\right)$$

$$=\inf_{x_1\neq x_2}\min(N_{x_1}^{\gamma}(X-\{x_2\}),N_{x_2}^{\gamma}(X-\{x_1\}))$$

$$\leq \inf_{x_1\neq x_2}N_{x_1}^{\gamma}(X-\{x_2\})=\inf_{x_2\in X}\inf_{x_1\in X-\{x_2\}}N_{x_1}^{\gamma}(X-\{x_2\})$$

$$=\inf_{x_2\in X}\tau_{\gamma}(X-\{x_2\})=\inf_{x\in X}\tau_{\gamma}(X-\{x\})=[\forall x(\{x\}\in F_{\gamma})].$$

Thus $[(X,\tau)\in T_1^{\gamma}]=[\forall x(\{x\}\in F_{\gamma})]$.

Definition 3.2. The fuzzifying γ-local base $\gamma\beta_x$ of x is a function from $P(X)$ into I such that the following conditions are satisfied:

(1) $\models \gamma\beta_x\subseteq N_x^{\gamma}$;

(2) $\models A\in N_x^{\gamma}\rightarrow \exists B(B\in\gamma\beta_x\wedge x\in B\subseteq A)$.

Lemma 3.8. $\models A\in N_x^{\gamma}\leftrightarrow \exists B(B\in\gamma\beta_x\wedge x\in B\subseteq A)$.

Proof. From the condition (1) in Definition 3.2 and Theorem 2.2 (2) then $N_x^\gamma(A) \geq N_x^\gamma(B) \geq \gamma\beta_x(B)$ for each $B \subseteq X$ such that $x \in B \subseteq A$. So, $N_x^\gamma(A) \geq \sup_{x \in B \subseteq A} \gamma\beta_x(B)$. From condition (2) in Definition 2.3, $N_x^\gamma(A) \leq \sup_{x \in B \subseteq A} \gamma\beta_x(B)$. Hence, $N_x^\gamma(A) = \sup_{x \in B \subseteq A} \gamma\beta_x(B)$.

Theorem 3.9. *If $\gamma\beta_x$ is a fuzzifying γ-local basis of x, then*

$$\models (X,\tau) \in T_2^\gamma \leftrightarrow \forall x \forall y \big((x \in X \wedge y \in X \wedge x \neq y) \to \big(\exists B \big(B \in \gamma\beta_x \wedge y \notin Cl_\gamma(B)\big)\big)\big).$$

Proof. Applying Lemma 3.8 we have

$$\begin{aligned}
&\Big[\forall x \forall y \big((x \in X \wedge y \in X \wedge x \neq y) \to \big(\exists B \big(B \in \gamma\beta_x \wedge y \notin Cl_\gamma(B)\big)\big)\big)\Big] \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \min\big(\gamma\beta_x(B), N_y^\gamma(X-B)\big) \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \min\left(\gamma\beta_x(B), \sup_{y \in C \subseteq (X-B)} \gamma\beta_y(C)\right) \\
&= \inf_{x \neq y} \sup_{B \subseteq X} \sup_{y \in C \subseteq (X-B)} \min\big(\gamma\beta_x(B), \gamma\beta_y(C)\big) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \sup_{x \in D \subseteq B, y \in E \subseteq C} \min\big(\gamma\beta_x(D), \gamma\beta_y(E)\big) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \min\left(\sup_{x \in D \subseteq B} \gamma\beta_x(D), \sup_{y \in E \subseteq C} \gamma\beta_y(E)\right) \\
&= \inf_{x \neq y} \sup_{B \cap C = \phi} \min\big(N_x^\gamma(B), N_y^\gamma(C)\big) \\
&= \big[(X,\tau) \in T_2^\gamma\big].
\end{aligned}$$

Theorem 3.10. *Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models (X,\tau) \in R_1^\gamma \to (X,\tau) \in R_0^\gamma$.

(2) If $T_0(X,\tau) = 1$, *then*

(a) $\models (X,\tau) \in R_0 \to (X,\tau) \in R_0^\gamma$.

(b) $\models (X,\tau) \in R_1 \to (X,\tau) \in R_1^\gamma$.

Proof. (1) From Lemma 3.3 (1) we have

$$\begin{aligned}
\big[(X,\tau) \in R_0^\gamma\big] &= \inf_{x \neq y} \min\big(1, 1 - K_\gamma(x,y) + H_\gamma(x,y)\big) \\
&\geq \inf_{x \neq y} \min\big(1, 1 - K_\gamma(x,y) + M_\gamma(x,y)\big) = \big[(X,\tau) \in R_1^\gamma\big].
\end{aligned}$$

(2) Since $T_0(X,\tau)=1$, then for each $x,y\in X$ and $x\neq y$ we have, $K(x,y)=1$ and so, $K_\gamma(x,y)=1$.

(a) Applying Lemma 3.1 (2) we have

$$\begin{aligned}[(X,\tau)\in R_0] &= \inf_{x\neq y}\min(1,1-K(x,y)+H(x,y)) \\ &\geq \inf_{x\neq y}\min(1,1-K(x,y)+H_\gamma(x,y)) \\ &= \inf_{x\neq y}\min(1,1-K_\gamma(x,y)+H_\gamma(x,y)) \\ &= [(X,\tau)\in R_0^\gamma].\end{aligned}$$

(b) Applying Lemma 3.1 (3) we have

$$\begin{aligned}[(X,\tau)\in R_1] &= \inf_{x\neq y}\min(1,1-K(x,y)+M(x,y)) \\ &\geq \inf_{x\neq y}\min(1,1-K(x,y)+M_\gamma(x,y)) \\ &= \inf_{x\neq y}\min(1,1-K_\gamma(x,y)+M_\gamma(x,y)) \\ &= [(X,\tau)\in R_1^\gamma].\end{aligned}$$

Theorem 3.11. *Let (X,τ) be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_1^\gamma \rightarrow (X,\tau)\in R_0^\gamma$

(2) $\models (X,\tau)\in T_1^\gamma \rightarrow ((X,\tau)\in R_0^\gamma \wedge (X,\tau)\in T_0^\gamma)$.

(3) If $T_0^\gamma(X,\tau)=1$, *then* $\models (X,\tau)\in T_1^\gamma \leftrightarrow ((X,\tau)\in R_0^\gamma \wedge (X,\tau)\in T_0^\gamma)$.

Proof. (1) By some calculations we have

$$T_1^\gamma(X,\tau)=\inf_{x\neq y}[H_\gamma(x,y)]\leq \inf_{x\neq y}\min(1,1-[K_\gamma(x,y)]+[H_\gamma(x,y)])=R_0^\gamma(X,\tau).$$

(2) It is obtained from (1) and from Theorem 3.4 (1).

(3) Since $T_0^\gamma(X,\tau)=1$, then for every $x,y\in X$ such that $x\neq y$ we have $[K_\gamma(x,y)]=1$. Therefore,

$$\begin{aligned}[(X,\tau)\in R_0^\gamma \wedge (X,\tau)T_0^\gamma] = [(X,\tau)\in R_0^\gamma] &= \inf_{x\neq y}\min(1,1-[K_\gamma(x,y)]+[H_\gamma(x,y)]) \\ &= \inf_{x\neq y}[H_\gamma(x,y)]=[(X,\tau)\in T_1^\gamma].\end{aligned}$$

Theorem 3.12. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models ((X,\tau)\in R_0^\gamma \mathbin{\dot\wedge} (X,\tau)\in T_0^\gamma) \to (X,\tau)\in T_1^\gamma$.

(2) If $T_0^\gamma(X,\tau)=1$, *then* $\models ((X,\tau)\in R_0^\gamma \mathbin{\dot\wedge} (X,\tau)\in T_0^\gamma) \leftrightarrow (X,\tau)\in T_1^\gamma$.

Proof. (1)

$$\begin{aligned}
\left[(X,\tau)\in R_0^\gamma \mathbin{\dot\wedge} (X,\tau)\in T_0^\gamma\right] &= \max\left(0, R_0^\gamma(X,\tau)+T_0^\gamma(X,\tau)-1\right)\\
&= \max\left(0, \inf_{x\neq y}\min\left(1,1-\left[K_\gamma(x,y)\right]+\left[H_\gamma(x,y)\right]\right)+\inf_{x\neq y}\left[K_\gamma(x,y)\right]-1\right)\\
&\le \max\left(0, \inf_{x\neq y}\left(\min\left(1,1-\left[K_\gamma(x,y)\right]+\left[H_\gamma(x,y)\right]\right)+\left[K_\gamma(x,y)\right]-1\right)\right)\\
&= \inf_{x\neq y}\left[H_\gamma(x,y)\right]=\left[(X,\tau)\in T_1^\gamma\right].
\end{aligned}$$

(2)

$$\begin{aligned}
\left[(X,\tau)\in R_0^\gamma \mathbin{\dot\wedge} (X,\tau)\in T_0^\gamma\right] = \left[R_0^\gamma(X,\tau)\right] &= \inf_{x\neq y}\min\left(1,1-\left[K_\gamma(x,y)\right]+\left[H_\gamma(x,y)\right]\right)\\
&= \inf_{x\neq y}\left[H_\gamma(x,y)\right]=\left[(X,\tau)\in T_1^\gamma\right],
\end{aligned}$$

because $T_0^\gamma(X,\tau)=1$, we have for each $x,y\in X$ such that $x\neq y$ we have $\left[K_\gamma(x,y)\right]$ $=1$.

Theorem 3.13. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_0^\gamma \to (X,\tau)\in T_1^\gamma)$;

(2) $\models (X,\tau)\in R_0^\gamma \to ((X,\tau)\in T_0^\gamma \to (X,\tau)\in T_1^\gamma)$.

Proof. (1) From Theorems 3.11 (1) and 3.12 (1) we have

$$\begin{aligned}
&\left[(X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_0^\gamma \to (X,\tau)\in T_1^\gamma)\right]\\
&= \min\left(1,1-\left[(X,\tau)\in T_0^\gamma\right]+\min\left(1,1-\left[(X,\tau)\in R_0^\gamma\right]+\left[(X,\tau)\in T_1^\gamma\right]\right)\right)\\
&= \min\left(1,1-\left[(X,\tau)\in T_0^\gamma\right]+1-\left[(X,\tau)\in R_0^\gamma\right]+\left[(X,\tau)\in T_1^\gamma\right]\right)\\
&= \min\left(1,1-\left(\left[(X,\tau)\in T_0^\gamma\right]+\left[(X,\tau)\in R_0^\gamma\right]-1\right)+\left[(X,\tau)\in T_1^\gamma\right]\right)=1.
\end{aligned}$$

(2) From Theorems 3.4 (1) and 3.12 (1) the proof is similar to (1).

Theorem 3.14. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_2^\gamma \to (X,\tau)\in R_1^\gamma$.

(2) $\models (X,\tau)\in T_2^\gamma \to ((X,\tau)\in R_1^\gamma \wedge (X,\tau)\in T_0^\gamma)$.

(3) *If* $T_0^\gamma(X,\tau)=1$, *then* $\models (X,\tau)\in T_2^\gamma \leftrightarrow ((X,\tau)\in R_1^\gamma \wedge (X,\tau)\in T_0^\gamma)$.

Proof. (1) We have

$$T_2^\gamma(X,\tau)=\inf_{x\neq y}[M_\gamma(x,y)]\le \inf_{x\neq y}[K_\gamma(x,y)\to M_\gamma(x,y)]=R_1^\gamma(X,\tau).$$

(2) It is obtained from (1) and from Corollary 3.5.

(3) Since $T_0^\gamma(X,\tau)=1$, then for each $x,y\in X$ such that $x\neq y$ we have $[K_\gamma(x,y)]=1$. Therefore,

$$T_2^\gamma(X,\tau)=\inf_{x\neq y}[M_\gamma(x,y)]=\inf_{x\neq y}[K_\gamma(x,y)\to M_\gamma(x,y)]=R_1^\gamma(X,\tau)$$

$$=R_1^\gamma(X,\tau)\wedge T_0^\gamma(X,\tau).$$

Theorem 3.15. Let (X,τ) be a fuzzifying topological space. Then

(1) $\models ((X,\tau)\in R_1^\gamma \dot\wedge (X,\tau)\in T_0^\gamma)\to (X,\tau)\in T_2^\gamma$;

(2) If $T_0^\gamma(X,\tau)=1$, then $\models ((X,\tau)\in R_1^\gamma \dot\wedge (X,\tau)\in T_0^\gamma)\leftrightarrow (X,\tau)\in T_2^\gamma$.

Proof. (1) By some calculations we have

$$[(X,\tau)\in R_1^\gamma \dot\wedge (X,\tau)\in T_0^\gamma]=\max(0,R_1^\gamma(X,\tau)+T_0^\gamma(X,\tau)-1)$$

$$=\max\left(0,\inf_{x\neq y}\min(1,1-[K_\gamma(x,y)]+[M_\gamma(x,y)])+\inf_{x\neq y}[K_\gamma(x,y)]-1\right)$$

$$\le \max\left(0,\inf_{x\neq y}\left(\min(1,1-[K_\gamma(x,y)]+[M_\gamma(x,y)])+[K_\gamma(x,y)]-1\right)\right.$$

$$=\inf_{x\neq y}[M_\gamma(x,y)]=T_2^\gamma(X,\tau).$$

(2) Since $T_0^\gamma(X,\tau)=1$, then for each $x,y\in X$ such that $x\neq y$ we have $[K_\gamma(x,y)]=1$. Therefore,

$$[(X,\tau)\in R_1^\gamma \dot\wedge (X,\tau)\in T_0^\gamma]=[(X,\tau)\in R_1^\gamma]=\inf_{x\neq y}\min(1,1-[K_\gamma(x,y)]+[M_\gamma(x,y)])$$

$$=\inf_{x\neq y}[M_\gamma(x,y)]=T_2^\gamma(X,\tau).$$

Theorem 3.16. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_1^\gamma \to (X,\tau)\in T_2^\gamma)$;

(2) $\models (X,\tau)\in R_1^\gamma \to ((X,\tau)\in T_0^\gamma \to (X,\tau)\in T_2^\gamma)$.

Proof. (1) From Theorems 3.14 (1) and 3.15 (1) we have

$$[(X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_1^\gamma \to (X,\tau)\in T_2^\gamma)]$$

$$= \min\left(1, 1 - T_0^\gamma(X,\tau) + \min\left(1, 1 - R_1^\gamma(X,\tau) + T_2^\gamma(X,\tau)\right)\right)$$

$$= \min\left(1, 1 - [(X,\tau)\in T_0^\gamma] + 1 - [(X,\tau)\in R_1^\gamma] + [(X,\tau)\in T_2^\gamma]\right)$$

$$= \min\left(1, 1 - ([(X,\tau)\in T_0^\gamma] + [(X,\tau)\in R_1^\gamma] - 1) + [(X,\tau)\in T_2^\gamma]\right) = 1.$$

(2) From Corollary 3.5 and Theorem 3.15 (1) the proof is similar to (1).

Theorem 3.17. *Let* (X,τ) *be a fuzzifying topological. If* $[(X,\tau)\in T_0^\gamma] = 1$, *then*

1. $\models ((X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_0^\gamma \to (X,\tau)\in T_1^\gamma)) \wedge ((X,\tau)\in T_1^\gamma \to \neg((X,\tau)\in T_0^\gamma \to \neg((X,\tau)\in R_0^\gamma)))$;

2. $\models ((X,\tau)\in R_0^\gamma \to ((X,\tau)\in T_0^\gamma \to (X,\tau)\in T_1^\gamma)) \wedge ((X,\tau)\in T_1^\gamma \to \neg((X,\tau)\in T_0^\gamma \to \neg((X,\tau)\in R_0^\gamma)))$;

3. $\models ((X,\tau)\in T_0^\gamma \to ((X,\tau)\in R_0^\gamma \to (X,\tau)\in T_1^\gamma)) \wedge ((X,\tau)\in T_1^\gamma \to \neg((X,\tau)\in R_0^\gamma \to \neg((X,\tau)\in T_0^\gamma)))$;

4. $\models ((X,\tau)\in R_0^\gamma \to ((X,\tau)\in T_0^\gamma \to (X,\tau)\in T_1^\gamma)) \wedge ((X,\tau)\in T_1^\gamma \to \neg((X,\tau)\in R_0^\gamma \to \neg((X,\tau)\in T_0^\gamma)))$;

Proof. For simplicity we put $[(X,\tau)\in T_0^\gamma] = \eta, [(X,\tau)\in R_0^\gamma] = \zeta$ and $[(X,\tau)\in T_1^\gamma] = \xi$. Now, applying Theorem 3.12 (2), the proof is obtained with some relations in fuzzy logic as follows.

(1) $(\eta \to (\zeta \to \xi)) \wedge (\xi \to \neg(\eta \to \neg\zeta)) = (\eta \to \neg(\zeta \wedge \neg\xi)) \wedge (\xi \to \neg(\eta \to \neg\zeta))$

$$= \neg(\eta \wedge \neg(\neg(\zeta \wedge \neg\xi))) \wedge \neg(\xi \wedge (\eta \to \neg\zeta))$$

$$= \neg(\eta \wedge \zeta \wedge \neg\xi) \wedge \neg(\xi \wedge \neg(\eta \wedge \zeta))$$

$$=(\eta\wedge\zeta\to\xi)\wedge(\xi\to\eta\wedge\zeta)=\eta\wedge\zeta\leftrightarrow\xi=1$$

Since $\wedge$ is commutative one can have the proof of statements (2)-(4) in a similar way as (1).

Theorem 3.18. *Let* (X,τ) *be a fuzzifying topological space. If* $[(X,\tau)\in T_0^\gamma]=1$*, then*

1. $\vDash\big((X,\tau)\in T_0^\gamma\to((X,\tau)\in R_1^\gamma\to(X,\tau)\in T_2^\gamma)\big)$
$\wedge\big((X,\tau)\in T_2^\gamma\to\neg((X,\tau)\in T_0^\gamma\to\neg((X,\tau)\in R_1^\gamma))\big)$

2. $\vDash\big((X,\tau)\in R_1^\gamma\to((X,\tau)\in T_0^\gamma\to(X,\tau)\in T_2^\gamma)\big)$
$\wedge\big((X,\tau)\in T_2^\gamma\to\neg((X,\tau)\in T_0^\gamma\to\neg((X,\tau)\in R_1^\gamma))\big)$;

3. $\vDash\big((X,\tau)\in T_0^\gamma\to((X,\tau)\in R_1^\gamma\to(X,\tau)\in T_2^\gamma)\big)$
$\wedge\big((X,\tau)\in T_2^\gamma\to\neg((X,\tau)\in R_1^\gamma\to\neg((X,\tau)\in T_0^\gamma))\big)$;

4. $\vDash\big((X,\tau)\in R_1^\gamma\to((X,\tau)\in T_0^\gamma\to(X,\tau)\in T_2^\gamma)\big)$
$\wedge\big((X,\tau)\in T_2^\gamma\to\neg((X,\tau)\in R_1^\gamma\to\neg((X,\tau)\in T_0^\gamma))\big)$;

Proof. The proof is similar to that of Theorem 3.17.

Lemma 3.19. (1) *If* $D\subseteq B$*, then*

$$\sup_{A\cap B=\phi}N_x^\gamma(A)=\sup_{A\cap B=\phi,D\subseteq B}N_x^\gamma(A)$$

(2) $\sup_{A\cap B=\phi}\inf_{y\in D}N_y^\gamma(X-A)=\sup_{A\cap B=\phi,D\subseteq B}\tau_\gamma(B)$.

Proof. (1) Since $D\subseteq B$, then we have

$$\sup_{A\cap B=\phi}N_x^\gamma(A)=\sup_{A\cap B=\phi}N_x^\gamma(A)\wedge[D\subseteq B]=\sup_{A\cap B=\phi,D\subseteq B}N_x^\gamma(A).$$

(2) Let $y\in D$ and $A\cap B=\phi$. Then

$$\sup_{A\cap B=\phi,D\subseteq B}\tau_\gamma(B)=\sup_{A\cap B=\phi,D\subseteq B}\tau_\gamma(B)\wedge[y\in D]=\sup_{y\in D\subseteq B\subseteq X-A}\tau_\gamma(B)$$

$$=\sup_{y\in B\subseteq X-A}\tau_\gamma(B)=N_y^\gamma(X-A)$$

$$=\inf_{y\in D} N_y^\gamma(X-A)=\sup_{A\cap B=\phi}\inf_{y\in D} N_y^\gamma(X-A).$$

Definition 3.3. Let (X,τ) be a fuzzifying opological space.

$$\gamma T_3^{(1)}(X,\tau):=\forall x\forall D\big((x\in X\wedge D\in F\wedge x\notin D)\to\exists A\big(A\in N_x^\gamma\wedge(Cl_\gamma(A)\cap D=\phi)\big)\big).$$

Theorem 3.20. $\models(X,\tau)\in T_3^\gamma\leftrightarrow(X,\tau)\in\gamma T_3^{(1)}$.

Proof. Now,

$$(X,\tau)\in T_3^{(1)}=\inf_{x\notin D}\min\left(1,1-\tau(X-D)+\sup_{A\in P(X)}\min\left(N_x^\gamma(A),\inf_{y\in D}(1-Cl_\gamma(A)(y))\right)\right)$$

$$=\inf_{x\notin D}\min\left(1,1-\tau(X-D)+\sup_{A\in P(X)}\min\left(N_x^\gamma(A),\inf_{y\in D}N_y^\gamma(X-A)\right)\right).$$

and

$$\left[(X,\tau)\in T_3^\gamma\right]=\inf_{x\notin D}\min\left(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\left(N_x^\gamma(A),\tau_\gamma(B)\right)\right).$$

So, the result holds if we prove that

$$\sup_{A\in P(X)}\min\left(N_x^\gamma(A),\inf_{y\in D}N_y^\gamma(X-A)\right)=\sup_{A\cap B=\phi,D\subseteq B}\min\left(N_x^\gamma(A),\tau_\gamma(B)\right).\qquad(1)$$

In fact, in the left site of (1) when $A\cap D\neq\phi$ then there exists $y\in X$ such that $y\in D$ and $y\in A$. Namely, $y\in D$ and $y\notin X-A$. So, $\inf_{y\in D}N_y^\gamma(X-A)=0$ and thus (1) becomes

$$\sup_{A\in P(X),A\cap B=\phi}\min\left(N_x^\gamma(A),\inf_{y\in D}N_y^\gamma(X-A)\right)=\sup_{A\cap B=\phi,D\subseteq B}\min\left(N_x^\gamma(A),\tau_\gamma(B)\right),$$

which is obtained from Lemma 3.19.

Definition 3.4. Let (X,τ) be a fuzzifying topological space.

$$\gamma T_3^{(2)}(X,\tau):=\forall x\forall B\big((x\in B\wedge B\in\tau)\to\big(\exists A\big(A\in N_x^\gamma\wedge Cl_\gamma(A)\subseteq B\big)\big)\big).$$

Theorem 3.21. $\models(X,\tau)\in T_3^\gamma\leftrightarrow(X,\tau)\in\gamma T_3^{(2)}$.

Proof. From Theorem 3.20, we have

$$\left[(X,\tau)\in T_3^{\gamma}\right]=\inf_{x\in D}\min\left(1,1-\tau(X-D)+\sup_{A\in P(X)}\min\left(N_x^{\gamma}(A),\inf_{y\in D}N_y^{\gamma}(X-A)\right)\right)$$

Now, if we put $B=X-D$, then

$$\left[(X,\tau)\in T_3^{(2)}\right]=\inf_{x\in B}\min\left(1,1-\tau(B)+\sup_{A\in P(X)}\min\left(N_x^{\gamma}(A),\inf_{y\in(X-B)}N_y^{\gamma}(X-A)\right)\right)$$

$$=\inf_{x\in D}\min\left(1,1-\tau(X-D)+\sup_{A\in P(X)}\min\left(N_x^{\gamma}(A),\inf_{y\in D}N_y^{\gamma}(X-A)\right)\right)$$

$$=\left[(X,\tau)\in T_3^{\gamma}\right].$$

Definition 3.5. Let (X,τ) be a fuzzifying topological space and φ be a subbase of τ then $(X,\tau)\in\gamma T_3^{(3)}:=\forall x\forall D(x\in D\wedge D\in\varphi\rightarrow\exists B(B\in N_x^{\gamma}\wedge Cl_{\gamma}(B)\subseteq D)$.

Theorem 3.22. $\models(X,\tau)\in T_3^{\gamma}\leftrightarrow(X,\tau)\in\gamma T_3^{(3)}$.

Proof. Since $[\varphi\subseteq\tau]=1$, and with regards to Theorem 3.20 and 3.21 we have $\gamma T_3^{(3)}(X,\tau)\geq\gamma T_3^{(2)}(X,\tau)=T_3^{\gamma}(X,\tau)$. So, it remains to prove that $\gamma T_3^{(3)}(X,\tau)\leq\gamma T_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x\in A$,

$$\min\left(1,1-\tau(A)+\sup_{B\in P(X)}\min\left(N_x^{\gamma}(B),\inf_{y\in X-A}N_y^{\gamma}(X-B)\right)\right)\geq\left[(X,\tau)\in\gamma T_3^{(3)}\right].$$

Set $\left[(X,\tau)\in\gamma T_3^{(3)}\right]=\delta$. Then, for any $x\in X$ and any $D_{\lambda_i}\in P(X)$, $\lambda_i\in I_\lambda$ (I_λ denotes a finite index set), $\lambda\in\Lambda$,

$$\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda}D_{\lambda_i}=A$$

we have

$$1-\varphi\left(D_{\lambda_i}\right)+\sup_{B\in P(X)}\min\left(N_x^{\gamma}(B),\inf_{y\in X-D_{\lambda_i}}N_y^{\gamma}(X-B)\right)\geq\delta>\delta-\varepsilon,$$

where ε is any positive number. Thus

$$\sup_{B\in P(X)}\min\left(N_x^{\gamma}(B),\inf_{y\in X-D_{\lambda_i}}N_y^{\gamma}(X-B)\right)>\varphi\left(D_{\lambda_i}\right)-1+\delta-\varepsilon.$$

Set $\beta_{\lambda_i}=\left\{B:B\subseteq D_{\lambda_i}\right\}$. Then

$$\inf_{\lambda_i \in I_\lambda} \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)$$

$$= \sup_{f \in \Pi\{\beta_{\lambda_i} : \lambda_i \in I_\lambda\}} \inf_{\lambda_i \in I_\lambda} \min\left(N_x^\gamma(f(\lambda_i)), \inf_{y \in (X - D_{\lambda_i})} N_y^\gamma(X - f(\lambda_i)) \right)$$

$$= \sup_{f \in \Pi\{\beta_{\lambda_i} : \lambda_i \in I_\lambda\}} \min\left(\inf_{\lambda_i \in I_\lambda} N_x^\gamma(f(\lambda_i)), \inf_{\lambda_i \in I_\lambda} \inf_{y \in (X - D_{\lambda_i})} N_y^\gamma(X - f(\lambda_i)) \right)$$

$$= \sup_{f \in \Pi\{\beta_{\lambda_i} : \lambda_i \in I_\lambda\}} \min\left(\inf_{\lambda_i \in I_\lambda} N_x^\gamma(f(\lambda_i)), \inf_{y \in \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - f(\lambda_i)) \right)$$

$$= \sup_{B \in P(X)} \min\left(\inf_{\lambda_i \in I_\lambda} N_x^\gamma(B), \inf_{y \in \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)$$

$$= \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - B) \right),$$

where $B = f(\lambda_i)$.

Similarly, we can prove

$$\inf_{\lambda \in \Lambda} \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)$$

$$= \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in \bigcup_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)$$

$$\le \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in \bigcap_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)$$

$$\le \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in X - A} N_y^\gamma(X - B) \right),$$

so we have

$$\sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in X - A} N_y^\gamma(X - B) \right)$$

$$\ge \inf_{\lambda \in \Lambda} \inf_{\lambda_i \in I_\lambda} \sup_{B \in P(X)} \min\left(N_x^\gamma(B), \inf_{y \in (X - D_{\lambda_i})} N_y^\gamma(X - B) \right)\Bigg)$$

$$\ge \inf_{\lambda \in \Lambda} \inf_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \varepsilon.$$

For any I_λ and Λ that satisfy

$$\bigcup_{\lambda\in\Lambda}\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$$

the above inequality is true. So,

$$\sup_{B\in P(X)} \min\left(N_x^{\gamma}(B), \inf_{y\in X-A} N_y^{\gamma}(X-B)\right) \geq \sup_{\bigcup_{\lambda\in\Lambda} D_\lambda = A} \inf_{\lambda\in\Lambda} \sup_{\bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i}=D_\lambda} \inf_{\lambda_i\in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \varepsilon .$$

$$= \tau(A) - 1 + \delta - \varepsilon ,$$

i.e.,

$$\min\left(1, 1-\tau(A) + \sup_{B\in P(X)} \min\left(N_x^{\gamma}(B), \inf_{y\in X-A} N_y^{\gamma}(X-B)\right)\right) \geq \delta - \varepsilon .$$

Because ε is any positive number, when $\varepsilon \to 0$ we have

$$\left[(X,\tau)\in \gamma T_3^{(2)}\right] \geq \delta = \left[(X,\tau)\in \gamma T_3^{(1)}\right].$$

So, $\models (X,\tau)\in T_3^{\gamma} \leftrightarrow (X,\tau)\in \gamma T_3^{(1)}$.

Definition 3.6. Let (X,τ) be any fuzzifying topological space and let

1. $(X,\tau)\in \gamma ST_3^{(1)} = \forall x \forall D\Big(\big((x\in X)\wedge(D\in F_\gamma)\wedge(x\notin D)\big)$
$\to \exists A\big(A\in N_x \wedge (Cl(A)\cap D \equiv \phi)\big)\Big)$;

2. $(X,\tau)\in \gamma ST_3^{(2)} = \forall x \forall B\Big(\big((x\in B)\wedge(B\in\tau)\big) \to \exists A\big(A\in N_x^{\gamma} \wedge (Cl(A)\subseteq B)\big)\Big)$;

3. $(X,\tau)\in \gamma T_4^{(1)} = \forall A \forall B\Big(\big((A\in\tau)\wedge(B\in F)\wedge(A\cap B\equiv\phi)\big)$;
$\to \exists G\big((G\in\tau)\wedge(A\subseteq G)\wedge(Cl_\gamma(G)\cap B \equiv \phi)\big)\Big)$;

4. $(X,\tau)\in \gamma T_4^{(2)} = \forall A \forall B\Big(\big((A\in F)\wedge(B\in\tau)\wedge(A\subseteq B)\big)$
$\to \exists G\big((G\in\tau)\wedge(A\subseteq G)\wedge(Cl_\gamma(G)\subseteq B)\big)\Big)$

5. $(X,\tau)\in \gamma ST_4^{(1)} = \forall A \forall B\Big(\big((A\in\tau)\wedge(B\in F_\gamma)\wedge(A\cap B\equiv\phi)\big)$
$\to \exists G\big((G\in\tau)\wedge(A\subseteq G)\wedge(Cl_\gamma(G)\cap B\equiv\phi)\big)\Big)$;

6. $(X,\tau)\in \gamma ST_4^{(2)} = \forall A \forall B\Big(\big((A\in F)\wedge(B\in\tau_\gamma)\wedge(A\subseteq B)\big)$
$\to \exists G\big((G\in\tau)\wedge(A\subseteq G)\wedge(Cl_\gamma(G)\subseteq B)\big)\Big)$.

By a similar proof of Theorems 3.20 and 3.21 we have the following theorem.

Theorem 3.23. *Let (X,τ) be any fuzzifying topological space. Then*

(1) $\models (X,\tau)\in T_3^{\gamma^S} \leftrightarrow (X,\tau)\in \gamma ST_3^{(1)}$;

(2) $\models (X,\tau)\in T_3^{\gamma^S} \leftrightarrow (X,\tau)\in \gamma ST_3^{(2)}$;

(3) $\models (X,\tau)\in T_4^{\gamma} \leftrightarrow (X,\tau)\in \gamma T_4^{(1)}$;

(4) $\models (X,\tau)\in T_4^{\gamma} \leftrightarrow (X,\tau)\in \gamma T_4^{(2)}$;

(5) $\models (X,\tau)\in T_4^{\gamma^S} \leftrightarrow (X,\tau)\in \gamma ST_4^{(1)}$;

(6) $\models (X,\tau)\in T_4^{\gamma^S} \leftrightarrow (X,\tau)\in \gamma ST_4^{(2)}$.

4. Relation among separation axioms

Theorem 4.1. $\models \left((X,\tau)\in T_3^{\gamma} \wedge (X,\tau)\in T_1\right)\rightarrow (X,\tau)\in T_2^{\gamma}$;

Proof. From Theorem 2.2 [5] we have, $T_1(X,\tau)=\inf_{z\in X}\tau(X-\{z\})$. So,

$$\left[(X,\tau)\in T_3^{\gamma}\right]+\left[(X,\tau)\in T_1\right]$$

$$=\inf_{x\notin D}\min\left(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\left(N_x^{\gamma}(A),\tau_{\gamma}(B)\right)\right)+\inf_{z\in X}\tau(X-\{z\})$$

$$\leq \inf_{x\in X,x\neq y}\inf_{y\in X}\min\left(1,1-\tau(X-\{y\})+\sup_{A\cap B=\phi}\min\left(N_x^{\gamma}(A),N_y^{\gamma}(B)\right)\right)+\inf_{z\in X}\tau(X-\{z\})$$

$$=\inf_{x\in X,x\neq y}\left(\inf_{y\in X}\min\left(1,1-\tau(X-\{y\})+\sup_{A\cap B=\phi}\min\left(N_x^{\gamma}(A),N_y^{\gamma}(B)\right)\right)+\inf_{z\in X}\tau(X-\{z\})\right)$$

$$\leq \inf_{x\in X,x\neq y}\inf_{y\in X}\left(\min\left(1,1-\tau(X-\{y\})+\sup_{A\cap B=\phi}\min\left(N_x^{\gamma}(A),N_y^{\gamma}(B)\right)\right)+\tau(X-\{y\})\right)$$

$$\leq \inf_{x\neq y}\left(1+\sup_{A\cap B=\phi}\min\left(N_x^{\gamma}(A),N_y^{\gamma}(B)\right)\right)$$

$$\leq 1+\inf_{x\neq y}\sup_{A\cap B=\phi}\min\left(N_x^{\gamma}(A),N_y^{\gamma}(B)\right)=1+\left[(X,\tau)\in T_2^{\gamma}\right],$$

namely, $\left[(X,\tau)\in T_2^{\gamma}\right]\geq\left[(X,\tau)\in T_3^{\gamma}\right]+\left[(X,\tau)\in T_1\right]-1$. Thus, $\left[(X,\tau)\in T_2^{\gamma}\right]\geq \max\left(0,\left[(X,\tau)\in T_3^{\gamma}\right]+\left[(X,\tau)\in T_1\right]-1\right)$.

Theorem 4.2. $\models \left((X,\tau)\in T_4^\gamma \,\dot{\wedge}\, (X,\tau)\in T_1\right)\to (X,\tau)\in T_3^\gamma$;

Proof. It is equivalent to prove that $\left[(X,\tau)\in T_3^\gamma\right]\geq\left[(X,\tau)\in T_4^\gamma\right]+\left[(X,\tau)\in T_1\right]-1$. In fact,

$$\left[(X,\tau)\in T_4^\gamma\right]+\left[(X,\tau)\in T_1\right]=\inf_{E\cap D=\phi}\min\Big(1,1-\min\big(\tau(X-E),\tau(X-D)\big)$$

$$+\sup_{A\cap B=\phi,E\subseteq A,D\subseteq B}\min\big(\tau_\gamma(A),\tau_\gamma(B)\big)\Big)+\inf_{z\in X}\tau\left(X-\{z\}\right)$$

$$\leq\inf_{x\notin D}\min\Big(1,1-\min\big(\tau\left(X-\{x\}\right),\tau(X-D)\big)$$

$$+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)+\inf_{z\in X}\tau\left(X-\{z\}\right)$$

$$\leq\inf_{x\notin D}\min\Big(1,\max\Big(1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big),$$

$$1-\tau\left(X-\{x\}\right)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\Big)\Big)+\inf_{z\in X}\tau\left(X-\{z\}\right)$$

$$=\inf_{x\notin D}\max\Big(\min\Big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big),$$

$$\min\left(1,1-\tau\left(X-\{x\}\right)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\right)\Big)+\inf_{z\in X}\tau\left(X-\{z\}\right)$$

$$\leq\inf_{x\notin D}\max\Big(\min\Big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)+\tau\left(X-\{x\}\right),$$

$$\min\left(1,1-\tau\left(X-\{x\}\right)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\right)\Big)+\tau\left(X-\{x\}\right)$$

$$\leq\inf_{x\notin D}\max\Big(\min\Big(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)+\tau\left(X-\{x\}\right),$$

$$1+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\Big)\Big)$$

$$\leq\inf_{x\notin D}\left(\min\left(1,1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\right)+1\right)$$

$$\leq\inf_{x\notin D}\left(1-\tau(X-D)+\sup_{A\cap B=\phi,D\subseteq B}\min\big(N_x^\gamma(A),\tau_\gamma(B)\big)\right)+1=\left[(X,\tau)\in T_3^\gamma\right]+1.$$

By a similar procedures of Theorems 4.1-4.2 we have the following theorems respectively.

Theorem 4.3. $\models\left((X,\tau)\in T_3^{\gamma^S}\,\dot{\wedge}\,(X,\tau)\in T_1^\gamma\right)\to(X,\tau)\in T_2$;

Theorem 4.4. $\models\left((X,\tau)\in T_4^{\gamma^S}\,\underset{\cdot}{\wedge}\,(X,\tau)\in T_1^{\gamma}\right)\to(X,\tau)\in T_3^{\gamma^S}$;

From the above discussion one can have the following diagram

$$\begin{array}{ccccccccc}
 & & & & T_4^{\gamma^S}\underset{\cdot}{\wedge}T_1^{\gamma} & \leftarrow & T_4^{\gamma^S}\underset{\cdot}{\wedge}T_1^{\gamma} & & \\
 & & & & & & \downarrow & & \\
 & & & & \downarrow & & T_3^{\gamma^S} & \leftarrow & T_4^{\gamma^S} \\
 & & & & & & \downarrow & & \downarrow \\
T_0 & \leftarrow & T_1 & \leftarrow & T_2 & \leftarrow & T_3 & \leftarrow & T_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_0^{\gamma} & \leftarrow & T_1^{\gamma} & \leftarrow & T_2^{\gamma} & \leftarrow & T_3^{\gamma} & \leftarrow & T_4^{\gamma}
\end{array}$$

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