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β- SEPARATION AXIOMS BASED ON ŁUKASIEWICZ LOGIC

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Abstract. In this paper, we introduce topological notions defined by means of β-open sets when these are planted into the framework of Ying's fuzzifying topological spaces (by Łukasiewicz logic in [0, 1]). We introduce $T_0^\beta-$, $T_1^\beta-$, $T_2^\beta-$ (β- Hausdorff)-, T_3^β (β-regular)- and T_4^β (β-normal)-separation axioms. Furthermore, the $R_0^\beta-$ and $R_1^\beta-$ separation axioms are studied and their relations with the $T_1^\beta-$ and $T_2^\beta-$ separation axioms are introduced. Moreover, we clarify the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

Keywords: Łukasiewicz logic; semantics; fuzzifying topology; fuzzifying separation axioms; β-separation axioms.

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1. Introduction and Preliminaries

In the last few years fuzzy topology, as an important research field in fuzzy set theory, has been developed into a quite mature discipline [7-9, 13-14, 23]. In contrast to classical topology, fuzzy topology is endowed with richer structure, to a certain extent, which is manifested with different ways to generalize certain classical concepts. So far, according to Ref. [8], the kind of topologies defined by Chang [4] and Goguen [5] is called the topologies of fuzzy subsets, and further is naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets

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(resp. an L-topological space) is a family τ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and τ satisfies the basic conditions of classical topologies [11]. On the other hand, Höhle in [6] proposed the terminology L-fuzzy topology to be an L-valued mapping on the traditional powerset $P(X)$ of X. The authors in [10, 23] defined an L-fuzzy topology to be an L-valued mapping on the L-powerset L^X of X.

In 1952, Rosser and Turquette [20] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories ? As an attempt to give a partial answer to this problem in the case of point set topology, Ying in 1991 [24-25] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. In fact, fuzzifying topologies are a special case of the L-fuzzy topologies in [10, 19] since all the t-norms on I are included as a special class of tensor products in these paper. Ying uses one particular tensor product, namely Łukasiewicz conjunction. Thus his fuzzifying topologies are a special class of all the I-fuzzy topologies considered in the categorical frameworks [10, 19]. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. So far, there has been significant research on fuzzifying topologies [1-12, 17-18, 21]. For example, Shen [21] introduced and studied T_0-, T_1-, T_2 (Hausdorff)-, T_3(regular)- and T_4(normal)-separation axioms in fuzzifying topology. In [12], the concepts of the R_0- and R_1-separation axioms in fuzzifying topology were added and their relations with the T_1- and T_2- separation axioms, were studied. Also, in [1] the concepts of fuzzifying β-open set and fuzzifying β-continuity were introduced and studied. In classical topology , β-separation axioms have been studied in [2, 15-16, 22]. As well as, they have been studied in fuzzy topology in [3]. In the present paper, we explore the problem proposed by Rosser and Turquette [20] in fuzzy β-separation axioms.

A basic structure of the paper is as follows. First, we offer some definitions and results which will be needed in this paper. Afterwards, in Section 2, in the framework of fuzzifying topology, the concept of β-separation axioms $T_0^\beta-$, $T_1^\beta-$, T_2^β (β-Hausdorff)-, T_3^β (β-regular)- and T_4^β (β-normal) are discussed. In Section 3, on the bases of fuzzifying topology the $R_0^\beta-$ and $R_1^\beta-$ separation axioms are introduced and their relations with the $T_1^\beta-$ and $T_2^\beta-$ separation axioms are studied. Furthermore , we give the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. Finally, in a conclusion, we summarize the main results obtained and raise some related problems for further study. Thus we fill a gap in the existing literature on fuzzifying topology. We will use the terminologies and notations in [1, 12, 21, 24, 25] without any explanation. We will use the symbol $\otimes$ instead of the second "AND" operation $\dot\wedge$ as dot is hardly visible. This mean that $[\beta] \leq [\varphi \to \psi] \Leftrightarrow [\beta] \otimes [\varphi] \leq [\psi]$.

A fuzzifying topology on a set X [6, 24] is a mapping $\tau \in \Im(P(X))$ such that:

(1) $\tau(X) = 1, \tau(\phi) = 1$;

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda} \tau\left(A_\lambda\right)$.

The family of all fuzzifying β-open sets [1], denoted by $\tau_\beta \in \Im(P(X))$, is defined as

$$A \in \tau_\beta := \forall x(x \in A \to x \in Cl(Int(Cl(A)))), \text{ i. e., } \tau_\beta(A) = \bigwedge_{x\in A} Cl(Int(Cl(A)))(x)$$

The family of all fuzzifying β-closed sets [1], denoted by $\digamma_\beta \in \Im(P(X))$, is defined as $A \in \digamma_\beta := X - A \in \tau_\beta$. The fuzzifying β-neighborhood system of a point $x \in X$ [1] is denoted by $N_x^\beta \in \Im(P(X))$ and defined as $N_x^\beta(A) = \bigvee_{x\in B\subseteq A} \tau_\beta(B)$.The fuzzifying β-closure of a set $A \subseteq X$ [1], denoted by $Cl_\beta \in \Im(X)$, is defined as $Cl_\beta(A)(x) = 1 - N_x^\beta(X - A)$. Let (X, τ) be a fuzzifying topological space. The binary fuzzy predicates $K, H, M \in \Im(X \times X), V \in \Im(X \times P(X))$ and $W \in \Im(P(X) \times P(X))$ [12] are defined as follows:

(1) $K(x, y) := \exists A((A \in N_x \wedge y \notin A) \vee (A \in N_y \wedge x \notin A))$;

(2) $H(x, y) := \exists B\exists C((B \in N_x \wedge y \notin B) \wedge (C \in N_y \wedge x \notin C))$;

(3) $M(x, y) := \exists B\exists C(B \in N_x \wedge C \in N_y \wedge B \cap C \equiv \emptyset)$;

(4) $V(x, D) := \exists A\exists B(A \in N_x \wedge B \in \tau \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$;

(5) $W(A,B) := \exists G \exists H(G \in \tau \wedge H \in \tau \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$.

Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i \in \Im(\Omega), i = 0,1,2,3,4$ [21] (see the rewritten form in [12]) and $R_i \in \Im(\Omega), i = 0,1$ [12] are defined as follows:

(1) $(X,\tau) \in T_0 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K(x,y)$;

(2) $(X,\tau) \in T_1 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H(x,y)$;

(3) $(X,\tau) \in T_2 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M(x,y)$;

(4) $(X,\tau) \in T_3 := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V(x,D)$;

(5) $(X,\tau) \in T_4 := \forall A \forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W(A,B)$;

(6) $(X,\tau) \in R_0 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x,y) \longrightarrow H(x,y))$;

(7) $(X,\tau) \in R_1 := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K(x,y) \longrightarrow M(x,y))$.

2. Fuzzifying β- separation axioms and their equivalents

For simplicity we give the following definition.

Definition 2.1. Let (X,τ) be a fuzzifying topological space. The binary fuzzy predicates $K^\beta, H^\beta, M^\beta \in \Im(X \times X), V^\beta \in \Im(X \times P(X))$ and $W^\beta \in \Im(P(X) \times P(X))$ are defined as follows:

(1) $K^\beta(x,y) := \exists A((A \in N_x^\beta \wedge y \notin A) \vee (A \in N_y^\beta \wedge x \notin A))$;

(2) $H^\beta(x,y) := \exists B \exists C((B \in N_x^\beta \wedge y \notin B) \wedge (C \in N_y^\beta \wedge x \notin C))$;

(3) $M^\beta(x,y) := \exists B \exists C(B \in N_x^\beta \wedge C \in N_y^\beta \wedge B \cap C \equiv \emptyset)$;

(4) $V^\beta(x,D) := \exists A \exists B(A \in N_x^\beta \wedge B \in \tau_\beta \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$;

(5) $W^\beta(A,B) := \exists G \exists H(G \in \tau_\beta \wedge H \in \tau_\beta \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$.

Definition 2.2. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates $T_i^\beta \in \Im(\Omega), i = 0,1,2,3,4$ and $R_i^\beta \in \Im(\Omega), i = 0,1$ are defined as follows:

(1) $(X,\tau) \in T_0^\beta := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K^\beta(x,y)$;

(2) $(X,\tau) \in T_1^\beta := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H^\beta(x,y)$;

(3) $(X,\tau) \in T_2^\beta := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M^\beta(x,y)$;

(4) $(X,\tau) \in T_3^\beta := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V^\beta(x,D)$;

(5) $(X,\tau) \in T_4^\beta := \forall A \forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W^\beta(A,B)$;

(6) $(X,\tau) \in T_3^{\beta'} := \forall x \forall D(x \in X \wedge D \in F_\beta \wedge x \notin D) \longrightarrow V(x,D)$;

(7) $(X,\tau) \in T_4^{\beta'} := \forall A \forall B(A \in F_\beta \wedge B \in F_\beta \wedge A \cap B = \emptyset) \longrightarrow W(A,B)$;

(8) $(X,\tau) \in R_0^{\beta} := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\beta(x,y) \longrightarrow H^\beta(x,y))$;

(9) $(X,\tau) \in R_1^{\beta} := \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^\beta(x,y) \longrightarrow M^\beta(x,y))$.

Theorem 2.3.

Let (X,τ) be a fuzzifying topological space. Then we have

$\models (X,\tau) \in T_0^\beta \longleftrightarrow \forall x \forall y(x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\beta(\{y\})) \vee \neg(y \in Cl_\beta(\{x\}))))$.

Proof. Since for any x, A, B, $\models A \subseteq B \to (A \in N_x^\beta \to B \in N_x^\beta)$ (see [1, Theorem 4.2 (2)]), then we have

$$\begin{aligned}
[(X,\tau) \in T_0^\beta] &= \bigwedge_{x \neq y} \max\Big(\bigvee_{y \notin A} N_x^\beta(A), \bigvee_{x \notin A} N_y^\beta(A)\Big) \\
&= \bigwedge_{x \neq y} \max(N_x^\beta(X - \{y\}), N_y^\beta(X - \{x\})) \\
&= \bigwedge_{x \neq y} \max(1 - Cl_\beta(\{y\})(x), 1 - Cl_\beta(\{x\})(y)) \\
&= \bigwedge_{x \neq y} (\neg(Cl_\beta(\{y\})(x)) \vee \neg(Cl_\beta(\{x\})(y))) \\
&= [\forall x \forall y(x \in X \wedge y \in X \wedge x \neq y \longrightarrow (\neg(x \in Cl_\beta(\{y\})) \vee \neg(y \in Cl_\beta(\{x\}))))].
\end{aligned}$$

Theorem 2.4.

For any fuzzifying topological space (X,τ) we have

$\models \forall x(\{x\} \in F_\beta) \leftrightarrow (X,\tau) \in T_1^\beta$.

Proof. Since $\tau_\beta(A) = \bigwedge_{x \in A} N_x^\beta(A)$ (Corollary 4.1 in [1]), then for any x_1, x_2 with $x_1 \neq x_2$, we have

$$\begin{aligned}
[\forall x(\{x\} \in F_\beta)] &= \bigwedge_{x \in X} F_\beta(\{x\}) = \bigwedge_{x \in X} \tau_\beta(X - \{x\}) \leq \bigwedge_{x \in X} \bigwedge_{y \in X - \{x\}} N_y^\beta(X - \{x\}) \\
&\leq \bigwedge_{y \in X - \{x_2\}} N_y^\beta(X - \{x_2\}) \leq N_{x_1}^\beta(X - \{x_2\}) = \bigvee_{x_2 \notin A} N_{x_1}^\beta(A).
\end{aligned}$$

Similarly, we have, $[\forall x(\{x\}\in F_\beta)]\le \bigvee_{x_1\notin B} N^{\beta}_{x_2}(B)$. Then

$$\begin{aligned}
[\forall x(\{x\}\in F_\beta)] &\le \bigwedge_{x_1\neq x_2}\min(\bigvee_{x_2\notin A} N^{\beta}_{x_1}(A), \bigvee_{x_1\notin B} N^{\beta}_{x_2}(B))\\
&= \bigwedge_{x_1\neq x_2}\ \bigvee_{x_1\notin B,\, x_2\notin A}\min(N^{\beta}_{x_1}(A), N^{\beta}_{x_2}(B))\\
&= [(X,\tau)\in T^{\beta}_1].
\end{aligned}$$

On the other hand

$$\begin{aligned}
[(X,\tau)\in T^{\beta}_1] &= \bigwedge_{x_1\neq x_2}\min(\bigvee_{x_2\notin A} N^{\beta}_{x_1}(A), \bigvee_{x_1\notin B} N^{\beta}_{x_2}(B))\\
&= \bigwedge_{x_1\neq x_2}\min(N^{\beta}_{x_1}(X-\{x_2\}), N^{\beta}_{x_2}(X-\{x_1\}))\\
&\le \bigwedge_{x_1\neq x_2} N^{\beta}_{x_1}(X-\{x_2\}) = \bigwedge_{x_2\in X}\ \bigwedge_{x_1\in X-\{x_2\}} N^{\beta}_{x_1}(X-\{x_2\})\\
&= \bigwedge_{x_2\in X}\tau_\beta(X-\{x_2\}) = \bigwedge_{x\in X}\tau_\beta(X-\{x\})\\
&= [\forall x(\{x\}\in F_\beta)].
\end{aligned}$$

Therefore $[\forall x(\{x\}\in F_\beta)] = [(X,\tau)\in T^{\beta}_1]$.

Definition 2.5.

Let (X,τ) be a fuzzifying topological space. The fuzzifying β-derived set $D_\beta(A)$ of A is defined as follows: $x\in D_\beta(A) := \forall B(B\in N^{\beta}_x \to B\cap(A-\{x\})\neq\phi)$.

Lemma 2.6.

$D_\beta(A)(x) = 1 - N^{\beta}_x((X-A)\cup\{x\})$.

Proof.

From Theorem 4.2 (2) [1] we have $D_\beta(A)(x) = 1 - \bigvee_{B\cap(A-\{x\})=\phi} N^{\beta}_x(B) = 1 - N^{\beta}_x((X - A)\cup\{x\})$.

Theorem 2.7.

For any finite set $A\subseteq X$, $\models T^{\beta}_1(X,\tau)\to D_\beta(A)\equiv\phi$.

Proof.

From Theorem 4.2 (2) [1] we have

$$\begin{aligned}\bigwedge_{y\in X-A} N_y^{\beta}((X-A)\cup\{y\}) &\geq \bigwedge_{y\in X-A} N_y^{\beta}(X-A) = \bigwedge_{y\in X-A} N_y^{\beta}(\bigcap_{x\in A}(X-\{x\}) \\ &\geq \bigwedge_{y\in X-A}\ \bigwedge_{x\in A} N_y^{\beta}(X-\{x\}) \geq \bigwedge_{x\neq y} N_y^{\beta}(X-\{x\}).\end{aligned}$$

Also

$$\begin{aligned}\bigwedge_{y\in A} N_y^{\beta}((X-A)\cup\{y\}) &= \bigwedge_{y\in A} N_y^{\beta}(X-(A-\{y\})) = \bigwedge_{y\in A} N_y^{\beta}(\bigcap_{x\in A-\{y\}}(X-\{x\}) \\ &\geq \bigwedge_{y\in A}\ \bigwedge_{x\in A-\{y\}} N_y^{\beta}(X-\{x\}) \geq \bigwedge_{x\neq y} N_y^{\beta}(X-\{x\}).\end{aligned}$$

Therefore

$$\begin{aligned}[D_{\beta}(A)\equiv\phi] &= \bigwedge_{x\in X} N_x^{\beta}((X-A)\cup\{x\}) \\ &= \min(\bigwedge_{y\in X-A} N_y^{\beta}((X-A)\cup\{y\}), \bigwedge_{y\in A} N_y^{\beta}((X-A)\cup\{y\})) \\ &\geq \bigwedge_{x\neq y} N_y^{\beta}(X-\{x\}) = \bigwedge_{x\in X}\ \bigwedge_{x\in X-\{y\}} N_y^{\beta}(X-\{x\}) \\ &= \bigwedge_{x\in X} \tau_{\beta}(X-\{x\}) = \bigwedge_{x\in X} F_{\beta}(\{x\}) = T_1^{\beta}(X,\tau).\end{aligned}$$

Definition 2.8.

The fuzzifying β-local basis β_x^{β} of x is a function from $P(X)$ into $I=[0,1]$ satisfying the following conditions:

(1) $\models \beta_x^{\beta} \subseteq N_x^{\beta}$, and (2) $\models A \in N_x^{\beta} \longrightarrow \exists B(B\in \beta_x^{\beta} \wedge x \in B \subseteq A)$.

Lemma 2.9.

$\models A \in N_x^{\beta} \longleftrightarrow \exists B(B\in \beta_x^{\beta} \wedge x \in B \subseteq A)$.

Proof.

From condition (1) in Definition 2.8 and Theorem 4.2 (2) in [1] we have $N_x^{\beta}(A) \geq N_x^{\beta}(B) \geq \beta_x^{\beta}(B)$ for each $B \in P(X)$ such that $x \in B \subseteq A$. So $N_x^{\beta}(A) \geq \bigvee_{x\in B\subseteq A} \beta_x^{\beta}(B)$. From condition (2) in Definition 2.8 we have $N_x^{\beta}(A) \leq \bigvee_{x\in B\subseteq A} \beta_x^{\beta}(B)$. Hence $N_x^{\beta}(A) = \bigvee_{x\in B\subseteq A} \beta_x^{\beta}(B)$.

Theorem 2.10.

If β_x^β is a fuzzifying β-local basis of x, then

$\models (X,\tau) \in T_1^\beta \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow \exists A(A \in \beta_x^\beta \wedge y \notin A))$.

Proof. For any x, y with $x \neq y$, $\bigvee_{y\notin A} \beta_x^\beta(A) \leq \bigvee_{y\notin A} N_x^\beta(A)$, $\bigvee_{x\notin B} \beta_y^\beta(B) \leq \bigvee_{x\notin B} N_y^\beta(B)$. So $\min(\bigvee_{y\notin A} \beta_x^\beta(A), \bigvee_{x\notin B} \beta_y^\beta(B)) \leq \min(\bigvee_{y\notin A} N_x^\beta(A), \bigvee_{x\notin B} N_y^\beta(B)) = \bigvee_{y\notin A, x\notin B} \min(N_x^\beta(A), N_y^\beta(B))$, i.e., $\bigwedge_{x\neq y} \bigvee_{y\notin A} \beta_x^\beta(A) \leq \bigwedge_{x\neq y} \bigvee_{y\notin A, x\notin B} \min(N_x^\beta(A), N_y^\beta(B)) = [(X,\tau) \in T_1^\beta]$. On the other hand, for any B with $x \in B \subseteq X - \{y\}$ we have $y \notin B$. So $\bigvee_{y\notin A} \beta_x^\beta(A) \geq \beta_x^\beta(B)$. According to Definition 2.4 we have $\bigvee_{y\notin A} \beta_x^\beta(A) \geq \bigvee_{x\in B\subseteq X-\{y\}} \beta_x^\beta(B) = N_x^\beta(X - \{y\})$. Furthermore, from Corollary 4.1 [1] we have $\bigwedge_{x\neq y} \bigvee_{y\notin A} \beta_x^\beta(A) \geq \bigwedge_{x\neq y} N_x(X - \{y\}) = \bigwedge_{y\in X} \bigwedge_{x\in X-\{y\}} N_x(X - \{y\}) = \bigwedge_{y\in X} \tau_\beta(X - \{y\}) = \bigwedge_{y\in X} F_\beta(\{y\}) = [(X,\tau) \in T_1^\beta]$.

Theorem 2.11.

If β_x^β is a fuzzifying β-local basis of x, then

$\models (X,\tau) \in T_2^\beta \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \longrightarrow \exists B(B \in \beta_x^\beta \wedge y \in \neg(Cl_\beta(B))))$.

Proof.

$$
\begin{aligned}
[\forall x \forall y (x \in X \wedge y \in X \wedge x &\neq y \longrightarrow \exists B(B \in \beta_x^\beta \wedge y \in \neg(Cl_\beta(B))))] \\
&= \bigwedge_{x\neq y} \bigvee_{B\in P(X)} \min(\beta_x^\beta(B), \neg(1 - N_y^\beta(X - B))) \\
&= \bigwedge_{x\neq y} \bigvee_{B\in P(X)} \min(\beta_x^\beta(B), N_y^\beta(X - B)) \\
&= \bigwedge_{x\neq y} \bigvee_{B\in P(X)} \bigvee_{y\in C\subseteq X-B} \min(\beta_x^\beta(B), \beta_y^\beta(C)) \\
&= \bigwedge_{x\neq y} \bigvee_{B\cap C=\emptyset} \bigvee_{x\in D\subseteq B,\, y\in E\subseteq C} \min(\beta_x^\beta(D), \beta_y^\beta(E)) \\
&= \bigwedge_{x\neq y} \bigvee_{B\cap C=\emptyset} \min(\bigvee_{x\in D\subseteq B} \beta_x^\beta(D), \bigvee_{y\in E\subseteq C} \beta_y^\beta(E)) \\
&= \bigwedge_{x\neq y} \bigvee_{B\cap C=\emptyset} \min(N_x^\beta(B), N_y^\beta(C)) = [(X,\tau) \in T_2^\beta].
\end{aligned}
$$

Definition 2.12.

The binary fuzzy predicate $\rhd^{\beta} \in \Im(N(X)\times X)$, is defined as $S \rhd^{\beta} x := \forall A(A \in N_x^{\beta} \longrightarrow S \lessapprox A)$, where $N(X)$ is the set of all nets of X, $[S \rhd^{\beta} x]$ stands for the degree to which S β-converges to x and " $\lessapprox$ " is the binary crisp predicates "almost in ".

Theorem 2.13.

Let (X,τ) be a fuzzifying topological space and $S \in N(X)$.

$\models (X,\tau) \in T_2^{\beta} \longleftrightarrow \forall S \forall x \forall y((S \subseteq X)\wedge(x \in X)\wedge(y \in X)\wedge(S \rhd^{\beta} x)\wedge(S \rhd^{\beta} y) \longrightarrow x = y).$

Proof.

$[(X,\tau) \in T_2^{\beta}] = \bigwedge_{x\neq y} \bigvee_{A\cap B=\emptyset} (N_x^{\beta}(A) \wedge N_y^{\beta}(B)),$

$[\forall S \forall x \forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^{\beta} x) \wedge (S \rhd^{\beta} y) \longrightarrow x = y)]$

$= \bigwedge_{x\neq y} \bigwedge_{S\subseteq X} (\bigvee_{S \lessapprox A} N_x^{\beta}(A) \vee \bigvee_{S \lessapprox B} N_y^{\beta}(B)) = \bigwedge_{x\neq y} \bigwedge_{S\subseteq X} \bigvee_{S \lessapprox A} \bigvee_{S \lessapprox B} (N_x^{\beta}(A) \vee N_y^{\beta}(B)).$

(1) If $A \cap B = \emptyset$, then for any $S \in N(X)$, we have $S \lessapprox A$ or $S \lessapprox B$. Therefore, we obtain $N_x^{\beta}(A) \wedge N_y^{\beta}(B) \leq \bigvee_{S \lessapprox A} N_x^{\beta}(A)$ or $N_x^{\beta}(A) \wedge N_y^{\beta}(B) \leq \bigvee_{S \lessapprox B} N_x^{\beta}(B)$.
Consequently, $\bigvee_{A\cap B=\emptyset} (N_x^{\beta}(A) \wedge N_y^{\beta}(B)) \leq \bigwedge_{S\subseteq X} (\bigvee_{S \lessapprox A} N_x^{\beta}(A) \vee \bigvee_{S \lessapprox B} N_y^{\beta}(B))$, and
$[(X,\tau) \in T_2^{\beta}] \leq [\forall S \forall x \forall y((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^{\beta} x) \wedge (S \rhd^{\beta} y) \to x = y)]$.

(2) First, for any $x,\ y$ with $x \neq y$, if $\bigvee_{A\cap B=\emptyset} (N_x^{\beta}(A) \wedge N_y^{\beta}(B)) < t$, then $N_x^{\beta}(A) < t$ or $N_y^{\beta}(B) < t$ provided $A \cap B = \emptyset$, i.e., $A \cap B \neq \emptyset$ when $A \in (N_x^{\beta})_t$ and $B \in (N_y^{\beta})_t$. Now, set a net $S^* : (N_x^{\beta})_t \times (N_y^{\beta})_t \longrightarrow X,\ (A,B) \longmapsto x_{(A,B)} \in A \cap B$. Then for any $A \in (N_x^{\beta})_t,\ B \in (N_y^{\beta})_t$, we have $S^* \lessapprox A$ and $S^* \lessapprox B$. Therefore, if $S^* \lessapprox A$ and $S^* \lessapprox B$, then $A \notin (N_x^{\beta})_t,\ B \notin (N_y^{\beta})_t$, i.e., $N_x^{\beta}(A) \vee N_y^{\beta}(B)) < t$. Consequently $\bigvee_{S^* \lessapprox A} \bigvee_{S^* \lessapprox B} (N_x^{\beta}(A) \vee N_y^{\beta}(B)) \leq t$. Moreover $\bigwedge_{S\subseteq X} \bigvee_{S \lessapprox A} \bigvee_{S \lessapprox B} (N_x^{\beta}(A) \vee N_y^{\beta}(B)) \leq t$.
Second, for any positive integer i, there exists $x_i,\ y_i$ with $x_i \neq y_i$, and

$$\bigvee_{A\cap B=\emptyset} (N_{x_i}^{\beta}(A) \wedge N_{y_i}^{\beta}(B)) < [(X,\tau) \in T_2^{\beta}] + 1/i,$$

and hence

$$\bigwedge_{S\subseteq X} \bigvee_{S \lessapprox A} \bigvee_{S \lessapprox B} (N_{x_i}^{\beta}(A) \vee N_{y_i}^{\beta}(B)) < [(X,\tau) \in T_2^{\beta}] + 1/i.$$

So we have

$$\begin{aligned}[\forall S \forall x \forall y((S \subseteq X) \quad \wedge \quad & (x \in X) \wedge (y \in X) \wedge (S \rhd^{\beta} x) \wedge (S \rhd^{\beta} y) \longrightarrow x = y)] \\ = \quad & \bigwedge_{x \neq y} \bigwedge_{S \subseteq X} \bigvee_{S \subsetneqq A} \bigvee_{S \subsetneqq B} (N_x^{\beta}(A) \vee N_y^{\beta}(B)) \leq [(X, \tau) \in T_2^{\beta}].\end{aligned}$$

Lemma 2.14.

Let (X, τ) be a fuzzifying topological space.

(1) If $D \subseteq B$, then $\bigvee_{A \cap B = \emptyset} N_x^{\beta}(A) = \bigvee_{A \cap B = \emptyset,\ D \subseteq B} N_x^{\beta}(A)$,

(2) $\bigvee_{A \cap B = \emptyset} \bigwedge_{y \in D} N_y^{\beta}(X - A) = \bigvee_{A \cap B = \emptyset,\ D \subseteq B} \tau_{\beta}(B)$.

Proof.

(1) Since $D \subseteq B$ then

$$\bigvee_{A \cap B = \emptyset} N_x^{\beta}(A) = \bigvee_{A \cap B = \emptyset} N_x^{\beta}(A) \wedge [D \subseteq B] = \bigvee_{A \cap B = \emptyset,\ D \subseteq B} N_x^{\beta}(A).$$

(2) Let $y \in D$ and $A \cap B = \emptyset$. Then

$$\begin{aligned}\bigvee_{A \cap B = \emptyset,\ D \subseteq B} \tau_{\beta}(B) \quad &= \quad \bigvee_{A \cap B = \emptyset,\ D \subseteq B} \tau_{\beta}(B) \wedge [y \in D] \\ &= \quad \bigvee_{y \in D \subseteq B \subseteq X - A} \tau_{\beta}(B) = \bigvee_{y \in B \subseteq X - A} \tau_{\beta}(B) \\ &= \quad N_y^{\beta}(X - A) = \bigwedge_{y \in D} N_y^{\beta}(X - A) \\ &= \quad \bigvee_{A \cap B = \emptyset} \bigwedge_{y \in D} N_y^{\beta}(X - A). \quad \blacksquare\end{aligned}$$

Definition 2.15.

Let (X, τ) be a fuzzifying topological space.

$\beta T_3^{(1)}(X, \tau) := \forall x \forall D(x \in X \wedge D \in F \wedge x \notin D \longrightarrow \exists A(A \in N_x^{\beta} \wedge (D \subseteq X - Cl_{\beta}(A))))$.

Theorem 2.16.

$\models (X, \tau) \in T_3^{\beta} \longleftrightarrow (X, \tau) \in \beta T_3^{(1)}$.

Proof.

$$\begin{aligned}\beta T_3^{(1)}(X,\tau) &= \bigwedge_{x\notin D}\min(1, 1-\tau(X-D)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in D}(1-Cl_\beta(A)(y))))\\ &= \bigwedge_{x\notin D}\min(1, 1-\tau(X-D)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in D}N_y^\beta(X-A)))\end{aligned}$$

and $\quad T_3^\beta(X,\tau) = \bigwedge_{x\notin D}\min(1, 1-\tau(X-D)+\bigvee_{A\cap B=\emptyset,\ D\subseteq B}\min(N_x^\beta(A), \tau_\beta(B)))$.

So, the result holds if we prove that

$$\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in D}N_y^\beta(X-A)) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B}\min(N_x^\beta(A), \tau_\beta(B)) \qquad (*)$$

It is clear that, on the left-hand side of $(*)$ in the case of $A\cap D\neq\emptyset$ there exists $y\in X$ such that $y\in D$ and $y\notin X-A$. So, $\bigwedge_{y\in D}N_y^\beta(X-A)=0$ and thus $(*)$ becomes

$$\bigvee_{A\in P(X),\ A\cap B=\emptyset}\min(N_x^\beta(A), \bigwedge_{y\in D}N_y^\beta(X-A)) = \bigvee_{A\cap B=\emptyset,\ D\subseteq B}\min(N_x^\beta(A), \tau_\beta(B)),$$

which is obtained from Lemma 2.14.

Definition 2.17.

Let (X,τ) be a fuzzifying topological space.

$\beta T_3^{(2)}(X,\tau) := \forall x\forall B(x\in B\wedge B\in\tau \longrightarrow \exists A(A\in N_x^\beta \wedge Cl_\beta(A)\subseteq B))$.

Theorem 2.18.

$\models (X,\tau)\in T_3^\beta \longleftrightarrow (X,\tau)\in \beta T_3^{(2)}$.

Proof. From Theorem 2.16 we have

$$T_3^\beta(X,\tau) = \bigwedge_{x\notin D}\min(1, 1-\tau(X-D)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in D}N_y^\beta(X-A))).$$

Now,

$$\begin{aligned}\beta T_3^{(2)}(X,\tau) &= \bigwedge_{x\in B}\min(1, 1-\tau(B)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in X-B}(1-Cl_\beta(A)(y))))\\ &= \bigwedge_{x\in B}\min(1, 1-\tau(B)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in X-B}(1-(1-N_y^\beta(X-A)))))\\ &= \bigwedge_{x\in B}\min(1, 1-\tau(B)+\bigvee_{A\in P(X)}\min(N_x^\beta(A), \bigwedge_{y\in X-B}N_y^\beta(X-A))).\end{aligned}$$

Put $B = X - D$ we have

$$\beta T_3^{(2)}(X,\tau) = \bigwedge_{x\notin D} \min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)} \min(N_x^\beta(A), \bigwedge_{y\in D} N_y^\beta(X-A)))$$
$$= T_3^\beta(X,\tau).$$

Definition 2.19.

Let φ be a subbase of τ then

$$\beta T_3^{(3)}(X,\tau) := \forall x \forall D(x \in D \wedge D \in \varphi \longrightarrow \exists B(B \in N_x^\beta \wedge Cl_\beta(B) \subseteq D)).$$

Theorem 2.20.

$\models (X,\tau) \in T_3^\beta \longleftrightarrow (X,\tau) \in \beta T_3^{(3)}$.

Proof. Since $[\varphi \subseteq \tau] = 1$, then from Theorems 2.16 we have

$$\beta T_3^{(3)}(X,\tau) \geq \beta T_3^{(2)}(X,\tau) = T_3^\beta(X,\tau).$$

So, it suffices to prove that $\beta T_3^{(3)}(X,\tau) \leq \beta T_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x \in A$,

$$\min(1, 1-\tau(A) + \bigvee_{B\in P(X)} \min(N_x^\beta(B), \bigwedge_{y\in X-A} N_y^\beta(X-B))) \geq \beta T_3^{(3)}(X,\tau).$$

Set $\beta T_3^{(3)}(X,\tau) = \delta$. Then for any $x \in X$ and any $D_{\lambda_i} \in P(X), x \in D_{\lambda_i}, \lambda_i \in I_\lambda$ (I_λ denotes a finite index set), $\lambda \in \Lambda$, $\bigcup_{\lambda\in\Lambda} \bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$ we have

$$1 - \varphi(D_{\lambda_i}) + \bigvee_{B\in P(X)} \min(N_x^\beta(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^\beta(X-B)) \geq \delta > \delta - \epsilon,$$

where ϵ is any positive number. Thus

$$\bigvee_{B\in P(X)} \min(N_x^\beta(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^\beta(X-B)) > \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

Set $\gamma_{\lambda_i} = \{B : B \subseteq D_{\lambda_i}\}$. From the completely distributive law we have

$$\begin{aligned}
\bigwedge_{\lambda_i \in I_\lambda} \bigvee_{B \in P(X)} \min(N_x^\beta(B) \quad , \quad & \bigwedge_{y \in X - D_{\lambda_i}} N_y^\beta(X - B)) \\
&= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \bigwedge_{\lambda_i \in I_\lambda} \min(N_x^\beta(f(\lambda_i)), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\beta(X - f(\lambda_i))) \\
&= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\beta(f(\lambda_i)), \bigwedge_{\lambda_i \in I_\lambda} \bigwedge_{y \in X - D_{\lambda_i}} N_y^\beta(X - f(\lambda_i))) \\
&= \bigvee_{f \in \Pi\{\gamma_{\lambda_i} : \lambda_i \in I_\lambda\}} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\beta(f(\lambda_i)), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - f(\lambda_i))) \\
&= \bigvee_{B \in P(X)} \min(\bigwedge_{\lambda_i \in I_\lambda} N_x^\beta(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - B)) \\
&= \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - B)),
\end{aligned}$$

where $B = f(\lambda_i)$.

Similarly, we can prove

$$\begin{aligned}
\bigwedge_{\lambda \in \Lambda} \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - B)) &= \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in \bigcup_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - B)) \\
&\leq \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in \bigcap_{\lambda \in \Lambda} \bigcup_{\lambda_i \in I_\lambda} X - D_{\lambda_i}} N_y^\beta(X - B)) \\
&\leq \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in X - A} N_y^\beta(X - B)),
\end{aligned}$$

so we have

$$\begin{aligned}
\bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in X - A} N_y^\beta(X - B)) &\geq \bigwedge_{\lambda \in \Lambda} \bigwedge_{\lambda_i \in I_\lambda} \bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^\beta(X - B)) \\
&\geq \bigwedge_{\lambda \in \Lambda} \bigwedge_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.
\end{aligned}$$

For any I_λ and Λ that satisfy $\bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$ the above inequality is true. So,

$$\begin{aligned}
\bigvee_{B \in P(X)} \min(N_x^\beta(B), \bigwedge_{y \in X - A} N_y^\beta(X - B)) &\geq \bigvee_{\cup_{\lambda \in \Lambda} D_\lambda = A} \bigwedge_{\lambda \in \Lambda} \bigvee_{\cap_{\lambda_i \in I_\lambda} D_{\lambda_i} = D_\lambda} \bigwedge_{\lambda_i \in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon \\
&= \tau(A) - 1 + \delta - \epsilon.
\end{aligned}$$

i.e., $\min(1, 1-\tau(A)+\bigvee_{B\in P(X)} \min(N_x^\beta(B), \bigwedge_{y\in X-A} N_y^\beta(X-B))) \geq \delta-\epsilon.$

Because ϵ is any arbitrary positive number, when $\epsilon \longrightarrow 0$ we have

$\beta T_3^{(2)}(X,\tau) \geq \delta = \beta T_3^{(3)}(X,\tau)$. So, $\models (X,\tau) \in T_3^\beta \longleftrightarrow (X,\tau) \in \beta T_3^{(3)}$.

Definition 2.21.

Let (X,τ) be any fuzzifying topological space.

(1) $\beta' T_3^{(1)}(X,\tau) := \forall x \forall D(x \in X \wedge D \in F_\beta \wedge x \notin D \longrightarrow \exists A(A \in N_x \wedge (D \subseteq X - Cl(A))))$;

(2) $\beta' T_3^{(2)}(X,\tau) := \forall x \forall B(x \in B \wedge B \in \tau_\beta \longrightarrow \exists A(A \in N_x \wedge Cl(A) \subseteq B))$;

(3) $\beta T_4^{(1)}(X,\tau) := \forall A \forall B(A \in \tau \wedge B \in F \wedge A \cap B \equiv \emptyset \to \exists G(G \in \tau \wedge A \subseteq G \wedge Cl_\beta(G) \cap B \equiv \phi))$;

(4) $\beta T_4^{(2)}(X,\tau) := \forall A \forall B(A \in F \wedge B \in \tau \wedge A \subseteq B \to \exists G(G \in \tau \wedge A \subseteq G \wedge Cl_\beta(G) \subseteq B))$;

(5) $\beta' T_4^{(1)}(X,\tau) := \forall A \forall B(A \in \tau \wedge B \in F_\beta \wedge A \cap B \equiv \emptyset \to \exists G(G \in \tau \wedge A \subseteq G \wedge Cl(G) \cap B \equiv \phi))$;

(6) $\beta' T_4^{(2)}(X,\tau) := \forall A \forall B(A \in F \wedge B \in \tau_\beta \wedge A \subseteq B \to \exists G(G \in \tau \wedge A \subseteq G \wedge Cl(G) \subseteq B))$.

By a similar proof of Theorem 2.16 and 2.18 we have the following theorem.

Theorem 2.22.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau) \in T_3^{\beta'} \longleftrightarrow (X,\tau) \in \beta' T_3^{(i)}$;

(2) $\models (X,\tau) \in T_4^{\beta} \longleftrightarrow (X,\tau) \in \beta T_4^{(i)}$;

(3) $\models (X,\tau) \in T_4^{\beta'} \longleftrightarrow (X,\tau) \in \beta' T_4^{(i)}$, where $i = 1, 2$.

3. Relation among fuzzifying separation axioms

Lemma 3.1.

(1) $\models K(x,y) \to K^\beta(x,y)$,

(2) $\models H(x,y) \to H^\beta(x,y)$,

(3) $\models M(x,y) \to M^\beta(x,y)$,

(4) $\models V(x,D) \to V^\beta(x,D)$,

(5) $\models W(A,B) \to W^\beta(A,B)$.

Proof. Since $\models \tau \subseteq \tau_\beta$, then $N_x(A) \leq N_x^\beta(A)$ for any $A \in P(X)$. Then the proof is immediate.

Theorem 3.2.

$\models (X,\tau) \in T_i \longrightarrow (X,\tau) \in T_i^\beta$, where $i = 0, 1, 2, 3, 4$.

Proof. It is obtained from Lemma 3.1.

Theorem 3.3.

If $T_0(X,\tau) = 1$, then

(1) $\models (X,\tau) \in R_0 \longrightarrow (X,\tau) \in R_0^\beta$,

(2) $\models (X,\tau) \in R_1 \longrightarrow (X,\tau) \in R_1^\beta$,

Proof. Since $T_0(X,\tau) = 1$, then for each $x, y \in X$ and $x \neq y$, we have $[K(x,y)] = 1$ and so $[K^\beta(x,y)] = 1$.

(1) Using Lemma 3.1 (1) and (2) we obtain

$$\begin{aligned}[(X,\tau) \in R_0] &= \bigwedge_{x\neq y}[K(x,y) \to H(x,y)] \leq \bigwedge_{x\neq y}[K(x,y) \to H^\beta(x,y)] \\ &\leq \bigwedge_{x\neq y}[K^\beta(x,y) \to H^\beta(x,y)] = R_0^\beta(X,\tau).\end{aligned}$$

(2) Using Lemma 3.1 (1) and (3) the proof is similar to (1).

Lemma 3.4.

(1) $\models M^\beta(x,y) \longrightarrow H^\beta(x,y)$;

(2) $\models H^\beta(x,y) \longrightarrow K^\beta(x,y)$;

(3) $\models M^\beta(x,y) \longrightarrow K^\beta(x,y)$.

Proof. (1) Since $\{B, C \in P(X) : B \cap C \equiv \emptyset\} \subseteq \{B, C \in P(X) : y \notin B \text{ and } x \notin C\}$, then $[M^\beta(x,y)] = \bigvee_{B\cap C=\emptyset} \min(N_x^\beta(B), N_y^\beta(C)) \leq \bigvee_{y\notin B,\, x\notin C} \min(N_x^\beta(B), N_y^\beta(C)) = [H^\beta(x,y)]$.

(2) $$\begin{aligned}[K^\beta(x,y)] &= \max(\bigvee_{y\notin A} N_x^\beta(A), \bigvee_{x\notin A} N_y^\beta(A)) \geq \bigvee_{y\notin A} N_x^\beta(A) \geq \bigvee_{y\notin A,\, x\notin B}(N_x^\beta(A) \wedge N_y^\beta(B)) \\ &= [H^\beta(x,y)].\end{aligned}$$

(3) From (1) and (2) it is obvious.

Theorem 3.5.

Let (X,τ) be a fuzzifying topological space. Then we have

(1) $\models (X,\tau) \in T_1^\beta \longrightarrow (X,\tau) \in T_0^\beta$;

(2) $\models (X,\tau) \in T_2^\beta \longrightarrow (X,\tau) \in T_1^\beta$;

(3) $\models (X,\tau) \in T_2^\beta \longrightarrow (X,\tau) \in T_0^\beta$.

Proof. The proof of (1) and (2) are obtained from Lemma 3.4 (2) and (1), respectively.

(3) From (1) and (2) above the result is obtained.

Theorem 3.6.

$\models (X,\tau) \in R_1^\beta \longrightarrow (X,\tau) \in R_0^\beta$.

Proof. From Lemma 3.4 (2), the proof is immediate.

Theorem 3.7.

For any fuzzifying topological space (X,τ) we have

(1) $\models (X,\tau) \in T_1^\beta \longrightarrow (X,\tau) \in R_0^\beta$;

(2) $\models (X,\tau) \in T_1^\beta \longrightarrow (X,\tau) \in R_0^\beta \wedge (X,\tau) \in T_0^\beta$;

(3) If $T_0^\beta(X,\tau) = 1$, then $\models (X,\tau) \in T_1^\beta \longleftrightarrow (X,\tau) \in R_0^\beta \wedge (X,\tau) \in T_0^\beta$.

Proof.

(1) $T_1^\beta(X,\tau) = \bigwedge_{x\neq y}[H^\beta(x,y)] \leq \bigwedge_{x\neq y}[K^\beta(x,y) \longrightarrow H^\beta(x,y)] = R_0^\beta(X,\tau)$.

(2) It is obtained from (1) and from Theorem 3.5 (1).

(3) Since $T_0^\beta(X,\tau) = 1$, for every $x, y \in X$ such that $x \neq y$, then we have $[K^\beta(x,y)] = 1$. Therefore

$$\begin{aligned}
[(X,\tau) \in R_0^\beta \wedge (X,\tau) \in T_0^\beta] &= [(X,\tau) \in R_0^\beta] \\
&= \bigwedge_{x\neq y} \min(1, 1 - [K^\beta(x,y)] + [H^\beta(x,y)]) \\
&= \bigwedge_{x\neq y} [H^\beta(x,y)] = T_1^\beta(X,\tau).
\end{aligned}$$

Theorem 3.8.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau) \in R_0^\beta \otimes (X,\tau) \in T_0^\beta \longrightarrow (X,\tau) \in T_1^\beta$, and

(2) If $T_0^\beta(X,\tau) = 1$, then $\models (X,\tau) \in R_0^\beta \otimes (X,\tau) \in T_0^\beta \longleftrightarrow (X,\tau) \in T_1^\beta$.

Proof.

$$
\begin{aligned}
(1)\ \ [(X,\tau)\in R_0^{\beta}\otimes(X,\tau)\in T_0^{\beta}] &= \max(0, R_0^{\beta}(X,\tau)+T_0^{\beta}(X,\tau)-1)\\
&= \max(0, \bigwedge_{x\neq y}\min(1,1-[K^{\beta}(x,y)]+[H^{\beta}(x,y)])+\bigwedge_{x\neq y}[K^{\beta}(x,y)]-1)\\
&\leq \max(0, \bigwedge_{x\neq y}\{\min(1,1-[K^{\beta}(x,y)]+[H^{\beta}(x,y)])+[K^{\beta}(x,y)]\}-1)\\
&= \bigwedge_{x\neq y}[H^{\beta}(x,y)] = T_1^{\beta}(X,\tau).
\end{aligned}
$$

$$
\begin{aligned}
(2)[(X,\tau)\in R_0^{\beta}\otimes(X,\tau)\in T_0^{\beta}] &= [(X,\tau)\in R_0^{\beta}]\\
&= \bigwedge_{x\neq y}\min(1,1-[K^{\beta}(x,y)]+[H^{\beta}(x,y)])\\
&= \bigwedge_{x\neq y}[H^{\beta}(x,y)] = T_1^{\beta}(X,\tau),
\end{aligned}
$$

because $T_0^{\beta}(X,\tau)=1$, implies that for each x,y such that $x\neq y$ we have $[K^{\beta}(x,y)]=1$.

Theorem 3.9. Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau)\in T_0^{\beta}\longrightarrow((X,\tau)\in R_0^{\beta}\longrightarrow(X,\tau)\in T_1^{\beta})$, and

(2) $\models (X,\tau)\in R_0^{\beta}\longrightarrow((X,\tau)\in T_0^{\beta}\longrightarrow(X,\tau)\in T_1^{\beta})$.

Proof. It obtained From Theorems 3.7 (1) and 3.8 (1) and the fact that $[\beta]\leq[\varphi\to\psi]\Leftrightarrow$ $[\beta]\otimes[\varphi]\leq[\psi]$.

Theorem 3.10.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau)\in T_2^{\beta}\longrightarrow(X,\tau)\in R_1^{\beta}$;

(2) $\models (X,\tau)\in T_2^{\beta}\longrightarrow(X,\tau)\in R_i^{\beta}\wedge(X,\tau)\in T_i^{\beta}$, where $i=0,1$;

(3) If $T_0^{\beta}(X,\tau)=1$, then

(i) $\models (X,\tau)\in T_2^{\beta}\longleftrightarrow(X,\tau)\in R_1^{\beta}\wedge(X,\tau)\in T_0^{\beta}$.

(ii) $\models (X,\tau)\in T_2^{\beta}\longleftrightarrow(X,\tau)\in R_1^{\beta}\wedge(X,\tau)\in T_1^{\beta}$.

Proof.

It is similar to the proof of Theorem 3.7.

Theorem 3.11.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau)\in R_1^{\beta}\otimes (X,\tau)\in T_0^{\beta}\longrightarrow (X,\tau)\in T_2^{\beta}$, and

(2) If $T_0^{\beta}(X,\tau)=1$, then $\models (X,\tau)\in R_1^{\beta}\otimes (X,\tau)\in T_0^{\beta}\longleftrightarrow (X,\tau)\in T_2^{\beta}$.

Proof.

It is similar to the proof of Theorem 3.8.

Theorem 3.12.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau)\in T_0^{\beta}\longrightarrow ((X,\tau)\in R_1^{\beta}\longrightarrow (X,\tau)\in T_2^{\beta})$, and

(2) $\models (X,\tau)\in R_1^{\beta}\longrightarrow ((X,\tau)\in T_0^{\beta}\longrightarrow (X,\tau)\in T_2^{\beta})$.

Proof.

It is similar to the proof of Theorem 3.9.

Theorem 3.13.

If $T_0^{\beta}(X,\tau)=1$, then

(1) $\models ((X,\tau)\in T_0^{\beta}\longrightarrow ((X,\tau)\in R_0^{\beta}\longrightarrow (X,\tau)\in T_1^{\beta}))\wedge ((X,\tau)\in T_1^{\beta}\longrightarrow \neg((X,\tau)\in T_0^{\beta}\longrightarrow \neg((X,\tau)\in \beta_0^{\beta})))$;

(2) $\models ((X,\tau)\in R_0^{\beta}\longrightarrow ((X,\tau)\in T_0^{\beta}\longrightarrow (X,\tau)\in T_1^{\beta}))\wedge ((X,\tau)\in T_1^{\beta}\longrightarrow \neg((X,\tau)\in T_0^{\beta}\longrightarrow \neg((X,\tau)\in \beta_0^{\beta})))$;

(3) $\models ((X,\tau)\in T_0^{\beta}\longrightarrow ((X,\tau)\in R_0^{\beta}\longrightarrow (X,\tau)\in T_1^{\beta}))\wedge ((X,\tau)\in T_1^{\beta}\longrightarrow \neg((X,\tau)\in R_0^{\beta}\longrightarrow \neg((X,\tau)\in T_0^{\beta})))$;

(4) $\models ((X,\tau)\in R_0^{\beta}\longrightarrow ((X,\tau)\in T_0^{\beta}\longrightarrow (X,\tau)\in T_1^{\beta}))\wedge ((X,\tau)\in T_1^{\beta}\longrightarrow \neg((X,\tau)\in R_0^{\beta}\longrightarrow \neg((X,\tau)\in T_0^{\beta})))$.

Proof. For simplicity we put, $T_0^{\beta}(X,\tau)=\beta$, $R_0^{\beta}(X,\tau)=\beta$ and $T_1^{\beta}(X,\tau)=\gamma$. Now, applying Theorem 3.8 (2), the proof is obtained with some relations in fuzzy logic as follows:

$$(1) \qquad 1 = (\beta \otimes \beta \longleftrightarrow \gamma) = (\beta \otimes \beta \longrightarrow \gamma) \wedge (\gamma \longrightarrow \beta \otimes \beta)$$

$$= \neg((\beta \otimes \beta) \otimes \neg\gamma) \wedge \neg(\gamma \otimes \neg(\beta \otimes \beta))$$

$$= \neg(\beta \otimes \neg(\neg(\beta \otimes \neg\gamma))) \wedge \neg(\gamma \otimes (\beta \longrightarrow \neg\beta))$$

$$= (\beta \longrightarrow \neg(\beta \otimes \neg\gamma)) \wedge (\gamma \longrightarrow \neg(\beta \longrightarrow \neg\beta))$$

$$= (\beta \longrightarrow (\beta \longrightarrow \gamma) \wedge (\gamma \longrightarrow \neg(\beta \longrightarrow \neg\beta))),$$

since $\otimes$ is commutative one can have the proof of statements (2) - (4) in a similar way as (1).

By a similar procedure to Theorem 3.13 one can have the following theorem.

Theorem 3.14.

If $T_0^\beta(X, \tau) = 1$, then

(1) $\models ((X, \tau) \in T_0^\beta \longrightarrow ((X, \tau) \in R_1^\beta \longrightarrow (X, \tau) \in T_2^\beta)) \wedge ((X, \tau) \in T_2^\beta \longrightarrow \neg((X, \tau) \in T_0^\beta \longrightarrow \neg((X, \tau) \in R_1^\beta)));$

(2) $\models ((X, \tau) \in R_1^\beta \longrightarrow ((X, \tau) \in T_0^\beta \longrightarrow (X, \tau) \in T_2^\beta)) \wedge ((X, \tau) \in T_2^\beta \longrightarrow \neg((X, \tau) \in T_0^\beta \longrightarrow \neg((X, \tau) \in \beta_1^\beta)));$

(3) $\models ((X, \tau) \in T_0^\beta \longrightarrow ((X, \tau) \in R_1^\beta \longrightarrow (X, \tau) \in T_2^\beta)) \wedge ((X, \tau) \in T_2^\beta \longrightarrow \neg((X, \tau) \in R_1^\beta \longrightarrow \neg((X, \tau) \in T_0^\beta)));$

(4) $\models ((X, \tau) \in R_1^\beta \longrightarrow ((X, \tau) \in T_0^\beta \longrightarrow (X, \tau) \in T_2^\beta)) \wedge ((X, \tau) \in T_2^\beta \longrightarrow \neg((X, \tau) \in R_1^\beta \longrightarrow \neg((X, \tau) \in T_0^\beta))).$

Theorem 3.16.

$\models (X, \tau) \in T_3^\beta \otimes (X, \tau) \in T_1 \longrightarrow (X, \tau) \in T_2^\beta.$

Proof.

From Theorem 2.2 [26] we have, $T_1(X, \tau) = \bigwedge_{y \in X} \tau(X - \{y\})$. Therefore

$$T_3^\beta(X, \tau) + T_1(X, \tau)$$

$$= \bigwedge_{x \notin D} \min\left(1, 1 - \tau(X - D) + \bigvee_{A \cap B = \emptyset,\, D \subseteq B} \min(N_x^\beta(A), \tau_\beta(B))\right) + \bigwedge_{y \in X} \tau(X - \{y\})$$

$$
\begin{aligned}
&\leq \bigwedge_{x\in X,\, x\neq y} \bigwedge_{y\in X} \min\left(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^\beta(A), N_y^\beta(B))\right) + \bigwedge_{y\in X} \tau(X-\{y\}) \\
&= \bigwedge_{x\in X,\, x\neq y} \left(\bigwedge_{y\in X} \min(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^\beta(A), N_y^\beta(B))) + \bigwedge_{y\in X} \tau(X-\{y\})\right) \\
&\leq \bigwedge_{x\in X,\, x\neq y} \bigwedge_{y\in X} \left(\min(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^\beta(A), N_y^\beta(B))) + \tau(X-\{y\})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bigwedge_{x\neq y} \left(1 + \bigvee_{A\cap B=\emptyset} \min(N_x^\beta(A), N_y^\beta(B))\right) \\
&= 1 + \bigwedge_{x\neq y} \bigvee_{A\cap B=\emptyset} \min(N_x^\beta(A), N_y^\beta(B)) = 1 + T_2^\beta(X,\tau),
\end{aligned}
$$

namely, $T_2^\beta(X,\tau) \geq T_3^\beta(X,\tau) + T_1(X,\tau) - 1$. Thus $T_2^\beta(X,\tau) \geq \max(0, T_3^\beta(X,\tau) + T_1(X,\tau) - 1)$.

Theorem 3.17.

$\models (X,\tau) \in T_4^\beta \otimes (X,\tau) \in T_1 \longrightarrow (X,\tau) \in T_3^\beta$.

Proof. It is equivalent to prove that $T_3^\beta(X,\tau) \geq T_4^\beta(X,\tau) + T_1(X,\tau) - 1$. In fact,

$$
\begin{aligned}
&T_4^\beta(X,\tau) + T_1(X,\tau) \\
&= \bigwedge_{E\cap D=\emptyset} \min\Bigg(1, 1-\min(\tau(X-E), \tau(X-D)) \\
&\qquad + \bigvee_{A\cap B=\emptyset,\, E\subseteq A,\, D\subseteq B} \min(\tau_\beta(A), \tau_\beta(B))\Bigg) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
&\leq \bigwedge_{x\notin D} \min\Bigg(1, 1-\min(\tau(X-\{x\}), \tau(X-D)) \\
&\qquad + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^\beta(A), \tau_\beta(B))\Bigg) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
&= \bigwedge_{x\notin D} \min\Bigg(1, \max\Bigg(1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^\beta(A), \tau_\beta(B)), 1-\tau(X-\{x\})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
= &\bigwedge_{x\notin D} \max\Bigg(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big), \min\Big(1, 1-\tau(X-\{x\}) \\
&\qquad\qquad + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
\leq &\bigwedge_{x\notin D} \max\Bigg(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big) + \tau(X-\{x\}), \\
&\qquad\qquad \min\Big(1, 1-\tau(X-\{x\}) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big) + \tau(X-\{x\})\Bigg) \\
\leq &\bigwedge_{x\notin D} \max\Bigg(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big) + \tau(X-\{x\}), 1\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\leq &\bigwedge_{x\notin D} \Bigg(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big) + 1\Bigg) \\
= &\bigwedge_{x\notin D} \min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^{\beta}(A), \tau_{\beta}(B))\Big) + 1 \\
= &\ T_3^{\beta}(X,\tau) + 1.
\end{aligned}
$$

By a similar procedures of Theorems 3.16 and 3.17 we have the following theorems

Theorem 3.18.

Let (X,τ) be a fuzzifying topological space.

(1) $\models (X,\tau)\in T_3^{\beta'} \otimes (X,\tau)\in T_1^{\beta} \longrightarrow (X,\tau)\in T_2$.

(2) $\models (X,\tau)\in T_4^{\beta'} \otimes (X,\tau)\in T_1^{\beta} \longrightarrow (X,\tau)\in T_3^{\beta'}$.

From the above discussion one can have the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & T_3^{\beta'}\otimes T_1^{\beta} & \leftarrow & T_4^{\beta'}\otimes T_1^{\beta} & & \\
 & & & & & & \downarrow & & \\
 & & & & \downarrow & & T_3^{\beta'} & \leftarrow & T_4^{\beta'} \\
 & & & & & & \downarrow & & \downarrow \\
T_0 & \leftarrow & T_1 & \leftarrow & T_2 & \leftarrow & T_3 & \leftarrow & T_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_0^{\beta} & \leftarrow & T_1^{\beta} & \leftarrow & T_2^{\beta} & \leftarrow & T_3^{\beta} & \leftarrow & T_4^{\beta}
\end{array}
$$

Conclusion: The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous valued-logic). It continue various investigations into fuzzy topology in a legitimate way and extend some fundamental results in general topology to fuzzifying topology. An important virtue of our approach (in which we follow Ying) is that we define topological notions as fuzzy predicates (by formulae of Łukasiewicz fuzzy logic) and prove the validity of fuzzy implications (or equivalences). Unlike the (more wide-spread) style of defining notions in fuzzy mathematics as crisp predicates of fuzzy sets, fuzzy predicates of fuzzy sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy implications are more general than the corresponding theorems on crisp predicates of fuzzy sets. The main contributions of the present paper are to study β-separation axioms in fuzzifying topology and give the relations of these axioms with each other as well as the relations with other fuzzifying separation axiom. The role or the meaning of each theorem in the present paper is obtained from its generalization to a corresponding theorem in crisp setting. For example: in crisp setting, a topological space (X, τ) is T_1^{β} if and only if for each $z \in X, z \in F_{\beta}$, where F_{β} is the family of β-closed sets. This fact can be rewritten as follows: the truth value of a topological space (X, τ) to be T_1^{β} equal the infimum of the truth values of its singletons to be β-closed, where the set of truth values is $\{0, 1\}$. Now, is this theorem still valid in fuzzifying settings, i.e., if the set of truth values is $[0, 1]$? The answer of this question is positive and is given in Theorem 2.4 above.

There are some problems for further study:

(1) What is the justification for fuzzifying β-separation axioms in the setting of (2, L) topologies.

(2) Obviously, fuzzifying topological spaces in [19] form a fuzzy category. Perhaps, this will become a motivation for further study of the fuzzy category.

(3) What is the justification for fuzzifying β-separation axioms in (M, L)-topologies etc.

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