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α-continuity and $c\alpha$-continuity in fuzzifying topology

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Abstract

The concepts of α-continuity and $c\alpha$-continuity are considered and studied in fuzzifying topology and by making use of these concepts, some decompositions of fuzzy continuity are introduced. It is proved that the family of all α-sets in fuzzifying topology may not be a fuzzifying topology.

Keywords: Fuzzifying topology; Fuzzy continuity; α-continuity; $c\alpha$-continuity

1. Introduction

In [4], Ying introduced the concept of fuzzifying topology with the semantic method of continuous-valued logic. All the conventions in [4–6] are good in this paper. In general topology, Njastad [7] introduced the concept of α-sets and in [3] the concept of α-continuity is studied. It is worth mentioning that these concepts are introduced in fuzzy topology by Singal and Rajvanshi [9]. In [8], the concept of $D(c,\alpha)$-continuity is introduced which will be renamed also in the present paper as $c\alpha$-continuity. In the present paper, we extend and study the concepts of α-continuity and $c\alpha$-continuity in fuzzifying topology and by making use of them some decompositions of fuzzy continuity are introduced. A counterexample is given to prove that the family of all α-open sets need not be a fuzzifying topology which contradicts a well-known result in crisp setting. Finally, the concept of $c\alpha$-neighborhood system is presented and a fuzzifying topology induced by it is introduced.

2. Preliminaries

For the fuzzy logical and the corresponding set theoretical notations we refer to [4,5]. We note that the set of truth values is the unit interval and we do often not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. For the definitions and results in fuzzifying topology which are used in the sequel we refer to [4–6].

We now give some definitions and results are introduced in [2] which are useful in the rest of the present paper.

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Definition 2.1. For any $\tilde{A} \in \mathscr{F}(X)$,

$$\models (\tilde{A})^\circ \equiv X \sim \overline{(X \sim \tilde{A})}.$$

Lemma 2.1. *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then*
(1) $\models \bar{\tilde{A}} \subseteq \bar{\tilde{B}}$;
(2) $\models (\tilde{A})^\circ \subseteq (\tilde{B})^\circ$.

Lemma 2.2. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A}, \tilde{B}$,
(1) $\models X^\circ \equiv X$;
(2) $\models (\tilde{A})^\circ \subseteq \tilde{A}$;
(3) $\models (\tilde{A} \cap \tilde{B})^\circ \equiv (\tilde{A})^\circ \cap (\tilde{B})^\circ$;
(4) $\models (\tilde{A})^{\circ\circ} \supseteq (\tilde{A})^\circ$.

One can add the following lemma

Lemma 2.3. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A} \in \mathscr{F}(X)$,
(1) $\models X \sim (\tilde{A})^{\circ-\circ} \equiv (X \sim \tilde{A})^{-\circ-}$;
(2) *if* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then* $\models (\tilde{A})^{\circ-\circ} \subseteq (\tilde{B})^{\circ-\circ}$.

3. Fuzzifying α-open sets and fuzzifying $c\alpha$-open sets

Definition 3.1. Let (X, τ) be a fuzzifying topological space.
(1) The family of fuzzifying α- (resp. $c\alpha$-) open sets is denoted by $\alpha\tau$ (resp. $c\alpha\tau$) $\in \mathscr{F}(P(X))$ and defined as follows:

$$A \in \alpha\tau := \forall x (x \in A \to x \in A^{\circ-\circ}) \quad (\text{resp. } A \in c\alpha\tau := \forall x (x \in A \cap A^{\circ-\circ} \to x \in A^\circ)).$$

(2) The family of fuzzifying α- (resp. $c\alpha$-) closed sets is denoted by αF (resp. $c\alpha F$) $\in \mathscr{F}(P(X))$ and defined as follows:

$$A \in \alpha F \ (\text{resp. } c\alpha F) := X \sim A \in \alpha\tau \ (\text{resp. } c\alpha\tau).$$

Lemma 3.1. *For any* $\alpha, \beta, \gamma, \delta \in I$,

$$(1 - \alpha + \beta) \wedge (1 - \gamma + \delta) \leqslant 1 - (\alpha \wedge \gamma) + (\beta \wedge \delta).$$

Lemma 3.2. *For any* $A \in P(X)$,

$$\models A^\circ \subseteq A^{\circ-\circ}.$$

Proof. From Theorem 5.3 [7] we have $[A^\circ \subseteq A^{\circ-}] = 1$ and from Lemma 2.1(2), $[A^{\circ\circ} \subseteq A^{\circ-\circ}] = 1$. Since from Lemma 2.2 we have $[A^{\circ\circ} \equiv A^\circ] = 1$, the result holds. $\square$

Theorem 3.1. *Let* (X, τ) *be a fuzzifying topological space. Then*
(1) (a) $\alpha\tau(X) = 1$, $\alpha\tau(\emptyset) = 1$;
(b) *for any* $\{A_\lambda : \lambda \in \Lambda\}$, $\alpha\tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geqslant \bigwedge_{\lambda \in \Lambda} \alpha\tau(A_\lambda)$;
(2) (a) $c\alpha\tau(X) = 1$, $c\alpha\tau(\phi) = 1$;
(b) $c\alpha\tau(A \cap B) \geqslant c\alpha\tau(A) \wedge c\alpha\tau(B)$.

Proof. The proof of (a) in (1) and (a) in (2) are straightforward.
(1) (b) From Lemma 2.3, $\models A_\lambda^{\circ-\circ} \subseteq (\bigcup_{\lambda\in\Lambda} A_\lambda)^{\circ-\circ}$. So,

$$\alpha\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \inf_{x\in\bigcup_{\lambda\in\Lambda}A_\lambda}\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{\circ-\circ}(x) = \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda}\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{\circ-\circ}(x)$$

$$\geqslant \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda} A_\lambda^{\circ-\circ}(x) = \bigwedge_{\lambda\in\Lambda}\alpha\tau(A_\lambda).$$

(2) (b) Applying Lemmas 2.2 (3), 2.3 (2) and 3.1 we have

$$\begin{aligned}
c\alpha\tau(A)\wedge c\alpha\tau(B) &= \inf_{x\in A}(1-A^{\circ-\circ}(x)+A^\circ(x))\wedge\inf_{x\in B}(1-B^{\circ-\circ}(x)+B^\circ(x))\\
&\leqslant \inf_{x\in A\cap B}((1-A^{\circ-\circ}(x)+A^\circ(x))\wedge(1-B^{\circ-\circ}(x)+B^\circ(x)))\\
&\leqslant \inf_{x\in A\cap B}(1-(A^{\circ-\circ}\cap B^{\circ-\circ})(x)+(A^\circ\cap B^\circ)(x))\\
&\leqslant \inf_{x\in A\cap B}(1-(A\cap B)^{\circ-\circ}(x)+(A\cap B)^\circ(x))\\
&= c\alpha\tau(A\cap B). \qquad \square
\end{aligned}$$

From Theorem 3.1, we can have the following theorem

Theorem 3.2. *Let (X,τ) be a fuzzifying topological space. Then*
(1) (a) $\alpha F(X)=1$, $\alpha F(\emptyset)=1$;
(b) $\alpha F(\bigcap_{\lambda\in\Lambda} A_\lambda)\geqslant\bigwedge_{\lambda\in\Lambda}\alpha F(A_\lambda)$;
(2) (a) $c\alpha F(X)=1$, $c\alpha F(\emptyset)=1$;
(b) $c\alpha F(A\cup B)\geqslant c\alpha F(A)\wedge c\alpha F(B)$.

Theorem 3.3. *Let (X,τ) be a fuzzifying topological space. Then*
(1) (a) $\models \tau\subseteq\alpha\tau$;
(b) $\models \tau\subseteq c\alpha\tau$;
(2) (a) $\models F\subseteq\alpha F$;
(b) $\models F\subseteq c\alpha F$.

Proof. From Theorem 2.2(3) [8] and Lemma 3.2, we have
(1) (a) $[A\in\tau]=[A\subseteq A^\circ]\leqslant[A\subseteq A^{\circ-\circ}]=[A\in\alpha\tau]$.
(b) $[A\in\tau]=[A\subseteq A^\circ]\leqslant[A\cap A^{\circ-\circ}\subseteq A^\circ]=[A\in c\alpha\tau]$.
(2) The proof is obtained from (1). $\square$

Remark 3.1. In crisp setting, i.e., if the underlying fuzzifying topology is the ordinary topology, one can have
(1) $\models (A\in\alpha\tau\wedge A\in c\alpha\tau)\to A\in\tau$;
(2) $\models (A\in\alpha\tau\wedge B\in\alpha\tau)\to A\cap B\in\alpha\tau$.

But these statements may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 3.1. Let $X=\{a,b,c\}$ and let τ be a fuzzifying topology on X defined as follows:

$$\tau(X)=\tau(\phi)=\tau(\{a\})=\tau(\{a,c\})=1;\ \tau(\{b\})=\tau(\{a,b\})=0 \quad \text{and } \tau(\{c\})=\tau(\{b,c\})=\tfrac{1}{8}.$$

One can have that
(1) $\alpha\tau(\{a,b\})=\frac{7}{8}$, $c\alpha\tau(\{a,b\})=\frac{1}{8}$ and hence, $\alpha\tau(\{a,b\})\wedge c\alpha\tau(\{a,b\})=\frac{7}{8}\wedge\frac{1}{8}=\frac{1}{8}\nleqslant 0=\tau(\{a,b\})$ and
(2) $\alpha\tau(\{a,b\})=\frac{7}{8}$, $\alpha\tau(\{b,c\})=\frac{1}{8}$ and $\alpha\tau(\{b\})=0$ and hence $\alpha\tau(\{a,b\})\wedge\alpha\tau(\{b,c\})=\frac{1}{8}\nleqslant 0=\alpha\tau(\{b\}) =\alpha\tau(\{a,b\}\cap\{b,c\})$.

Theorem 3.4. *Let (X,τ) be a fuzzifying topological space.*
(1) $\models A\in\tau\to(A\in\alpha\tau\wedge A\in c\alpha\tau)$;
(2) *If* $[A\in\alpha\tau]=1$ *or* $[A\in c\alpha\tau]=1$, *then* $\models A\in\tau\leftrightarrow(A\in\alpha\tau\wedge A\in c\alpha\tau)$.

Proof. (1) Obtained from Theorem 3.3 (1).
(2) If $[A\in\alpha\tau]=1$, then for each $x\in A$, $A^{\circ-\circ}(x)=1$ and so for each $x\in A$, $1-A^{\circ-\circ}(x)+A^{\circ}(x)=A^{\circ}(x)$. Thus, from Lemma 3.2, $\models A^{\circ}\subseteq A^{\circ-\circ}$ and so we have, $[A\in\alpha\tau]\wedge[A\in c\alpha\tau]=[A\in c\alpha\tau]=[A\in\tau]$. If $[A\in c\alpha\tau]=1$ then for each $x\in A$, $1-A^{\circ-\circ}(x)+A^{\circ}(x)=1$ and so for each $x\in A$, we have $A^{\circ-\circ}(x)=A^{\circ}(x)$. Thus $[A\in\alpha\tau]\wedge[A\in c\alpha\tau]=[A\in\alpha\tau]=[A\in\tau]$. □

Theorem 3.5. *Let (X,τ) be a fuzzifying topological space. Then*

$$\models(A\in\alpha\tau \mathbin{\dot\wedge} A\in c\alpha\tau)\to A\in\tau.$$

Proof.

$$\begin{aligned}\alpha\tau(A)\mathbin{\dot\wedge} c\alpha\tau(A) &= \inf_{x\in A}A^{\circ-\circ}(x)\mathbin{\dot\wedge}\inf_{x\in A}(1-A^{\circ-\circ}(x)+A^{\circ}(x))\\ &=\max\left(0,\inf_{x\in A}A^{\circ-\circ}(x)+\inf_{x\in A}(1-A^{\circ-\circ}(x)+A^{\circ}(x))-1\right)\\ &\leqslant\inf_{x\in A}A^{\circ}(x)=[A\in\tau].\qquad\square\end{aligned}$$

4. Fuzzifying α- (resp. $c\alpha$-) neighborhood structure of a point

Definition 4.1. Let $x\in X$. The α- (resp. $c\alpha$-) neighborhood of x is denoted by αN_x (resp. $c\alpha N_x)\in\mathscr{F}(P(X))$ and defined as

$$\alpha N_x(A)=\sup_{x\in B\subseteq A}\alpha\tau(B)\quad\left(\text{resp. } c\alpha N_x(A)=\sup_{x\in B\subseteq A}c\alpha\tau(B)\right).$$

Theorem 4.1. (1) $\models A\in\alpha\tau\leftrightarrow\forall x(x\in A\to\exists B(B\in\alpha\tau\wedge x\in B\subseteq A))$;
(2) $\models A\in\alpha\tau\leftrightarrow\forall x(x\in A\to\exists B(B\in\alpha N_x\wedge B\subseteq A))$.

Proof. (1) Now,

$$[\forall x(x\in A\to\exists B(B\in\alpha\tau\wedge x\in B\subseteq A))]=\inf_{x\in A}\sup_{x\in B\subseteq A}\alpha\tau(B).$$

It is clear that $\inf_{x\in A}\sup_{x\in B\subseteq A}\alpha\tau(B)\geqslant\alpha\tau(A)$. On the other hand, let $\beta_x=\{B: x\in B\subseteq A\}$. Then, for any $f\in\prod_{x\in A}\beta_x$ we have $\bigcup_{x\in A}f(x)=A$ and so $\alpha\tau(A)=\alpha\tau(\bigcup_{x\in A}f(x))\geqslant\inf_{x\in A}\alpha\tau(f(x))$. Thus,

$$\alpha\tau(A)\geqslant\sup_{f\in\prod_{x\in A}\beta_x}\inf_{x\in A}\alpha\tau(f(x))=\inf_{x\in A}\sup_{x\in B\subseteq A}\alpha\tau(B).$$

(2) From (1) we have

$$
\begin{aligned}
[\forall x(x \in A \to \exists B(B \in \alpha N_x \wedge B \subseteq A))] &= \inf_{x \in A} \sup_{B \subseteq A} \alpha N_x(B) \\
&= \inf_{x \in A} \sup_{B \subseteq A} \sup_{x \in C \subseteq B} \alpha\tau(C) \\
&= \inf_{x \in A} \sup_{x \in C \subseteq A} \alpha\tau(C) = [A \in \alpha\tau]. \qquad \square
\end{aligned}
$$

Corollary 4.1. $\inf_{x \in A} \alpha N_x(A) = \alpha\tau(A)$.

Theorem 4.2. *The mapping* $\alpha N : X \to \mathscr{F}^N(P(X))$, $x \mapsto \alpha N_x$ *where* $\mathscr{F}^N(P(X))$ *is the set of all normal fuzzy subsets of* $P(X)$ *has the following properties*:
(1) *for any* $x, A, \models A \in \alpha N_x \to x \in A$;
(2) *for any* $x, A, B, \models A \subseteq B \to (A \in \alpha N_x \to B \in \alpha N_x)$;
(3) *for any* $x, A, \models A \in \alpha N_x \to \exists H(H \in \alpha N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \alpha N_y))$.

Proof. One can easily have that for each $x \in X$, $\alpha N_x(X) = 1$, i.e. each αN_x is normal.
(1) If $\alpha N_x(A) = 0$, the result holds. Suppose $\alpha N_x(A) > 0$, then $\sup_{x \in H \subseteq A} \alpha\tau(H) > 0$ and so there exists $H_\circ$ such that $x \in H_\circ \subseteq A$. Thus $[x \in A] = 1 \geqslant \alpha N_x(A)$.
(2) Immediate.

(3) $[\exists H(H \in \alpha N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \alpha N_y))]$

$$
\begin{aligned}
&= \sup_{H \subseteq A} \left(\alpha N_x(H) \wedge \inf_{y \in H} \alpha N_y(H) \right) \\
&= \sup_{H \subseteq A} (\alpha N_x(H) \wedge \alpha\tau(H)) \\
&= \sup_{H \subseteq A} \alpha\tau(H) \geqslant \sup_{x \in H \subseteq A} \alpha\tau(H) = [A \in \alpha N_x]. \qquad \square
\end{aligned}
$$

Theorem 4.3. *The mapping* $c\alpha N : X \to \mathscr{F}^N(P(X))$, $x \mapsto c\alpha N_x$, *where* $\mathscr{F}^N(P(X))$, *is the set of all normal fuzzy subsets of* $P(X)$ *has the following properties*:
(1) *for any* $x, A, \models A \in c\alpha N_x \to x \in A$;
(2) *for any* $x, A, B, \models A \subseteq B \to (A \in c\alpha N_x \to B \in c\alpha N_x)$;
(3) *for any* $x, A, B, \models A \in c\alpha N_x \wedge B \in c\alpha N_x \to A \cap B \in c\alpha N_x$.

Conversely, if a mapping $c\alpha N$ *satisfies* (2) *and* (3), *then* $c\alpha N$ *assigns a fuzzifying topology on X, denoted by* $\tau_{c\alpha N} \in \mathscr{F}(P(X))$ *and defined as*

$$A \in \tau_{c\alpha N} := \forall x(x \in A \to A \in c\alpha N_x).$$

Proof. It is clear that each $c\alpha N_x$ is normal.

The proof of (1) and (2) are similar to the corresponding results in Theorem 4.2.

(3) From Theorem 3.1(2)(b) we have

$$
\begin{aligned}
[A \cap B \in c\alpha N_x] &= \sup_{x \in H \subseteq A \cap B} c\alpha\tau(H) = \sup_{x \in H_1 \subseteq A, x \in H_2 \subseteq B} c\alpha\tau(H_1 \cap H_2) \\
&\geqslant \sup_{x \in H_1 \subseteq A, x \in H_2 \subseteq B} (c\alpha\tau(H_1) \wedge c\alpha\tau(H_2)) = \sup_{x \in H_1 \subseteq A} c\alpha\tau(H_1) \wedge \sup_{x \in H_2 \subseteq B} c\alpha\tau(H_2) \\
&= c\alpha N_x(A) \wedge c\alpha N_x(B).
\end{aligned}
$$

Conversely, we need to prove that $\tau_{c\alpha N}=\inf_{x\in A} c\alpha N_x(A)$ is a fuzzifying topology. From Theorem 3.2 [7] and since $\tau_{c\alpha N}$ satisfies properties (2) and (3), $\tau_{c\alpha N}$ is a fuzzifying topology. □

Theorem 4.4. *Let (X,τ) be a fuzzifying topological space. Then* $\models c\alpha\tau\subseteq\tau_{c\alpha N}$.

Proof. Let $B\in P(X)$; $\tau_{c\alpha N}(B)=\inf_{x\in B} c\alpha N_x(B)=\inf_{x\in B}\sup_{x\in A\subseteq B} c\alpha\tau(A)\geqslant c\alpha\tau(B)$. □

5. α- (resp. $c\alpha$-) closure and α- (resp. $c\alpha$-) interior

Definition 5.1. (1) The α- (resp. $c\alpha$-) closure of A is denoted by $\alpha-cl$ (resp. $c\alpha-cl)\in\mathscr{F}(P(X))$ and defined as follows:

$$\alpha-cl(A)(x)=\inf_{x\notin B\supseteq A}(1-\alpha F(B))\quad (\text{resp. } c\alpha-cl(A)(x)=\inf_{x\notin B\supseteq A}(1-c\alpha F(B))).$$

(2) The α- (resp. $c\alpha$-) interior of A is denoted by $\alpha-int(A)$ (resp. $c\alpha-int(A))\in\mathscr{F}(P(X))$ and defined as follows:

$$\alpha-int(A)(x)=\alpha N_x(A)\quad (\text{resp. } c\alpha-int(A)(x)=c\alpha N_x(A)).$$

Theorem 5.1.
(1) (a) $\alpha-cl(A)(x)=1-\alpha N_x(X\sim A)$;
(b) $\models\alpha-cl(\phi)\equiv\emptyset$;
(c) $\models A\subseteq\alpha-cl(A)$;
(d) $\models x\in\alpha-cl(A)\leftrightarrow\forall B(B\in\alpha N_x\to A\cap B\neq\emptyset)$;
(e) $\models A\equiv\alpha-cl(A)\leftrightarrow A\in\alpha F$;
(f) $\models B\doteq\alpha-cl(A)\to B\in\alpha F$.
(2) (a) $c\alpha-cl(A)(x)=1-c\alpha N_x(X\sim A)$;
(b) $\models c\alpha-cl(\emptyset)\equiv\emptyset$;
(c) $\models A\subseteq c\alpha-cl(A)$;
(d) $\models x\in c\alpha-cl(A)\leftrightarrow\forall B(B\in c\alpha N_x\to A\cap B\neq\emptyset)$;
(e) $\models A\equiv c\alpha-cl(A)\leftrightarrow A\in F\tau_{c\alpha N}$;
(f) $\models B\doteq c\alpha-cl(A)\to B\in F\tau_{c\alpha N}$.

Proof. (1) (a)

$$\alpha-cl(A)(x)=\inf_{x\notin B\supseteq A}(1-\alpha F(B))=\inf_{x\in X\sim B\subseteq X\sim A}(1-\alpha\tau(X\sim B))$$
$$=1-\sup_{x\in X\sim B\subseteq X\sim A}\alpha\tau(X\sim B)=1-\alpha N_x(X\sim A);$$

(b) $\alpha-cl(\emptyset)(x)=1-\alpha N_x(X\sim\emptyset)=0$;

(c) It is clear that for any $A\in P(X)$ and any $x\in X$, if $x\notin A$, then $\alpha N_x(A)=0$. If $x\in A$, then $\alpha-cl(A)(x)=1-\alpha N_x(X\sim A)=1-0=1$. Then $[A\subseteq\alpha-cl(A)]=1$.

(d) $[\forall B(B\in\alpha N_x\to A\cap B\neq\emptyset)]=\inf_{B\subseteq X\sim A}(1-\alpha N_x(B))=1-\alpha N_x(X\sim A)=[x\in\alpha-cl(A)]$.

(e) From Corollary 4.1 and from (a), (c) above we have

$$[A\equiv\alpha-cl(A)]=\inf_{x\in X\sim A}(1-(\alpha-cl(A))(x))$$
$$=\inf_{x\in X\sim A}\alpha N_x(X\sim A)=\alpha\tau(X\sim A)=[A\in\alpha F].$$

(f) If $[A \subseteq B] = 0$, then $[B \doteq \alpha - cl(A)] = 0$. Now, we suppose $[A \subseteq B] = 1$, and have, $[B \subseteq \alpha - cl(A)] = 1 - \sup_{x \in B \sim A} \alpha N_x(X \sim A)$, $[\alpha - cl(A) \subseteq B] = \inf_{x \in X \sim B} \alpha N_x(X \sim A)$. So, $[B \doteq \alpha - cl(A)] = \max(0, \inf_{x \in X \sim B} \alpha N_x (X \sim A) - \sup_{x \in B \sim A} \alpha N_x(X \sim A))$. If $[B \doteq \alpha - cl(A)] > t$, then $\inf_{x \in X \sim B} \alpha N_x(X \sim A) > t + \sup_{x \in B \sim A} \alpha N_x(X \sim A)$. For any $x \in X \sim B$, $\sup_{x \in C \subseteq X \sim A} \alpha\tau(C) > t + \sup_{x \in B \sim A} \alpha N_x(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$ and $\alpha\tau(C_x) > t + \sup_{x \in B \sim A} \alpha N_x(X \sim A)$. Now we want to prove that $C_x \subseteq X \sim B$. If not, then there exists $x' \in B \sim A$ with $x' \in C_x$. Hence, we obtain $\sup_{x \in B \sim A} \alpha N_x(X \sim A) \geqslant \alpha N_{x'}(X \sim A) \geqslant \alpha\tau(C_x) > t + \sup_{x \in B \sim A} \alpha N_x(X \sim A)$, a contradiction. Therefore, $\alpha F(B) = \alpha\tau(X \sim B) = \inf_{x \in X \sim B} \alpha N_x(X \sim B) \geqslant \inf_{x \in X \sim B} \alpha\tau(C_x) \geqslant t + \sup_{x \in B \sim A} \alpha N_x(X \sim A) \geqslant t$. Since t is arbitrary, it holds that $[B \doteq \alpha - cl(A)] \leqslant [B \in \alpha F]$.

(2) The proof is similar to (1). □

Theorem 5.2. *For any* x, A, B,

(1) (a) $\models \alpha - int(A) \equiv X \sim \alpha - cl(X \sim A)$;
 (b) $\models \alpha - int(X) \equiv X$;
 (c) $\models \alpha - int(A) \subseteq A$;
 (d) $\models B \doteq \alpha - int(A) \to B \in \alpha\tau$;
 (e) $\models B \in \alpha\tau \wedge B \subseteq A \to B \subseteq \alpha - int(A)$;
 (f) $\models A \equiv \alpha - int(A) \leftrightarrow A \in \alpha\tau$;

(2) (a) $\models c\alpha - int(A) \equiv X \sim c\alpha - cl(X \sim A)$;
 (b) $\models c\alpha - int(X) \equiv X$;
 (c) $\models c\alpha - int(A) \subseteq A$;
 (d) $\models B \doteq c\alpha - int(A) \to B \in \tau_{c\alpha N}$;
 (e) $\models B \in \tau_{c\alpha N} \wedge B \subseteq A \to B \subseteq c\alpha - int(A)$;
 (f) $\models A \equiv c\alpha - int(A) \leftrightarrow A \in \tau_{c\alpha N}$.

Proof. (1) (a) From Theorem 5.1(a) $\alpha - cl(X \sim A)(x) = 1 - \alpha N_x(A) = 1 - (\alpha - int(A))(x)$. Then, $[\alpha - int(A) \equiv X \sim \alpha - cl(X \sim A)] = 1$.

(b) and (c) are obtained from (a) above and from Theorem 5.1(1)(b) and (1)(c).

(d) From (a) above and from Theorem 5.1(1)(f) we have

$$[B \doteq \alpha - int(A)] = [X \sim B \doteq \alpha - cl(X \sim A)] \leqslant [X \sim B \in \alpha F] = [B \in \alpha\tau].$$

(e) If $[B \subseteq A] = 0$, then the result holds. If $[B \subseteq A] = 1$, then we have that $[B \subseteq \alpha - int(A)] = \inf_{x \in B}(\alpha - int(A))(x) = \inf_{x \in B} \alpha N_x(A) \geqslant \inf_{x \in B} \alpha N_x(B) = \alpha\tau(B) = [B \in \alpha\tau \wedge B \subseteq A]$.

(f) From Corollary 4.1, we have

$$\begin{aligned}[A \equiv \alpha - int(A)] &= \min(\inf_{x \in A}(\alpha - int(A))(x), \inf_{x \in X \sim A}(1 - (\alpha - int(A))(x)) \\ &= \inf_{x \in A}(\alpha - int(A))(x) = \inf_{x \in A} \alpha N_x(A) = \alpha\tau(A) = [A \in \alpha\tau].\end{aligned}$$

(2) The proof is similar to (1). □

6. α-Continuous functions and $c\alpha$-continuous functions

Definition 6.1. Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces.

(1) A unary fuzzy predicate $\alpha C \in \mathscr{F}(Y^X)$ called fuzzy α-continuity, is given as

$$\alpha C(f) := \forall u(u \in U \to f^{-1}(u) \in \alpha\tau).$$

(2) A unary fuzzy predicate $c\alpha C \in \mathscr{F}(Y^X)$ called fuzzy $c\alpha$-continuity, is given as

$$c\alpha C(f) := \forall u(u \in U \to f^{-1}(u) \in c\alpha\tau). \qquad \square$$

Definition 6.2. Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. For any $f \in Y^X$, we define the unary fuzzy predicates $\alpha H_j, c\alpha H_j \in \mathscr{F}(Y^X)$ where $j = 1,2,\ldots,5$ as follows:
(1) (a) $\alpha H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in \alpha F_X)$; (b) $c\alpha H_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in c\alpha F_X)$, where F_Y is the family of closed subsets of Y; and αF_X and $c\alpha F_X$ are the families of α-closed and $c\alpha$-closed subsets of X, respectively.
(2) (a) $\alpha H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in \alpha N^X_x)$; (b) $c\alpha H_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in c\alpha N^X_x)$, where N^Y is the neighborhood system of Y; and αN^X and $c\alpha N^X$ are the α-neighborhood and $c\alpha$-neighborhood systems of X, respectively.
(3) (a) $\alpha H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in \alpha N^X_x))$; (b) $c\alpha H_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists v(f(v) \subseteq u \to v \in c\alpha N^X_x))$;
(4) (a) $\alpha H_4(f) := \forall A(f(\alpha - cl_X(A)) \subseteq cl_Y(f(A)))$; (b) $c\alpha H_4(f) := \forall A(f(c\alpha - cl_X(A)) \subseteq cl_Y(f(A)))$;
(5) (a) $\alpha H_5(f) := \forall B(\alpha - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))$; (b) $c\alpha H_5(f) := \forall B(c\alpha - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))$.

Theorem 6.1. (1) $\models f \in \alpha C \leftrightarrow f \in \alpha H_j,\ j = 1,2,3,4,5$;
(2) $\models f \in c\alpha C \leftrightarrow f \in c\alpha H_1$.

Proof. We will prove (1) only since the proof of (2) is similar to the corresponding result in (1)(a). We prove that $\models f \in \alpha C \leftrightarrow f \in \alpha H_1$:

$$\begin{aligned}
[f \in \alpha H_1] &= \inf_{B \in P(Y)} \min(1, 1 - F_Y(B) + \alpha F_X(f^{-1}(B))) \\
&= \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \alpha\tau(X \sim f^{-1}(B))) \\
&= \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + \alpha\tau(f^{-1}(Y \sim B))) \\
&= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \alpha\tau(f^{-1}(u))) \\
&= [f \in \alpha C].
\end{aligned}$$

(b) We want to prove that $\models f \in \alpha C \leftrightarrow f \in \alpha H_2$.

First, we prove that $\alpha H_2(f) \geqslant \alpha C(f)$. If $N^Y_{f(x)}(u) \leqslant \alpha N^X_x(f^{-1}(u))$ the result holds. Suppose $N^Y_{f(x)}(u) > \alpha N^X_x(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$$\begin{aligned}
N^Y_{f(x)}(u) - \alpha N^X_x(f^{-1}(u)) &= \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} \alpha\tau(B) \\
&\leqslant \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} \alpha\tau(f^{-1}(A)) \\
&\leqslant \sup_{f(x) \in A \subseteq u} (U(A) - \alpha\tau(f^{-1}(A))).
\end{aligned}$$

So,

$$1 - N^Y_{f(x)}(u) + \alpha N^X_x(f^{-1}(u)) \geqslant \inf_{f(x) \in A \subseteq u} (1 - U(A) + \alpha\tau(f^{-1}(A)))$$

and, thus,

$$\min(1, 1 - N^Y_{f(x)}(u) + \alpha N^X_x(f^{-1}(u))) \geqslant \inf_{f(x)\in A\subseteq U} \min(1, 1 - U(A) + \alpha\tau(f^{-1}(A)))$$
$$\geqslant \inf_{v\in P(Y)} \min(1, 1 - U(v) + \alpha\tau(f^{-1}(v))) = \alpha C(f).$$

Hence,

$$\inf_{x\in X}\inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \alpha N^X_x(f^{-1}(u))) \geqslant [f \in \alpha C].$$

Secondly, we prove that $\alpha C(f) \geqslant \alpha H_2(f)$. From Corollary 4.1, we have

$$\alpha C(f) = \inf_{u\in P(Y)} \min(1, 1 - U(u) + \alpha\tau(f^{-1}(u)))$$
$$\geqslant \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{f(x)\in u} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} \alpha N^X_x(f^{-1}(u))\right)$$
$$= \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{x\in f^{-1}(u)} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} \alpha N^X_x(f^{-1}(u))\right)$$
$$\geqslant \inf_{x\in X}\inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \alpha N^X_x(f^{-1}(u))) = \alpha H_2(f).$$

(c) We prove that $\models f \in \alpha H_2 \leftrightarrow f \in \alpha H_3$. From Theorem 4.2(2) we have

$$\alpha H_3(f) = \inf_{x\in X}\inf_{u\in P(Y)} \min\left(1, 1 - N^Y_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} \alpha N^X_x(v)\right)$$
$$= \inf_{x\in X}\inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \alpha N^X_x(f^{-1}(u))) = \alpha H_2(f).$$

(d) We prove that $\models f \in \alpha H_4 \leftrightarrow f \in \alpha H_5$.

First, for any $B \in P(Y)$ one can deduce that $[f^{-1}(f(\alpha - cl_X(f^{-1}(B)))) \supseteq \alpha - cl_X(f^{-1}(B))] = 1$, $[cl_Y(f(f^{-1}(B))) \subseteq cl_Y(B)] = 1$ and $[f^{-1}(cl_Y(f(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] = 1$. Then from Lemma 1.2(2) [9] we have

$$[\alpha - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geqslant [f^{-1}(f(\alpha - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))]$$
$$\geqslant [f^{-1}(f(\alpha - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))]$$
$$\geqslant [f(\alpha - cl_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))].$$

Therefore,

$$\alpha H_5(f) = \inf_{B\in P(Y)} [\alpha - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))]$$
$$\geqslant \inf_{B\in P(Y)} [f^{-1}(f(\alpha - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))]$$
$$\geqslant \inf_{B\in P(Y)} [f^{-1}(f(\alpha - cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))]$$
$$\geqslant \inf_{B\in P(Y)} [f(\alpha - cl_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))]$$
$$\geqslant \inf_{A\in P(X)} [f(\alpha - cl_X(A)) \subseteq cl_Y(f(A))] = \alpha H_4(f).$$

Secondly, for each $A\in P(X)$, there exists $B\in P(Y)$ such that $f(A)=B$ and $f^{-1}(B)\supseteq A$. Hence,

$$\begin{aligned}[\alpha - cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))] &\leqslant [\alpha - cl_X(A))\subseteq f^{-1}(cl_Y(f(A)))]\\ &\leqslant [f(\alpha - cl_X(A))\subseteq f(f^{-1}(cl_Y(f(A))))]\\ &\leqslant [f(\alpha - cl_X(A)\subseteq cl_Y(f(A))].\end{aligned}$$

Thus,

$$\begin{aligned}\alpha H_4(f) &= \inf_{A\in P(X)}[\alpha - cl_X(A)\subseteq f^{-1}(cl_Y(f(A))]\geqslant \inf_{B\in P(Y),B=f(A)}[\alpha - cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))]\\ &\geqslant \inf_{B\in P(Y)}[\alpha - cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))]=\alpha H_5(f).\end{aligned}$$

(e) We want to prove that $\models f\in\alpha H_5\leftrightarrow f\in\alpha H_2$.

$$\begin{aligned}\alpha H_5(f) &= [\forall B(\alpha - cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B)))]\\ &= \inf_{B\in P(Y)}\inf_{x\in X}\min(1,1-(1-\alpha N_x(X\sim f^{-1}(B)))+1-N_{f(x)}(Y\sim B))\\ &= \inf_{B\in P(Y)}\inf_{x\in X}\min(1,1-N_{f(x)}(Y\sim B))+\alpha N_x(f^{-1}(Y\sim B)))\\ &= \inf_{u\in P(Y)}\inf_{x\in X}\min(1,1-N_{f(x)}(u))+\alpha N_x(f^{-1}(u)))=\alpha H_2(f). \qquad \square\end{aligned}$$

Theorem 6.2. (1) $\models f\in c\alpha H_2\leftrightarrow f\in c\alpha H_j,\ \ j=3,4,5;$
(2) $\models f\in c\alpha C\to f\in c\alpha H_2$.

Proof. (1) It is similiar to the proof of (c)–(e) in the proof of Theorem 6.1.
(2) It is similiar to the proof of the first part in (b) in Theorem 6.1. □

Remark 6.1. In the following theorem we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous or $c\alpha C$-continuous. Thus, the symbols $(\tau,U)-C(f),(\tau_{c\alpha N},U)-C(f),(\tau,U_{c\alpha N})-c\alpha C(f)$, etc., will be understood.

Applying Theorem 3.4(1) and Theorem 4.4 one can deduce the following theorem.

Theorem 6.3. (1) $\models f\in(\tau,U_{c\alpha N})-C\to f\in(\tau,U)-C;$
(2) $\models f\in(\tau,U)-c\alpha C\to f\in(\tau_{c\alpha N},U)-C;$
(3) $\models f\in(\tau,U)-C\to f\in(\tau,U)-c\alpha C.$

7. Decompositions of fuzzy continuity in fuzzifying topology

Theorem 7.1. *Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. Then for each $f\in Y^X,\models C(f)\to(\alpha C(f)\vee c\alpha C(f))$.*

Proof. The proof is obtained from Theorem 3.3(1). □

Remark 7.1. In crisp setting, i.e., if the underlying fuzzifying topology is the ordinary topology, one can have $\models\alpha C(f)\wedge c\alpha C(f)\to C(f)$.

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 7.1. Let (X,τ) be the fuzzifying topological space defined in Counterexample 3.1; consider the identity function f from (X,τ) onto (X,σ) where σ is a fuzzifying topology on X defined as follows:

$$\sigma A=\begin{cases}1, & A\in\{X,\emptyset,\{a,b\}\},\\ 0, & \text{o.w.}\end{cases}$$

Then $\frac{7}{8}\wedge\frac{1}{8}=\alpha C(f)\wedge c\alpha C(f)\nleqslant C(f)=0$.

Theorem 7.2. *Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces and let $f\in Y^X$. Then $\models C(f)\to(\alpha C(f)\leftrightarrow c\alpha C(f))$.*

Proof. $[\alpha C(f)\to c\alpha C(f)]=\min(1,1-\alpha C(f)+c\alpha C(f))\geqslant\alpha C(f)\wedge c\alpha C(f)$. Also, $[c\alpha C(f)\to\alpha C(f)]=\min(1,1-c\alpha C(f)+\alpha C(f))\geqslant c\alpha C(f)\wedge\alpha C(f)$. Then from Theorem 7.1 we have $c\alpha C(f)\wedge\alpha C(f)\geqslant C(f)$ and so the result holds. □

Theorem 7.3. *Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces and let $f\in Y^X$. If $[\alpha\tau(f^{-1}(u))]=1$ or $[c\alpha\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models C(f)\leftrightarrow(\alpha C(f)\wedge c\alpha C(f))$.*

Proof. Now, we need to prove that $C(f)=\alpha C(f)\wedge c\alpha C(f)$. Applying Theorem 3.4(2) we have

$$\begin{aligned}\alpha C(f)\wedge c\alpha C(f)&=\inf_{u\in P(Y)}\min(1,1-U(u)+\alpha\tau(f^{-1}(u)))\wedge\inf_{u\in P(Y)}\min(1,1-U(u)+c\alpha\tau(f^{-1}(u)))\\&=\inf_{u\in P(Y)}\min(1,(1-U(u)+\alpha\tau(f^{-1}(u)))\wedge(1-U(u)+c\alpha\tau(f^{-1}(u))))\\&=\inf_{u\in P(Y)}\min(1,1-U(u)+(\alpha\tau(f^{-1}(u)))\wedge c\alpha\tau(f^{-1}(u)))\\&=\inf_{u\in P(Y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f).\qquad\square\end{aligned}$$

Theorem 7.4. *Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces and let $f\in Y^x$. Then,*
(1) *if $[\alpha\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then*

$$\models\alpha C(f)\to(c\alpha C(f)\leftrightarrow C(f)),$$

(2) *if $[c\alpha\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then*

$$\models c\alpha C(f)\to(\alpha C(f)\leftrightarrow C(f)).$$

Proof. (1) Since $[\alpha\tau(f^{-1}(u))]=1$ and so $[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ-\circ}]=1$, then $[f^{-1}(u)\cap(f^{-1}(u))^{\circ-\circ}\subseteq(f^{-1}(u))^{\circ}]=[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]$. Thus,

$$\begin{aligned}c\alpha C(f)&=\inf_{u\in P(Y)}\min(1,1-U(u)+c\alpha\tau(f^{-1}(u)))\\&=\inf_{u\in P(Y)}\min(1,1-U(u)+[f^{-1}(u)\cap(f^{-1}(u))^{\circ-\circ}\subseteq(f^{-1}(u))^{\circ}])\\&=\inf_{u\in P(Y)}\min(1,1-U(u)+[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}])\\&=\inf_{u\in P(Y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f).\end{aligned}$$

(2) Since $[c\alpha\tau(f^{-1}(u))]=1$ one can deduce that $(f^{-1}(u))^{\circ-\circ}=(f^{-1}(u))^{\circ}$. So,

$$\begin{aligned}
\alpha C(f) &= \inf_{u\in P(Y)} \min(1, 1-U(u)+\alpha\tau(f^{-1}(u)))\\
&= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\subseteq (f^{-1}(u)))^{\circ-\circ}])\\
&= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\subseteq (f^{-1}(u)))^{\circ}])\\
&= \inf_{u\in P(Y)} \min(1, 1-U(u)+\tau(f^{-1}(u)))=C(f). \qquad \square
\end{aligned}$$

Theorem 7.5. *Let $(X,\tau),(Y,U),(Z,V)$ be three fuzzifying topological spaces. For any $f\in Y^X$, $g\in Z^Y$,*
(1) $\models \alpha C(f)\to(C(g)\to\alpha C(g\circ f))$;
(2) $\models C(g)\to(\alpha C(f)\to\alpha C(g\circ f))$;
(3) $\models c\alpha C(f)\to(C(g)\to c\alpha C(g\circ f))$;
(4) $\models C(g)\to(c\alpha C(f)\to c\alpha C(g\circ f))$.

Proof. (1) We need to prove that $[\alpha C(f)]\leqslant[C(g)\to\alpha C(g\circ f)]$. If $[C(g)]\leqslant[\alpha C(g\circ f)]$, the result holds, if $[c(g)]>[\alpha C(g\circ f)]$, then

$$\begin{aligned}
[C(g)]-[\alpha C(g\circ f)] &= \inf_{v\in P(Z)} \min(1,1-V(v)+U(g^{-1}(v)))\\
&\quad - \inf_{v\in P(Z)} \min(1,1-V(v)+\alpha\tau((g\circ f)^{-1}(v)))\\
&\leqslant \sup_{v\in P(Z)} (U(g^{-1}(v))-\alpha\tau((g\circ f)^{-1}(v)))\\
&\leqslant \sup_{u\in P(Y)} (U(u)-\alpha\tau(f^{-1}(u))).
\end{aligned}$$

Therefore,

$$\begin{aligned}
[C(g)\to\alpha C(g\circ f)] &= \min(1,1-[C(g)]+[\alpha C(g\circ f)])\\
&\geqslant \inf_{u\in P(Y)} \min(1,1-U(u)+\alpha\tau(f^{-1}(u)))=\alpha C(f).
\end{aligned}$$

(2)

$$\begin{aligned}
[C(g)\to(\alpha C(f)\to\alpha C(g\circ f))] &= [\neg(C(g)\dot\wedge\neg(\alpha C(f)\to\alpha C(g\circ f)))]\\
&= [\neg(C(g)\dot\wedge\neg\neg(\alpha C(f)\dot\wedge\neg(\alpha C(g\circ f))))]\\
&= [\neg(C(g)\dot\wedge\alpha C(f)\dot\wedge\neg(\alpha C(g\circ f)))]\\
&= [\neg(\alpha C(f)\dot\wedge C(g)\dot\wedge\neg\alpha C(g\circ f))]\\
&= [\neg(\alpha C(f)\dot\wedge\neg\neg(C(g)\dot\wedge\neg(\alpha C(g\circ f))))]\\
&= [\neg(\alpha C(f)\dot\wedge\neg(C(g)\to\alpha C(g\circ f)))]\\
&= [\alpha C(f)\to(C(g)\to\alpha C(g\circ f))]=1.
\end{aligned}$$

The proofs of (3) and (4) are similar to (1) and (2), respectively. $\square$

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