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Developable Surfaces through Spacelike Sweeping Surfaces in Minkowski 3–Space

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Abstract: We investigate a spacelike sweeping surface with rotation minimizing frames at Minkowski 3–Space $\mathbb{E}_1^3$. Then, we state many results related to the differential geometry of sweeping surfaces that are resulted from these frames. Subsequently, the problem of constructing spacelike/timelike developable surfaces from that sweeping surfaces is analysed.

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1 Introduction

Sweeping surfaces result from surfaces swept out using movement of the plane curve (profile curve or generatrix). This movement direction of this plane at the space has the same direction of the normal to this plane. Sweeping is an essential and famous tool that is used at geometric modeling. This idea depends on choosing several geometrical objects (such as generators), that is moved along the spine curve (trajectory) at the space. This evolution includes the movement at the space with the deformation of intrinsic resulted in the sweep object . The type of the sweep objects depends on choosing both generators and trajectories. Therefore, sweeping the curve through another curve creates the sweeping surface. Many familiar names of sweeping surface are known as tubular surface, pipe surface, string, well as canal surface ([1],[2],[3],[4],[5], [6])

One of the most appropriate methods to analyze the curve and surface at differential geometry is Serret–Frenet frame, in addition to other frame fields such as rotation minimizing frame (RMF) or Bishop frame [7]. Some applications of the Bishop frame can be found in ([8],[9],[10]). Like the Bishop frame at the Euclidean space, the Minkowski version of this frame is called the Minkowski Bishop frame which is used at Minkowski geometry. Investigating the space curve, shows that using the Minkowski Bishop frame through the curve is more appropriatethan using the Serret–Frenet frame type at Lorentzian space. Many researches interest in Minkowski Bishop frame, such as in ([10][11],[12],[13]).

In this paper, the Bishop frame along a spacelike curve is established and the local differential geometry of spacelike sweeping surface in Minkowski 3-space is developed. Then, several results related to the differential geometry of the spacelike sweeping surface that is generated by this frame are summarized. As a consequence, the necessary and sufficient condition of the spacelike sweeping surfaces to become spacelike/timelike developable ruled surfaces is given. The present paper focuses on the associated developable surface to become cylinder, cone or tangent surface. At the end, we present some examples to demonstrate timelike developable surfaces related to common line of curvature.

2 Preliminaries

In this section some important notions on Minkowski 3-space are introduced, for more definitions see ([14],[15]).

Suppose $\mathbb{R}^3 = \{(a_1,a_2,a_3) \mid, a_i \in \mathbb{R}$ (i=1, 2, 3)$\}$ is a 3-dimensional Cartesian space. For all $\mathbf{a} = (a_1,a_2,a_3)$, and $\mathbf{b} = (b_1,b_2,b_3) \in \mathbb{R}^3$, the pseudo scalar product of $\mathbf{a}$, and

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$\mathbf{b}$ is given by

$$<\mathbf{a},\mathbf{b}>=a_1b_1+a_2b_2-a_3b_3. \tag{1}$$

We call $(\mathbb{R}^3,<,>)$ Minkowski 3-space. It is written $\mathbb{E}_1^3$ rather than $(\mathbb{R}^3,<,>)$. The non-zero vector $\mathbf{a}\in\mathbb{E}_1^3$ is called spacelike, lightlike or timelike in case $<\mathbf{a},\mathbf{a}>>0$, $<\mathbf{a},\mathbf{a}>=0$ or $<\mathbf{a},\mathbf{a}><0$ in the same order. The norm of the vector $\mathbf{a}\in\mathbb{E}_1^3$ is identified to be $\|\mathbf{a}\|=\sqrt{|<\mathbf{a},\mathbf{a}>|}$. For any two vectors $\mathbf{a}$, $\mathbf{c}\in\mathbb{E}_1^3$, we define a vector $\mathbf{a}\times\mathbf{c}$ by

$$\mathbf{a}\times\mathbf{c}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}=((a_2c_3-a_3c_2),(a_3c_1-a_1c_3),-(a_1c_2-a_2c_1)), \tag{2}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the canonical basis of $\mathbb{E}_1^3$. We can easily check the following

$$\det(\mathbf{a},\mathbf{c},\mathbf{b})=<\mathbf{a}\times\mathbf{c},\mathbf{b}>, \tag{3}$$

so $\mathbf{a}\times\mathbf{c}$ is pseudo orthogonal to any $\mathbf{b}=(b_1,b_2,b_3)\in\mathbb{E}_1^3$.

Let $\beta=\beta(s)$ defines the unit speed spacelike curve, $\kappa(s)$ and $\tau(s)$ define the natural curvature and torsion of $\beta(s)$, in the same order. Suppose $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is a Serret–Frenet frame associated with $\beta(s)$. In this study, $\mathbf{B}(s)$ will be taken as timelike. In case of spacelike, the same processes will be used. For every point of $\beta(s)$, the corresponding Serret-Frenet formulae reads:

$$\begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix}=\begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & \tau(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}=\omega\times\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}, \tag{4}$$

with $\omega(s)=-\tau\mathbf{T}+\kappa\mathbf{B}$ defines Darboux vector of the Serret–Frenet frame. In this paper, dash defines the derivation respecting to s the arc-length parameter. It is easy to see that

$$\begin{array}{c}<\mathbf{T},\mathbf{T}>=<\mathbf{N},\mathbf{N}>=1,\ <\mathbf{B},\mathbf{B}>=-1,\\ \mathbf{T}\times\mathbf{N}=-\mathbf{B},\ \mathbf{T}\times\mathbf{B}=-\mathbf{N},\ \mathbf{N}\times\mathbf{B}=\mathbf{T}.\end{array} \tag{5}$$

Definition 2.1. The pseudo orthogonal moving frame $\{\xi_1$, ξ_3, $\xi_3\}$, through the non null space curve $\alpha(s)$, defines the rotation minimizing frame (RMF) respecting to ξ_1 in case its angular velocity ω insures $<\omega,\xi_1>=0$ or as equivalent, the derivatives of ξ_2 and ξ_3 are both parallel to ξ_1. Analogously, characterization in case ξ_2 or ξ_3 is selected to be the reference direction.

As stated in the Definition 2.1 , it is observed that the Serret–Frenet frame is RMF respecting to the principal normal $\mathbf{N}$, but not respecting to both the tangent $\mathbf{T}$ and binormal $\mathbf{B}$. Even in case the Serret–Frenet frame is not RMF respecting to $\mathbf{T}$, it is easy to derive such a RMF from it. The new normal plane vectors $(\mathbf{N}_1,\mathbf{N}_2)$ are given along the rotation of $(\mathbf{N},\mathbf{B})$ by

$$\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\vartheta & \sinh\vartheta\\ 0 & \sinh\vartheta & \cosh\vartheta\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}, \tag{6}$$

with a particular spacelike angle $\vartheta(s)\geq 0$. Also, the set $\{\mathbf{T}_1, \mathbf{N}_1, \mathbf{N}_2\}$ will be called as RMF or Bishop frame. The RMF vector satisfies the following

$$\begin{array}{c}<\mathbf{T}_1,\mathbf{T}_1>=<\mathbf{N}_1,\mathbf{N}_1>=1,\ <\mathbf{N}_2,\mathbf{N}_2>=-1,\\ \mathbf{T}_1\times\mathbf{N}_1=-\mathbf{N}_2,\ \mathbf{N}_1\times\mathbf{N}_2=\mathbf{T}_1,\ \mathbf{N}_2\times\mathbf{T}_1=\mathbf{N}_1.\end{array} \tag{7}$$

Then, the alternative frame equations are

$$\begin{pmatrix}\mathbf{T}_1'\\ \mathbf{N}_1'\\ \mathbf{N}_2'\end{pmatrix}=\begin{pmatrix}0 & \kappa_1(s) & -\kappa_2(s)\\ -\kappa_1(s) & 0 & 0\\ -\kappa_2(s) & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\widetilde{\omega}\times\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}, \tag{8}$$

where $\widetilde{\omega}(s)=-\kappa_2\mathbf{N}_1+\kappa_1\mathbf{N}_2$ is RMF Darboux vector. Here, the Bishop curvatures are defined by $\kappa_1(s)=\kappa\cosh\vartheta$, and $\kappa_2(s)=\kappa\sinh\vartheta$. One can show that

$$\left.\begin{array}{l}\kappa_1^2-\kappa_2^2=\kappa^2, \text{ and } \vartheta=\tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right);\ \kappa_1\neq 0,\\ \vartheta(s)=-\int\limits_{s_0}^{s}\tau ds+\vartheta_0,\ \vartheta_0=\vartheta(0).\end{array}\right\} \tag{9}$$

Comparing Eq. 4 with Eq. 8 we observe that the relative velocity is

$$\widetilde{\omega}(s)-\omega(s)=\tau\mathbf{T}. \tag{10}$$

It is clear that the Serret–Frenet frame includes the additional rotation around the tangent, where the speed is the same as the torsion $\tau(s)$. This note clarifies the integration form of Eq. 9 to compute the RMF using the correction of the unwanted rotation of the Serret–Frenet frame. Thus, the Serret–Frenet frame coincides with the RMF of a planar curve, where $\tau=0$. It is fully compatible with Klok's result [8].

A surface M in $\mathbb{E}_1^3$ is denoted as

$$M:\Upsilon(s,u)=(x_1((s,u),x_2((s,u),(x_3((s,u)),\ \ (s,u)\in D\subseteq\mathbb{R}^2. \tag{11}$$

Suppose $\mathbf{U}$ is the standard unit normal vector field at the surface M determined as $\mathbf{U}=\frac{\Upsilon_s\times\Upsilon_u}{\|\Upsilon_s\times\Upsilon_u\|}$, where, $\Upsilon_i=\frac{\partial\Upsilon}{\partial i}$. Therefore the metric (first fundamental form) I of the surface M is given as

$$I=g_{11}ds^2+2g_{12}dsdu+g_{22}du^2, \tag{12}$$

where $g_{11}=<\Upsilon_s,\Upsilon_s>$, $g_{12}=<\Upsilon_s,\Upsilon_u>$ and , $g_{22}=<\Upsilon_u,\Upsilon_u>$. The second fundamental form II of M is

$$II=h_{11}ds^2+2h_{12}dsdu+h_{22}du^2, \tag{13}$$

where $h_{11}=<\Upsilon_{ss},\mathbf{N}>$, $h_{12}=<\Upsilon_{su},\mathbf{N}>$ and $h_{22}=<\Upsilon_{uu},\mathbf{N}>$.

Definition 2.2. The sweeping surface through $\beta(s)$ is the surface given by

$$M : \Upsilon(s,u) = \beta(s) + T(s)\mathbf{x}(u) = \alpha(s) + x_1(u)\mathbf{N}_1(s) + x_2(u)\mathbf{N}_2(s), \quad (14)$$

where $\beta(s)$ named the (at least C^1-continuous) spine curve, $0 \leq s \leq T$, s indicates the arc length parameter. $\mathbf{x}(u)$ indicates the planar profile (cross-section) curve defined as the parametric presentation $\mathbf{x}(u) = (0, x_1(u), x_2(u))^t$, the symbol 't' indicates the transposition, in addition to other $u \in I \subseteq \mathbb{R}$. A semi orthogonal matrix $T(s) = \{\mathbf{T}_1, \mathbf{N}_1, \mathbf{N}_2\}$ defines the RMF through $\beta(s)$. Kinematically, the sweeping surface M is created by the movement of the profile curve $\mathbf{x}(u)$ through the spine curve $\beta(s)$ with the orientation of $F(s)$. The profile curve $\mathbf{x}(u)$ is in 2D or 3D space that passes along the spine curve $\beta(s)$ through sweeping. Interestingly, RMF allows for a simple characterization of spine curve.

Definition 2.3. The surface at Minkowski 3-space $\mathbb{E}_1^3$ is called the timelike surface in case an induced metric at the surface is the Lorentz metric and is called the spacelike surface in case an induced metric at the surface is the positive definite Riemannian metric, which means the normal vector oat spacelike (timelike) surface is the timelike (spacelike) vector.

3 Spacelike sweeping surface

We investigate the spacelike sweeping surface at Minkowski 3-space $\mathbb{E}_1^3$ in addition to consider the planar profile spacelike curve as $\mathbf{x}(u) = (0, \sinh u, \cosh u)$. Using Eq. 11 , we have

$$M : \Upsilon(s,u) = \beta(s) + \sinh u\mathbf{N}_1 + \cosh u\mathbf{N}_2. \quad (15)$$

Using Eq. 8 , it is calculated that

$$\left.\begin{aligned} \Upsilon_s(s,u) &= (1 - \kappa_1 \sinh u - \kappa_2 \cosh u)\,\mathbf{T}, \\ \Upsilon_u(s,u) &= \cosh u\mathbf{N}_1 + \sinh u\mathbf{N}_2. \end{aligned}\right\} \quad (16)$$

In consequence of Eqs. 15 and 16 we got:

Proposition 3.1. Considering the point $\mathbf{x}$ in a normal plane of the spine curve $\beta(s)$, tangent vector of its trajectory $\beta(s) + T(s)\mathbf{x}(u)$, which is generated by the RMF, is usually parallel to the tangent vector of the spine curve.

By simple calculations, we have the following:

$$\left.\begin{aligned} g_{11} &= (1 - \kappa_1 \sinh u - \kappa_2 \cosh u)^2, \\ g_{12} &= 0, \; g_{22} = 1. \end{aligned}\right\} \quad (17)$$

The surface M has the unit normal vector as

$$\mathbf{U}(s,u) = \frac{\Upsilon_u \times \Upsilon_s}{\|\Upsilon_u \times \Upsilon_s\|} = \cosh u\mathbf{N}_2 + \sinh u\mathbf{N}_1. \quad (18)$$

It is noted that $\|\mathbf{U}(s,u)\|^2 = -1$ which means M is the spacelike surface. Furthermore, we got:

$$\left.\begin{aligned} \Upsilon_{ss} &= -\left(\kappa_1' \sinh u + \kappa_2' \cosh u\right)\mathbf{T}_1 \\ &\quad + (1 - \kappa_1 \sinh u - \kappa_2 \cosh u)(\kappa_1 \mathbf{N}_1 - \kappa_2 \mathbf{N}_2), \\ \Upsilon_{su} &= -(\kappa_1 \cosh u + \kappa_2 \sinh u)\mathbf{T}_1, \\ \Upsilon_{uu} &= \sinh u\mathbf{N}_1 + \cosh u\mathbf{N}_2. \end{aligned}\right\} \quad (19)$$

Thus, we arrive by means of Eqs. 18 , and 19 , at

$$\left.\begin{aligned} h_{11} &= (1 - \kappa_1 \sinh u - \kappa_2 \cosh u)(\kappa_1 \sinh u + \kappa_2 \cosh u), \\ h_{12} &= 0, \; h_{22} = -1. \end{aligned}\right\} \quad (20)$$

Thus, the parametric curves of M are lines of curvature, that is, $g_{12} = h_{12} = 0$. Furthermore, the isoparametric curve

$$\gamma(u) := \Upsilon(u, s_0) = \beta(s_0) + \sinh u\mathbf{N}_1(s_0) + \cosh u\mathbf{N}_2(s_0), \quad (21)$$

defines the planar unit speed spacelike line of curvature. The spacelike unit tangent vector to $\gamma(u)$ is given here by

$$\mathbf{t}_\gamma(u) = \cosh u\mathbf{N}_1(s_0) + \sinh u\mathbf{N}_2(s_0), \quad (22)$$

and then the unit principal normal vector of $\gamma(u)$ is given by

$$\mathbf{n}_\gamma(u) = \mathbf{T}_1(s_0) \times \mathbf{t}_\gamma(u) = \mathbf{U}(s_0, u). \quad (23)$$

Consequently, surface normal $\mathbf{U}$ and normal $\mathbf{n}_\gamma(u)$ of $\gamma(u)$ are identical, the curve $\gamma(u)$ is the planar geodesic spacelike line of curvature and can't be asymptotic curve.

3.1 Local singularities and convexity

At this section, singularities and principal curvatures of M are investigated. The point of the spacelike surface M is named singular if and only if

$$\|\Upsilon_u \times \Upsilon_s\| = 1 - \kappa_1 \sinh u - \kappa_2 \cosh u = 0,$$

which implies that

$$\sinh u = \frac{\kappa_1 \pm \kappa_2 \sqrt{1 + \kappa_1^2 - \kappa_2^2}}{\kappa_1^2 - \kappa_2^2}$$

and

$$\cosh u = \frac{-\kappa_2 \mp \kappa_1 \sqrt{1 + \kappa_1^2 - \kappa_2^2}}{\kappa_1^2 - \kappa_2^2},$$

with $\kappa_1^2 - \kappa_2^2 \neq 0$. Hence, the singular point of M is the image of the curve

$$\alpha(s) = \beta(s) + \frac{\kappa_1 \pm \kappa_2 \sqrt{1 + \kappa_1^2 - \kappa_2^2}}{\kappa_1^2 - \kappa_2^2}\mathbf{N}_1 + \frac{-\kappa_2 \mp \kappa_1 \sqrt{1 + \kappa_1^2 - \kappa_2^2}}{\kappa_1^2 - \kappa_2^2}\mathbf{N}_2. \quad (24)$$

with $\kappa_1^2 - \kappa_2^2 \neq 0$. Consequently, the next corollary will be given:

Corollary 3.1. The spacelike sweeping surface as defined in Eq. 15 has no singular point if following equation holds

$$1 - \kappa_1 \sinh u - \kappa_2 \cosh u \neq 0, \tag{25}$$

for every s and u.

To study the shape of M, the distribution of the Gaussian curvature $K(s,u) = -\chi_1\chi_2$ will be examined. Here, the $\chi_i(s,u)$ $(i = 1,2)$ are the principal curvatures of the sweeping surface. In view of Eqs. 6 and 18 , the normal

$$\mathbf{U}(s,u) = \sinh\psi\mathbf{N} + \cosh\psi\mathbf{B};\ (\psi = u+\theta), \tag{26}$$

lies at the normal plane of the spine curve $\beta(s)$. The principal curvature χ_1 is

$$\chi_1 := \frac{\|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}\|}{\|\dot{\mathbf{x}}\|^3} = \cosh^2 u - \sinh^2 u = 1; \left(. = \frac{d}{du}\right). \tag{27}$$

Furthermore, the curvature of the isoparametric curves u=const is

$$\chi(s,u) := \frac{\|\Upsilon_s \times \Upsilon_{ss}\|}{\|\Upsilon_s\|^3} = \frac{\kappa}{1 - \kappa_1 \sinh u - \kappa_2 \cosh u}. \tag{28}$$

At the same time, the relation between the principal curvature χ_2 and the curvature $\chi(s,u)$ via Meusnier's formula [3] is

$$\chi_2 = \chi(s,u)\sinh\psi, \tag{29}$$

Using Eqs. 27 and 29, we get

$$K(s,u) = -\chi(s,u)\sinh\psi \tag{30}$$

For the shape characterization of M, we tried finding curves at M which resulted from using the parabolic points, points that have zero Gaussian curvature. These kinds of curves divide the surface to elliptic parts ($K > 0$, locally convex) and hyperbolic parts ($K < 0$, thus nonconvex). Under Eq. 30 , there are two potential situations that cause parabolic points:

– Case (1) occurs where $\chi(s,u) = 0$, $\kappa = 0$. In this case, if the spine curve $\beta(s)$ is degenerate to the straight line. Therefore, flat or an infection point of $\beta(s)$ leads to the isoparametric parabolic curve u=constant at M.
– Case (2) occurs when $\psi = 0$, $\mathbf{N}(s,u) \parallel \mathbf{B}$. Therefore, the curve $\beta(s)$ is both the line of curvature and asymptotic of the sweeping surface.

Corollary 3.2. A spacelike sweeping surface defined as in Eq. 15 has no singular points in case the spine curve is the non-asymptotic curve.

4 Developable surfaces

Developable surfaces may be presented as specific cases of ruled surfaces. Such surfaces are used widely, for example, in the manufacture of automobile body parts, airplane wings, and ship hulls ([16], [17],[18],[19], [20]). Then, we analyze this case that a profile curve $\mathbf{x}(u)$ degenerates to the line. Therefore, a timelike developable surface is written as

$$\Omega^{\perp}: \mathbf{P}(s,u) = \beta(s) + u\mathbf{N}_2(s),\ u \in \mathbb{R}, \tag{31}$$

We also have

$$\Omega: \mathbf{Q}(s,u) = \beta(s) + u\mathbf{N}_1(s),\ u \in \mathbb{R}, \tag{32}$$

spacelike surface. It is possible to show $\mathbf{P}(s,0) = \beta(s)$ (resp. $\mathbf{Q}(s,0) = \beta(s)$), such that this surface Ω (resp. $\Omega^{\perp}$) interpolates the curve $\beta(s)$. Furthermore, since

$$\mathbf{P}_s \times \mathbf{P}_u := -(1 - u\kappa_2)\mathbf{N}_1(s), \tag{33}$$

then $\Omega^{\perp}$ is the normal developable surface of Ω along $\beta(s)$. Therefore, the surface $\Omega^{\perp}$ (resp. Ω) interpolates the curve $\beta(s)$, where $\beta(s)$ is the spacelike line of curvature. Under the above notations, we have the following theories:

Theorem 4.1. Suppose M is a spacelike sweeping surface defined using Eq. 31. Then we have:
(1) developable surfaces Ω and $\Omega^{\perp}$ intersect orthogonally along $\beta(s)$,
(2) spacelike curve $\beta(s)$ is a line of curvature on Ω and $\Omega^{\perp}$.

Theorem 4.2. (Existence and uniqueness). Considering the previous notations there is a unique timelike developable surface presented using Eq.31.

Proof. For the existence, we have the developable represented by Eq. 31. Moreover, because $\Omega^{\perp}$ is the ruled surface, it is assumed that

$$\left.\begin{aligned} &\Omega^{\perp}: \mathbf{P}(s,u) = \beta(s) + u\mathbf{r}(s),\ u \in \mathbb{R}, \\ &\mathbf{r}(s) = r_1(s)\mathbf{N}_1 + r_2(s)\mathbf{N}_2 + r_3(s)\mathbf{T}, \\ &\|\mathbf{r}(s)\|^2 = r_1^2 - r_2^2 + r_3^2 = -1,\ \mathbf{r}'(s) \neq \mathbf{0}, \end{aligned}\right\} \tag{34}$$

where the components $r_i = r_i(s)$ (i=1, 2, 3) are scalar functions of the arc length parameter s of the base curve $\beta(s)$. It can be immediately seen that $\Omega^{\perp}$ is developable if and only if

$$\begin{aligned} &\det(\beta', \mathbf{r}, \mathbf{r}') = 0 \Leftrightarrow \\ &\qquad r_1 r_2' - r_2 r_1' - \kappa r_3 (r_2 \cosh\vartheta + r_1 \sinh\vartheta) = 0. \end{aligned} \tag{35}$$

On the other hand, in view of Eq. 33, we have

$$(\mathbf{P}_s \times \mathbf{P}_u)(s,u) = -\phi(s,u)\mathbf{N}_1, \tag{36}$$

where $\phi(s,u)$ is a differentiable function. Furthermore, the normal vector $\mathbf{P}_s \times \mathbf{P}_u$ about the point $(s,0)$ is

$$(\mathbf{P}_s \times \mathbf{P}_u)(s,0) = -r_2\mathbf{N}_1 - r_1\mathbf{N}_2. \quad (37)$$

Thus, from Eqs. 35 and 37, one finds that:

$$r_1 = 0, \text{ and } r_2 = \phi(s,0), \quad (38)$$

which follows from Eq. 35 that $\kappa r_2 r_3 \cosh\vartheta = 0$, which leads to $r_2 r_3 = 0$, with $\kappa \neq 0$. In case $(s,0)$ defines the regular point (which means $\phi(s,0) \neq 0$), $r_2(s) \neq 0$, and $r_3 = 0$. Then, $\mathbf{r}(s) = \mathbf{N}_2$. In other words, the direction of $\mathbf{r}(s)$ equals the direction of $\mathbf{N}_2(s)$.

Moreover, consider $\Omega^{\perp}$ has the singular point at $(s_0,0)$. Therefore $\phi(s_0,0) = r_2(s_0) = r_1(s_0) = 0$, and we have $\mathbf{r}(s_0) = r_3(s_0)\mathbf{T}(s_0)$. In case $\Omega^{\perp}$ is developable along $\beta(s)$ that is regular, if the singular point $\beta(s_0) \in \Omega^{\perp}$, then there is a point $\beta(s)$ in any neighborhood of $\beta(s_0)$ that is the uniqueness of the $\Omega^{\perp}$ holds at $\beta(s)$. Passing to the limit $s \to s_0$, uniqueness of the developable at s_0. Consider there is an open interval $J \subseteq I$ such that $\Omega^{\perp}$ is singular at $\beta(s)$ for all $s \in J$. Therefore $\mathbf{P}(s,u) = \beta(s) + ur_3(s)\mathbf{T}(s)$ for all $s \in J$. In other words, $r_1(s) = r_2(s) = 0$ for $s \in J$. Then, we have

$$(\mathbf{P}_s \times \mathbf{P}_u)(s,u) = -\kappa u r_3^2 (\cosh\vartheta\mathbf{N}_2 - \sinh\vartheta\mathbf{N}_1).$$

Hence the previous vector is directed to $\mathbf{N}_1$, which means $\mathbf{P}_s \times \mathbf{P}_u \parallel \mathbf{N}_2(s)$ if and only if $\vartheta = 0$ for all $s \in J$. Here, $\mathbf{r}(s) = \pm\mathbf{N}_2$. In other words, uniqueness holds .
We can classify singularities of the timelike developable surface $\Omega^{\perp}$ using κ_2 ([20][21],[22]):

Theorem 4.3. Suppose $\Omega^{\perp}$ is the timelike developable described using Eq. 31, Then
(1) $\Omega^{\perp}$ is locally diffeomorphic to Cuspidal edge at (s_0,u_0) if and only if $\kappa_2(s_0) = 0$ and $\kappa_2'(s_0) \neq 0$;
(2) $\Omega^{\perp}$ is locally diffeomorphic to Swallowtail at (s_0,u_0) if and only if $\kappa_2(s_0) \neq 0$, and $\frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} \neq 0$.

Proof. If there is a parameter s_0 such that $\kappa_2(s_0) = 0$, and $u_0' = \frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} \neq 0$ ($\kappa_2'(s_0) \neq 0$), then $\Omega^{\perp}$ is locally diffeomorphic to Cuspidal edge at (s_0,u_0). Then, assertion (1) holds. In Addition, if there is a parameter s_0 that is $u_0 = \frac{1}{\kappa_2(s_0)} \neq 0$, $u_0' = \frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} = 0$, and $\left(\frac{1}{\kappa_2(s_0)}\right)'' \neq 0$, then $\Omega^{\perp}$ is locally diffeomorphic to Swallowtail at (s_0,u_0), assertion (2) holds.

Example 4.1. Given the spacelike helix

$$\beta(s) = (a\cosh\frac{s}{c}, \frac{bs}{c}, a\sinh\frac{s}{c}),$$

where $a > 0$, $b \neq 0$, and $b^2 - a^2 = c^2$. Clearly,

$$\left.\begin{array}{r}\mathbf{T}(s) = (\frac{a}{c}\sinh\frac{s}{c}, \frac{b}{c}, \frac{a}{c}\cosh\frac{s}{c}),\\ \mathbf{N}(s) = (\cosh\frac{s}{c}, 0, \sinh\frac{s}{c}),\\ \mathbf{B}(s) = (-\frac{b}{c}\sinh\frac{s}{c}, -\frac{a}{c}, -\frac{b}{c}\cosh\frac{s}{c}),\\ \kappa(s) = \frac{a}{c^2}, \text{ and } \tau(s) = -\frac{b}{c^2}.\end{array}\right\}$$

Then $\vartheta(s) = \frac{b}{c^2}s + \vartheta_0$. If we choose $\vartheta_0 = 0$, $a = \sqrt{2}$, and $b = \sqrt{3}$ for example, we have

$$\kappa_1(s) = \sqrt{2}\cosh\sqrt{3}s, \text{ and } \kappa_2(s) = \sqrt{2}\sinh\sqrt{3}s.$$

We also have that

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\sqrt{3}s & \sinh\sqrt{3}s\\ 0 & \sinh\sqrt{3}s & \cosh\sqrt{3}s\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix},$$

Then, we have that

$$\mathbf{N}_1 = \begin{pmatrix}N_{11}\\ N_{12}\\ N_{13}\end{pmatrix} = \begin{pmatrix}\cosh(\sqrt{3}s)\cosh s - \sqrt{3}\sinh(\sqrt{3}s)\sinh s\\ -\sqrt{2}\sinh(\sqrt{3}s)\\ \cosh(\sqrt{3}s)\sinh s - \sqrt{3}\sinh(\sqrt{3}s)\cosh s\end{pmatrix},$$

$$\mathbf{N}_2 = \begin{pmatrix}N_{21}\\ N_{22}\\ N_{23}\end{pmatrix} = \begin{pmatrix}\sinh(\sqrt{3}s)\cosh s - \sqrt{3}\cosh(\sqrt{3}s)\sinh s\\ -\sqrt{2}\cosh(\sqrt{3}s)\\ \sinh(\sqrt{3}s)\sinh s - \sqrt{3}\cosh(\sqrt{3}s)\cosh s\end{pmatrix}.$$

(1) If $s_0 = 0$, then $\kappa_2(s_0) = 0$, and $\kappa_2'(s_0) \neq 0$. The spacelike developable surface

$$\mathbf{P}(s,u) = \left(\sqrt{2}\cosh s + uN_{21}, \sqrt{3}s + uN_{22}, \sqrt{2}\sinh s + uN_{23}\right),$$

$$u \in \mathbb{R},$$

is locally diffeomorphic to Cuspidal edge, (Figure 1); $-2 \leq s \leq 2$, and $-1 \leq u \leq 1$.
(2) If $s_0 = 0$, then $\kappa_1(s_0) \neq 0$, and $\kappa_2'(s_0) = 0$. The timelike developable surface

$$\mathbf{Q}(s,u) = \left(\sqrt{2}\cosh s + uN_{11}, \sqrt{3}s + uN_{12}, \sqrt{2}\sinh s + uN_{13}\right),$$

$$u \in \mathbb{R},$$

is locally diffeomorphic to Swallowtail, (Figure 2); $-.4 \leq s \leq .4$, and $-.3 \leq u \leq .3$

Fig. 1: Locally diffeomorphic to Cuspidal edge

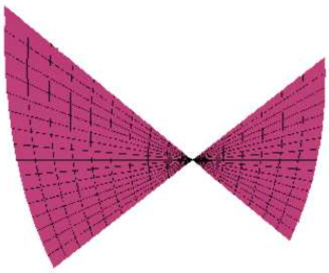

Fig. 2: Locally diffeomorphic to Swallowtail

Now, we the conditions in case the developable surface Ω defines cylinder, cone or tangent surface, in the same order. Since $\Omega^{\perp}$ is a developable surface, then

$$\det(\beta',\mathbf{N}_2,\mathbf{N}_2')=0 \Leftrightarrow <\beta',\mathbf{N}_2\times\mathbf{N}_2'>=0. \quad (39)$$

The first case when,

$$\mathbf{N}_2\times\mathbf{N}_2'=\mathbf{0} \Leftrightarrow \kappa\sinh\vartheta\mathbf{N}_1=\mathbf{0}. \quad (40)$$

Here, M is specified as the cylindrical surface. Because $\mathbf{N}_1$ is not zero spacelike unit vector, $\Omega^{\perp}$ is the timelike cylindrical surface if and only if $\sinh\vartheta=0 \Leftrightarrow \vartheta(s)=0$. In this case, $\vartheta'=0$, therefore $\tau=0$. Specifically, the curve is the planar spacelike curve and the corresponding surface is the binormal timelike surface.

Corollary 4.1. A timelike developable surface $\Omega^{\perp}$ defines the cylinder surface if and only if $\vartheta(s)=0$.

Using similar procedure, we have:

$$\mathbf{N}_2\times\mathbf{N}_2'\neq\mathbf{0}. \quad (41)$$

It leads to $\Omega^{\perp}$ is non-cylindrical surface. Then, the first derivative of the directrix is

$$\beta'(s)=\mathbf{C}'(s)+\mu(s)\mathbf{N}_2'(s)+\mu'(s)\mathbf{N}_2(s), \quad (42)$$

above $\mathbf{C}'$ defines first derivative of the striction curve $\mu(s)$ defines the smooth function [3]. Using Eq. 42 and Eq. 39 results in:

$$<\mathbf{N}_2\times\mathbf{N}_2',\mathbf{C}'>=0. \quad (43)$$

Similarly, the two possible cases that satisfy Eq. 43 are: first, when the first derivative of the striction curve is $\mathbf{C}'=0$. In geometry, it means the striction curve degenerates to the point, and $\Omega^{\perp}$ is the cone. In this case, from Eq. 42, we get $\mu\kappa_2=-1$, $\mu'=0$, which imply that

$$\mu=const.=-\frac{1}{\kappa\sinh\vartheta} \Leftrightarrow \kappa\sinh\vartheta=\kappa_0\sinh\vartheta_0, \quad (44)$$

where $\vartheta_0=\vartheta(0)$, and $\kappa_0=\kappa(0)$. Then, in case ϑ is a constant, $\tau=0$, the curve is the planar curve with the constant curvature. Similarly, in case κ is constant, we have $\tau=0$ and θ is a constant. Therefore the curve $\alpha(s)$ defines arc of a spacelike circle.

Corollary 4.2. A timelike developable surface $\Omega^{\perp}$ defines the cone if and only if $\kappa\sinh\vartheta=\kappa_0\sinh\vartheta_0$; $\vartheta_0=\vartheta(0)$, and $\kappa_0=\kappa(0)$.

The second case is $\mathbf{C}'\neq 0$, which means $\kappa\sinh\vartheta\neq\kappa_0\sinh\vartheta_0$. From Eq. 43, $\mathbf{C}'$ is perpendicular to $\mathbf{N}_2\times\mathbf{N}_2'$, then $\mathbf{C}'$ is at the plane generated by $\mathbf{N}_2$ and $\mathbf{N}_2'$. The condition of $\mathbf{C}$ to be striction curve is when $\mathbf{C}'$ and $\mathbf{N}_2'$ are perpendicular to each other. Then, it is concluded that the ruling is parallel to the first derivative of the striction curve, that is tangent of the striction curve. This ruled surface is defined as the tangent ruled surface.

Corollary 4.3. A timelike developable surface $\Omega^{\perp}$ is the tangent surface if and only if $\kappa\sinh\vartheta\neq\kappa_0\sinh\vartheta_0$; $\vartheta_0=\vartheta(0)$, and $\kappa_0=\kappa(0)$.

4.1 The spacelike developable surface Ω

We consider the case of the spacelike developable surface Ω. In similar arguments, we have

$$\det(\beta',\mathbf{N}_1,\mathbf{N}_1')=0 \Leftrightarrow <\beta',\mathbf{N}_1\times\mathbf{N}_1'>=0. \quad (45)$$

The first case when,

$$\mathbf{N}_1\times\mathbf{N}_1'=\mathbf{0} \Leftrightarrow \kappa\cosh\vartheta\mathbf{N}_2=\mathbf{0} \Leftrightarrow \kappa\cosh\vartheta=0. \quad (46)$$

In the equation above, since $\cosh\vartheta\neq 0$, we get $\kappa=0$. Consequently, there is no spacelike cylindrical ruled surface as defined by Eq. 32. Indeed, we can also have the following:

Corollary 4.4. Let Ω the spacelike developable surface defined using Eq. 32. Therefore

(1) Ω is the cone if and only if $\kappa\cosh\vartheta=\kappa_0\cosh\vartheta_0$; $\vartheta_0=\vartheta(0)$, and $\kappa_0=\kappa(0)$,

(2) Ω is the tangent surface if and only if $\kappa\cosh\vartheta\neq\kappa_0\cosh\vartheta_0$; $\vartheta_0=\vartheta(0)$, and $\kappa_0=\kappa(0)$.

Thus, it is associated with the difference at some signs of equations and previous results.

4.2 Examples

The construction of timelike developable surfaces will be discussed with the given curve as the spacelike line of curvature.

Example 4.2. Using Example 4.1, we have the following: If we choose $\vartheta_0=-1$, we have $\kappa\sinh\vartheta\neq\kappa_0\sinh\vartheta_0$. According to Corollary 4.3, the surface

$$\Omega^{\perp}:\mathbf{P}(s,u)=(a\cosh\frac{s}{c},,\frac{bs}{c},a\sinh\frac{s}{c})+u\mathbf{N}_2(s),$$

is a timelike tangent developable surface, where

$$\mathbf{N}_2(s)=\begin{pmatrix}\sinh\left(\frac{sb}{c^2}\right)\cosh\frac{s}{c}-\frac{b}{c}\cosh\left(\frac{sb}{c^2}\right)\sinh\frac{s}{c}\\ -\frac{a}{c}\cosh\left(\frac{sb}{c^2}\right)\\ \sinh\left(\frac{sb}{c^2}\right)\sinh\frac{s}{c}-\frac{b}{c}\cosh\left(\frac{sb}{c^2}\right)\cosh\frac{s}{c}\end{pmatrix}.$$

Choosing $a = \sqrt{3}$, $b = \sqrt{5}$, the surface $\Omega^{\perp}$ as shown in Figure 3. Figure 4 presents the surface with $\vartheta_0 = -0.7$

Example 4.3. Let

$$\beta(s) = (\sin s, \cos s, 0),$$

be a spacelike curve. Clearly,

$$\left.\begin{array}{c} \mathbf{T}(s) = (\cos s, \sin s, 0), \\ \mathbf{N}(s) = (-\sin s, \cos s, 0), \\ \mathbf{B}(s) = (0, 0, -1), \\ \kappa(s) = 1,\ \tau(s) = 0, \text{ and } \vartheta(s) = \vartheta_0. \end{array}\right\}$$

Therefore, the timelike surface family is written as

$$\Omega^{\perp} : \mathbf{P}(s,u) = (\sin s - u \sinh \vartheta_0, \cos s + u \sinh \vartheta_0, -u \cosh \vartheta_0).$$

If we choose $\vartheta_0 = 0$, and $-1 \leq s \leq 1$ then we have a timelike cylinder (Figure 5). Figure 6 shows the timelike cone with $\vartheta_0 = 1$, and $0 \leq s \leq 2\pi$.

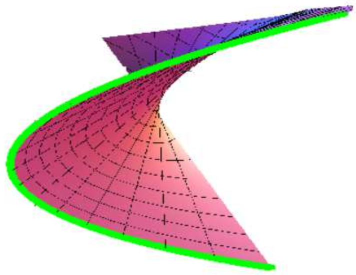

Fig. 3: $\Omega^{\perp}$ with $\vartheta_0 = -1$.

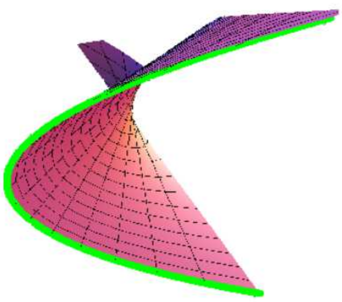

Fig. 4: $\Omega^{\perp}$ with $\vartheta_0 = -0.7$.

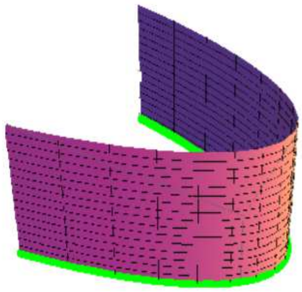

Fig. 5: Timelike cylinder

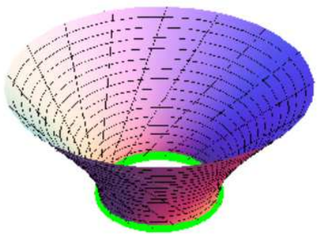

Fig. 6: Timelike cone.

5 Conclusion

We introduced the spacelike sweeping surface with rotation minimizing frames (RMF) at Minkowski 3–Space $\mathbb{E}^3_1$. We also showed that parametric curves on these surfaces are lines of curvature. Then, we derived the necessary and sufficient condition of this spacelike sweeping surface to become a spacelike/timelike developable ruled surface. Moreover, we analyzed necessary and sufficient conditions in case the resulting spacelike/timelike developable surface is the cylinder, cone or tangent surface.

Conflict of Interest

The authors declare that they have no conflict of interest.

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