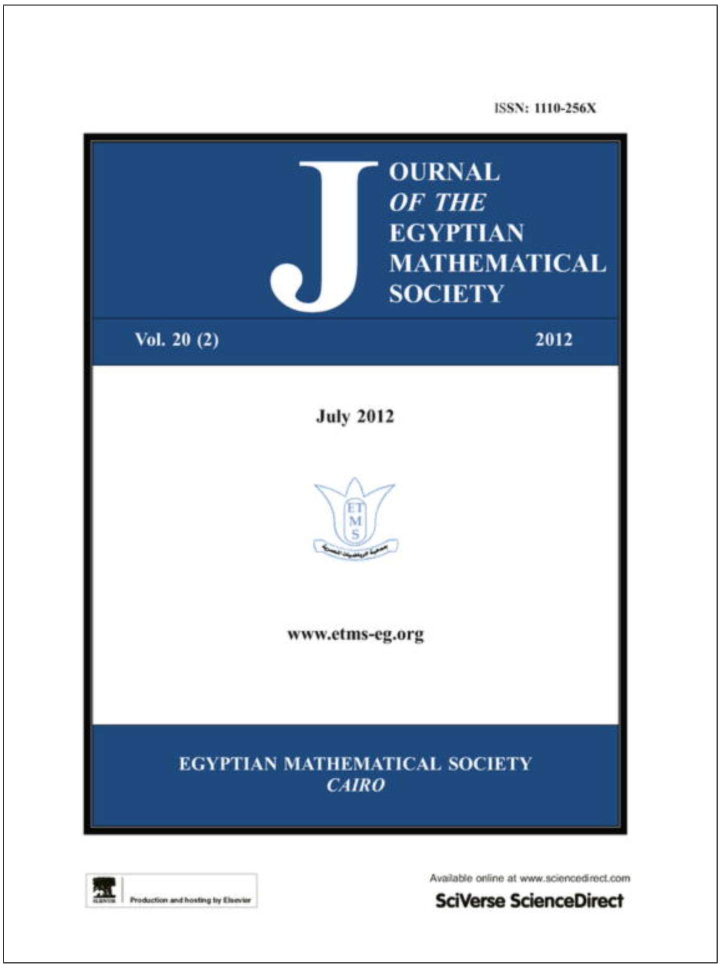
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ORIGINAL ARTICLE

α-Irresoluteness and α-compactness based on continuous valued logic ☆

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Abstract This paper considers fuzzifying topologies, a special case of *I*-fuzzy topologies (bifuzzy topologies), introduced by Ying [1]. It investigates topological notions defined by means of α-open sets when these are planted into the framework of Ying's fuzzifying topological spaces (by Łukasiewicz logic in [0, 1]) . The concept of α-irresolute functions and α-compactness in the framework of fuzzifying topology are introduced and some of their properties are obtained. We use the finite intersection property to give a characterization of fuzzifying α-compact spaces. Furthermore, we study the image of fuzzifying α-compact spaces under fuzzifying α-continuity and fuzzifying α-irresolute maps.

1. Introduction

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [2–7]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [3], the kind of topologies defined by Chang [8] and Goguen [9] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family τ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [10]. On the other hand, Höhle in [11] proposed the terminology L-fuzzy topology to be an L-valued mapping on the traditional powerset $P(X)$ of X. The authors in [5,6,12,13] defined an L-fuzzy topology to be an L-valued mapping on the L-powerset L^X of X.

In 1952, Rosser and Turquette [14] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories? As an attempt to give a partial answer to this problem in the case of point set topology, Ying in 1991 [1,15,16] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In fact, fuzzifying topologies are a special case of the L-fuzzy topologies in [12,13] since all the t-norms on I are included as a special class of tensor products in these paper. Ying uses one particular tensor product, namely Łukasiewicz conjunction. Thus his fuzzifying topologies are a special class of all the I-fuzzy topologies considered in the categorical

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frameworks [12,13]. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [17–20]. For example, Ying [21] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [17] the concepts of fuzzifying α-open set and fuzzifying α-continuity were introduced and studied. Also, Sayed [22] introduced and studied the concept of fuzzifying α-Hausdorff separation axiom. In classical topology, α-irresolute mappings and α-compact spaces have been studied in [23–25], respectively. As well as, they have been studied in fuzzy topology in [26–28], respectively. In [29] it was shown that α-compactness (due to Ganter, Steinlage and Warren) of fuzzy topological spaces is the categorical compactness (in the sense of Herrlich et al. [30]) which arises from a factorization structure on the category of fuzzy topological and fuzzy continuous. Also, some characterizations of α-compactness were given. In this paper, the concept of α-irresolute mappings between fuzzifying topological spaces has been studied. Furthermore, the concept of α-compactness in the framework of fuzzifying topology has been reported. The finite intersection property used to give a characterization of the fuzzifying α-compact spaces. Moreover, we study the image of fuzzifying α-compact spaces under fuzzifying α -continuity and fuzzifying α-irresolute mappings. Thus we fill a gap in the existing literature on fuzzifying topology. We use the terminologies and notations in [1,15–17,21] without any explanation. We will use the symbol $\otimes$ instead of the second "AND" operation $\wedge$ as dot is hardly visible. This mean that $[\alpha] \leqslant [\varphi \to \psi] \iff [\alpha] \otimes [\varphi] \leqslant [\psi]$. Also, we need the following two facts: $[\varphi \to \psi] \otimes [\varphi] \leqslant [\psi]$ and $[(\alpha \to \gamma) \wedge (\beta \to \gamma)] = [(\alpha \vee \beta) \to \gamma]$.

A fuzzifying topology on a set X [7,11] is a mapping $\tau \in \Im(P(X))$ such that:

(1) $\tau(X) = 1, \tau(\phi) = 1$;
(2) for any $A, B, \tau(A \cap B) \geqslant \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) \geqslant \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

The family of all fuzzifying α-open sets [17], denoted by $\tau_\alpha \in \Im(P(X))$, is defined as
$A \in \tau_\alpha := \forall x(x \in A \to x \in Int(Cl(Int(A))))$, i.e., $\tau_\alpha(A) = \bigwedge_{x \in A} Int(Cl(Int(A)))(x)$

The family of all fuzzifying α-closed sets [17], denoted by $\mathsf{F}_\alpha \in \Im(P(X))$, is defined as $A \in \mathsf{F}_\alpha := X - A \in \tau_\alpha$. The fuzzifying α-neighborhood system of a point $x \in X$ [17] is denoted by $N_x^\alpha \in \Im(P(X))$ and defined as $N_x^\alpha(A) = \bigvee_{x \in B \subseteq A} \tau_\alpha(B)$. The fuzzifying α-closure of a set $A \subseteq X$ [17], denoted by $Cl_\alpha \in \Im(X)$, is defined as $Cl_\alpha(A)(x) = 1 - N_x^\alpha(X - A)$. If (X, τ) and (Y, σ) are two fuzzifying topological spaces and $f \in Y^X$, the unary fuzzy predicate $C_\alpha \in \Im(Y^X)$, called fuzzifying α-continuity [17], is given as $C_\alpha(f) := \forall B(B \in \sigma \to f^{-1}(B) \in \tau_\alpha)$. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $T_2^\alpha \in \Im(\Omega)$, called fuzzifying α-Hausdorffness [22], is given as follows:

$$T_2^\alpha(X, \tau) = (\forall x)(\forall y)((x \in X \wedge y \in Y \wedge x \neq y) \to (\exists B)(\exists C)(B \in N_x^\alpha \wedge C \in N_y^\alpha \wedge B \cap C \equiv \phi)).$$

A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness [21], is given as follows:

(1) $\Gamma(X, \tau) := (\forall \Re)(K_\circ(\Re, X) \to (\exists \wp)((\wp \leqslant \Re) \wedge K(\wp, A) \otimes FF(\wp)))$.For K, $K_\circ$ (resp. $\leqslant$ and FF) see ([15], Definition 4.4)(resp. ([15], Theorem 4.3) and ([21], Definition 1.1 and Lemma 1.1)).
(2) If $A \subseteq X$, then $\Gamma(A) := \Gamma(A, \tau/A)$.

2. Fuzzifying α-irresolute mappings

The purpose of this section is to introduce and study the concept of α-irresolute mappings in fuzzifying topological spaces.

Definition 2.1. Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. A unary fuzzy predicate $I_\alpha \in \Im(Y^X)$, called fuzzifying α-irresoluteness, is given as follows:

$$I_\alpha(f) := \forall B(B \in \sigma_\alpha \to f^{-1}(B) \in \tau_\alpha).$$

The following theorem generalize the well known result in general topology which state that the concept α-continuous mappings is strictly weaker than that of α-irresolute mappings [23].

Theorem 2.1. *Let (X, τ) and (Y, σ) be two fuzzifying topological spaces and let $f \in Y^X$. Then*

$$\models f \in I_\alpha \to f \in C_\alpha.$$

Proof. From ([17], Theorem 3.3 (1) (a)) we have $\sigma(B) \leqslant \sigma_\alpha(B)$ and the result holds. □

Lemma 2.1. *Let (X, τ), (Y, σ) and (Z, ν) be three fuzzifying topological spaces and let $f \in Y^X$ and $g \in Z^Y$. Then*

$$\models I_\alpha(f) \to (C_\alpha(g) \to C_\alpha(g \circ f))$$

Proof. It suffices $[I_\alpha(f)] \leqslant [C_\alpha(g) \to C_\alpha(g \circ f)]$. If $[C_\alpha(g)] \leqslant [C_\alpha(g \circ f)]$, the results holds. If $[C_\alpha(g)] \geqslant [C_\alpha(g \circ f)]$, then

$$\begin{aligned}[C_\alpha(g)] - [C_\alpha(g \circ f)] &= \bigwedge_{V \in P(Z)} \min(1, 1 - \nu(V) + \sigma_\alpha(g^{-1}(V))) \\ &\quad - \bigwedge_{V \in P(Z)} \min(1, 1 - \nu(V) + \tau_\alpha(f^{-1}(g^{-1}(V)))) \\ &\leqslant \bigvee_{V \in P(Z)} (\sigma_\alpha(g^{-1}(V)) - \tau_\alpha(f^{-1}(g^{-1}(V)))).\end{aligned}$$

Therefore

$$\begin{aligned}[C_\alpha(g) \to C_\alpha(g \circ f)] &= \min(1, 1 - [C_\alpha(g)] + [C_\alpha(g \circ f)]) \\ &\geqslant \bigwedge_{U \in P(Y)} \min(1, 1 - \sigma_\alpha(U) + \tau_\alpha(f^{-1}(U))) \\ &= [I_\alpha(f)]. \quad \square\end{aligned}$$

The above lemma is a generalization of the following well known result in general topology ([31], Proposition 4).

Corollary 2.1. *If $f:(X,\tau)\to(Y,\sigma)$ is α-irresolute and $g:(Y,\sigma)\to(Z,v)$ is α-continuous, then $g\circ f:(X,\tau)\to(Z,v)$ is α-continuous.*

Definition 2.2. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces and let $f\in Y^X$. We define the unary fuzzy predicates $\omega_k\in\Im(Y^X)$, where $k=1,\dots,5$, as follows:

(1) $f\in\omega_1=\forall B\big(B\in \mathsf{F}^Y_\alpha\to f^{-1}(B)\in \mathsf{F}^X_\alpha\big)$, where F^X_α and F^Y_α are the fuzzifying α-closed subsets of X and Y, respectively;
(2) $f\in\omega_2=\forall x\forall U\Big(U\in N^{\alpha^Y}_{f(x)}\to f^{-1}(U)\in N^{\alpha^X}_x\Big)$, where N^{α^X} and N^{α^Y} are the family of fuzzifying α-neighborhood systems of X and Y, respectively;
(3) $f\in\omega_3=\forall x\forall U\Big(U\in N^{\alpha^Y}_{f(x)}\to\exists V\Big(f(V)\subseteq U\to V\in N^{\alpha^X}_x\Big)\Big)$;
(4) $f\in\omega_4=\forall A\big(f\big(Cl^X_\alpha(A)\big)\subseteq Cl^Y_\alpha(f(A))\big)$;
(5) $f\in\omega_5=\forall B(Cl^X_\alpha(f^{-1}(B))\subseteq f^{-1}(Cl^Y_\alpha(B)))$.

Theorem 2.2. $\models f\in I_\alpha\leftrightarrow f\in\omega_k$, *k = 1, …, 5.*

Proof.

(a) We will prove that $\models f\in I_\alpha\leftrightarrow f\in\omega_1$.

$$\begin{aligned}[f\in\omega_1]&=\bigwedge_{B\in P(Y)}\min\big(1,1-\mathsf{F}^Y_\alpha(B)+\mathsf{F}^X_\alpha(f^{-1}(B))\big)\\&=\bigwedge_{B\in P(Y)}\min\big(1,1-\sigma_\alpha(Y-B)+\tau_\alpha(f^{-1}(Y-B))\big)\\&=\bigwedge_{B\in P(Y)}\min\big(1,1-\sigma_\alpha(Y-B)+\tau_\alpha(X-f^{-1}(B))\big)\\&=\bigwedge_{U\in P(Y)}\min\big(1,1-\sigma_\alpha(U)+\tau_\alpha(f^{-1}(U))\big)=[f\in I_\alpha].\end{aligned}$$

(b) We will prove that $\models f\in I_\alpha\leftrightarrow f\in\omega_2$. First, we prove that $[f\in\omega_2]\geqslant[f\in I_\alpha]$. If $N^{\alpha^Y}_{f(x)}(U)\leqslant N^{\alpha^X}_x(f^{-1}(U))$, then $\min(1,1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U)))=1\geqslant[f\in I_\alpha]$. Suppose $N^{\alpha^Y}_{f(x)}(U)>N^{\alpha^X}_x(f^{-1}(U))$. It is clear that, if $f(x)\in A\subseteq U$, then $x\in f^{-1}(A)\subseteq f^{-1}(U)$. Then

$$\begin{aligned}N^{\alpha^Y}_{f(x)}(U)-N^{\alpha^X}_x(f^{-1}(U))&=\bigvee_{f(x)\in A\subseteq U}\sigma_\alpha(A)-\bigvee_{x\in B\subseteq f^{-1}(U)}\tau_\alpha(B)\\&\leqslant\bigvee_{f(x)\in A\subseteq U}\sigma_\alpha(A)-\bigvee_{f(x)\in A\subseteq U}\tau_\alpha(f^{-1}(A))\\&\leqslant\bigvee_{f(x)\in A\subseteq U}(\sigma_\alpha(A)-\tau_\alpha(f^{-1}(A)))\end{aligned}$$

So $1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U))\geqslant\bigwedge_{f(x)\in A\subseteq U}(1-\sigma_\alpha(A)+\tau_\alpha(f^{-1}(A)))$. Therefore

$$\begin{aligned}&\min\Big(1,1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U))\Big)\\&\geqslant\bigwedge_{f(x)\in A\subseteq U}\min(1,1-\sigma_\alpha(A)+\tau_\alpha(f^{-1}(A)))\\&\geqslant\bigwedge_{V\in P(Y)}\min(1,1-\sigma_\alpha(V)+\tau_\alpha(f^{-1}(V)))=[f\in I_\alpha].\end{aligned}$$

Hence $\bigwedge_{x\in X}\bigwedge_{U\in P(Y)}\min(1,1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U)))\geqslant[f\in I_\alpha]$. Second, we prove that $[f\in I_\alpha]\geqslant[f\in\omega_2]$. From Corollary 4.1 [17] we have

$$\begin{aligned}[f\in I_\alpha]&=\bigwedge_{U\in P(Y)}\min(1,1-\sigma_\alpha(U)+\tau_\alpha(f^{-1}(U)))\\&\geqslant\bigwedge_{U\in P(Y)}\min\left(1,1-\bigwedge_{f(x)\in U}N^{\alpha^Y}_{f(x)}(U)+\bigwedge_{x\in f^{-1}(U)}N^{\alpha^X}_x(f^{-1}(U))\right)\\&\geqslant\bigwedge_{U\in P(Y)}\min\left(1,1-\bigwedge_{x\in f^{-1}(U)}N^{\alpha^Y}_{f(x)}(U)+\bigwedge_{x\in f^{-1}(U)}N^{\alpha^X}_x(f^{-1}(U))\right)\\&\geqslant\bigwedge_{x\in X}\bigwedge_{U\in P(Y)}\min\Big(1,1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U))\Big)=[f\in\omega_2].\end{aligned}$$

(c) We prove that $[f\in\omega_2]=[f\in\omega_3]$. Since $f(V)\subseteq U$ implies $V\subseteq f^{-1}(U)$, then from Theorem 4.2 (2) [17] we have

$$\begin{aligned}[f\in\omega_3]&=\bigwedge_{x\in X}\bigwedge_{U\in P(Y)}\min\left(1,1-N^{\alpha^Y}_{f(x)}(U)+\bigvee_{V\in P(X),f(V)\subseteq U}N^{\alpha^X}_x(V)\right)\\&=\geqslant\bigwedge_{x\in X}\bigwedge_{U\in P(X)}\min\Big(1,1-N^{\alpha^Y}_{f(x)}(U)+N^{\alpha^X}_x(f^{-1}(U))\Big)=[f\in\omega_2].\end{aligned}$$

(d) We will prove that $[f\in\omega_4]=[f\in\omega_5]$. First, we prove $[f\in\omega_4]\leqslant[f\in\omega_5]$. Since for any fuzzy set $\widetilde{A}$ we have $[f^{-1}(f(\widetilde{A}))\supseteq\widetilde{A}]=1$. Then for any $B\in P(Y)$, we have $[f^{-1}(f(Cl^X_\alpha(f^{-1}(B))))\supseteq Cl^X_\alpha(f^{-1}(B))]=1$. Also, since $[f(f^{-1}(B))\subseteq B]=1$, then $\big[Cl^Y_\alpha(f(f^{-1}(B)))\subseteq Cl^Y_\alpha(B)\big]=1$. From Lemma 1.2 (2) [15], we obtain

$$\begin{aligned}\big[Cl^X_\alpha(f^{-1}(B))\subseteq f^{-1}(Cl^Y_\alpha(B))\big]&\geqslant\big[f^{-1}(f(Cl^X_\alpha(f^{-1}(B))))\subseteq f^{-1}(Cl^Y_\alpha(B))\big]\\&\geqslant\big[f^{-1}(f(Cl^X_\alpha(f^{-1}(B))))\subseteq f^{-1}(Cl^Y_\alpha(f(f^{-1}(B))))\big]\\&\geqslant\big[f(Cl^X_\alpha(f^{-1}(B)))\subseteq Cl^Y_\alpha(f(f^{-1}(B)))\big].\end{aligned}$$

Hence

$$\begin{aligned}[f\in\omega_5]&=\bigwedge_{B\in P(Y)}\big[Cl^X_\alpha(f^{-1}(B))\subseteq f^{-1}(Cl^Y_\alpha(B))\big]\\&\geqslant\bigwedge_{B\in P(Y)}\big[f(Cl^X_\alpha(f^{-1}(B)))\subseteq Cl^Y_\alpha(f(f^{-1}(B)))\big]\\&\geqslant\bigwedge_{A\in P(X)}\big[f(Cl^X_\alpha(A))\subseteq Cl^Y_\alpha(f(A))\big]=[f\in\omega_4].\end{aligned}$$

Second, for each $A\in P(X)$, there exists $B\in P(Y)$ such that $f(A)=B$ and $f^{-1}(B)\supseteq A$. Hence from Lemma 1.2 (1) [15] we have

$$\begin{aligned}[f\in\omega_4]&=\bigwedge_{A\in P(X)}\big[f(Cl^X_\alpha(A))\subseteq Cl^Y_\alpha(f(A))\big]\\&\geqslant\bigwedge_{A\in P(X)}\big[f(Cl^X_\alpha(A))\subseteq f(f^{-1}(Cl^Y_\alpha(f(A))))\big]\\&\geqslant\bigwedge_{A\in P(X)}\big[Cl^X_\alpha(A)\subseteq f^{-1}(Cl^Y_\alpha(f(A)))\big]\\&\geqslant\bigwedge_{B\in P(Y),B=f(A)}\big[Cl^X_\alpha(f^{-1}(B))\subseteq f^{-1}(Cl^Y_\alpha(B))\big]\\&\geqslant\bigwedge_{B\in P(Y)}\big[Cl^X_\alpha(f^{-1}(B))\subseteq f^{-1}(Cl^Y_\alpha(B))\big]=[f\in\omega_5].\end{aligned}$$

(e) We want to prove that $\models f\in\omega_2\leftrightarrow f\in\omega_5$.

$$[f\in\omega_5]=\bigwedge_{B\in P(Y)}[Cl_\alpha^X(f^{-1}(B))\subseteq f^{-1}(Cl_\alpha^Y(B))]$$
$$=\bigwedge_{B\in P(Y)}\bigwedge_{x\in X}\min\left(1,1-\left(1-N_x^{\alpha^X}(X-f^{-1}(B))\right)+1-N_{f(x)}^{\alpha^Y}(Y-B)\right)$$
$$=\bigwedge_{B\in P(Y)}\bigwedge_{x\in X}\min\left(1,1-N_{f(x)}^{\alpha^Y}(Y-B)+N_x^{\alpha^X}(f^{-1}(Y-B))\right)$$
$$=\bigwedge_{U\in P(Y)}\bigwedge_{x\in X}\min\left(1,1-N_{f(x)}^{\alpha^Y}(U)+N_x^{\alpha^X}(f^{-1}(U))\right)=[f\in\omega_2].\ \square$$

3. Fuzzifying α-compact spaces

Definition 3.1. A fuzzifying topological space (X,τ) is said to be fuzzifying α-topological space if $\tau_\alpha(A\cap B)\geqslant\tau_\alpha(A)\wedge\tau_\alpha(B)$.

In a topological space (X,τ), a family $\Re$ of subsets of X is said to be α-covering of X if and only if $\Re$ covers X and $\Re\subseteq\alpha(X)$, where $\alpha(X)$ is the class of all α-sets in X. We generalize this notion to the fuzzifying setting in the following definition:

Definition 3.2. A binary fuzzy predicate $K_\alpha\in\Im(\Im(P(X))\times P(X))$, called fuzzifying α-open covering, is given as $K_\alpha(\Re,A):=K(\Re,A)\otimes(\Re\subseteq\tau_\alpha)$.

In classical topology, a space (X,τ) is α-compact if and only if every α-covering of X has a finite subcover. Also, a subset A of a space (X,τ) is said to be α-compact if and only if $(A,\tau_{/A})$ is α-compact, where $\tau_{/A}$ denotes the induced topology on A. The following definition generalize these notions.

Definition 3.3. Let Ω be the class of all fuzzifying topological spaces. A unary fuzzy predicate $\Gamma_\alpha\in\Im(\Omega)$, called fuzzifying α-compactness, is given as follows:

(1) $(X,\tau)\in\Gamma_\alpha:=(\forall\Re)(K_\alpha(\Re,X)\to(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,X)\otimes FF(\wp)))$;
(2) If $A\subseteq X$, then $\Gamma_\alpha(A):=\Gamma_\alpha(A,\tau_{/A})$.

Lemma 3.1. $\models K_\circ(\Re,A)\to K_\alpha(\Re,A)$.

Proof. Since $\models\tau\subseteq\tau_\alpha$, (see [17, Theorem 3.3 (1) (a)]), then we have $[\Re\subseteq\tau]\leqslant[[\Re\subseteq\tau_\alpha]$. Therefore, $[K_\circ(\Re,A)]\leqslant[K_\alpha(\Re,A)]$. □

Since in general topology every α-compact space is compact ([24], Remark 3.1), we have the following theorem in fuzzifying topology.

Theorem 3.1. $\models(X,\tau)\in\Gamma_\alpha\to(X,\tau)\in\Gamma$.

Proof. From Lemma 3.1 the proof is immediate. □

The following theorem generalize the notions which state that:

(1) A subset A of a topological space (X,τ) is α-compact relative to (X,τ) if and only if every cover of A by α-open sets of (X,τ) has a finite subcover.
(2) A subset A of a topological space (X,τ) is α-compact relative to (X,τ) if and only if it is compact in (X,τ_α).

Theorem 3.2. *For any fuzzifying topological space (X,τ) and $A\subseteq X$ we have*

(1) $\models\Gamma_\alpha(A)\leftrightarrow(\forall\Re)(K_\alpha(\Re,A)\to(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,A)\otimes FF(\wp)))$, *where K_α is related to τ.*
(2) $\Gamma_\alpha(A)=\Gamma(A,\tau_\alpha)$.

Proof. For any $\Re\in\Im(\Im(X))$, we set $\overline{\Re}\in\Im(\Im(A))$ defined as $\overline{\Re}(A\cap B)=\Re(B),B\subseteq X$. Then $K(\overline{\Re},A)=\bigwedge_{x\in A}\bigvee_{x\in C}\overline{\Re}(C)=\bigwedge_{x\in A}\bigvee_{x\in C=A\cap B}\Re(B)=\bigwedge_{x\in A}\bigvee_{x\in B}\Re(B)=K(\Re,A)$, because $x\in A$ and $x\in B$ if and only if $x\in A\cap B$. Therefore

$$[\overline{\Re}\subseteq\tau_\alpha/_A]=\bigwedge_{C\subseteq A}\min(1,1-\overline{\Re}(C)+\tau_\alpha/_A(C))$$
$$=\bigwedge_{C\subseteq A}\min\left(1,1-\bigvee_{C=A\cap B,B\subseteq X}\Re(B)+\bigvee_{C=A\cap B,B\subseteq X}\tau_\alpha(B)\right)$$
$$\geqslant\bigwedge_{C\subseteq A,C=A\cap B,B\subseteq X}\min(1,1-\Re(B)+\tau_\alpha(B))$$
$$\geqslant\bigwedge_{B\subseteq X}\min(1,1-\Re(B)+\tau_\alpha(B))=[\Re\subseteq\tau_\alpha].$$

For any $\wp\leqslant\overline{\Re}$, we define $\wp'\in\Im(P(X))$ as $\wp'(B)=\begin{cases}\wp(B) & \text{if } B\subseteq A\\ 0 & \text{otherwise.}\end{cases}$ Then $\wp'\leqslant\Re,FF(\wp')=FF(\wp)$ and $K(\wp',A)=K(\wp,A)$. Furthermore, we have

$$[\Gamma_\alpha(A)\otimes K_\alpha(\Re,A)]\leqslant[\Gamma_\alpha(A)\otimes K'_\alpha(\overline{\Re},A)]$$
$$\leqslant[(\forall\overline{\Re})(K'_\alpha(\overline{\Re},A)\to(\exists\wp)((\wp\leqslant\overline{\Re})\wedge K(\wp,A)\otimes FF(\wp)))]\otimes[K'_\alpha(\overline{\Re},A)]$$
$$\leqslant[K'_\alpha(\overline{\Re},A)\to(\exists\wp)((\wp\leqslant\overline{\Re})\wedge K(\wp,A)\otimes FF(\wp))]\otimes[K'_\alpha(\overline{\Re},A)]$$
$$\leqslant[(\exists\wp)((\wp\leqslant\overline{\Re})\wedge K(\wp,A)\otimes FF(\wp))]$$
$$\leqslant[(\exists\wp')((\wp'\leqslant\Re)\wedge K(\wp',A)\otimes FF(\wp'))]$$
$$\leqslant[(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))]$$

Then $\Gamma_\alpha(A)\leqslant[K_\alpha(\Re,A)]\to[(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))]$, where $K'_\alpha(\overline{\Re},A)=[K(\overline{\Re},A)\otimes(\overline{\Re}\subseteq\tau_\alpha/_A)]$. Therefore

$$\Gamma_\alpha(A)\leqslant\bigwedge_{\Re\in\Im(P(X))}[K_\alpha(\Re,A)\to(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))]$$
$$=[(\forall\Re)(K_\alpha(\Re,A)\to(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta)))].$$

Conversely, for any $\Re\in\Im(P(A))$, if $[\Re\subseteq\tau_\alpha/_A]=\bigwedge_{B\subseteq A}\min(1,1-\Re(B)+\tau_\alpha/_A(B))=\lambda$, then for any $n\in N$ and $B\subseteq A$, $\bigvee_{B=A\cap C,C\subseteq X}\tau_\alpha(C)=\tau_\alpha/_A(B)>\lambda+\Re(B)-1-\frac{1}{n}$, and there exists $C_B\subseteq X$ such that $C_B\cap A=B$ and $\tau_\alpha(C_B)>\lambda+\Re(B)-1-\frac{1}{n}$. Now, we define $\overline{\Re}\in\Im(P(X))$ as $\overline{\Re}(C)=\max_{B\subseteq A}(0,\lambda+\Re(B)-1-\frac{1}{n})$. Then $[\overline{\Re}\subseteq\tau_\alpha]=1$ and

$$K(\overline{\Re},A)=\bigwedge_{x\in A}\bigvee_{x\in C\subseteq X}\overline{\Re}(C)=\bigwedge_{x\in A}\bigvee_{x\in B}\overline{\Re}(C_B)$$
$$\geqslant \bigwedge_{x\in A}\bigvee_{x\in B}\left(\lambda+\Re(B)-1-\frac{1}{n}\right)$$
$$=\bigwedge_{x\in A}\bigvee_{x\in B}\Re(B)+\lambda-1-\frac{1}{n}=K(\Re,A)+\lambda-1-\frac{1}{n}.$$

Also, we have

$$K_\alpha(\overline{\Re},A)=[K(\overline{\Re},A)\otimes(\overline{\Re}\subseteq\tau_\alpha)]=[K(\overline{\Re},A)]$$
$$\geqslant \max\left(0,K(\Re,A)+\lambda-1-\frac{1}{n}\right)$$
$$\geqslant \max(0,K(\Re,A)+\lambda-1)-\frac{1}{n}$$
$$=\max(0,K(\Re,A)+[\Re\subseteq\tau_\alpha/_A]-1)-\frac{1}{n}$$
$$=K(\Re,A)\otimes[\Re\subseteq\tau_\alpha/_A]-\frac{1}{n}=K'_\alpha(\Re,A)-\frac{1}{n}.$$

For any $\wp\leqslant\overline{\Re}$, we set $\wp'\in\Im(P(A))$ as $\wp'(B)=\wp(C_B),B\subseteq A$. Then $\wp'\leqslant\Re, FF(\wp')=FF(\wp)$ and $K(\wp',A)=K(\wp,A)$.

Therefore

$$[(\forall\Re)(K_\alpha(\Re,A)\to(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,A)\otimes FF(\wp))]$$
$$\otimes[K'_\alpha(\Re,A)]-\frac{1}{n}\leqslant[(\forall\Re)(K_\alpha(\Re,A)$$
$$\to(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,A)\otimes FF(\wp))]$$
$$\otimes\left([K'_\alpha(\Re,A)]-\frac{1}{n}\right)\leqslant[K_\alpha(\overline{\Re},A)$$
$$\to(\exists\wp)((\wp\leqslant\overline{\Re})\wedge K(\wp,A)\otimes FF(\wp))]$$
$$\otimes[K_\alpha(\overline{\Re},A)]\leqslant[(\exists\wp)((\wp\leqslant\overline{\Re})\wedge K(\wp,A)$$
$$\otimes FF(\wp))]\leqslant[(\exists\wp')((\wp'\leqslant\Re)\wedge K(\wp',A)$$
$$\otimes FF(\wp'))]\leqslant[(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)$$
$$\otimes FF(\beta))].$$

Let $n\to\infty$. We obtain

$$[(\forall\Re)(K_\alpha(\Re,A)\to(\exists\wp)((\wp$$
$$\leqslant\Re)\wedge K(\wp,A)\otimes FF(\wp)))]\otimes[K'_\alpha(\Re,A)]$$
$$\leqslant[(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))].$$

Then

$$[(\forall\Re)(K_\alpha(\Re,A)\to(\exists\wp)((\wp\leqslant\Re)\wedge K(\wp,A)\otimes FF(\wp))]$$
$$\leqslant[K'_\alpha(\Re,A)\to(\exists\beta)((\beta$$
$$\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))]$$
$$\leqslant\bigwedge_{\Re\in\Im(P(X))}[K'_\alpha(\Re,A)$$
$$\to(\exists\beta)((\beta\leqslant\Re)\wedge K(\beta,A)\otimes FF(\beta))]$$
$$=\Gamma_\alpha(A).$$

(2) Obvious. □

The following definition is given in [21].

Definition 3.4. Let X be a set. A unary fuzzy predicate $fI\in\Im(\Im(P(X)))$, called fuzzy finite intersection property, is given as follows:

$$fI(\Re)\ :=\ (\forall\beta)((\beta\leqslant\Re)\wedge FF(\beta)\to(\exists x)(\forall B)((B\in\beta)$$
$$\to(x\in B))).$$

In a topological space (X,τ), the following are equivalent:

(1) X is α-compact;
(2) Each family of α-closed sets in X has the finite intersection property;
(3) Each family of α-closed sets in X whose intersection is a subset of an α-set B contains a finite subfamily whose intersection is a subset of B.

We extend this notion in the following theorem:

Theorem 3.3. *Let (X,τ) be a fuzzifying topological space.*

$$\pi_1\ :=\ (\forall\Re)((\Re\in\Im(P(X)))\wedge(\Re\subseteq\mathsf{F}_\alpha)\otimes fI(\Re)$$
$$\to(\exists x)(\forall A)(A\in\Re\to x\in A));$$

$$\pi_2\ :=\ (\forall\Re)(\exists B)(((\Re\subseteq\mathsf{F}_\alpha)\wedge(B\in\tau_\alpha))\otimes(\forall\wp)$$
$$\left((\wp\leqslant\Re)\otimes FF(\wp)\to\neg\left(\bigcap\wp\subseteq B\right)\right)\to\neg\left(\bigcap\Re\subseteq B\right)\right).$$

Then $\models\Gamma_\alpha(X,\tau)\leftrightarrow\pi_i$, $i=1,2$.

Proof.

(a) We prove $\Gamma_\alpha(X,\tau)=[\pi_1]$. For any $\Re\in\Im(P(X))$, we set $\Re^c(X-A)=\Re(A)$. Then

$$[\Re\subseteq\tau_\alpha]=\bigwedge_{A\in P(X)}\min(1,1-\Re(A)+\tau_\alpha(A))$$
$$=\bigwedge_{X-A\in P(X)}\min(1,1-\Re^c(X-A)+\mathsf{F}_\alpha(X-A))$$
$$=[\Re^c\subseteq\mathsf{F}_\alpha],$$

$$FF(\Re)=1-\bigwedge\{\delta\in[0,1]:F(\Re_\delta)\}$$
$$=1-\bigwedge\left\{\delta\in[0,1]:F(\Re^c_\delta)\right\}=FF(\Re^c)$$

and

$$\beta\leqslant\Re^c\iff\beta(M)\leqslant\Re^c(M)\iff\beta^c(X-M)$$
$$\leqslant\Re(X-M)\iff\beta^c\leqslant\Re.$$

Therefore

$$\begin{aligned}
\Gamma_\alpha(X,\tau) &= [(\forall\Re)(K_\alpha(\Re,X) \\
&\to (\exists\wp)((\wp \leqslant \Re) \wedge K(\wp,X) \otimes FF(\wp)))] \\
&= [(\forall\Re)((\Re \subseteq \tau_\alpha) \otimes K(\Re,X) \\
&\to (\exists\wp)((\wp \leqslant \Re) \wedge K(\wp,X) \otimes FF(\wp)))] \\
&= [(\forall\Re)((\Re \subseteq \tau_\alpha) \to (K(\Re,X) \\
&\to (\exists\wp)((\wp \leqslant \Re) \wedge K(\wp,X) \otimes FF(\wp))))] \\
&= [(\forall\Re)((\Re^c \subseteq \mathsf{F}_\alpha) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \\
&\to (\exists\wp)((\wp \leqslant \Re) \wedge K(\wp,X) \otimes FF(\wp))))] \\
&= [(\forall\Re)((\Re^c \subseteq \mathsf{F}_\alpha) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \\
&\to (\exists\beta^c)((\beta^c \leqslant \Re) \wedge K(\beta^c,X) \otimes FF(\beta^c))))] \\
&= [(\forall\Re)((\Re^c \subseteq \mathsf{F}_\alpha) \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \\
&\to (\exists\beta)((\beta \leqslant \Re^c) \wedge FF(\beta) \otimes K(\beta^c,X))))] \\
&= [(\forall\Re)(\Re^c \subseteq \mathsf{F}_\alpha \to ((\forall x)(\exists A)(A \in \Re \wedge x \in A) \\
&\to (\exists\beta)((\beta \leqslant \Re^c) \wedge FF(\beta) \otimes (\forall x)(\exists B)(B \in \beta^c \wedge x \in B))] \\
&= [(\forall\Re)(\Re^c \subseteq \mathsf{F}_\alpha \to (\neg((\exists\beta)(\beta \leqslant \Re^c \wedge FF(\beta) \\
&\otimes (\forall x)(\exists B)(B \in \beta^c \wedge x \in B)) \\
&\to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A))))] \\
&= [(\forall\Re)((\Re^c \subseteq \mathsf{F}_\alpha) \to (fI(\Re^c) \\
&\to \neg((\forall x)(\exists A)(A \in \Re \wedge x \in A)))] \\
&= [(\forall\Re)((\Re^c \subseteq \mathsf{F}_\alpha) \otimes fI(\Re^c) \\
&\to (\exists x)(\forall A)(A \in \Re^c \to x \in A))] \\
&= [\pi_1].
\end{aligned}$$

(b) We prove $[\pi_1] = [\pi_2]$. Let $X - B \in P(X)$. For any $\Re \in \Im(P(X))$.

$$\begin{aligned}
[(\Re \subseteq \mathsf{F}_\alpha) \wedge (B \in \tau_\alpha)] &= [(\Re \subseteq \mathsf{F}_\alpha) \wedge (X - B \in \mathsf{F}_\alpha)] \\
&= \bigwedge_{A \in P(X)} \min(1, 1 - \Re(A) + \mathsf{F}_\alpha(A)) \wedge \mathsf{F}_\alpha(X - B) \\
&= \bigwedge_{A \in P(X)} \min(1, 1 - \Re(A) + \mathsf{F}_\alpha(A)) \\
&\wedge \bigwedge_{A \in P(X)} \min(1, 1 - [A \in \{X - B\}] + \mathsf{F}_\alpha(A)) \\
&= \bigwedge_{A \in P(X)} \min(1, 1 - [(\Re \cup \{X - B\})(A)] + \mathsf{F}_\alpha(A)) \\
&= [(\Re \cup \{X - B\}) \subseteq \mathsf{F}_\alpha].
\end{aligned}$$

Therefore, for any $\beta \in \Im(P(X))$, let $\wp = \beta \setminus \{X - B\} \in \Im(P(X))$. $\wp(A) = \begin{cases} \beta(A), & A \neq X - B \\ 0, & A = X - B \end{cases}$. Then $\wp \leqslant \beta$, $\wp \cup \{X - B\} \geqslant \beta$, $[FF(\wp)] = [FF(\beta)]$, $[\wp \leqslant \Re] = [\beta \leqslant (\Re \cup \{X - B\})]$ and

$$\begin{aligned}
&[(\forall\wp)((\wp \leqslant \Re) \otimes FF(\wp) \to (\exists x)(\forall A)(A \in (\wp \cup \{X - B\}) \to (x \in A))] \\
&= \bigwedge_{\wp \leqslant \Re} \min\left(1, 1 - [FF(\wp)] + \bigvee_{x \in X} \bigwedge_{A \in P(X)} ((\wp \cup \{X - B\})(A)\right. \\
&\to A(x))) \leqslant \bigwedge_{\beta \leqslant (\Re \cup \{X - B\})} \min(1, 1 - [FF(\beta)] \\
&+ \bigvee_{x \in X} \bigwedge_{A \in P(X)} (\beta(A) \to A(x))) = fI(\Re \cup \{X - B\}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\pi_1 \otimes \Big[((\Re \subseteq \mathsf{F}_\alpha) \wedge (B \in \tau_\alpha)) \otimes (\forall\wp)\Big((\wp \leqslant \Re) \otimes FF(\wp) \to \neg\Big(\bigcap \wp \subseteq B\Big)\Big)\Big] \\
&= \pi_1 \otimes [(\Re \cup \{X - B\} \subseteq \mathsf{F}_\alpha) \otimes (\forall\wp)((\wp \leqslant \Re) \otimes FF(\wp) \\
&\to (\exists x)(\forall A)(A \in (\wp \cup \{X - B\}) \to x \in A))] \\
&= \pi_1 \otimes [(\Re \cup \{X - B\} \subseteq \mathsf{F}_\alpha) \otimes fI(\Re \cup \{X - B\})] \\
&\leqslant [(\exists x)(\forall A)(A \in (\Re \cup \{X - B\}) \to x \in A)] = \Big[\neg\Big(\bigcap \Re \subseteq B\Big)\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi_1 &\leqslant \bigwedge_{\Re \in \Im(P(X))} \bigvee_{B \subseteq X} ((\Re \subseteq \mathsf{F}_\alpha \wedge B \in \tau_\alpha) \otimes (\forall\wp)((\wp \leqslant \Re) \otimes FF(\wp) \\
&\to \neg\Big(\bigcap \wp \subseteq B\Big)\Big) \to \neg\Big(\bigcap \Re \subseteq B\Big)\Big) = \pi_2.
\end{aligned}$$

Conversely

$$\begin{aligned}
\pi_2 \otimes [(\Re \subseteq \mathsf{F}_\alpha) \otimes fI(\Re)] &= \pi_2 \otimes [((\Re \setminus \{B\}) \cup \{B\}) \subseteq \mathsf{F}_\alpha] \\
&\otimes [fI((\Re \setminus \{B\}) \cup \{B\})] \\
&= \pi_2 \otimes [(\Re' \subseteq \mathsf{F}_\alpha) \wedge (X - B \in \tau_\alpha) \\
&\otimes (\forall\wp)((\wp \leqslant \Re') \otimes FF(\wp) \\
&\to (\exists x)(\forall A)(A \in (\wp \cup \{B\}) \\
&\to x \in A))] \\
&= \pi_2 \otimes [(\Re' \subseteq \mathsf{F}_\alpha) \wedge (X - B \in \tau_\alpha) \\
&\otimes (\forall\wp)((\wp \leqslant \Re') \otimes FF(\wp) \\
&\to \neg\Big(\bigcap \wp \subseteq X - B\Big)\Big)\Big] \\
&\leqslant \Big[\neg\Big(\bigcap \Re' \subseteq X - B\Big)\Big] \\
&= [(\exists x)(\forall A)((A \in (\Re' \cup \{B\}) \\
&\to (x \in A))] = [(\exists x)(\forall A)(A \in \Re \\
&\to (x \in A))].
\end{aligned}$$

Therefore

$$\pi_2 \leqslant \bigwedge_{\Re \in \Im(P(X))} [(\Re \subseteq \mathsf{F}_\alpha) \otimes fI(\Re) \to (\exists x)(\forall A)(A \in \Re \to (x \in A))] = \pi_1. \quad \square$$

4. Some properties of fuzzifying α-compact spaces

Theorem 4.1. *For any fuzzifying topological space (X, τ) and $A \subseteq X$,*

(1) $\models \Gamma_\alpha(X, \tau) \otimes A \in \mathsf{F}_\alpha \to \Gamma_\alpha(A)$;
(2) $\models \Gamma_\alpha(X, \tau) \otimes A \in \mathsf{F}_\alpha \to \Gamma(A)$;
(3) $\models \Gamma_\alpha(X, \tau) \otimes A \in \mathsf{F} \to \Gamma_\alpha(A)$;
(4) $\models \Gamma_\alpha(X, \tau) \otimes A \in \mathsf{F} \to \Gamma(A)$.

Proof.

(1) For any $\Re \in \Im(P(A))$, we define $\overline{\Re} \in \Im(P(X))$ as $\overline{\Re}(B) = \begin{cases} \Re(B) & \text{if } B \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$ Then $FF(\overline{\Re}) = 1 - \bigwedge\{\beta \in [0,1] : F(\overline{\Re}_\beta)\} = 1 - \bigwedge\{\beta \in [0,1] : F(\Re_\beta)\} = FF(\Re)$

and

$$\bigvee_{x\in X}\bigwedge_{x\notin B\subseteq X}(1-\overline{\Re}(B))=\bigvee_{x\in X}\left(\left(\bigwedge_{x\not\subseteq B\subseteq A}(1-\overline{\Re}(B))\right)\wedge\left(\bigwedge_{x\notin B\subsetneq A}(1-\overline{\Re}(B))\right)\right)$$
$$=\bigvee_{x\in X}\left(\bigwedge_{x\not\subseteq B\subseteq A}(1-\overline{\Re}(B))\right)\wedge\bigvee_{x\in X}\left(\bigwedge_{x\notin B\subsetneq A}(1-\overline{\Re}(B))\right)$$
$$=\bigvee_{x\in X}\left(\bigwedge_{x\notin B\subseteq A}(1-\Re(B))\right)$$
$$=\bigvee_{x\in A}\left(\bigwedge_{x\notin B\subseteq A}(1-\Re(B))\right)\vee\bigvee_{x\notin A}\left(\bigwedge_{x\notin B\subseteq A}(1-\Re(B))\right)$$

If $x\notin A$, then for any $x'\in A$ we have

$$\bigwedge_{x\notin B\subseteq A}(1-\Re(B))=\bigwedge_{B\subseteq A}(1-\Re(B))\leqslant\bigwedge_{x\prime\notin B\subseteq A}(1-\Re(B)).$$

Therefore

$$\bigvee_{x\in X}\bigwedge_{x\notin B\subseteq X}(1-\overline{\Re}(B))=\bigvee_{x\in A}\bigwedge_{x\notin B\subseteq A}(1-\Re(B)),$$

$$[fI(\overline{\Re})]=[(\forall\overline{\beta})((\overline{\beta}\leqslant\overline{\Re})\wedge FF(\overline{\beta})\to(\exists x)(\forall B)((B\in\overline{\Re})\to(x\in B)))]$$
$$=\bigwedge_{\overline{\beta}\leqslant\overline{\Re}}\min\left(1,1-FF(\overline{\beta})+\bigvee_{x\in X}\bigwedge_{x\notin B\subseteq X}(1-\overline{\Re}(B))\right)$$
$$=\bigwedge_{\beta\leqslant\Re}\min\left(1,1-FF(\beta)+\bigvee_{x\in A}\bigwedge_{x\notin B\subseteq A}(1-\Re(B))\right)$$
$$=[fI(\Re)].$$

We want to prove $\mathsf{F}_\alpha(A)\otimes[\Re\subseteq\mathsf{F}_\alpha/_A]\leqslant[\overline{\Re}\subseteq\mathsf{F}_\alpha]$. In fact, $F_\alpha\left(\bigcap_{\lambda\in\Lambda}A_\lambda\right)\geqslant\bigwedge_{\lambda\in\Lambda}F_\alpha(A_\lambda)$ (see[17, Theorem 3.2]). Thus

$$\mathsf{F}_\alpha(A)\otimes[\Re\subseteq\mathsf{F}_\alpha/_A]$$
$$=\max\left(0,\mathsf{F}_\alpha(A)+\bigwedge_{B\subseteq A}\min(1,1-\Re(B)+\mathsf{F}_\alpha/_A(B))-1\right)$$
$$\leqslant\bigwedge_{B\subseteq A}(1-\Re(B))+(\mathsf{F}_\alpha(A)+\mathsf{F}_\alpha/_A(B)-1)$$
$$\leqslant\bigwedge_{B\subseteq A}(1-\Re(B))+(\mathsf{F}_\alpha(A)\wedge\mathsf{F}_\alpha/_A(B))$$
$$=\bigwedge_{B\subseteq A}(1-\Re(B))+\left(\mathsf{F}_\alpha(A)\wedge\bigvee_{B'\cap A=B,B'\subseteq X}\mathsf{F}_\alpha(B')\right)$$
$$=\bigwedge_{B\subseteq A}(1-\Re(B))+\bigvee_{B'\cap A=B,B'\subseteq X}(\mathsf{F}_\alpha(A)\wedge\mathsf{F}_\alpha(B'))$$
$$\leqslant\bigwedge_{B\subseteq A}(1-\Re(B))+\bigvee_{B'\cap A=B,B'\subseteq X}(\mathsf{F}_\alpha(A\cap B'))$$
$$\leqslant\bigwedge_{B\subseteq A}(1-\Re(B))+\mathsf{F}_\alpha(B)$$
$$=\bigwedge_{B\subseteq A}\min(1,1-\Re(B)+\mathsf{F}_\alpha(B))$$
$$\leqslant\bigwedge_{B\subseteq A}\min(1,1-\overline{\Re}(B)+\mathsf{F}_\alpha(B))$$
$$=[\overline{\Re}\subseteq\mathsf{F}_\alpha].$$

Furthermore, from Theorem 3.3 we have

$$\Gamma_\alpha(X,\tau)\otimes\mathsf{F}_\alpha(A)\otimes[\Re\subseteq\mathsf{F}_\alpha/_A]\otimes fI(\Re)$$
$$\leqslant\Gamma_\alpha(X,\tau)\otimes[\overline{\Re}\subseteq\mathsf{F}_\alpha]\otimes fI(\overline{\Re})\leqslant\bigvee_{x\in X}\bigwedge_{x\notin B\subseteq A}(1-\overline{\Re}(B))$$
$$=\bigvee_{x\in A}\bigwedge_{x\notin B\subseteq A}(1-\Re(B)).$$

Then

$$\Gamma_\alpha(X,\tau)\otimes\mathsf{F}_\alpha(A)\leqslant[\Re\subseteq\mathsf{F}_\alpha/_A]\otimes fI(\Re)\to\bigvee_{x\in A}\bigwedge_{x\notin B\subseteq A}(1-\Re(B))$$
$$\leqslant\bigwedge_{\Re\in\Im(P(A))}\left([\Re\subseteq\mathsf{F}_\alpha/_A]\otimes fI(\Re)\to\bigvee_{x\in A}\bigwedge_{x\notin B\subseteq A}(1-\Re(B))\right)$$
$$=\Gamma_\alpha(A).$$

(2) From (1) above and Theorem 3.1, the result holds.
(3) From Theorem 3.3 [17] we have $\models\mathsf{F}\subseteq\mathsf{F}_\alpha$ or $[A\in\mathsf{F}]\leqslant[A\in\mathsf{F}_\alpha]$. Then we obtain

$$[\Gamma_\alpha(X,\tau)\otimes A\in\mathsf{F}]\leqslant[\Gamma_\alpha(X,\tau)\otimes A\in\mathsf{F}_\alpha]\leqslant\Gamma_\alpha(A).$$

(4) From (3) above and Theorem 3.1 the result holds. □

As a corollary of the above theorem we have the following well known theorem for classical topological spaces (see [25], Corollary 3.5).

Theorem 4.2. *Every α-closed subset of an α-compact space is α - compact.*

Theorem 4.3. *Let (X,τ) and (Y,σ) be any two fuzzifying topological spaces and $f\in Y^X$ is surjection. Then*

$$\models\Gamma_\alpha(X,\tau)\otimes C_\alpha(f)\to\Gamma(f(X))$$

Proof. (1) For $\beta\in\Im(P(Y))$, we define $\Re\in\Im(P(X))$ as $\Re(A)=f^{-1}(\beta)(A)=\beta(f(A))$. Then $K(\Re,X)=\bigwedge_{x\in X}\bigvee_{x\in A}\Re(A)=\bigwedge_{x\in X}\bigvee_{x\in A}\beta(f(A))=\bigwedge_{x\in X}\bigvee_{f(x)\in B}\beta(B)=\bigwedge_{y\in f(X)}\bigvee_{y\in B}\beta(B)=K(\beta,f(X))$ and some calculations lead to

$$[\beta\subseteq\sigma]\otimes[C_\alpha(f)]=\bigwedge_{B\subseteq Y}\min(1,1-\beta(B)+\sigma(B))$$
$$\otimes\bigwedge_{B\subseteq Y}\min(1,1-\sigma(B)+\tau_\alpha(f^{-1}(B)))$$
$$=\max(0,\bigwedge_{B\subseteq Y}\min(1,1-\beta(B)+\sigma(B))$$
$$+\bigwedge_{B\subseteq Y}\min(1,1-\sigma(B)+\tau_\alpha(f^{-1}(B)))-1)$$
$$\leqslant\bigwedge_{B\subseteq Y}\max(0,\min(1,1-\beta(B)+\sigma(B))$$
$$+\min(1,1-\sigma(B)+\tau_\alpha(f^{-1}(B)))-1)$$
$$\leqslant\bigwedge_{B\subseteq Y}\min(1,1-\beta(B)+\tau_\alpha(f^{-1}(B)))$$
$$=\bigwedge_{A\subseteq X}\bigwedge_{f^{-1}(B)=A}\min(1,1-\beta(B)+\tau_\alpha(f^{-1}(B)))$$
$$=\bigwedge_{A\subseteq X}\bigwedge_{f^{-1}(B)=A}\min(1,1-\beta(B)+\tau_\alpha(A))$$
$$=\bigwedge_{A\subseteq X}\min\left(1,1-\bigvee_{f^{-1}(B)=A}\beta(B)+\tau_\alpha(A)\right)$$
$$=\bigwedge_{A\subseteq X}\min(1,1-\Re(A)+\tau_\alpha(A))=[\Re\subseteq\tau_\alpha].$$

For any $\wp \leqslant \Re$, we set $\overline{\wp} \in \Im(P(Y))$ defined as $\overline{\wp}(f(A)) = f(\wp)(f(A)) = \wp(A), A \subseteq X$. Then $\overline{\wp}(f(A)) = f(\wp)(f(A)) \leqslant f(\Re)(f(A)) = f(f^{-1}(\beta)(f(A)) \leqslant \beta(f(A)), FF(\wp) = 1 - \bigwedge\{\delta \in [0,1] : F(\wp_{[\delta]})\} = 1 - \bigwedge\{\delta \in [0,1] : F(f(\wp)_{[\delta]})\} = FF(f(\wp) \leqslant FF(\overline{\wp})$ and $K(\overline{\wp}, f(X)) = \bigwedge_{y\in f(X)} \bigvee_{y\in B} \overline{\wp}(B) = \bigwedge_{y\in f(X)} \bigvee_{y\in B=f(A)} \wp(A) \geqslant \bigwedge_{y\in f(X)} \bigvee_{f^{-1}(y)\in A} \wp(A) = \bigwedge_{x\in X} \bigvee_{x\in A} \wp(A) = K(\wp, X)$.

Furthermore

$$\begin{aligned}
&[\Gamma_\alpha(X,\tau)] \otimes [C_\alpha(f)] \otimes [K'_\circ(\beta, f(X))] \\
&= [\Gamma_\alpha(X,\tau)] \otimes [C_\alpha(f)] \otimes [K(\beta, f(X))] \otimes [\beta \subseteq \sigma] \\
&\leqslant [\Gamma_\alpha(X,\tau)] \otimes [\Re \subseteq \tau_\alpha] \otimes [K(\Re, X)] \\
&= [\Gamma_\alpha(X,\tau)] \otimes [K_\alpha(\Re, X)] \\
&\leqslant [(\exists \wp)((\wp \leqslant \Re) \wedge K(\wp, X) \otimes FF(\wp))] \\
&\leqslant [(\exists \wp)((\wp \leqslant \Re) \wedge K(\overline{\wp}, f(X)) \otimes FF(\overline{\wp}))] \\
&\leqslant [(\exists \wp')((\wp' \leqslant \Re) \wedge K(\wp', f(X)) \otimes FF(\wp'))],
\end{aligned}$$

where $K'_\circ$ is related to σ. Therefore

$$\begin{aligned}
[\Gamma_\alpha(X,\tau)] \otimes [C_\alpha(f)] \leqslant K'_\circ(\beta, f(X)) &\to (\exists \wp')((\wp' \leqslant \Re) \wedge K(\wp', f(X)) \\
&\otimes FF(\wp')) \leqslant \bigwedge_{\beta \in \Im(P(X))} \big(K'_\circ(\beta, f(X)) \\
&\to (\exists \wp')((\wp' \leqslant \Re) \wedge K(\wp', f(X)) \otimes FF(\wp'))) \\
&= [\Gamma(f(X))]. \quad \square
\end{aligned}$$

Theorem 4.4. *Let (X,τ) and (Y,σ) be any two fuzzifying topological space and $f \in Y^X$ is surjection. Then $\models \Gamma_\alpha(X,\tau) \otimes I_\alpha(f) \to \Gamma_\alpha(f(X))$.*

Proof. From the proof of Theorem 4.3 we have for any ß$\in \Im(P(Y))$ we define $\Re \in \Im(P(X))$ as $\Re(A) = f^{-1}(\beta)(A) = \beta(f(A))$. Then $K(\Re, X) = K(\beta, f(X))$ and $[\beta \subseteq \sigma_\alpha] \otimes [I_\alpha(f)] \leqslant [\Re \subseteq \tau_\alpha]$. For any $\wp \leqslant \Re$, we set $\overline{\wp} \in \Im(P(Y))$ defined as $\overline{\wp}(f(A)) = f(\wp)(f(A)) = \wp(A), A \subseteq X$ and we have $FF(\wp) \leqslant FF(\overline{\wp}), K(\overline{\wp}, f(X)) \geqslant K(\wp, X)$. Therefore

$$\begin{aligned}
&[\Gamma_\alpha(X,\tau)] \otimes [I_\alpha(f)] \otimes [K'_\alpha(\beta, f(X))] \\
&= [\Gamma_\alpha(X,\tau)] \otimes [I_\alpha(f)] \otimes [K(\beta, f(X))] \otimes [\beta \subseteq \sigma_\alpha] \\
&\leqslant [\Gamma_\alpha(X,\tau)] \otimes [\Re \subseteq \tau_\alpha] \otimes [K(\Re, X)] \\
&= [\Gamma_\alpha(X,\tau)] \otimes [K_\alpha(\Re, X)] \\
&\leqslant [(\exists \wp)((\wp \leqslant \Re) \wedge K(\wp, X) \otimes FF(\wp))] \\
&\leqslant [(\exists \wp)((\wp \leqslant \Re) \wedge K(\overline{\wp}, f(X)) \otimes FF(\overline{\wp}))] \\
&\leqslant [(\exists \wp')((\wp' \leqslant \beta) \wedge K(\wp', f(X)) \otimes FF(\wp'))],
\end{aligned}$$

where K'_α is related to σ. Therefore

$$\begin{aligned}
[\Gamma_\alpha(X,\tau)] \otimes [I_\alpha(f)] \leqslant K'_\alpha(\beta, f(X)) &\to (\exists \wp')((\wp' \leqslant \beta) \wedge K(\wp', f(X)) \otimes FF(\wp')) \\
&\leqslant \inf_{\beta \in \Im(P(X))} \big(K'_\alpha(\beta, f(X)) \\
&\to (\exists \wp')((\wp' \leqslant \beta) \wedge K(\wp', f(X)) \otimes FF(\wp'))) \\
&= [\Gamma_\alpha(f(X))]. \quad \square
\end{aligned}$$

As a corollary of the above two theorems we have the following theorem [24].

Theorem 4.5. *Let (X,τ), (Y,σ) be two topological spaces and $f:(X,\tau) \to (Y,\sigma)$ be a surjective mapping. If f is α-continuous (resp. α-irresolute) and X is α-compact, then Y is compact (resp. α-compact).*

Theorem 4.6. *Let (X,τ) be any fuzzifying α-topological space and $A, B \subseteq X$. Then*

(1) $T_2^\alpha(X,\tau) \otimes (\Gamma_\alpha(A) \wedge \Gamma_\alpha(B)) \wedge A \cap B = \phi \models^{ws} T_2^\alpha(X,\tau) \to (\exists U)(\exists V)((U \in \tau_\alpha) \wedge (V \in \tau_\alpha) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V = \phi))$;
(2) $T_2^\alpha(X,\tau) \otimes \Gamma_\alpha(A) \models^{ws} T_2^\alpha(X,\tau) \to A \in \mathsf{F}_\alpha$, *where* $\varphi \models^{ws} \psi$ *means that* $[\varphi] > 0$ *implies* $[\psi] = 1$ *(see* [21], Definition 3.1]).

Proof.

(1) Assume $A \cap B = \phi$ and $T_2^\alpha(X,\tau) = t$. Let $x \in A$. For any $y \in B$ and $\lambda < t$ we have $\bigvee\{\tau_\alpha(P) \wedge \tau_\alpha(Q) : x \in P, y \in Q, P \cap Q = \phi\} = \bigvee\{\tau_\alpha(P) \wedge \tau_\alpha(Q) : x \in P \subseteq U, y \in Q \subseteq V, U \cap V = \phi\} = \bigvee_{U\cap V=\phi} \left\{\bigvee_{x\in P\subseteq U} \tau_\alpha(P) \wedge \bigvee_{y\in Q\subseteq V} \tau_\alpha(Q)\right\} = \bigvee_{U\cap V=\phi} \left\{N_x^\alpha(U) \wedge N_y^\alpha(V)\right\} \geqslant \bigwedge_{x\neq y} \bigvee_{U\cap V=\phi} \{N_x^\alpha(U) \wedge N_y^\alpha(V)\} = T_2^\alpha(X,\tau) = t > \lambda$, i.e., there exist P_y, Q_y such that $x \in P_y, y \in Q_y, P_y \cap Q_y = \phi$ and $\tau_\alpha(P_y) > \lambda, \tau_\alpha(Q_y) > \lambda$. Set $\beta(Q_y) = \tau_\alpha(Q_y)$ for $y \in B$. Since $[\beta \subseteq \tau_\alpha] = 1$, we have

$$[K_\alpha(\beta, B)] = [K(\beta, B)] = \bigwedge_{y\in B} \bigvee_{y\in C} \beta(C) \geqslant \bigwedge_{y\in B} \beta(Q_y) = \bigwedge_{y\in B} \tau_\alpha(Q_y) \geqslant \lambda.$$

On the other hand, Since $[T_2^\alpha(X,\tau) \otimes (\Gamma_\alpha(A) \wedge \Gamma_\alpha(B))] > 0$, $T_2^\alpha(X,\tau) + (\Gamma_\alpha(A) \wedge \Gamma_\alpha(B)) - 1 > 0$ or $1 - t < \Gamma_\alpha(A) \wedge \Gamma_\alpha(B) \leqslant \Gamma_\alpha(A)$. From Theorem 3.2 we have for any $\lambda \in (1 - \Gamma_\alpha(A), t)$, it holds that $1 - \lambda < \Gamma_\alpha(A) \leqslant 1 - [K_\alpha(\beta, B)] + \bigvee_{\wp \leqslant \beta}\{K(\wp, B)] \otimes FF(\wp)\} \leqslant 1 - \lambda + \bigvee_{\wp \leqslant \beta}\{K(\wp, B)] \otimes FF(\wp)\}$, i.e., $\bigvee_{\wp \leqslant \beta}\{K(\wp, B)] \otimes FF(\wp)\} > 0$ and there exists $\wp \leqslant \beta$ such that $K(\wp, B) + FF(\wp) - 1 > 0$, i.e., $1 - FF(\wp) < K(\wp, B)$. Then, $\bigwedge\{\theta : F(\wp_\theta)\} < K(\wp, B)$. Now, there exists θ_1 such that $\theta_1 < K(\wp, B)$ and $F(\wp_{\theta_1})$. Since $\wp \leqslant \beta$, we may write $\wp_{\theta_1} = \{Q_{y_1}, \ldots, Q_{y_n}\}$. We put $U_x = \{P_{y_1} \cap \ldots \cap P_{y_n}\}, V_x = \{Q_{y_1} \cap \ldots \cap Q_{y_n}\}$ and have $V_x \supseteq B, U_x \cap V_x = \phi, \tau_\alpha(U_x) \geqslant \tau_\alpha(P_{y_1}) \wedge \ldots \wedge \tau_\alpha(P_{y_n}) > \lambda$ because (X,τ) is fuzzifying α-topological space. Also, $\tau_\alpha(V_x) \geqslant \tau_\alpha(Q_{y_1}) \wedge \ldots \wedge \tau_\alpha(Q_{y_n}) > \lambda$. In fact, $\bigwedge_{y\in B} \bigvee_{y\in D} \wp(D) = K(\wp, B) > \theta_1$, and for any $y \in B$, there exists D such that $y \in D$ and $\wp(D) > \theta_1, D \in \wp_{\theta_1}$. Similarly, if $\lambda \in (1 - [\Gamma_\alpha(A) \wedge \Gamma_\alpha(B)], t)$, then we can find $x_1, \ldots, x_m \in A$ with $U_\circ = U_{x_1} \cup \ldots \cup U_{x_m} \supseteq A$. By putting $V_\circ = V_{x_1} \cap \ldots \cap V_{x_m}$ we obtain $V_\circ \supseteq B, U_\circ \cap V_\circ = \phi$ and

$$\begin{aligned}
(\exists U)(\exists V)((U \in \tau_\alpha) \wedge (V \in \tau_\alpha) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (U \cap V \\
= \phi)) \geqslant \tau_\alpha(U_\circ) \wedge \tau_\alpha(V_\circ) \\
\geqslant \min_{i=1,\ldots,n} \tau_\alpha(U_{x_i}) \wedge \min_{i=1,\ldots,n} \tau_\alpha(V_{x_i}) > \lambda.
\end{aligned}$$

Finally, we let $\lambda \to t$ and complete the proof.

(2) Assume $[T_2^\alpha(X,\tau) \otimes \Gamma_\alpha(A)] > 0$. Then for any $x \in X - A$ we have from (1) above that

$$\bigvee_{x \in U \subseteq X-A} \tau_\alpha(U) \geqslant \bigvee\{\tau_\alpha(U) \wedge \tau_\alpha(V) : x \in U, A \subseteq V, U \cap V = \phi\} \geqslant [T_2^\alpha(X,\tau)].$$

Since $\tau_\alpha(A) = \bigwedge_{x \in A} N_x^\alpha(A)$ (see [17], Corollary 4.1), then we have that

$$\mathsf{F}_\alpha(A) = \tau_\alpha(X - A) = \bigwedge_{x \in X-A} N_x^\alpha(X - A) = \bigwedge_{x \in X-A} \bigvee_{x \in U \subseteq X-A} \tau_\alpha(U) \geqslant [T_2^\alpha(X,\tau)]. \quad \square$$

As a corollary of the above theorem we have the following result in general topology.

Theorem 4.7.

(1) *For any disjoint α-compact subsets A and B of an α-Hausdorff space X, there exist disjoint α-open sets U and V such that $A \subset U$ and $B \subset V$.*
(2) *Every α-compact subset of α-Hausdorff space is α-closed.*

Definition 4.1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. A unary fuzzy predicate $Q_\alpha \in \Im(Y^X)$, called fuzzifying α-closedness, is given as follows:

$$Q_\alpha(f) := \forall B\big(B \in \mathsf{F}_\alpha^X \to f^{-1}(B) \in \mathsf{F}_\alpha^Y\big),$$

where F_α^X and F_α^Y are the fuzzy families of τ, σ-α-closed in X and Y, respectively.

Theorem 4.8. *Let (X,τ) be a fuzzifying topological space (Y,σ) be a fuzzifying α-topological space and $f \in Y^X$. Then $\models \Gamma_\alpha(X,\tau) \otimes T_2^\alpha(Y,\sigma) \otimes I_\alpha(f) \to Q_\alpha(f)$.*

Proof. For any $A \subseteq X$, we have the following:

(i) From Theorem 4.1 we have $\left[\Gamma_\alpha(X,\tau) \otimes \mathsf{F}_\alpha^X(A)\right] \leqslant \Gamma_\alpha(A)$;

(ii)
$$\begin{aligned} I_\alpha(f_{\backslash A}) &= \bigwedge_{U \in P(Y)} \min\left(1, 1 - \sigma_\alpha(U) + \tau_\alpha/_A((f/_A)^{-1}(U))\right) \\ &= \bigwedge_{U \in P(Y)} \min\left(1, 1 - \sigma_\alpha(U) + \tau_\alpha/_A(A \cap f^{-1}(U))\right) \\ &= \bigwedge_{U \in P(Y)} \min\left(1, 1 - \sigma_\alpha(U) + \bigvee_{A \cap f^{-1}(U) = B \cap A} \tau_\alpha(B)\right) \\ &\geqslant \bigwedge_{U \in P(Y)} \min\left(1, 1 - \sigma_\alpha(U) + \tau_\alpha(f^{-1}(U))\right) \\ &= I_\alpha(f). \end{aligned}$$

(iii) From Theorem 4.4, we have $[\Gamma_\alpha(A) \otimes I_\alpha(f_{\backslash A})] \leqslant \Gamma_\alpha(f(A))$.

(iv) From Theorem 4.6 (2) we have $T_2^\alpha(Y,\sigma) \otimes \Gamma_\alpha(f(A)) \models^{ws} T_2^\alpha(Y,\sigma) \to f(A) \in \mathsf{F}_\alpha^Y$, which implies $\models T_2^\alpha(Y,\sigma) \otimes \Gamma_\alpha(f(A)) \to f(A) \in \mathsf{F}_\alpha^Y$.By combining (i)–(iv) we have

$$\begin{aligned} [\Gamma_\alpha(X,\tau) \otimes T_2^\alpha(Y,\sigma) \otimes I_\alpha(f)] &\leqslant [(\mathsf{F}_\alpha^X(A) \to \Gamma_\alpha(A)) \otimes I_\alpha(f_{\backslash A}) \\ &\quad \otimes T_2^\alpha(Y,\sigma)] \leqslant [(\mathsf{F}_\alpha^X(A) \to (\Gamma_\alpha(A)) \\ &\quad \otimes I_\alpha(f_{\backslash A}))) \otimes T_2^\alpha(Y,\sigma)] \\ &\leqslant \left[\mathsf{F}_\alpha^X(A) \to \Gamma_\alpha(f(A)) \otimes T_2^\alpha(Y,\sigma)\right] \\ &= \left[\mathsf{F}_\alpha^X(A) \to \mathsf{F}_\alpha^Y(f(A))\right] \\ &\leqslant \bigwedge_{A \subseteq X} \left(\left[\mathsf{F}_\alpha^X(A) \to \mathsf{F}_\alpha^Y(f(A))\right]\right) \\ &= Q_\alpha(f). \quad \square \end{aligned}$$

As a crisp setting from the above theorem we have

Theorem 4.9. *An α-irresolute map from α-compact space to α-Hausdorff space is α-closed.*

5. Conclusion

The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous valued-logic). It continue various investigations into fuzzy topology in a legitimate way and extend some fundamental results in general topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of ukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more wide-spread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contributions of the paper are to study α-compact spaces in fuzzifying topology and the behavior of α-compact spaces under various types of mappings. There are some problems for further study:

(1) What is the justification for fuzzifying α-compactness in the setting of (2, L) topologies.
(2) Obviously, fuzzifying topological spaces in [13] form a fuzzy category. Perhaps, this will become a motivation for further study of the fuzzy category.
(3) It would be interesting to give examples and results considering sums, hereditary and productivity, etc.

References

[1] M.S. Ying, A new approach for fuzzy topology (I), Fuzzy Sets and Systems 39 (1991) 303–321.
[2] U. Höhle, Many Valued Topology and its Applications, Kluwer Academic Publishers, Dordrecht, 2001.
[3] U. Höhle, S.E. Rodabaugh, Mathematics of fuzzy sets: logic, topology, and measure theory, Handbook of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, 1999.

[4] U. Höhle, S.E. Rodabaugh, A. Ŝostak, (Eds.), Special Issue on Fuzzy Topology, Fuzzy Sets and Systems 73 (1995) 1–183.
[5] T. Kubiak, On Fuzzy Topologies, Ph.D. Thesis, Adam Mickiewicz University, Poznan, Poland, 1985.
[6] Y.M. Liu, M.K. Luo, Fuzzy Topology, World Scientific, Singapore, 1998.
[7] G.J. Wang, Theory of L-Fuzzy Topological Spaces, Shanxi Normal University Press, Xian, 1988 (in Chinese).
[8] C.L. Chang, Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 24 (1968) 182–190.
[9] J.A. Goguen, The fuzzy Tychonoff theorem, Journal of Mathematical Analysis and Applications 43 (1973) 182–190.
[10] J.L. Kelley, General Topology, Van Nostrand, New York, 1955.
[11] U. Höhle, Uppersemicontinuous fuzzy sets and applications, Journal of Mathematical Analysis and Applications 78 (1980) 659–673.
[12] U. Höhle, A. Ŝostak, Axiomatic foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Handbook of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, 1999, pp. 123–272.
[13] S.E. Rodabaugh, Categorical foundations of variable-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Handbook of Fuzzy Sets Series, vol. 3, Kluwer Academic Publishers, Dordrecht, 1999, pp. 273–388.
[14] J.B. Rosser, A.R. Turquette, Many-Valued Logics, North-Holland, Amsterdam, 1952.
[15] M.S. Ying, A new approach for fuzzy topology (II), Fuzzy Sets and Systems 47 (1992) 221–223.
[16] M.S. Ying, A new approach for fuzzy topology (III), Fuzzy Sets and Systems 55 (1993) 193–207.
[17] F.H. Khedr, F.M. Zeyada, O.R. Sayed, α-continuity and cα-continuity in fuzzifying topology, Fuzzy Sets and Systems 116 (2000) 325–337.
[18] D. Qiu, Fuzzifying topological linear spaces, Fuzzy Sets and Systems 147 (2004) 249–272.
[19] D. Qiu, Characterizations of fuzzy finite automata, Fuzzy Sets and Systems 141 (2004) 391–414.
[20] O.R. Sayed, Semi-compactness in fuzzifying topology, Proceedings of Pakistan Academic Sciences 42 (3) (2005) 177–186, 299–303.
[21] M.S. Ying, Compactness in fuzzifying topology, Fuzzy Sets and Systems 55 (1993) 79–92.
[22] O.R. Sayed, On Fuzzifying Topological Spaces, Ph.D. Thesis, Assiut University, Egypt, 2002.
[23] S.N. Maheshwari, S.S. Thakur, On α-irresolute mappings, Tamkang Journal of Mathematical 11 (1980) 209–214.
[24] S.N. Maheshwari, S.S. Thakur, On α-compact spaces, Bulletin of the Institute of Mathematics. Acadrmia Sinica 13 (4) (1985) 341–347.
[25] T. Noiri, G. Di Maio, Properties of α-compact spaces, III Convegno Topologia, Trieste, Giugno 1986, Supp. Rend. Circ. Mat. Palermo, Seric II 18 (1988) 359–369.
[26] R. Prasad, S.S. Thakur, R.K. Saraf, Fuzzy α-irresolute mappings, Fuzzy Mathematics 2 (1994) 335–339.
[27] T.E. Gantner, R.C. Steinlage, R.H. Worren, Compactness in fuzzy topological spaces, Journal of Mathematical Analysis and Applications 62 (1978) 547–562.
[28] S.S. Thakur, R.K. Saraf, α-Compact fuzzy topological spaces, Mathematica Bohemica 120 (3) (1995) 299–303.
[29] I.W. Alderton, α-Compactness of fuzzy topological spaces: a categorical approach, Quastiones Mathematicae 20 (1997) 269–282.
[30] H. Herrlich, G. Salicrup, G.E. Strecker, Factorizations, denseness, separation, and relatively compact objects, Topology Applications 27 (1987) 157–196.
[31] I.L. Reilly, M.K. Vamanamurthy, On α-continuity in topological spaces, Acta Mathematica Hungarica 45 (1–2) (1985) 27–32.