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ISSN 1066-8950

THE JOURNAL OF FUZZY MATHEMATICS

REPRINTED FROM
THE JOURNAL OF FUZZY MATHEMATICS

INTERNATIONAL FUZZY MATHEMATICS INSTITUTE

Los Angeles California U.S.A.

The Journal of Fuzzy Mathematics Vol. 7, No. 3, 1999
Los Angeles

β-continuity and $D(c, \beta)$-continuity in Fuzzifying Topology

K. M. Abd El-Hakeim, F. M. Zeyada and O. R. Sayed

Department of Mathematics, Faculty of Sciences
Assiut University, Assiut 71516, Egypt

Abstract:

The concepts of β-continuity and $D(c, \beta)$-continuity are considered and studied in fuzzifying topology and by making use of these concepts, some decompositions of fuzzy continuity are introduced.

Keywords:

Fuzzifying topology, fuzzy continuity, β-continuity, $c\beta$-continuity.

1. Introduction

In [6], M. S. Ying introduced the concept of fuzzifying topology with the semantic method of continuous valued logic. All the conventions in [6, 7, 8] are good in this paper. In classical topology, M. E. Abd El-Monsef et-al [1] introduced the concepts of β-open sets and β-continuous functions which are studied in D. Andrijevic [2] under the name semi-preopen sets and semi-preopen functions. In J. Dontchev and M. przemski [3] it is noted that a subset A is β-open iff $A \subseteq$ semi-int(semi-cl(A)) and thus the term semi-preopen is replaced by the term pre-semiopen. In fact we use the term β-open sets and β-continuity in the present paper. In [3] the concept of the family of $D(c, \beta)$-open sets is defined and $D(c, \beta)$-continuous functions are introduced. In the present paper the term $c\beta$-open sets and $c\beta$-continuity are used instead of the terms $D(c, \beta)$-open sets and $D(c, \beta)$-continuity respectively. In the present paper we extend and study the concepts of β-continuity and $c\beta$-continuity in fuzifying topology and by making use of them some decompositions of fuzzy continuity are introduced.

2. Preliminaries

For the fuzzy logical and corresponding set theoretical notations we refer to [6, 7]. We note that the set of truth values is the unit interval and we do often not distinguish the

Received December, 1997.

1066-8950/99 $5.50

(0) [illegible] 25

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connectives and their truth value functions and state strictly our results on formalization as M. S. Ying do. For the definitions and results in fuzzifying topology which are used in the sequel we refer to [6, 7, 8].

We give now some definitions and results are introduced in [5] which are useful in the rest of the present paper.

Definition 2.1. For any $\tilde{A} \in \mathfrak{F}(X)$

$$\models (\tilde{A})^{\circ} \equiv X \sim \overline{(X \sim \tilde{A})}.$$

Lemma 2.1. *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then*

(1) $\models \overline{\tilde{A}} \subseteq \overline{\tilde{B}}$;

(2) $\models (\tilde{A})^{\circ} \subseteq (\tilde{B})^{\circ}$.

Lemma 2.2. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A}, \tilde{B}$

(1) $\models X^{\circ} \equiv X$;

(2) $\models (\tilde{A})^{\circ} \subseteq \tilde{A}$;

(3) $\models (\tilde{A} \cap \tilde{B})^{\circ} \equiv (\tilde{A})^{\circ} \cap (\tilde{B})^{\circ}$

(4) $\models (\tilde{A})^{-} \supseteq (\tilde{A})^{\circ}$.

One can add the following lemma.

Lemma 2.3. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A} \in \mathfrak{F}(X)$,

(1) $\models X \sim (\tilde{A})^{-\circ-} \equiv (X \sim \tilde{A})^{\circ-\circ}$;

(2) *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then* $\models (\tilde{A})^{-\circ-} \subseteq (\tilde{B})^{-\circ-}$.

3. Fuzzifying β-open sets and $c\beta$-open sets

Definition 3.1. Let (X, τ) be a fuzzifying topological space.

(1) The family of fuzzifying β-(resp. $c\beta$-)open sets is denoted by $\beta\tau$ (resp. $c\beta\tau$) $\in \mathfrak{F}(P(X))$ and defined as follows:

$$A \in \beta\tau := \forall x (x \in A \to x \in A^{-\circ-}) \text{ (resp. } A \in c\beta\tau := \forall x (x \in A \cap A^{-\circ-} \to x \in A^{\circ})).$$

(2) The family of fuzzifying β-(resp. $c\beta$-)closed sets is denoted by βF (resp. $c\beta F$) $\in \mathfrak{F}(P(X))$ and defined as follows:

$$A \in \beta F \text{ (resp } c\beta F\text{)} := X \sim A \in \beta\tau \text{ (resp. } c\beta F\text{)}.$$

Lemma 3.1. *For any* $\alpha, \beta, \gamma, \delta \in I$,

$$(1-\alpha+\beta)\wedge(1-\gamma+\delta) \leq 1-(\alpha\wedge\gamma)+(\beta\wedge\delta).$$

Lemma 3.2. *For any* $A \in P(X)$,

$\models A^{\circ} \subseteq A^{-\circ-}$.

Proof. From Theorem 5.3 [6] we have $[A \subseteq \overline{A}] = 1$ and from Lemma 2.1 (2) we have $[A^{\circ} \subseteq A^{-\circ}] = 1$. Also from Theorem 5.3 [6] we have $[A^{\circ} \subseteq A^{-\circ-}] = 1$.

Theorem 3.1. *Let* (X, τ) *be a fuzzifying topological space. then*

(1) (a) $\beta\tau(X) = 1, \beta\tau(\varnothing) = 1$;

(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\beta\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \beta\tau(A_\lambda)$;

(2) (a) $c\beta\tau(X) = 1, c\beta\tau(\varnothing) = 1$;

(b) $c\beta\tau(A \cap B) \geq c\beta\tau(A) \wedge c\beta\tau(B)$.

Proof. The proof of (a) in (1) and (a) in (2) are straightforward.

(1) (b) From Lemma 2.3,

$\models A_\lambda^{-\circ-} \subseteq \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}$.

So,

$$\beta\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \inf_{x\in\bigcup_{\lambda\in\Lambda} A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}(x) = \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}(x)$$

$$\geq \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} A_\lambda^{-\circ-}(x) = \bigwedge_{\lambda\in\Lambda} \beta\tau(A_\lambda).$$

(2) (b) Applying Lemmas 2.2(3), 2.3(2), 3.1 and 3.2 we have

$$\begin{aligned} c\beta\tau(A) \wedge c\beta\tau(B) &= \inf_{x\in A}(1 - A^{-\circ-}(x) + A^{\circ}(x)) \wedge \inf_{x\in B}(1 - B^{-\circ-}(x) + B^{\circ}(x)) \\ &\leq \inf_{x\in A\cap B}((1 - A^{-\circ-}(x) + A^{\circ}(x)) \wedge (1 - B^{-\circ-}(x) + B^{\circ}(x))) \\ &\leq \inf_{x\in A\cap B}(1 - (A^{-\circ-} \cap B^{-\circ-})(x) + (A^{\circ} \cap B^{\circ})(x)) \\ &\leq \inf_{x\in A\cap B}(1 - (A \cap B)^{-\circ-}(x) + (A \cap B)^{\circ}(x)) \\ &= c\beta\tau(A \cap B). \end{aligned}$$

From Theorem 3.1, we can have the following theorem.

Theorem 3.2. *Let* (X, τ) *be a fuzzifying topological space, then*

(1) (a) $\beta F(X)=1, \beta F(\emptyset)=1$;

(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\beta F\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \beta F(A_\lambda)$;

(2) (a) $c\beta F(X)=1, c\beta F(\emptyset)=1$;

(b) $c\beta F(A \cup B) \geq c\beta F(A) \wedge c\beta F(B)$.

Theorem 3.3. Let (X, τ) be a fuzzifying topological space, then

(1) (a) $\models \tau \subseteq \beta\tau$;

(b) $\models \tau \subseteq c\beta\tau$;

(2) (a) $\models F \subseteq \beta F$;

(b) $\models F \subseteq c\beta F$.

Proof. From Theorem 2.2(3) [5] and Lemma 3.2, we have

(1) (a) $[A \in \tau]=[A \subseteq A^{\circ}] \leq [A \subseteq A^{-\circ-}]=[A \in \beta\tau]$;

(B) $[A \in \tau]=[A \subseteq A^{\circ}] \leq [A \subseteq A^{-\circ-} \subseteq A^{\circ}]=[A \in c\beta\tau]$;

(2) The proof is obtained from (1).

Remark 3.1. In crisp setting, i.e., if the underlying fuzzifying topologyis the ordinary topology, one can have

$$\models (A \in \beta\tau \wedge A \in c\beta\tau) \rightarrow A \in \tau.$$

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 3.1. Let $X=(a, b, c)$ and let τ be a fuzzifying topology defined as follows: $\tau(X)=\tau(\emptyset)=\tau(\{a\})=\tau(\{a, c\})=1$, $\tau(\{b\})=\tau(\{a, b\})=0$ and $\tau(\{c\})=\tau(\{b, c\}) =\frac{1}{8}$. One can have that $\beta\tau(\{a, b\})=\frac{7}{8}$, $c\beta\tau(\{a, b\})=\frac{1}{8}$ and hence $\beta\tau(\{a, b\}) \wedge c\beta\tau(\{a, b\}) =\frac{7}{8} \wedge \frac{1}{8}=\frac{1}{8} \nleq 0=\tau(\{a, b\})$.

Theorem 3.4. *Let (X, τ) be a fuzzifying topological space.*

(1) $\models A \in \tau \rightarrow (A \in \beta\tau \wedge A \in c\beta\tau)$;

(2) *if* $[A \in \beta\tau]=1$ *or* $[A \in c\beta\tau]=1$ *then* $\models A \in \tau \leftrightarrow (A \in \beta\tau \wedge A \in c\beta\tau)$.

Proof. (1) It is obtained from Theorem 3.3 (1).

(2) If $[A \in \beta\tau]=1$, then for each $x \in A$, $A^{-\circ-}(x)=1$ and so for each $x \in A$, $1+A^{\circ}(x) -A^{-\circ-}(x)=A^{\circ}(x)$. Thus from Lemma 3.2 $\models A^{\circ} \subseteq A^{-\circ-}$ and so we have $[A \in \beta\tau] \wedge [A \in c\beta\tau]=[A \in c\beta\tau]=[A \in \tau]$. If $[A \in c\beta\tau]=1$ then for each $x \in A, 1-A^{-\circ-}(x)+A^{\circ}(x) =1$ and so for each $x \in A$, we have $A^{-\circ-}(x)=A^{\circ}(x)$. Thus $[A \in \beta\tau] \wedge [A \in c\beta\tau]=[A \in \beta\tau]= [A \in \tau]$.

Theorem 3.5. Let (X, τ) be a fuzzifying topological space. then

$$\models \left(A \in \beta\tau \wedge A \in c\beta\tau\right) \rightarrow A \in \tau.$$

Proof.

$$\begin{aligned}\beta\tau(A) \wedge c\beta\tau(A) &= \inf_{x\in A} A^{-\circ-}(A) \wedge \inf_{x\in A}\left(1 - A^{-\circ-}(x) + A^{\circ}(x)\right)\\ &= \max\left(0, \inf_{x\in A} A^{-\circ-}(x) + \inf_{x\in A}\left(1 - A^{-\circ-}(x) + A^{\circ}(x)\right) - 1\right)\\ &\le \inf_{x\in A} A^{\circ}(x) = [A \in \tau].\end{aligned}$$

4. Fuzzifying β-(resp. $c\beta$-) neighborhood structure of a point

Definition 4.1. Let $x \in X$. The β-(resp. $c\beta$-) neighborhood of x is denoted by βN_x (resp. $c\beta N_x$)$\in \mathfrak{F}(P(X))$ and defined as

$$\beta N_x(A) = \sup_{x\in B\subseteq A} \beta\tau(B) \text{ (resp. } c\beta N_x(A) = \sup_{x\in B\subseteq A} c\beta\tau(B)\text{)}.$$

Theorem 4.1.

(1) $\models A \in \beta\tau \leftrightarrow \forall x(x \in A \rightarrow \exists B(B \in \beta\tau \wedge x \in B \subseteq A))$;

(2) $\models A \in \beta\tau \leftrightarrow \forall x(x \in A \rightarrow \exists B(B \in \beta N_x \wedge B \subseteq A))$.

Proof. (1) Now, $[\forall x(x \in A \rightarrow \exists B(B \in \beta\tau \wedge x \in B \subseteq A))] = \inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B)$. It is clear that $\inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B) \ge \beta\tau(A)$. In the other hand, let $\gamma_x = \{B : x \in B \subseteq A\}$. Then, for any $f \in \prod_{x\in A}\gamma_x$ we have $\bigcup_{x\in A} f(x) = A$ and so $\beta\tau(A) = \beta\tau(\bigcup_{x\in A} f(x)) \ge \inf_{x\in A} \beta\tau(f(x))$. Thus $\beta\tau(A) \ge \sup_{f\in\prod_{x\in A}\gamma_x} \inf_{x\in A} \beta\tau(f(x)) = \inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B)$.

(2) From (1) we have $[\forall x(x \in A \rightarrow \exists B(B \in \beta N_x \wedge B \subseteq A))] = \inf_{x\in A} \sup_{B\subseteq A} \beta N_x(B) = \inf_{x\in A} \sup_{B\subseteq A} \sup_{x\in C\subseteq B} \beta\tau(C) = \inf_{x\in A} \sup_{x\in C\subseteq A} \beta\tau(C) = [A \in \beta\tau]$.

Corollary 4.1. $\inf_{x\in A} \beta N_x(A) = \beta\tau(A)$.

Theorem 4.2. *The mapping $\beta N : X \rightarrow \mathfrak{F}^N(P(X))$, $x \rightarrow \beta N_x$ where $\mathfrak{F}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$ has the following properties:*

(1) *for any* $x, A, \models A \in \beta N_x \rightarrow x \in A$;

(2) *for any* $x, A, B, \models A \subseteq B \rightarrow (A \in \beta N_x \rightarrow B \in \beta N_x)$;

(3) *for any* $x, A, \models A \in \beta N_x \rightarrow \exists H(H \in \beta N_x \wedge H \subseteq A \wedge \forall y(y \in H \rightarrow H \in \beta N_y))$.

Proof. One can easily have that for each $x \in X$, $\beta N_x(X) = 1$, i.e., each βN_x is normal.

(1) If $\beta N_x(A) = 0$, the result holds. Suppose $\beta N_x(A) > 0$, then $\sup_{x\in H\subseteq A} \beta\tau(H) > 0$ and so there exists H_0 such that $x \in H_0 \subseteq A$. Thus $[x \in A] = 1 \ge \beta N_x(A)$.

(2) Immediate.

(3) $[\exists H(H \in \beta N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \beta N_y))] = \sup_{H \subseteq A}(\beta N_x(H) \wedge \inf_{y \in H} \beta N_y(H)) = \sup_{H \subseteq A}(\beta N_x(H) \wedge \beta\tau(H)) = \sup_{H \subseteq A} \beta\tau(H) \geq \sup_{x \in H \subseteq A} \beta\tau(H) = [A \in \beta N_x]$.

Theorem 4.3. *The mapping $c\beta N : X \to \mathfrak{T}^N(P(X))$, $x \to c\beta N_x$ where $\mathfrak{T}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$ has the following properties:*

(1) *for any $x, A, \models A \in c\beta N_x \to x \in A$;*

(2) *for any $x, A, B, \models A \subseteq B \to (A \in c\beta N_x \to B \in c\beta N_x)$;*

(3) *for any $x, A, B, \models A \in c\beta N_x \wedge B \in c\beta N_x \to A \cap B \in c\beta N_x$.*

Conversely, if a mapping $c\beta N$ satisfies (2), (3), then $c\beta N$ assigns a fuzzifying topology on X, denoted by $\tau_{c\beta N} \in \mathfrak{F}(P(X))$ and defined as

$$A \in \tau_{c\beta N} := \forall x(x \in A \to A \in c\beta N_x).$$

Proof. It is clear that each $c\beta N_x$ is normal. The proof of (1) and (2) are similar to the corresponding results in Theorem 4.2.

(3) From Theorem 3.1 (2) (b) we have $[A \cap B \in c\beta N_x] = \sup_{x \in H \subseteq A \cap B} c\beta\tau(H) = \sup_{x \in H_1 \subseteq A, x \in H_2 \subseteq B} c\beta\tau(H_1 \cap H_2) \geq \sup_{x \in H_1 \subseteq A, x \in H_2 \subseteq B}(c\beta\tau(H_1) \wedge c\beta\tau(H_2)) = \sup_{x \in H_1 \subseteq A} c\beta\tau(H_1) \wedge \sup_{x \in H_2 \subseteq B} c\beta\tau(H_2) = c\beta N_x(A) \wedge c\beta N_x(B)$.

Conversely, we need to prove that $\tau_{c\beta N} = \inf_{x \in A} c\beta N_x(A)$ is a fuzzifying topology. From Theorem 3.2 [4] and since $\tau_{c\beta N}$ satisfies the properties (2) and (3), then $\tau_{c\beta N}$ is a fuzzifying topology.

Theorem 4.4. *Let (X, τ) be a fuzzifying topological space. Then $\models c\beta\tau \subseteq \tau_{c\beta N}$.*

Proof. Let $B \in P(X)$, $\tau_{c\beta N}(B) = \inf_{x \in B} c\beta N_x(B) = \inf_{x \in B} \sup_{x \in A \subseteq B} c\beta\tau(A) \geq c\beta\tau(B)$.

5. β-(resp. $c\beta$-)closure and β-(resp. $c\beta$-)interior

Definition 5.1. (1) The β-(resp. $c\beta$-)closure of A is denoted by β-cl(resp. $c\beta$-cl) $\in \mathfrak{T}(P(X))$ and defined as follows:

$$\beta-\text{cl}(A)(x) = \inf_{x \notin H \supseteq A}(1 - \beta F(B)) \ \text{(resp. } c\beta-\text{cl}(A)(x) = \inf_{x \notin H \supseteq A}(1 - c\beta F(B))).$$

(2) The β-(resp. $c\beta$-)interior of A is denoted by β-int(resp. $c\beta$-int) $\in \mathfrak{T}(P(X))$ and defined as follows:

$$\beta-\text{int}(A)(x) = \beta N_x(A) \ \text{(resp. } c\beta-\text{int}(A)(x) = c\beta N_x(A)).$$

Theorem 5.1.

(1) (a) $\beta-\text{cl}(A)(x) = 1 - \beta N_x(X \sim A)$;

(b) $\models \beta-\text{cl}(\varnothing) \equiv \varnothing$;

(c) $\models A \subseteq \beta - \mathrm{cl}(A)$;
(d) $\models x \in \beta - \mathrm{cl}(A) \leftrightarrow \forall B(B \in \beta N_x \to A \cap B \neq \varnothing)$;
(e) $\models A \equiv \beta - \mathrm{cl}(A) \leftrightarrow A \in \beta F$;
(f) $\models B \doteq \beta - \mathrm{cl}(A) \leftrightarrow B \in \beta F$;

(2) (a) $c\beta - \mathrm{cl}(A)(x) = 1 - c\beta N_x(X \sim A)$;
(b) $\models c\beta - \mathrm{cl}(\varnothing) \equiv \varnothing$;
(c) $\models A \subseteq c\beta - \mathrm{cl}(A)$;
(d) $\models x \in c\beta - \mathrm{cl}(A) \leftrightarrow \forall B(B \in c\beta N_x \to A \cap B \neq \varnothing)$;
(e) $\models A \equiv c\beta - \mathrm{cl}(A) \leftrightarrow A \in Fr_{c\beta N}$;
(f) $\models B \doteq c\beta - \mathrm{cl}(A) \leftrightarrow B \in Fr_{c\beta N}$.

Proof. (1) (a) $\beta - \mathrm{cl}(A)(x) = \inf_{x \notin B \supseteq A}(1 - \beta F(B)) = \inf_{x \in X - B \subseteq X - A}(1 - \beta\tau(X \sim B)) = 1 - \sup_{x \in X - B \subseteq X - A} \beta\tau(X \sim B) = 1 - \beta N_x(X \sim A)$.

(b) $\beta - \mathrm{cl}(\varnothing)(x) = 1 - \beta N_x(X \sim \varnothing) = 0$.

(c) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $\beta N_x(A) = 0$. If $x \in A$, then $\beta - \mathrm{cl}(A)(x) = 1 - \beta N_x(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq \beta - \mathrm{cl}(A)] = 1$.

(d) $[\forall B(B \in \beta N_x \to A \cap B \neq \varnothing)] = \inf_{B \subseteq X - A}(1 - \beta N_x(B)) = 1 - \beta N_x(X \sim A) = [x \in \beta - \mathrm{cl}(A)]$.

(e) From Corollary 4.1 and from (a), (c) above we have $[A \equiv \beta - \mathrm{cl}(A)] = \inf_{x \in X - A}(1 - (\beta - \mathrm{cl}(A))(x)) \inf_{x \in X - A} \beta N_x(X \sim A) = \beta\tau(X \sim A) = [A \in \beta F]$.

(f) If $[A \subseteq B] = 0$, then $[B \doteq \beta - \mathrm{cl}(A)] = 0$. Now,suppose $[A \subseteq B] = 1$, we have $[B \subseteq \beta - \mathrm{cl}(A)] = 1 - \sup_{x \in B - A} \beta N_x(X \sim A)$, $[\beta - \mathrm{cl}(A) \subseteq B] = \inf_{x \in X - B} \beta N_x(X \sim A)$. So $[B \doteq \beta - \mathrm{cl}(A)] = \max(0, \inf_{x \in X - B} \beta N_x(X \sim A) - \sup_{x \in B - A} \beta N_x(X \sim A))$. If $[B \doteq \beta - \mathrm{cl}(A)] > t$, then $\inf_{x \in X - B} \beta N_x(X \sim A) > t + \sup_{x \in B - A} \beta N_x(X \sim A)$. For any $x \in X - B$, $\sup_{x \in C \subseteq X - A} \beta\tau(C) > t + \sup_{x \in B - A} \beta N_x(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$ and $\beta\tau(C_x) > t + \sup_{x \in B - A} \beta N_x(X \sim A)$. Now we want to prove that $C_x \subseteq X \sim B$. If not, then there exists $x' \in B \sim A$ with $x' \in C_x$. Hence we obtain $\sup_{x \in B - A} \beta N_x(X \sim A) \geq \beta N_{x'}(X \sim A) > t + \sup_{x \in B - A} \beta N_x(X \sim A)$,a contradiction.Therefore $\beta F(B) = \beta\tau(X \sim B) = \inf_{x \in X - B} \beta N_x(X \sim B) \geq \inf_{x \in X - B} \beta\tau(C) > t + \sup_{x \in B - A} \beta N_x(X \sim A) > t$. As t is arbitrary, it holds that $[B \doteq \beta - \mathrm{cl}(A)] \leq [B \in \beta F]$.

(2) The proof is similar to (1).

Theorem 5.2. *For any x, A, B,*

(1) (a) $\models \beta - \mathrm{int}(A) \equiv X \sim B - \mathrm{cl}(X \sim A)$;
(b) $\models \beta - \mathrm{int}(A) \equiv X$;
(c) $\models \beta - \mathrm{int}(A) \subseteq A$;
(d) $\models B \doteq \beta - \mathrm{int}(A) \to B \in \beta\tau$;
(e) $\models B \in \beta\tau \wedge B \subseteq A \to B \subseteq \beta - \mathrm{int}(A)$;
(f) $\models A \equiv \beta - \mathrm{int}(A) \leftrightarrow A \in \beta\tau$;

(2) (a) $\models c\beta - \mathrm{int}(A) \equiv X \sim cB - \mathrm{cl}(X \sim A)$;
(b) $\models c\beta - \mathrm{int}(A) \equiv X$;
(c) $\models c\beta - \mathrm{int}(A) \subseteq A$;

(d) $\models B \doteq c\beta-\text{int}(A) \to B \in \tau_{c\beta N}$;

(e) $\models B \in \tau_{c\beta N} \wedge B \subseteq A \to B \subseteq c\beta-\text{int}(A)$;

(f) $\models A \equiv c\beta-\text{int}(A) \leftrightarrow A \in \tau_{c\beta N}$.

Proof. (1) (a) From Theorem 5.1 (a), $\beta-\text{cl}(A)(x) = 1-\beta N_x(X \sim A) = 1-(\beta-\text{int}(A))(x)$. Then $[\beta-\text{int}(A) \equiv X \sim \beta-\text{cl}(X \sim A)] = 1$.

(b) and (c) are obtained from (a) above and from Theorem 5.1 (1) (b) and (1) (c).

(d) From (a) above and from Theorem 5.1 (1) (f) we have $[B \doteq \beta-\text{int}(A)] = [X \sim B \doteq \beta-\text{cl}(X \sim A)] \le [X \sim B \in \beta F] = [B \in \beta\tau]$.

(e) If $[B \subseteq A] = 0$, then the result holds. If $[B \subseteq A] = 1$, then we have that $[B \subseteq \beta-\text{int}(A)] = \inf_{x\in B} \beta-\text{int}(A)(x) = \inf_{x\in B} \beta N_x(A) \ge \inf_{x\in B} \beta N_x(B) = \beta\tau(B) = [B \in \beta\tau \wedge B \subseteq A]$.

(f) From Corollary 4.1, we have $[A \equiv \beta-\text{int}(A)] = \min(\inf_{x\in A} \beta-\text{int}(A)(x), \inf_{x\in X-A}(1-(\beta-\text{int}(A))(x))) = \inf_{x\in A} \beta-\text{int}(A)(x) = \inf_{x\in A} \beta N_x(A) = \beta\tau(A) = [A \in \beta\tau]$.

(2) The proof is similar to (1).

6. β-continuous functions and $c\beta$-continuous functions

Definition 6.1. Let $(X, \tau), (Y, U)$ be two fuzzifying topologicalspaces.

(1) A unary fuzzy predicate $c\beta C \in \mathfrak{F}(Y^X)$ is called fuzzy β-continuous if

$$c\beta C(f) := \forall u(u \in U \to f^{-1}(u) \in \beta\tau);$$

(2) A unary fuzzy predicate $c\beta C \in \mathfrak{F}(Y^X)$ is called fuzzy $c\beta$-continuous if

$$c\beta C(f) := \forall u(u \in U \to f^{-1}(u) \in c\beta\tau).$$

Definition 6.2. Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces.For any $f \in Y^X$, we define the unary fuzzy predicates $\beta H_j, c\beta H_j \in \mathfrak{F}(Y^X)$ where $j = 1, 2, \cdots, 5$ as follows:

(1) (a) $\beta H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in \beta F_X)$;

(b) $c\beta H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in c\beta F_X)$;

where F_Y is the family of closed subsets of Y; and βF_X and $c\beta F_X$ are the families of β-closed and $c\beta$-closed subsets of X respectively.

(2) (a) $\beta H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in \beta N^X_x)$;

(b) $c\beta H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in c\beta N^X_x)$;

where N^Y is the neighborhood system of Y; and βN_X and $c\beta N_X$ are the β-neighborhood and $c\beta$-neighborhood systems of X respectively.

(3) (a) $\beta H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in \beta N^X_x))$;

(b) $c\beta H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in c\beta N^X_x))$;

(4) (a) $\beta H_4(f) := \forall A(f(\beta - \mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A)))$;

(b) $c\beta H_4(f) := \forall A(f(c\beta - \mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A)))$;

(5) (a) $\beta H_5(f) := \forall B((\beta - \mathrm{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}_Y(B)))$;

(b) $c\beta H_5(f) := \forall B((c\beta - \mathrm{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}_Y(B)))$.

Theorem 6.1.

(1) $\models f \in \beta C \leftrightarrow f \in \beta H_j$, $j = 1, 2, 3, 4, 5$;

(2) $\models f \in c\beta C \leftrightarrow f \in c\beta H_1$.

Proof. We will prove (1) only since the proof of (2) is similar to the corresponding result in (1).

(a) We prove that $\models f \in \beta C \leftrightarrow f \in \beta H_1$. $[f \in \beta H_1] = \inf_{B \in P(Y)} \min(1, 1 - F_Y(B) + \beta F_X(f^{-1}(B))) = \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \beta\tau(X \sim f^{-1}(B))) = \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \beta\tau(f^{-1}(Y \sim B))) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + \beta\tau(f^{-1}(u))) = [f \in \beta C]$.

(b) We want to prove that $\models f \in \beta C \leftrightarrow f \in \beta H_2$.

First we prove that $\beta H_2(f) \geq \beta C(f)$. If $N^Y_{f(x)}(u) \leq \beta N^X_x(f^{-1}(u))$, then the result holds. Suppose $N^Y_{f(x)}(u) > \beta N^X_x(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Hence we have $N^Y_{f(x)}(u) - \beta N^X_x(f^{-1}(u)) = \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} \beta\tau(B) \leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} \beta\tau(f^{-1}(A)) \leq \sup_{f(x) \in A \subseteq u} (U(A) - \beta\tau(f^{-1}(A)))$. So $1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u} (1 - U(A) + \beta\tau(f^{-1}(A)))$ and thus $\min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) \geq \inf_{f(x) \in A \subseteq u} \min(1, 1 - U(A) + \beta\tau(f^{-1}(A))) \geq \inf_{v \in P(Y)} \min(1, 1 - U(v) + \beta\tau(f^{-1}(v))) = \beta C(f)$. Hence, $\inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) \geq [f \in \beta C]$.

Secondly, we prove that $\beta C(f) \geq \beta H_2(f)$. From Corollary 4.1, we have $\beta C(f) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + \beta\tau(f^{-1}(u))) \geq \inf_{u \in P(Y)} \min\left(1, 1 - \inf_{f(x) \in u} N^Y_{f(x)}(u) + \inf_{x \in f^{-1}(u)} \beta N^X_x(f^{-1}(u))\right) = \inf_{u \in P(Y)} \min\left(1, 1 - \inf_{x \in f^{-1}(u)} N^Y_{f(x)}(u) + \inf_{x \in f^{-1}(u)} \beta N^X_x(f^{-1}(u))\right) \geq \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) = \beta H_2(f)$

(c) We prove that $\models f \in \beta H_2 \leftrightarrow f \in \beta H_3$. From Theorem 4.2 (2) we have $\beta H_3(f) = \inf_{x \in X} \inf_{u \in P(Y)} \min\left(1, 1 - N^Y_{f(x)}(u) + \sup_{v \in P(X), f(v) \subseteq u} \beta N^X_x(v)\right) = = \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) = \beta H_2(f)$.

(d) We prove that $\models f \in \beta H_4 \leftrightarrow f \in \beta H_5$. First, for any $B \in P(Y)$ one can deduce that $[f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \supseteq \beta - cl_X(f^{-1}(B))] = 1$, $[cl_Y(f(f^{-1}(B)) \subseteq cl_Y(B)] = 1$ and $[f^{-1}(cl_Y(f(f^{-1}(B)))) \supseteq f^{-1}(cl_Y(B))] = 1$. Then from Lemma 1.2 (2) [6] we have, $[\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq [f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] \geq [f^{-1}(f(\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \geq [f(\beta - cl_X(f^{-1}(B)) \subseteq cl_Y(f(f^{-1}(B))]$. Therefore, $\beta H_5(f) = \inf_{B \in P(Y)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq \inf_{B \in P(Y)} [f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))]$

$\geq \inf_{B\in P(Y)} [f^{-1}(f(\beta - cl_X(f^{-1}(B))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \geq \inf_{B\in P(Y)} [f(\beta - cl_X(f^{-1}(B)) \subseteq cl_Y(f(f^{-1}(B)))] \geq \inf_{A\in P(X)} [f(\beta - cl_X(A)) \subseteq cl_Y(f(A))] = \beta H_4(f)$.

Secondly, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, $[\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \leq [\beta - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] \leq [f(\beta - cl_X(A)) \subseteq f(f^{-1}(cl_Y(A)))] \leq [f(\beta - cl_X(A)) \subseteq cl_Y(f(A))]$. Thus, $\beta H_4(f) \geq \inf_{A\in P(X)} [\beta - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] \geq \inf_{B\in P(Y), B=f(A)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq \inf_{B\in P(Y)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] = \beta H_5(f)$.

(e) We want to prove that $\models f \in \beta H_5 \leftrightarrow f \in \beta H_2$. $\beta H_5(f) = [\forall B(\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))] = \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1 - (1 - \beta N_x(X \sim f^{-1}(B))) + 1 - \beta N_{f(x)}(Y \sim B)) = \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1 - \beta N_{f(x)}(Y \sim B) + \beta N_x(f^{-1}(Y \sim B))) = \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1 - \beta N_{f(x)}(u) + \beta N_x(f^{-1}(u))) = \beta H_2(f)$.

Remark 6.1. In the following theorem we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous or $c\alpha C$-continuous. Thus, the symbols $(\tau, U) - C(f)$, $(\tau_{\alpha N}, U) - C(f)$, $(\tau, U_{\alpha N}) - c\alpha C(f), \ldots$ etc will be understood. Applying Theorem 3.4 (1) and Theorem 4.4 one can deduce the following theorem.

Theorem 6.2.

(1) $\models f \in c\beta H_2 \leftrightarrow f \in c\beta H_j$, $j = 3, 4, 5$;

(2) $\models f \in c\beta C \rightarrow f \in c\beta H_2$.

Proof. (1) It is similiar to the proof of (c), (d) and (e) in Theorem 6.1. (2) It is similiar to the proof of the first part in (b) in Theorem 6.1.

Theorem 6.3.

(1) $\models f \in (\tau, U_{\beta N}) - C \rightarrow f \in (\tau, U) - C$;

(2) $\models f \in (\tau, U) - c\beta C \rightarrow f \in (\tau_{\beta N}, U) - C$;

(3) $\models f \in (\tau, U) - C \rightarrow f \in (\tau, U) - c\beta C$.

7. Decompositions of fuzzy continuity in fuzzifying topology.

Theorem 7.1. *Let (X, τ), (Y, U) be two fuzzifying topological spaces. Then for each $f \in Y^X$, $\models C(f) \rightarrow (\beta C(f) \wedge c\beta C(f))$.*

Proof. The proof is obtainned from Theorem 3.3 (1).

Remark 7.1. In crisp setting i.e., if the underlying fuzzifying topology is the ordinary topology, one can have $\models (\beta C(f) \wedge c\beta C(f)) \rightarrow C(f)$.

But this statement may not true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 7.1. Let (X,τ) be the fuzzifying topological space defined in Counterexample 3.1, consider the identity functions f from (X,τ) onto (X,σ) where σ is a fuzzifying topology on X defined as follows:

$$\sigma(A)=\begin{cases}1, & A\in\{X,\varnothing,\{a,b\}\}\\ 0, & o.w\end{cases}$$

Then $\frac{7}{8}\wedge\frac{1}{8}=\beta C(f)\wedge c\beta C(f)\nleq C(f)=0$. Hence the statement $\models(\beta C(f)\wedge c\beta C(f))\to C(f)$ may not be true in fuzzifying setting.

Theorem 7.2. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$ then $\models C(f)\to(\beta C(f)\leftrightarrow c\beta C(f))$.*

Proof. $[\beta C(f)\to c\beta C(f)]=\min(1,1-\beta C(f)+c\beta C(f))\geq\beta C(f)\wedge c\beta C(f)$. Also, $[c\beta C(f)\to\beta C(f)]=\min(1,1-c\beta C(f)+\beta C(f))\geq c\beta C(f)\wedge\beta C(f)$. Then from Theorem 7.1 we have $c\beta C(f)\wedge\beta C(f)\geq C(f)$ and so the result holds.

Theorem 7.3. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$. If $[\beta\tau(f^{-1}(u))]=1$ or $[c\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models C(f)\leftrightarrow(\beta C(f)\wedge c\beta C(f))$.*

Proof. Now, we need to prove that $C(f)=\beta C(f)\wedge c\beta C(f)$. Applying Theorem 3.4 (2) we have, $\beta C(f)\wedge c\beta C(f)=\inf_{u\in P(Y)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))\ \wedge\inf_{u\in P(y)}\min(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(y)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))\ \wedge(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(y)}\min(1,1-U(u)+(\beta\tau(f^{-1}(u)))\wedge c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min(1-U(u)+\tau(f^{-1}(u)))=C(f)$.

Theorem 7.4. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$. Then,*

(1) *if $[\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then*

$$\models\beta C(f)\to(c\beta C(f)\leftrightarrow C(f));$$

(2) *if $[c\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models c\beta C(f)\to(\beta C(f)\leftrightarrow C(f))$.*

Proof. (1) Since $[\beta\tau(f^{-1}(u))]=1$ and so $[f^{-1}(u)\subseteq(f^{-1}(u))^{-\circ-}]=1$, then $[f^{-1}(u)\cap(f^{-1}(u))^{-\circ-}\subseteq(f^{-1}(u))^{\circ}]=[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]$. Thus, $c\beta C(f)=\inf_{u\in P(y)}\min(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min\left(1,1-U(u)+\left[f^{-1}(u)\cap(f^{-1}(u))^{-\circ-}\subseteq(f^{-1}(u))^{\circ}\right]\right)=\inf_{u\in P(Y)}\min(1,1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}\right])=\inf_{u\in P(y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f)$

(2) Since $[c\beta\tau(f^{-1}(u))]=1$ one can deduce that $(f^{-1}(u))^{-\wedge-}=(f^{-1}(u))^{\wedge}$. So, $\beta C(f)=\inf_{u\in P(Y)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min\left(1,1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{-\wedge-}\right]\right)=\inf_{u\in P(Y)}\min\left(1,1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{\wedge}\right]\right)=\inf_{u\in P(Y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f)$.

Theorem 7.5. *Let* (X,τ), (Y,U) (Z,V) *be three fuzzifying topological spaces. For any* $f\in Y^X$, $g\in Z^Y$,

(1) $\models \beta C(f)\rightarrow(C(g)\rightarrow\beta C(g\circ f))$;

(2) $\models C(g)\rightarrow(\beta C(f)\rightarrow\beta C(g\circ f))$;

(3) $\models c\beta C(f)\rightarrow(C(g)\rightarrow c\beta C(g\circ f))$;

(4) $\models C(g)\rightarrow(c\beta C(f)\rightarrow c\beta C(g\circ f))$

Proof. (1) We need to prove that $[\alpha C(f)]\le[C(g)\rightarrow\beta C(g\circ f)]$. If $c(g)\le[\beta C(g\circ f)]$, the result holds, if $c(g)>[\beta C(g\circ f)]$, then $c(g)-[\beta C(g\circ f)]=\inf_{v\in P(Z)}\min(1,1-V(v)+U(g^{-1}(v)))-\inf_{v\in P(Z)}\min(1,1-V(v)+\beta\tau((g\circ f)^{-1}(v)))\le\sup_{v\in P(Z)}(U(g^{-1}(v))-\beta\tau((g\circ f)^{-1}(v)))\le\sup_{u\in P(Z)}(U(u)-\beta\tau(f^{-1}(u)))$. Therefore, $[C(g)\rightarrow\beta C(g\circ f)]=\min(1,1-[C(g)]+[\beta C(g\circ f)])\ge\inf_{u\in P(Z)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))=\beta C(f)$.

(2) $[C(g)\rightarrow(\beta C(f)\rightarrow\beta C(g\circ f))]=[\neg(C(g)\wedge\neg(\beta C(f)\rightarrow\beta C(g\circ f)))]=[\neg(C(g)\wedge\neg\neg(\beta C(f)\wedge\neg(\beta C(g\circ f))))]=\left[\neg(C(g)\wedge\beta C(f)\wedge\neg(\beta C(g\circ f)))\right]=[\neg(\beta C(f)\wedge C(g)\wedge\neg\beta C(g\circ f))]=[\neg(\beta C(f)\wedge\neg\neg(C(g)\wedge\neg(\beta C(g\circ f))))]=\left[\neg\left(\beta C(f)\wedge\neg(C(g)\rightarrow\beta C(g\circ f))\right)\right]=[\beta C(f)\rightarrow(C(g)\rightarrow\beta C(g\circ f))]=1$.

The proofs of (3) and (4) are similar to (1) and (2) respectively.

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$$A \in \beta F \text{ (resp } c\beta F\text{)} := X \sim A \in \beta\tau \text{ (resp. } c\beta F\text{)}.$$

Lemma 3.1. *For any* $\alpha, \beta, \gamma, \delta \in I$,

$$(1-\alpha+\beta) \wedge (1-\gamma+\delta) \leq 1-(\alpha \wedge \gamma)+(\beta \wedge \delta).$$

Lemma 3.2. *For any* $A \in P(X)$,

$\models A^{\circ} \subseteq A^{-\circ-}$.

Proof. From Theorem 5.3 [6] we have $[A \subseteq \overline{A}] = 1$ and from Lemma 2.1 (2) we have $[A^{\circ} \subseteq A^{-\circ}] = 1$. Also from Theorem 5.3 [6] we have $[A^{\circ} \subseteq A^{-\circ-}] = 1$.

Theorem 3.1. *Let* (X, τ) *be a fuzzifying topological space. then*

(1) (a) $\beta\tau(X) = 1, \beta\tau(\varnothing) = 1$;

(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\beta\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \beta\tau(A_\lambda)$;

(2) (a) $c\beta\tau(X) = 1, c\beta\tau(\varnothing) = 1$;

(b) $c\beta\tau(A \cap B) \geq c\beta\tau(A) \wedge c\beta\tau(B)$.

Proof. The proof of (a) in (1) and (a) in (2) are straightforward.

(1) (b) From Lemma 2.3,

$\models A_\lambda^{-\circ-} \subseteq \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}$.

So,

$$\beta\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \inf_{x\in\bigcup_{\lambda\in\Lambda} A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}(x) = \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ-}(x)$$

$$\geq \inf_{\lambda\in\Lambda} \inf_{x\in A_\lambda} A_\lambda^{-\circ-}(x) = \bigwedge_{\lambda\in\Lambda} \beta\tau(A_\lambda).$$

(2) (b) Applying Lemmas 2.2(3), 2.3(2), 3.1 and 3.2 we have

$$\begin{aligned}
c\beta\tau(A) \wedge c\beta\tau(B) &= \inf_{x\in A}(1 - A^{-\circ-}(x) + A^{\circ}(x)) \wedge \inf_{x\in B}(1 - B^{-\circ-}(x) + B^{\circ}(x)) \\
&\leq \inf_{x\in A\cap B}((1 - A^{-\circ-}(x) + A^{\circ}(x)) \wedge (1 - B^{-\circ-}(x) + B^{\circ}(x))) \\
&\leq \inf_{x\in A\cap B}(1 - (A^{-\circ-} \cap B^{-\circ-})(x) + (A^{\circ} \cap B^{\circ})(x)) \\
&\leq \inf_{x\in A\cap B}(1 - (A \cap B)^{-\circ-}(x) + (A \cap B)^{\circ}(x)) \\
&= c\beta\tau(A \cap B).
\end{aligned}$$

From Theorem 3.1, we can have the following theorem.

Theorem 3.2. *Let* (X, τ) *be a fuzzifying topological space, then*

(1) (a) $\beta F(X)=1, \beta F(\emptyset)=1$;

(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\beta F\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \beta F(A_\lambda)$;

(2) (a) $c\beta F(X)=1, c\beta F(\emptyset)=1$;

(b) $c\beta F(A \cup B) \geq c\beta F(A) \wedge c\beta F(B)$.

Theorem 3.3. Let (X, τ) be a fuzzifying topological space, then

(1) (a) $\models \tau \subseteq \beta\tau$;

(b) $\models \tau \subseteq c\beta\tau$;

(2) (a) $\models F \subseteq \beta F$;

(b) $\models F \subseteq c\beta F$.

Proof. From Theorem 2.2(3) [5] and Lemma 3.2, we have

(1) (a) $[A \in \tau] = [A \subseteq A^{\circ}] \leq [A \subseteq A^{-\circ-}] = [A \in \beta\tau]$;

(B) $[A \in \tau] = [A \subseteq A^{\circ}] \leq [A \subseteq A^{-\circ-} \subseteq A^{\circ}] = [A \in c\beta\tau]$;

(2) The proof is obtained from (1).

Remark 3.1. In crisp setting, i.e., if the underlying fuzzifying topologyis the ordinary topology, one can have

$$\models (A \in \beta\tau \wedge A \in c\beta\tau) \to A \in \tau.$$

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 3.1. Let $X = (a, b, c)$ and let τ be a fuzzifying topology defined as follows: $\tau(X) = \tau(\emptyset) = \tau(\{a\}) = \tau(\{a, c\}) = 1$, $\tau(\{b\}) = \tau(\{a, b\}) = 0$ and $\tau(\{c\}) = \tau(\{b, c\}) = \frac{1}{8}$. One can have that $\beta\tau(\{a, b\}) = \frac{7}{8}$, $c\beta\tau(\{a, b\}) = \frac{1}{8}$ and hence $\beta\tau(\{a, b\}) \wedge c\beta\tau(\{a, b\}) = \frac{7}{8} \wedge \frac{1}{8} = \frac{1}{8} \not\leq 0 = \tau(\{a, b\})$.

Theorem 3.4. *Let (X, τ) be a fuzzifying topological space.*

(1) $\models A \in \tau \to (A \in \beta\tau \wedge A \in c\beta\tau)$;

(2) *if* $[A \in \beta\tau] = 1$ *or* $[A \in c\beta\tau] = 1$ *then* $\models A \in \tau \leftrightarrow (A \in \beta\tau \wedge A \in c\beta\tau)$.

Proof. (1) It is obtained from Theorem 3.3 (1).

(2) If $[A \in \beta\tau] = 1$, then for each $x \in A$, $A^{-\circ-}(x) = 1$ and so for each $x \in A$, $1 + A^{\circ}(x) - A^{-\circ-}(x) = A^{\circ}(x)$. Thus from Lemma 3.2 $\models A^{\circ} \subseteq A^{-\circ-}$ and so we have $[A \in \beta\tau] \wedge [A \in c\beta\tau] = [A \in c\beta\tau] = [A \in \tau]$. If $[A \in c\beta\tau] = 1$ then for each $x \in A, 1 - A^{-\circ-}(x) + A^{\circ}(x) = 1$ and so for each $x \in A$, we have $A^{-\circ-}(x) = A^{\circ}(x)$. Thus $[A \in \beta\tau] \wedge [A \in c\beta\tau] = [A \in \beta\tau] = [A \in \tau]$.

Theorem 3.5. Let (X, τ) be a fuzzifying topological space. then

(1) (a) $\beta F(X)=1, \beta F(\varnothing)=1$;

(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\beta F\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \beta F(A_\lambda)$;

(2) (a) $c\beta F(X)=1, c\beta F(\varnothing)=1$;

(b) $c\beta F(A\cup B) \geq c\beta F(A) \wedge c\beta F(B)$.

Theorem 3.3. Let (X,τ) be a fuzzifying topological space, then

(1) (a) $\models \tau \subseteq \beta\tau$;

(b) $\models \tau \subseteq c\beta\tau$;

(2) (a) $\models F \subseteq \beta F$;

(b) $\models F \subseteq c\beta F$.

Proof. From Theorem 2.2(3) [5] and Lemma 3.2, we have

(1) (a) $[A\in\tau]=[A\subseteq A^{\circ}]\leq[A\subseteq A^{\circ-\circ}]=[A\in\beta\tau]$;

(B) $[A\in\tau]=[A\subseteq A^{\circ}]\leq[A\subseteq A^{-\circ-}\subseteq A^{\circ}]=[A\in c\beta\tau]$;

(2) The proof is obtained from (1).

Remark 3.1. In crisp setting, i.e., if the underlying fuzzifying topologyis the ordinary topology, one can have

$$\models (A\in\beta\tau \wedge A\in c\beta\tau)\rightarrow A\in\tau.$$

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 3.1. Let $X=(a,b,c)$ and let τ be a fuzzifying topology defined as follows: $\tau(X)=\tau(\varnothing)=\tau(\{a\})=\tau(\{a,c\})=1$, $\tau(\{b\})=\tau(\{a,b\})=0$ and $\tau(\{c\})=\tau(\{b,c\})=\frac{1}{3}$. One can have that $\beta\tau(\{a,b\})=\frac{2}{3}$, $c\beta\tau(\{a,b\})=\frac{1}{3}$ and hence $\beta\tau(\{a,b\})\wedge c\beta\tau(\{a,b\})=\frac{2}{3}\wedge\frac{1}{3}=\frac{1}{3}\nleq 0=\tau(\{a,b\})$.

Theorem 3.4. *Let (X,τ) be a fuzzifying topological space.*

(1) $\models A\in\tau\rightarrow(A\in\beta\tau\wedge A\in c\beta\tau)$;

(2) *if* $[A\in\beta\tau]=1$ *or* $[A\in c\beta\tau]=1$ *then* $\models A\in\tau\leftrightarrow(A\in\beta\tau\wedge A\in c\beta\tau)$.

Proof. (1) It is obtained from Theorem 3.3 (1).

(2) If $[A\in\beta\tau]=1$, then for each $x\in A$, $A^{\circ-\circ}(x)=1$ and so for each $x\in A$, $1+A^{\circ}(x)=A^{\circ-\circ}(x)=A^{\circ}(x)$. Thus from Lemma 3.2 $\models A^{\circ}\subseteq A^{\circ-\circ}$ and so we have $[A\in\beta\tau]\wedge[A\in c\beta\tau]=[A\in c\beta\tau]=[A\in\tau]$. If $[A\in c\beta\tau]=1$ then for each $x\in A, 1-A^{-\circ-}(x)+A^{\circ}(x)=1$ and so for each $x\in A$, we have $A^{-\circ-}(x)=A^{\circ}(x)$. Thus $[A\in\beta\tau]\wedge[A\in c\beta\tau]=[A\in\beta\tau]=[A\in\tau]$.

Theorem 3.5. Let (X,τ) be a fuzzifying topological space. then

$$\models \left(A \in \beta\tau \wedge A \in c\beta\tau\right) \to A \in \tau.$$

Proof.

$$\begin{aligned}\beta\tau(A) \wedge c\beta\tau(A) &= \inf_{x\in A} A^{-\circ-}(A) \wedge \inf_{x\in A}(1 - A^{-\circ-}(x) + A^{\circ}(x)) \\ &= \max\left(0, \inf_{x\in A} A^{-\circ-}(x) + \inf_{x\in A}(1 - A^{-\circ-}(x) + A^{\circ}(x)) - 1\right) \\ &\le \inf_{x\in A} A^{\circ}(x) = [A \in \tau].\end{aligned}$$

4. Fuzzifying β-(resp. $c\beta$-) neighborhood structure of a point

Definition 4.1. Let $x \in X$. The β-(resp. $c\beta$-) neighborhood of x is denoted by βN_x (resp. $c\beta N_x$)$\in \mathscr{F}(P(X))$ and defined as

$$\beta N_x(A) = \sup_{x\in B\subseteq A} \beta\tau(B) \text{ (resp. } c\beta N_x(A) = \sup_{x\in B\subseteq A} c\beta\tau(B)\text{)}.$$

Theorem 4.1.

(1) $\models A \in \beta\tau \leftrightarrow \forall x(x \in A \to \exists B(B \in \beta\tau \wedge x \in B \subseteq A))$;

(2) $\models A \in \beta\tau \leftrightarrow \forall x(x \in A \to \exists B(B \in \beta N_x \wedge B \subseteq A))$.

Proof. (1) Now, $[\forall x(x \in A \to \exists B(B \in \beta\tau \wedge x \in B \subseteq A))] = \inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B)$. It is clear that $\inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B) \ge \beta\tau(A)$. In the other hand, let $\gamma_x = \{B : x \in B \subseteq A\}$. Then, for any $f \in \Pi_{x\in A}\gamma_x$ we have $\bigcup_{x\in A} f(x) = A$ and so $\beta\tau(A) = \beta\tau(\bigcup_{x\in A} f(x)) \ge \inf_{x\in A} \beta\tau(f(x))$. Thus $\beta\tau(A) \ge \sup_{f\in\Pi_{x\in A}\gamma_x} \inf_{x\in A} \beta\tau(f(x)) = \inf_{x\in A} \sup_{x\in B\subseteq A} \beta\tau(B)$.

(2) From (1) we have $[\forall x(x \in A \to \exists B(B \in \beta N_x \wedge B \subseteq A))] = \inf_{x\in A} \sup_{B\subseteq A} \beta N_x(B)$ $= \inf_{x\in A} \sup_{B\subseteq A} \sup_{x\in C\subseteq B} \beta\tau(C) = \inf_{x\in A} \sup_{x\in C\subseteq A} \beta\tau(C) = [A \in \beta\tau]$.

Corollary 4.1. $\inf_{x\in A} \beta N_x(A) = \beta\tau(A)$.

Theorem 4.2. *The mapping $\beta N : X \to \mathfrak{F}^N(P(X))$, $x \to \beta N_x$ where $\mathfrak{F}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$ has the following properties:*

(1) *for any* $x, A, \models A \in \beta N_x \to x \in A$;

(2) *for any* $x, A, B, \models A \subseteq B \to (A \in \beta N_x \to B \in \beta N_x)$;

(3) *for any* $x, A, \models A \in \beta N_x \to \exists H(H \in \beta N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \beta N_y))$.

Proof. One can easily have that for each $x \in X$, $\beta N_x(X) = 1$, i.e., each βN_x is normal.

(1) If $\beta N_x(A) = 0$, the result holds. Suppose $\beta N_x(A) > 0$, then $\sup_{x\in H\subseteq A} \beta\tau(H) > 0$ and so there exists H_0 such that $x \in H_0 \subseteq A$. Thus $[x \in A] = 1 \ge \beta N_x(A)$.

(2) Immediate.

(3) $[\exists H(H \in \beta N_x \wedge H \subseteq A \wedge \forall y(y \in H \rightarrow H \in \beta N_y))] = \sup_{H\subseteq A}(\beta N_x(H) \wedge \inf_{y\in H} \beta N_y(H)) = \sup_{H\subseteq A}(\beta N_x(H) \wedge \beta\tau(H)) = \sup_{H\subseteq A} \beta\tau(H) \geq \sup_{x\in H\subseteq A} \beta\tau(H) = [A \in \beta N_x]$.

Theorem 4.3. *The mapping $c\beta N : X \rightarrow \mathfrak{F}^N(P(X))$, $x \rightarrow c\beta N_x$ where $\mathfrak{F}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$ has the following properties:*

(1) *for any $x, A, \models A \in c\beta N_x \rightarrow x \in A$;*

(2) *for any $x, A, B, \models A \subseteq B \rightarrow (A \in c\beta N_x \rightarrow B \in c\beta N_x)$;*

(3) *for any $x, A, B, \models A \in c\beta N_x \wedge B \in c\beta N_x \rightarrow A \cap B \in c\beta N_x$.*

Conversely, if a mapping $c\beta N$ satisfies (2), (3), then $c\beta N$ assigns a fuzzifying topology on X, denoted by $\tau_{c\beta N} \in \mathfrak{F}(P(X))$ and defined as

$$A \in \tau_{c\beta N} := \forall x(x \in A \rightarrow A \in c\beta N_x).$$

Proof. It is clear that each $c\beta N_x$ is normal. The proof of (1) and (2) are similar to the corresponding results in Theorem 4.2.

(3) From Theorem 3.1 (2) (b) we have $[A \cap B \in c\beta N_x] = \sup_{x\in H\subseteq A\cap B} c\beta\tau(H) = \sup_{x\in H_1\subseteq A, x\in H_2\subseteq B} c\beta\tau(H_1 \cap H_2) \geq \sup_{x\in H_1\subseteq A, x\in H_2\subseteq B}(c\beta\tau(H_1) \wedge c\beta\tau(H_2)) = \sup_{x\in H_1\subseteq A} c\beta\tau(H_1) \wedge \sup_{x\in H_2\subseteq B} c\beta\tau(H_2) = c\beta N_x(A) \wedge c\beta N_x(B)$.

Conversely, we need to prove that $\tau_{c\beta N} = \inf_{x\in A} c\beta N_x(A)$ is a fuzzifying topology. From Theorem 3.2 [4] and since $\tau_{c\beta N}$ satisfies the properties (2) and (3), then $\tau_{c\beta N}$ is a fuzzifying topology.

Theorem 4.4. *Let (X, τ) be a fuzzifying topological space. Then $\models c\beta\tau \subseteq \tau_{c\beta N}$.*

Proof. Let $B \in P(X)$, $\tau_{c\beta N}(B) = \inf_{x\in B} c\beta N_x(B) = \inf_{x\in B} \sup_{x\in A\subseteq B} c\beta\tau(A) \geq c\beta\tau(B)$.

5. β-(resp. $c\beta$-)closure and β-(resp. $c\beta$-)interior

Definition 5.1. (1) The β-(resp. $c\beta$-)closure of A is denoted by β-cl(resp. $c\beta$-cl) $\in \mathfrak{F}(P(X))$ and defined as follows:

$$\beta-\text{cl}(A)(x) = \inf_{x\notin B\supseteq A}(1-\beta F(B)) \ (\text{resp. } c\beta-\text{cl}(A)(x) = \inf_{x\notin B\supseteq A}(1-c\beta F(B))).$$

(2) The β-(resp. $c\beta$-)interior of A is denoted by β-int(resp. $c\beta$-int) $\in \mathfrak{F}(P(X))$ and defined as follows:

$$\beta-\text{int}(A)(x) = \beta N_x(A) \ (\text{resp. } c\beta-\text{int}(A)(x) = c\beta N_x(A)).$$

Theorem 5.1.

(1) (a) $\beta-\text{cl}(A)(x) = 1-\beta N_x(X \sim A)$;

(b) $\models \beta-\text{cl}(\varnothing) \equiv \varnothing$;

(c) $\models A \subseteq \beta-\mathrm{cl}(A)$;

(d) $\models x \in \beta-\mathrm{cl}(A) \leftrightarrow \forall B(B \in \beta N_x \rightarrow A \cap B \neq \varnothing)$;

(e) $\models A \equiv \beta-\mathrm{cl}(A) \leftrightarrow A \in \beta F$;

(f) $\models B \doteq \beta-\mathrm{cl}(A) \leftrightarrow B \in \beta F$;

(2) (a) $c\beta-\mathrm{cl}(A)(x) = 1 - c\beta N_x(X \sim A)$;

(b) $\models c\beta-\mathrm{cl}(\varnothing) \equiv \varnothing$;

(c) $\models A \subseteq c\beta-\mathrm{cl}(A)$;

(d) $\models x \in c\beta-\mathrm{cl}(A) \leftrightarrow \forall B(B \in c\beta N_x \rightarrow A \cap B \neq \varnothing)$;

(e) $\models A \equiv c\beta-\mathrm{cl}(A) \leftrightarrow A \in Fr_{c\beta N}$;

(f) $\models B \doteq c\beta-\mathrm{cl}(A) \leftrightarrow B \in Fr_{c\beta N}$.

Proof. (1) (a) $\beta-\mathrm{cl}(A)(x) = \inf_{x \notin B \supseteq A}(1 - \beta F(B)) = \inf_{x \in X-B \subseteq X-A}(1 - \beta\tau(X \sim B)) = 1 - \sup_{x \in X-B \subseteq X-A} \beta\tau(X \sim B) = 1 - \beta N_x(X \sim A)$.

(b) $\beta-\mathrm{cl}(\varnothing)(x) = 1 - \beta N_x(X \sim \varnothing) = 0$.

(c) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $\beta N_x(A) = 0$. If $x \in A$, then $\beta-\mathrm{cl}(A)(x) = 1 - \beta N_x(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq \beta-\mathrm{cl}(A)] = 1$.

(d) $[\forall B(B \in \beta N_x \rightarrow A \cap B \neq \varnothing)] = \inf_{B \subseteq X-A}(1 - \beta N_x(B)) = 1 - \beta N_x(X \sim A) = [x \in \beta-\mathrm{cl}(A)]$.

(e) From Corollary 4.1 and from (a), (c) above we have $[A \equiv \beta-\mathrm{cl}(A)] = \inf_{x \in X-A}(1 - (\beta-\mathrm{cl}(A))(x)) \inf_{x \in X-A} \beta N_x(X \sim A) = \beta\tau(X \sim A) = [A \in \beta F]$.

(f) If $[A \subseteq B] = 0$, then $[B \doteq \beta-\mathrm{cl}(A)] = 0$. Now, suppose $[A \subseteq B] = 1$, we have $[B \subseteq \beta-\mathrm{cl}(A)] = 1 - \sup_{x \in B-A} \beta N_x(X \sim A)$, $[\beta-\mathrm{cl}(A) \subseteq B] = \inf_{x \in X-B} \beta N_x(X \sim A)$. So $[B \doteq \beta-\mathrm{cl}(A)] = \max(0, \inf_{x \in X-B} \beta N_x(X \sim A) - \sup_{x \in B-A} \beta N_x(X \sim A))$. If $[B \doteq \beta-\mathrm{cl}(A)] > t$, then $\inf_{x \in X-B} \beta N_x(X \sim A) > t + \sup_{x \in B-A} \beta N_x(X \sim A)$. For any $x \in X \sim B$, $\sup_{x \in C \subseteq X-A} \beta\tau(C) > t + \sup_{x \in B-A} \beta N_x(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$ and $\beta\tau(C_x) > t + \sup_{x \in B-A} \beta N_x(X \sim A)$. Now we want to prove that $C_x \subseteq X \sim B$. If not, then there exists $x' \in B \sim A$ with $x' \in C_x$. Hence we obtain $\sup_{x \in B-A} \beta N_x(X \sim A) \geq \beta N_{x'}(X \sim A) > t + \sup_{x \in B-A} \beta N_x(X \sim A)$, a contradiction. Therefore $\beta F(B) = \beta\tau(X \sim B) = \inf_{x \in X-B} \beta N_x(X \sim B) \geq \inf_{x \in X-B} \beta\tau(C) > t + \sup_{x \in B-A} \beta N_x(X \sim A) > t$. As t is arbitrary, it holds that $[B \doteq \beta-\mathrm{cl}(A)] \leq [B \in \beta F]$.

(2) The proof is similar to (1).

Theorem 5.2. *For any x, A, B,*

(1) (a) $\models \beta-\mathrm{int}(A) \equiv X \sim B-\mathrm{cl}(X \sim A)$;

(b) $\models \beta-\mathrm{int}(A) \equiv X$;

(c) $\models \beta-\mathrm{int}(A) \subseteq A$;

(d) $\models B \doteq \beta-\mathrm{int}(A) \rightarrow B \in \beta\tau$;

(e) $\models B \in \beta\tau \wedge B \subseteq A \rightarrow B \subseteq \beta-\mathrm{int}(A)$;

(f) $\models A \equiv \beta-\mathrm{int}(A) \leftrightarrow A \in \beta\tau$;

(2) (a) $\models c\beta-\mathrm{int}(A) \equiv X \sim cB-\mathrm{cl}(X \sim A)$;

(b) $\models c\beta-\mathrm{int}(A) \equiv X$;

(c) $\models c\beta-\mathrm{int}(A) \subseteq A$;

(d) $\models B \doteq c\beta-\text{int}(A) \to B \in \tau_{c\beta N}$;

(e) $\models B \in \tau_{c\beta N} \wedge B \subseteq A \to B \subseteq c\beta-\text{int}(A)$;

(f) $\models A \equiv c\beta-\text{int}(A) \leftrightarrow A \in \tau_{c\beta N}$.

Proof. (1) (a) From Theorem 5.1 (a), $\beta-\text{cl}(A)(x) = 1-\beta N_x(X \sim A) = 1-(\beta-\text{int}(A))(x)$. Then $[\beta-\text{int}(A) \equiv X \sim \beta-\text{cl}(X \sim A)] = 1$.

(b) and (c) are obtained from (a) above and from Theorem 5.1 (1) (b) and (1) (c).

(d) From (a) above and from Theorem 5.1 (1) (f) we have $[B \doteq \beta-\text{int}(A)] = [X \sim B \doteq \beta-\text{cl}(X \sim A)] \le [X \sim B \in \beta F] = [B \in \beta\tau]$.

(e) If $[B \subseteq A] = 0$, then the result holds. If $[B \subseteq A] = 1$, then we have that $[B \subseteq \beta-\text{int}(A)] = \inf_{x\in B} \beta-\text{int}(A)(x) = \inf_{x\in B} \beta N_x(A) \ge \inf_{x\in B} \beta N_x(B) = \beta\tau(B) = [B \in \beta\tau \wedge B \subseteq A]$.

(f) From Corollary 4.1, we have $[A \equiv \beta-\text{int}(A)] = \min(\inf_{x\in A} \beta-\text{int}(A)(x), \inf_{x\in X-A}(1-(\beta-\text{int}(A))(x))) = \inf_{x\in A} \beta-\text{int}(A)(x) = \inf_{x\in A} \beta N_x(A) = \beta\tau(A) = [A \in \beta\tau]$.

(2) The proof is similar to (1).

6. β-continuous functions and $c\beta$-continuous functions

Definition 6.1. Let $(X,\tau), (Y,U)$ be two fuzzifying topologicalspaces.

(1) A unary fuzzy predicate $c\beta C \in \mathfrak{F}(Y^X)$ is called fuzzy β-continuous if

$$c\beta C(f) := \forall u(u \in U \to f^{-1}(u) \in \beta\tau);$$

(2) A unary fuzzy predicate $c\beta C \in \mathfrak{F}(Y^X)$ is called fuzzy $c\beta$-continuous if

$$c\beta C(f) := \forall u(u \in U \to f^{-1}(u) \in c\beta\tau).$$

Definition 6.2. Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces.For any $f \in Y^X$, we define the unary fuzzy predicates $\beta H_j, c\beta H_j \in \mathfrak{F}(Y^X)$ where $j = 1,2,\cdots,5$ as follows:

(1) (a) $\beta H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in \beta F_X)$;

(b) $c\beta H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in c\beta F_X)$;

where F_Y is the family of closed subsets of Y; and βF_X and $c\beta F_X$ are the families of β-closed and $c\beta$-closed subsets of X respectively.

(2) (a) $\beta H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in \beta N^X_x)$;

(b) $c\beta H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in c\beta N^X_x)$;

where N^Y is the neighborhood system of Y; and βN_X and $c\beta N_X$ are the β-neighborhood and $c\beta$-neighborhood systems of X respectively.

(3) (a) $\beta H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in \beta N^X_x))$;

(b) $c\beta H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in c\beta N^X_x))$;

(4) (a) $\beta H_4(f) := \forall A(f(\beta-\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A)))$;

(b) $c\beta H_4(f) := \forall A(f(c\beta-\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A)))$;

(5) (a) $\beta H_5(f) := \forall B((\beta-\mathrm{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}_Y(B)))$;

(b) $c\beta H_5(f) := \forall B((c\beta-\mathrm{cl}_X(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}_Y(B)))$.

Theorem 6.1.

(1) $\models f \in \beta C \leftrightarrow f \in \beta H_j,\ j = 1,2,3,4,5$;

(2) $\models f \in c\beta C \leftrightarrow f \in c\beta H_1$.

Proof. We will prove (1) only since the proof of (2) is similar to the corresponding result in (1).

(a) We prove that $\models f \in \beta C \leftrightarrow f \in \beta H_1$. $[f \in \beta H_1] = \inf_{B \in P(Y)} \min(1, 1 - F_Y(B) + \beta F_X(f^{-1}(B))) = \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \beta\tau(X \sim f^{-1}(B))) = \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \beta\tau(f^{-1}(Y \sim B))) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + \beta\tau(f^{-1}(u))) = [f \in \beta C]$.

(b) We want to prove that $\models f \in \beta C \leftrightarrow f \in \beta H_2$.

First we prove that $\beta H_2(f) \geq \beta C(f)$. If $N^Y_{f(x)}(u) \leq \beta N^X_x(f^{-1}(u))$, then the result holds. Suppose $N^Y_{f(x)}(u) > \beta N^X_x(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Hence we have $N^Y_{f(x)}(u) - \beta N^X_x(f^{-1}(u)) = \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} \beta\tau(B) \leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} \beta\tau(f^{-1}(A)) \leq \sup_{f(x) \in A \subseteq u}(U(A) - \beta\tau(f^{-1}(A)))$. So $1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u}(1 - U(A) + \beta\tau(f^{-1}(A)))$ and thus $\min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) \geq \inf_{f(x) \in A \subseteq u} \min(1, 1 - U(A) + \beta\tau(f^{-1}(A))) \geq \inf_{v \in P(Y)} \min(1, 1 - U(v) + \beta\tau(f^{-1}(v))) = \beta C(f)$. Hence, $\inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) \geq [f \in \beta C]$.

Secondly, we prove that $\beta C(f) \geq \beta H_2(f)$. From Corollary 4.1, we have $\beta C(f) = \inf_{u \in P(Y)} \min(1, 1 - U(u) + \beta\tau(f^{-1}(u))) \geq \inf_{u \in P(Y)} \min\left(1, 1 - \inf_{f(x) \in u} N^Y_{f(x)}(u) + \inf_{x \in f^{-1}(u)} \beta N^X_x(f^{-1}(u))\right) = \inf_{u \in P(Y)} \min\left(1, 1 - \inf_{x \in f^{-1}(u)} N^Y_{f(x)}(u) + \inf_{x \in f^{-1}(u)} \beta N^X_x(f^{-1}(u))\right) \geq \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) = \beta H_2(f)$

(c) We prove that $\models f \in \beta H_2 \leftrightarrow f \in \beta H_3$. From Theorem 4.2 (2) we have $\beta H_3(f) = \inf_{x \in X} \inf_{u \in P(Y)} \min\left(1, 1 - N^Y_{f(x)}(u) + \sup_{v \in P(X),\, f(v) \subseteq u} \beta N^X_x(v)\right) = = \inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \beta N^X_x(f^{-1}(u))) = \beta H_2(f)$.

(d) We prove that $\models f \in \beta H_4 \leftrightarrow f \in \beta H_5$. First, for any $B \in P(Y)$ one can deduce that $[f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \supseteq \beta - cl_X(f^{-1}(B))] = 1$, $[cl_Y(f(f^{-1}(B)) \subseteq cl_Y(B)] = 1$ and $[f^{-1}(cl_Y(f(f^{-1}(B))) \supseteq f^{-1}(cl_Y(B))] = 1$. Then from Lemma 1.2 (2) [6] we have, $[\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq [f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] \geq [f^{-1}(f(\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \geq [f(\beta - cl_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))]$. Therefore, $\beta H_5(f) = \inf_{B \in P(Y)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq \inf_{B \in P(Y)} [f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))]$

$\geq \inf_{B\in P(Y)} [f^{-1}(f(\beta - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \geq \inf_{B\in P(Y)} [f(\beta - cl_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))] \geq \inf_{A\in P(X)} [f(\beta - cl_X(A)) \subseteq cl_Y(f(A))] = \beta H_4(f)$.

Secondly, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, $[\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \leq [\beta - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] \leq [f(\beta - cl_X(A)) \subseteq f(f^{-1}(cl_Y(A)))] \leq [f(\beta - cl_X(A)) \subseteq cl_Y(f(A))]$. Thus, $\beta H_4(f) \geq \inf_{A\in P(X)} [\beta - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] \geq \inf_{B\in P(Y), B=f(A)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq \inf_{B\in P(Y)} [\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] = \beta H_5(f)$.

(e) We want to prove that $\models f \in \beta H_5 \leftrightarrow f \in \beta H_2$. $\beta H_5(f) = [\forall B(\beta - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))] = \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-\beta N_x(X \sim f^{-1}(B))) + 1 - \beta N_{f(x)}(Y \sim B)) = \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1 - \beta N_{f(x)}(Y \sim B) + \beta N_x(f^{-1}(Y \sim B))) = \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1 - \beta N_{f(x)}(u) + \beta N_x(f^{-1}(u))) = \beta H_2(f)$.

Remark 6.1. In the following theorem we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous or $c\alpha C$-continuous. Thus, the symbols $(\tau, U) - C(f)$, $(\tau_{\alpha N}, U) - C(f)$, $(\tau, U_{\alpha N}) - c\alpha C(f)$,… etc will be understood. Applying Theorem 3.4 (1) and Theorem 4.4 one can deduce the following theorem.

Theorem 6.2.

(1) $\models f \in c\beta H_2 \leftrightarrow f \in c\beta H_j$, $j = 3, 4, 5$;

(2) $\models f \in c\beta C \rightarrow f \in c\beta H_2$.

Proof. (1) It is similiar to the proof of (c), (d) and (e) in Theorem 6.1. (2) It is similiar to the proof of the first part in (b) in Theorem 6.1.

Theorem 6.3.

(1) $\models f \in (\tau, U_{\beta N}) - C \rightarrow f \in (\tau, U) - C$;

(2) $\models f \in (\tau, U) - c\beta C \rightarrow f \in (\tau_{\beta N}, U) - C$;

(3) $\models f \in (\tau, U) - C \rightarrow f \in (\tau, U) - c\beta C$.

7. Decompositions of fuzzy continuity in fuzzifying topology.

Theorem 7.1. *Let* (X, τ), (Y, U) *be two fuzzifying topological spaces. Then for each* $f \in Y^X$, $\models C(f) \rightarrow (\beta C(f) \wedge c\beta C(f))$.

Proof. The proof is obtainned from Theorem 3.3 (1).

Remark 7.1. In crisp setting i.e., if the underlying fuzzifying topology is the ordinary topology, one can have $\models (\beta C(f) \wedge c\beta C(f)) \rightarrow C(f)$.

But this statement may not true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 7.1. Let (X,τ) be the fuzzifying topological space defined in Counterexample 3.1, consider the identity functions f from (X,τ) onto (X,σ) where σ is a fuzzifying topology on X defined as follows:

$$\sigma(A)=\begin{cases}1, & A\in\{X,\varnothing,\{a,b\}\}\\ 0, & o.w\end{cases}$$

Then $\frac{7}{8}\wedge\frac{1}{8}=\beta C(f)\wedge c\beta C(f)\nleq C(f)=0$. Hence the statement $\models(\beta C(f)\wedge c\beta C(f))\rightarrow C(f)$ may not be true in fuzzifying setting.

Theorem 7.2. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$ then $\models C(f)\rightarrow(\beta C(f)\leftrightarrow c\beta C(f))$.*

Proof. $[\beta C(f)\rightarrow c\beta C(f)]=\min(1,1-\beta C(f)+c\beta C(f))\geq\beta C(f)\wedge c\beta C(f)$. Also, $[c\beta C(f)\rightarrow\beta C(f)]=\min(1,1-c\beta C(f)+\beta C(f))\geq c\beta C(f)\wedge\beta C(f)$. Then from Theorem 7.1 we have $c\beta C(f)\wedge\beta C(f)\geq C(f)$ and so the result holds.

Theorem 7.3. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$. If $[\beta\tau(f^{-1}(u))]=1$ or $[c\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models C(f)\leftrightarrow(\beta C(f)\wedge c\beta C(f))$.*

Proof. Now, we need to prove that $C(f)=\beta C(f)\wedge c\beta C(f)$. Applying Theorem 3.4 (2) we have, $\beta C(f)\wedge c\beta C(f)=\inf_{u\in P(Y)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))\ \wedge\inf_{u\in P(Y)}\min(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min(1,1-U(u)+\beta\tau(f^{-1}(u)))\ \wedge(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min(1,1-U(u)+(\beta\tau(f^{-1}(u)))\wedge c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min(1-U(u)+\tau(f^{-1}(u)))=C(f)$.

Theorem 7.4. *Let (X,τ), (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$. Then,*

(1) *if $[\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then*

$$\models\beta C(f)\rightarrow(c\beta C(f)\leftrightarrow C(f));$$

(2) *if $[c\beta\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models c\beta C(f)\rightarrow(\beta C(f)\leftrightarrow C(f))$.*

Proof. (1) Since $[\beta\tau(f^{-1}(u))]=1$ and so $[f^{-1}(u)\subseteq(f^{-1}(u))^{-\circ-}]=1$, then $[f^{-1}(u)\cap(f^{-1}(u))^{-\circ}\subseteq(f^{-1}(u))^{\circ}]=[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]$. Thus, $c\beta C(f)=\inf_{u\in P(Y)}\min(1,1-U(u)+c\beta\tau(f^{-1}(u)))=\inf_{u\in P(Y)}\min\left(1,1-U(u)+\left[f^{-1}(u)\cap(f^{-1}(u))^{-\circ}\subseteq(f^{-1}(u))^{\circ}\right]\right)=\inf_{u\in P(Y)}\min\left(1,1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}\right]\right)=\inf_{u\in P(Y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f)$

(2) Since $[c\beta\tau(f^{-1}(u))]=1$ one can deduce that $(f^{-1}(u))^{-\wedge-}=(f^{-1}(u))^{\circ}$. So, $\beta C(f)= \inf_{u\in P(y)} \min(1, 1-U(u)+\beta\tau(f^{-1}(u))) = \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{-\wedge-}\right]\right) = \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}\right]\right) = \inf_{u\in P(y)} \min(1, 1-U(u)+\tau(f^{-1}(u)))=C(f)$.

Theorem 7.5. *Let (X,τ), (Y,U) (Z,V) be three fuzzifying topological spaces. For any $f\in Y^X$, $g\in Z^Y$,*

(1) $\models \beta C(f)\to(C(g)\to\beta C(g\circ f))$;

(2) $\models C(g)\to(\beta C(f)\to\beta C(g\circ f))$;

(3) $\models c\beta C(f)\to(C(g)\to c\beta C(g\circ f))$;

(4) $\models C(g)\to(c\beta C(f)\to c\beta C(g\circ f))$

Proof. (1) We need to prove that $[\alpha C(f)]\le[C(g)\to\beta C(g\circ f)]$. If $c(g)\le[\beta C(g\circ f)]$, the result holds, if $c(g)>[\beta C(g\circ f)]$, then $c(g)-[\beta C(g\circ f)]= \inf_{v\in P(Z)} \min(1, 1-V(v)+U(g^{-1}(v))) - \inf_{v\in P(Z)} \min(1, 1-V(v)+\beta\tau((g\circ f)^{-1}(v))) \le \sup_{v\in P(Z)} (U(g^{-1}(v))-\beta\tau((g\circ f)^{-1}(v))) \le \sup_{u\in P(Z)} (U(u)-\beta\tau(f^{-1}(u)))$. Therefore, $[C(g)\to\beta C(g\circ f)]=\min(1, 1-[C(g)]+[\beta C(g\circ f)]) \ge \inf_{u\in P(Z)} \min(1, 1-U(u)+\beta\tau(f^{-1}(u)))=\beta C(f)$.

(2) $[C(g)\to(\beta C(f)\to\beta C(g\circ f))]=\left[\neg(C(g)\wedge\neg(\beta C(f)\to\beta C(g\circ f)))\right] =\left[\neg(C(g)\wedge \neg\neg(\beta C(f)\wedge\neg(\beta C(g\circ f))))\right]= \left[\neg(C(g)\wedge\beta C(f)\wedge\neg(\beta C(g\circ f)))\right] =\left[\neg(\beta C(f)\wedge C(g)\wedge\neg\beta C(g\circ f))\right]= \left[\neg(\beta C(f)\wedge\neg\neg(C(g)\wedge\neg(\beta C(g\circ f))))\right] =\left[\neg\left(\beta C(f)\wedge\neg(C(g)\to\beta C(g\circ f))\right)\right] =[\beta C(f)\to(C(g)\to\beta C(g\circ f))]=1$.

The proofs of (3) and (4) are similar to (1) and (2) respectively.

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