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SUPRA PRE-OPEN SETS AND SUPRA PRE-CONTINUITY ON TOPOLOGICAL SPACES

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Abstract. In this paper, a new class of sets and maps between topological spaces called supra pre-open sets and supra pre-continuous maps, respectively are introduced and studied. Furthermore, the concepts of supra pre-open maps and supra pre-closed maps in terms of supra pre-open sets and supra pre-closed sets, respectively, are introduced and several properties of them are investigated.

1. Introduction.

General topology is important in many fields of applied sciences as well as branches of mathematics. In reality it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design and engineering design (briefly CAGD), digital topology, information systems, non-commutative geometry and its application to particle physics and quantum physics etc. We recommend that the reader should refer to the following papers, respectively: [2-4, 6, 7, 8]. One can observe the influence made in these realms of applied research by general topological spaces, properties and structures. Rosen and Peters [8] have used topology as a body of mathematics that could unify diverse areas of CAGD and engineering design research. They have presented several examples of the application of topology to CAGD and design.

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The concept of supra topology is fundamental with respect to the investigation of general topological spaces. Also, generalization of openness, such as supra-open, supra α-open, supra pre-open, supra b-open, etc. are important in topological spaces. In 1983, Mashhour et al. [5] initiated the study of the so-called supra topological spaces and studied S-continuous maps and $S^{\star}$-continuous maps. We will use the term supra-continuous maps instead of S-continuous maps. In 2008, Devi et al. [1] introduced and studied a class of sets and maps between topological spaces called supra $\alpha-$open sets and supra $\alpha-$continuous maps, respectively. Recently, Sayed and Noiri [9] introduced and investigated the notions of supra b-continuity, supra b-openness and supra b-closedness in terms of supra b-open set and supra b-closed set, respectively. Now, we introduce the concept of supra pre-open sets and study some basic properties of it. Also, we introduce the concepts of supra pre-continuous maps, supra pre-open maps and supra pre-closed maps and investigate several properties for this class of maps. In particular, we study the relation between supra pre-continuous maps and supra pre-open maps (supra pre-closed maps).

Throughout this paper (X, τ) , (Y, σ) and (Z, υ) (or simply, X , Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. All sets are assumed to be subset of topological spaces. The closure and the interior of a set A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subcollection $\mu \subset 2^X$ is called a supra topology [5] on X if $X \in \mu$ and μ is closed under arbitrary union. (X, μ) is called a supra topological space. The elements of μ are called supra open in (X, μ) and the complement of a supra open set is called a supra closed set. The supra closure of a set A, denoted by $Cl^{\mu}(A)$, is the intersection of the supra closed sets including A. The supra interior of a set A, denoted by $Int^{\mu}(A)$, is the union of the supra open sets included in A. The supra topology μ on X is associated with the topology τ if $\tau \subset \mu$. A set A is called supra α-open[1] (resp. supra b-open [9]) if $A \subseteq Int^{\mu}(Cl^{\mu}(Int^{\mu}(A)))$ (resp. $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \cup Int^{\mu}(Cl^{\mu}(A))$).

2. Supra pre-open sets

In this section, we introduce a new class of generalized open sets called supra b-open sets and study some of their basic properties.

Definition 2.1. *A set A is supra pre-open if $A \subseteq Int^{\mu}(Cl^{\mu}(A))$. The complement of supra pre-open is called supra pre-closed. Thus A is supra pre-closed if and only if $Cl^{\mu}(Int^{\mu}(A)) \subseteq A$.*

Theorem 2.1. *(1) Every supra α-open is supra pre-open.*
(2) Every supra pre-open is supra b-open.

Proof. Obvious. ∎

The following examples show that supra pre-open is placed strictly between supra α-open and supra b-open.

Example 2.1. *Let (X, μ) be a supra topological space, where $X = \{a, b, c\}$ and $\mu = \{X, \phi, \{a\}, \{a, b\}, \{b, c\}\}$. Here $\{a, c\}$ is supra pre-open, but it is not supra α-open.*

Example 2.2. *Let (X, μ) be a supra topological space, where $X = \{a, b, c, d\}$ and $\mu = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Here $\{b, c\}$ is a supra b-open, but it is not supra pre-open.*

The following diagram show how supra pre-open sets are related to some similar types of supra-open sets.

$$supra - open \quad \rightarrow \quad supra\ \alpha - open \quad \rightarrow \quad supra\ pre - open \quad \rightarrow \quad supra\ b - open$$

Theorem 2.2. *(i) Arbitrary union of supra pre-open sets is always supra pre-open.*
(ii) Finite intersection of supra pre-open sets may fail to be supra pre-open.
(iii) X is a supra pre-open set

Proof. (i) Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of supra pre-open sets in a topological space X. Then for any $\lambda \in \Lambda$ we have $A_\lambda \subseteq Int^\mu(Cl^\mu(A_\lambda))$. Hence $\underset{\lambda\in\Lambda}{\cup} A_\lambda \subseteq \underset{\lambda\in\Lambda}{\cup} (Int^\mu(Cl^\mu(A_\lambda))) \subseteq Int^\mu(\underset{\lambda\in\Lambda}{\cup} (Cl^\mu(A_\lambda))) \subseteq Int^\mu(Cl^\mu(\underset{\lambda\in\Lambda}{\cup} A_\lambda))$. Therefore $\underset{\lambda\in\Lambda}{\cup} A_\lambda$ is a supra pre-open set.
(ii) In Example 2.1 both $\{a, c\}$ and $\{b, c\}$ are supra pre-open, but their intersection $\{c\}$ is not supra pre-open. ∎

Theorem 2.3. *(i) Arbitrary intersection of supra pre-closed sets is always supra pre-closed.*
(ii) Finite union of supra pre-closed sets may fail to be supra pre-closed.

Proof. (i) This follows immediately from Theorem 2.2.
(ii) In Example 2.1 both $\{a\}$ and $\{b\}$ are supra pre-closed, but their union $\{a, b\}$ is not supra pre-closed. ∎

Definition 2.2. *The supra pre-closure of a set* A*, denoted by* $Cl_p^\mu(A)$*, is the intersection of the supra pre-closed sets including* A*. The supra pre-interior of a set* A*,denoted by* $Int_p^\mu(A)$*, is the union of the supra pre-open sets included in* A*.*

Remark 2.1. *It is clear that* $Int_p^\mu(A)$ *is a supra pre-open set and* $Cl_p^\mu(A)$ *is supra pre-closed.*

Theorem 2.4. *(i)* $A \subseteq Cl_p^\mu(A)$*; and* $A = Cl_p^\mu(A)$ *iff* A *is a supra pre-closed set.*
(ii) $Int_p^\mu(A) \subseteq A$*; and* $Int_p^\mu(A) = A$ *iff* A *is a supra pre-open set.*
(iii) $X - Int_p^\mu(A) = Cl_p^\mu(X - A)$*.*
(iv) $X - Cl_p^\mu(A) = Int_p^\mu(X - A)$*.*
(v) If $A \subseteq B$*, then* $Cl_p^\mu(A) \subseteq Cl_p^\mu(B)$ *and* $Int_p^\mu(A) \subseteq Int_p^\mu(B)$*.*

Proof. Obvious. ∎

Theorem 2.5. *(a)* $Int_p^\mu(A) \cup Int_p^\mu(B) \subseteq Int_p^\mu(A \cup B)$*;*
(b) $Cl_p^\mu(A \cap B) \subseteq Cl_p^\mu(A) \cap Cl_p^\mu(B)$*.*

Proof. obvious. ∎

The inclusions in (a) and (b) in Theorem 2.5 can not replaced by equalities by Example 2.1. Where, if $A = \{b\}$ and $B = \{c\}$,then $Int_p^\mu(A) = Int_p^\mu(B) = \phi$ and $Int_p^\mu(A \cup B) = \{b, c\}$. Also, if $C = \{a, b\}$ and $D = \{a, c\}$, then $Cl_p^\mu(C) = Cl_p^\mu(D) = X$ and $Cl_p^\mu(C \cap D) = \{a\}$.

Proposition 2.1. *(1) The intersection of supra open and supra pre-open is supra pre-open.*
(2) The intersection of supra α*-open and supra pre-open is supra pre-open.*

3. Supra pre-continuous maps

In this section, we introduce a new type of continuous maps called a supra pre-continuous maps and obtain some of their properties and characterizations.

Definition 3.1. *Let* (X, τ) *and* (Y, σ) *be two topological spaces and* μ *be an associated supra topology with* τ*. A map* $f : (X, \tau) \to (Y, \sigma)$ *is called supra pre-continuous map if the inverse image of each open set in* Y *is a supra pre-open set in* X*.*

Theorem 3.1. *Every continuous map is supra pre-continuous.*

Proof. Let $f : X \to Y$ be a continuous map and A is open set in Y. Then $f^{-1}(A)$ is an open set in X. Since μ associated with τ, then $\tau \subseteq \mu$. Therefore $f^{-1}(A)$ is a supra open set in X which is a supra pre-open set in X . Hence f is supra pre-continuous map. ∎

The converse of the above theorem is not true as shown in the following example.

Example 3.1. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$ be a topology on X. The supra topology μ is defined as follows: $\mu = \{X, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a map defined as follows: $f(a) = b$, $f(b) = c$, $f(c) = a$. Since the inverse image of the open set $\{a, b\}$ is $\{a, c\}$ which is not an open set but it is a supra pre-open set. Then f is supra pre-continuous map but not continuous map.*

The following example shows that supra pre-continuous map need not be supra α-continuous map.

Example 3.2. *Consider the set $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a, c\}, \{b, d\}\}$ and the supra topology $\mu = \{X, \phi, \{a, c\}, \{b, d\}, \{a, c, d\}\}$. Also, let $Y = \{x, y, z\}$ with the topology $\sigma = \{Y, \phi, \{z\}, \{y, z\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = y$, $f(b) = f(c) = z, f(d) = x$. Clearly, f is a supra pre continuous map but it is not supra α-continuous.*

The following example shows that a supra b-continuous map need not be supra pre-continuous.

Example 3.3. *Consider the set $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b, c\}\}$ and the supra topology $\mu = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Also, let $Y = \{x, y, z\}$ with the topology $\sigma = \{Y, \phi, \{x, z\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = x$, $f(b) = y, f(c) = z$. Then the inverse image of the open set $\{x, z\}$ is $\{a, c\}$ which is not supra pre-open but it is a supra b-open set. Therefore f is a supra b-continuous map but it is not supra pre-continuous.*

Example 4.1 in [1] shows that supra α-continuous map need not be supra continuous.

From the above discussion we have the following diagram in which the converses of the implications need not be true (cont. is the abbreviation of continuity).

supra cont. $\to$ *supra α-cont.* $\to$ *supra pre-cont.* $\to$ *supra b-cont.*

Theorem 3.2. *Let (X,τ) and (Y,σ) be two topological spaces and μ be an associated supra topology with τ. Let f be a map from X into Y. Then the following are equivalent:*
(1) f is a supra pre-continuous map;
(2) The inverse image of a closed set in Y is a supra pre-closed set in X;
(3) $Cl_p^{\mu}(f^{-1}(A)) \subseteq f^{-1}(Cl(A))$ for every set A in Y;
(4) $f(Cl_p^{\mu}(A)) \subseteq Cl(f(A))$ for every set A in X;
(5) $f^{-1}(Int(B)) \subseteq Int_p^{\mu}(f^{-1}(B))$ for every B in Y.

Proof. (1)⇒(2): Let A be a closed set in Y, then $Y-A$ is an open set in Y. Then $f^{-1}(Y-A) = X - f^{-1}(A)$ is a supra pre-open set in X. It follows that $f^{-1}(A)$ is a supra pre-closed subset of X.
(2)⇒(3): Let A be any subset of Y. Since $Cl(A)$ is closed in Y, then $f^{-1}(Cl(A))$ is a supra pre-closed set in X. Therefore $Cl_p^{\mu}(f^{-1}(A)) \subseteq Cl_p^{\mu}(f^{-1}(Cl(A))) = f^{-1}(Cl(A))$.
(3)⇒(4): Let A be any subset of X. By (3) we have $f^{-1}(Cl(f(A))) \supseteq Cl_p^{\mu}(f^{-1}(f(A))) \supseteq Cl_p^{\mu}(A)$.Therefore $f(Cl_p^{\mu}(A)) \subseteq Cl(f(A))$.
(4)⇒(5): Let B be any subset of Y. By (4) $f(Cl_p^{\mu}(X - f^{-1}(B))) \subseteq Cl(f(X - f^{-1}(B)))$ and $f(X - Int_p^{\mu}(f^{-1}(B))) \subseteq Cl(Y-B) = Y - Int(B)$. Therefore we have $X - Int_p^{\mu}(f^{-1}(B)) \subseteq f^{-1}(Y - Int(B))$ and hence $f^{-1}(Int(B)) \subseteq Int_p^{\mu}(f^{-1}(B))$.
(5)⇒(1): Let B be an open set in Y and $f^{-1}(Int(B)) \subseteq Int_p^{\mu}(f^{-1}(B))$. Then $f^{-1}(B) \subseteq Int_p^{\mu}(f^{-1}(B))$. But, $Int_p^{\mu}(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = Int_p^{\mu}(f^{-1}(B))$. Therefore $f^{-1}(B)$ is supra pre-open set in Y. ∎

Theorem 3.3. *If $f : X \to Y$ is supra pre-continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is supra pre-continuous.*

Proof. Obvious. ∎

Theorem 3.4. *Let (X,τ) and (Y,σ) be two topological spaces and μ and ν be the associated supra topologies with τ and σ, respectively. Then $f : X \to Y$ is supra pre-continuous if one of the following holds:*
(1) $f^{-1}(Int_P^{\nu}(B)) \subseteq Int(f^{-1}(B))$ for every set B in Y.
(2) $Cl(f^{-1}(B)) \subseteq f^{-1}(Cl_p^{\nu}(B))$ for every set B in Y.
(3) $f(Cl(A)) \subseteq Cl_p^{\mu}(f(A))$ for every set A in X.

Proof. Let B be any open set of Y, if condition (1) is satisfied, then $f^{-1}(Int_p^{\nu}(B)) \subseteq Int(f^{-1}(B))$. We get $f^{-1}(B) \subseteq Int(f^{-1}(B))$. Therefore $f^{-1}(B)$ is an open set. Every open set is supra pre-open. Hence

f is supra pre-continuous.
If condition (2) is satisfied, then we can easily prove that f is supra pre-continuous.
Let condition (3) be satisfied and B be any open set in Y. Then $f^{-1}(B)$ is a set in X and $f(Cl(f^{-1}(B))) \subseteq Cl^{\mu}_p(f(f^{-1}(B)))$. This implies $f(Cl(f^{-1}(B))) \subseteq Cl^{\mu}_p(B)$. This is nothing but condition (2). Hence f is supra pre-continuous. ∎

4. Supra pre-open maps and supra pre-closed maps

Definition 4.1. *A map $f : X \to Y$ is called supra pre-open (resp. supra pre-closed) if the image of each open (resp. closed) set in X, is supra pre-open (resp. supra pre-closed) in Y.*

Theorem 4.1. *A map $f : X \to Y$ is supra pre-open if and only if $f(Int(A)) \subseteq Int^{\nu}_p(f(A))$ for each set A in X.*

Proof. Suppose that f is a supra pre-open map. Since $Int(A) \subseteq A$, then $f(Int(A)) \subseteq f(A)$. By hypothesis $f(Int(A))$ is a supra pre-open set and $Int^{\nu}_p(f(A))$ is the largest supra pre-open set contained in $f(A)$, then $f(Int(A)) \subseteq Int^{\nu}_p(f(A))$.

Conversely, suppose A is an open set in X. Then $f(Int(A)) \subseteq Int^{\nu}_p(f(A))$. Since $Int(A) = A$, then $f(A) \subseteq Int^{\nu}_p(f(A))$. Therefore $f(A)$ is a supra pre-open set in Y and f is supra pre-open. ∎

Theorem 4.2. *A map $f : X \to Y$ is supra pre-closed if and only if $Cl^{\nu}_p(f(A)) \subseteq f(Cl(A))$ for each set A in X.*

Proof. Suppose f is a supra pre-closed map. Since for each set A in X, $Cl(A)$ is closed set in X, then $f(Cl(A))$ is a supra pre-closed set in Y. Also, since $f(A) \subseteq f(Cl(A))$, then $Cl^{\nu}_p(f(A)) \subseteq f(Cl(A))$.

Conversely, Let A be a closed set in X. Since $Cl^{\nu}_p(f(A))$ is the smallest supra pre-closed set containing $f(A)$, then $f(A) \subseteq Cl^{\nu}_p(f(A)) \subseteq f(Cl(A)) = f(A)$. Thus $f(A) = Cl^{\nu}_p(f(A))$. Hence $f(A)$ is a supra pre-closed set in Y. Therefore f is a supra pre-closed map. ∎

Theorem 4.3. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps.*
(1) If $g \circ f$ is supra pre-open and f is continuous surjective, then g is supra pre-open.
(2) If $g \circ f$ is open and g is supra pre-continuous injective, then f is supra pre-open.

Proof. (1) Let A be an open set in Y. Then $f^{-1}(A)$ is an open set in X. Since $g \circ f$ is a supra pre-open map, then $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ (because f is surjective) is a supra pre-open set in Z. Therefore g is a supra pre-open map.
(2) Let A be an open set in X. Then, $g(f(A))$ is open set in Z. Therefore, $g^{-1}(g(f(A))) = f(A)$ (because g is injective) is a supra open set in Y. Hence f is a supra pre-open map. ∎

Theorem 4.4. *Let $f : X \to Y$ be a map. Then the following are equivalent:*
(1) f is a supra pre-open map;
(2) f is a supra pre-closed map;
(3) f^{-1} is a supra pre-continuous map.

Proof. (1) $\Longrightarrow$ (2). Suppose B is a closed set in X. Then $X - B$ is an open set in X. By (1), $f(X - B)$ is a supra pre-open set in Y. Since f is bijective, then $f(X - B) = Y - f(B)$. Hence $f(B)$ is a supra pre-closed set in Y. Therefore f is a supra pre-closed map.
(2) $\Longrightarrow$ (3). Let f is a supra pre-closed map and B be closed set in X. Since f is bijective, then $(f^{-1})^{-1}(B) = f(B)$ which is a supra pre-closed set in Y. By Theorem 3.2 f is a supra pre-continuous map.
(3) $\Longrightarrow$ (1). Let A be an open set in X. Since f^{-1} is a supra pre-continuous map, then $(f^{-1})^{-1}(A) = f(A)$ is a supra pre-open set in Y. Hence f is supra pre-open. ∎

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