JOURNAL OF ADVANCED STUDIES IN TOPOLOGY
Vol. 4, No. 4, 2013, pp. 46–58
ISSN 2090-8288 | EISSN 2090-388X

Almost Separation Axioms in Fuzzifying Topology

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ABSTRACT. In the present paper, we introduced topological notions defined by means of regular open sets when these are planted into the framework of Ying's fuzzifying topological spaces (in Łukasiewicz fuzzy logic). We used fuzzy logic to introduce almost separation axioms T_0^R-, T_1^R-, T_2^R (almost Hausdorff)-, T_3^R (almost-regular)- and T_4^R (almost-normal). Furthermore, the R_0^R- and R_1^R-separation axioms have been studied and their relations with the T_1^R- and T_2^R-separation axioms have been introduced. Moreover, we gave the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms.

ARTICLE INFORMATION
Received: May 15, 2013
Accepted: August 29, 2013
Communicated By: A. Ghareeb
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KEYWORDS. Łukasiewicz logic; Fuzzifying topology; Fuzzifying regular open set; Fuzzifying separation axioms.

1. Introduction and Preliminaries

Chang [1], Wong [2], Hutton [3], Lowen [4], Pu and Liu [5], and others have discussed various aspects of fuzzy topology with crisp methods. Ying [6, 7] introduced fuzzifying topology and elementarily developed fuzzy topology from a new direction with the semantic method of continuous valued logic. In the framework of fuzzifying topology, Shen [8] introduced and studied T_{0-}, T_{1-}, T_2 (Hausdorff)-, T_3 (regular)- and T_4(normal)-separation axioms in fuzzifying topology.

In [9], the concepts of the R_{0-} and R_{1-} separation axioms in fuzzifying topology were added and their relations with the T_{1-} and T_{2-} separation axioms, were studied, respectively. In [10, 11], the authors introduced and studied the concepts of fuzzifying regular neighborhood structure of a point, fuzzifying regular interior, fuzzifying regular closure, and fuzzifying regular convergence.

In the present paper, we introduce and study T_{0-}^R, T_{1-}^R, T_2^R (almost Hausdorff)-, T_3^R (almost-regular)- and T_4^R (almost-normal)-separation axioms in fuzzifying topology. Also, we introduce the R_{0-}^R and R_{1-}^R separation axioms and study their relations with the T_{1-}^R and T_{2-}^R separation axioms, respectively.

The contents of the paper are arranged as follows. First, in the framework of fuzzifying topology, the concept of almost separation axioms T_{0-}^R, T_{1-}^R, T_2^R (almost Hausdorff)-, T_3^R (almost-regular)- and T_4^R (almost-normal)- have been discussed on the bases of fuzzifying topology. Furthermore, the R_{0-}^R and R_{1-}^R separation axioms have been studied and their relations with the T_{1-}^R and T_{2-}^R separation axioms have been introduced. In Section 3, we give the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. In the last section, a conclusion is given. Thus we fill a gap in the existing literature on fuzzifying topology. We will use terminologies and notations in [9, 8, 6, 7, 10, 11] without any explanation. We will use the symbol $\otimes$ instead of the second "AND" operation "$\dot{\wedge}$" as dot is hardly visible. This means that

$$[\alpha] \leq [\varphi \to \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi].$$

A fuzzifying topology on a set X [6] is a function $\tau \in \Im(P(X))$ such that:

(1) $\tau(X) = \tau(\emptyset) = 1$;
(2) for any $A, B \in P(X), \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda \in P(X) : \lambda \in \Lambda\}, \tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \tau(A_\lambda)$.

The family of all fuzzifying regular open sets [10], denoted by $\tau_R \in \Im(P(X))$, is defined as $A \in \tau_R := A \equiv Int(Cl(A))$, i.e.,

$$[A \in \tau_R] = \min(\bigwedge_{x\in A} Int(Cl(A)(x)), \bigwedge_{x\in X-A} (1 - Int(Cl(A)(x)))).$$

The family of all fuzzifying regular closed sets [10], denoted by $F_R \in \Im(P(X))$, is defined as $F_R(A) = \tau_R(X - A)$. Let $x \in X$. The fuzzifying regular neighborhood system of x [10], denoted by $N_x^R \in \Im(P(X))$, is defined as $A \in N_x^R := \exists B((B \in \tau_R) \wedge (x \in B \subseteq A))$, i.e., $N_x^R(A) = \bigvee_{x\in B\subseteq A} \tau_R(B)$. The fuzzifying regular closure [10] of a set $A \subseteq X$, denoted by $Cl_R(A) \in \Im(X)$, is defined as $Cl_R(A)(x) = 1 - N_x^R(X - A)$. The binary fuzzy predicates $\rhd^R \in \Im(N(X) \times X)$ [11], is defined as:

$$S \rhd^R x := \forall A(A \in N_x^R \longrightarrow S \subsetapprox A),$$

where $[S \rhd^R x]$ stands for the degree to which S regular converges to x and " $\subsetapprox$ " is the binary crisp predicates " almost in ".

2. Fuzzifying almost separation axioms and their equivalents

For simplicity we give the following definition.

Definition 2.1. Let (X, τ) be a fuzzifying topological space. The binary fuzzy predicates $K^R, H^R, M^R \in \Im(X \times X), V^R \in \Im(X \times P(X))$ and $W^R \in \Im(P(X) \times P(X))$ are defined as follows:

(1) $K^R(x,y) := \exists A((A \in N_x^R \wedge y \notin A) \vee (A \in N_y^R \wedge x \notin A))$;
(2) $H^R(x,y) := \exists B\exists C((B \in N_x^R \wedge y \notin B) \wedge (C \in N_y^R \wedge x \notin C))$;
(3) $M^R(x,y) := \exists B\exists C(B \in N_x^R \wedge C \in N_y^R \wedge B \cap C \equiv \emptyset)$;
(4) $V^R(x,D) := \exists A\exists B(A \in N_x^R \wedge B \in \tau_R \wedge D \subseteq B \wedge A \cap B \equiv \emptyset)$;
(5) $W^R(A,B) := \exists G\exists H(G \in \tau_R \wedge H \in \tau_R \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H \equiv \emptyset)$.

Definition 2.2. Let Ω be the class of all fuzzifying topological spaces. The unary fuzzy predicates almost-$T_i \in \Im(\Omega)$, denoted by $T_i^R, i = 0,1,2,3,4$ and almost-$R_i \in \Im(\Omega)$, denoted by $R_i^R, i = 0,1$ are defined as follows:

(1) $(X,\tau) \in T_0^R := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow K^R(x,y)$;
(2) $(X,\tau) \in T_1^R := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow H^R(x,y)$;
(3) $(X,\tau) \in T_2^R := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow M^R(x,y)$;
(4) $(X,\tau) \in T_3^R := \forall x\forall D(x \in X \wedge D \in F \wedge x \notin D) \longrightarrow V^R(x,D)$;
(5) $(X,\tau) \in T_4^R := \forall A\forall B(A \in F \wedge B \in F \wedge A \cap B = \emptyset) \longrightarrow W^R(A,B)$;
(6) $(X,\tau) \in R_0^R := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^R(x,y) \longrightarrow H^R(x,y))$;
(7) $(X,\tau) \in R_1^R := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \longrightarrow (K^R(x,y) \longrightarrow M^R(x,y))$.

Remark 2.3. In crisp setting, one can have:

(1) $\models (X,\tau) \in T_i^R \longrightarrow (X,\tau) \in T_i$, where $i = 0,1,2,3,4$ and
(2) $\models (X,\tau) \in R_i^R \longrightarrow (X,\tau) \in R_i$, where $i = 0,1$.

But these statements may not be true in general in fuzzifying topology as shown by the following example.

Example 2.4. Let $X=\{a,b,c\}$ and τ be a fuzzifying topology on X defined as $\tau(X)=\tau(\emptyset)=1$, $\tau(\{a\})=\tau(\{a,b\})=\frac{1}{8}$, $\tau(\{b\})=\frac{1}{3}$, and $\tau(\{c\})=\tau(\{a,c\})=\tau(\{b,c\})=\frac{1}{4}$. Then one can have that:

(1) $[(X,\tau)\in T_0^R]=\frac{1}{3}>\frac{1}{4}=[(X,\tau)\in T_0]$;
(2) $[(X,\tau)\in T_1^R]=\frac{1}{4}>\frac{1}{8}=[(X,\tau)\in T_1]$;
(3) $[(X,\tau)\in T_2^R]=\frac{1}{4}>\frac{1}{8}=[(X,\tau)\in T_2]$;
(4) $[(X,\tau)\in T_3^R]=\frac{22}{24}>\frac{21}{24}=[(X,\tau)\in T_3]$;
(5) $[(X,\tau)\in T_4^R]=\frac{22}{24}>\frac{21}{24}=[(X,\tau)\in T_4]$;
(6) $[(X,\tau)\in R_0^R]=\frac{22}{24}>\frac{21}{24}=[(X,\tau)\in R_0]$;
(7) $[(X,\tau)\in R_1^R]=\frac{22}{24}>\frac{21}{24}=[(X,\tau)\in R_1]$.

Lemma 2.5. (1) $\models M^R(x,y)\longrightarrow H^R(x,y)$;
(2) $\models H^R(x,y)\longrightarrow K^R(x,y)$;
(3) $\models M^R(x,y)\longrightarrow K^R(x,y)$.

Proof. (1) Since $\{B,C\in P(X):B\cap C\equiv\emptyset\}\subseteq\{B,C\in P(X):y\notin B\wedge x\notin C\}$, then

$$[M^R(x,y)]=\bigvee_{B\cap C=\emptyset}\min(N_x^R(B),N_y^R(C))\le\bigvee_{y\notin B,\,x\notin C}\min(N_x^R(B),N_y^R(C))=[H^R(x,y)].$$

(2) $[K^R(x,y)]=\max(\bigvee_{y\notin A}N_x^R(A),\bigvee_{x\notin A}N_y^R(A))\ge\bigvee_{y\notin A}N_x^R(A)\ge\bigvee_{y\notin A,\,x\notin B}(N_x^R(A)\wedge N_y^R(B))=[H^R(x,y)]$.

(3) From (1) and (2) it is obvious. □

Theorem 2.6. (1) $\models(X,\tau)\in T_1^R\longrightarrow(X,\tau)\in T_0^R$;
(2) $\models(X,\tau)\in T_2^R\longrightarrow(X,\tau)\in T_1^R$;
(3) $\models(X,\tau)\in T_2^R\longrightarrow(X,\tau)\in T_0^R$.

Proof. The proof of (1) and (2) are obtained from Lemma 2.5 (2) and (1), respectively.

(3) From (1) and (2) above, the result is obtained. □

Theorem 2.7.

$$\models(X,\tau)\in T_0^R\longleftrightarrow\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\longrightarrow(\neg(x\in Cl_R(\{y\}))\vee\neg(y\in Cl_R(\{x\})))).$$

Proof. Since for any $x,A,B,\models A\subseteq B\to(A\in N_x^R\to B\in N_x^R)$ (see [10], Theorem 4.2 (2)), then we have

$$\begin{aligned}[(X,\tau)\in T_0^R]&=\bigwedge_{x\neq y}\max(\bigvee_{y\notin A}N_x^R(A),\bigvee_{x\notin A}N_y^R(A))\\&=\bigwedge_{x\neq y}\max(N_x^R(X-\{y\}),N_y^R(X-\{x\}))\\&=\bigwedge_{x\neq y}\max(1-Cl_R(\{y\})(x),1-Cl_R(\{x\})(y))\\&=\bigwedge_{x\neq y}(\neg(Cl_R(\{y\})(x))\vee\neg(Cl_R(\{x\})(y)))\\&=[\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\longrightarrow(\neg(x\in Cl_R(\{y\}))\vee\neg(y\in Cl_R(\{x\}))))].\end{aligned}$$

□

Remark 2.8. In crisp setting, one can have $\models (X,\tau)\in T_1^R \longrightarrow \forall x(\{x\}\in F_R)$
But this statement may not be true in general in fuzzifying topology as illustrated by the following example.

Example 2.9. Let (X,τ) be the fuzzifying topological space defined in Example 2.4. Then one can have that

$$[(X,\tau)\in T_1^R]=\frac{1}{4}>\frac{1}{8}=[\forall x(\{x\}\in F_R)].$$

Theorem 2.10. For any fuzzifying topological space (X,τ),

$$\models \forall x(\{x\}\in F_R)\longrightarrow (X,\tau)\in T_1^R.$$

Proof. Since $\tau_R(A)\le\bigwedge_{x\in A}N_x^R(A)$ (see [11], Theorem 3.3), then for any x_1, x_2, $x_1\neq x_2$, we have

$$\begin{aligned}[\forall x(\{x\}\in F_R)] &= \bigwedge_{x\in X}F_R(\{x\})=\bigwedge_{x\in X}\tau_R(X-\{x\})\le\bigwedge_{x\in X}\bigwedge_{y\in X-\{x\}}N_y^R(X-\{x\})\\ &\le\bigwedge_{y\in X-\{x_2\}}N_y^R(X-\{x_2\})\le N_{x_1}^R(X-\{x_2\})=\bigvee_{x_2\notin A}N_{x_1}^R(A).\end{aligned}$$

Similarly, we have, $[\forall x(\{x\}\in F_R)]\le\bigvee_{x_1\notin B}N_{x_2}^R(B)$. Then,

$$\begin{aligned}[\forall x(\{x\}\in F_R)] &\le \bigwedge_{x_1\neq x_2}\min(\bigvee_{x_2\notin A}N_{x_1}^R(A),\bigvee_{x_1\notin B}N_{x_2}^R(B))\\ &=\bigwedge_{x_1\neq x_2}\bigvee_{x_1\notin B,\,x_2\notin A}\min(N_{x_1}^R(A),N_{x_2}^R(B))=[(X,\tau)\in T_1^R].\end{aligned}$$

□

Theorem 2.11.

$$\models (X,\tau)\in T_2^R\longleftrightarrow\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^R x)\wedge(S\rhd^R y)\longrightarrow x=y).$$

Proof. $[(X,\tau)\in T_2^R]=\bigwedge_{x\neq y}\bigvee_{A\cap B=\emptyset}(N_x^R(A)\wedge N_y^R(B))$,
$[\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^R x)\wedge(S\rhd^R y)\longrightarrow x=y)]$
$=\bigwedge_{x\neq y}\bigwedge_{S\subseteq X}(\bigvee_{S\not\supseteq A}N_x^R(A)\vee\bigvee_{S\not\supseteq B}N_y^R(B))=\bigwedge_{x\neq y}\bigwedge_{S\subseteq X}\bigvee_{S\not\supseteq A}\bigvee_{S\not\supseteq B}(N_x^R(A)\vee N_y^R(B))$.
(1) If $A\cap B=\emptyset$, then for any S, we have $S\not\supseteq A$ or $S\not\supseteq B$, and

$$N_x^R(A)\wedge N_y^R(B)\le\bigvee_{S\not\supseteq A}N_x^R(A)\text{ or }N_x^R(A)\wedge N_y^R(B)\le\bigvee_{S\not\supseteq B}N_x^R(B).$$

Consequently $\bigvee_{A\cap B=\emptyset}(N_x^R(A)\wedge N_y^R(B))\le\bigwedge_{S\subseteq X}(\bigvee_{S\not\supseteq A}N_x^R(A)\vee\bigvee_{S\not\supseteq B}N_y^R(B))$.

Also,

$$[(X,\tau)\in T_2^R]\le[\forall S\forall x\forall y((S\subseteq X)\wedge(x\in X)\wedge(y\in X)\wedge(S\rhd^R x)\wedge(S\rhd^R y)\longrightarrow x=y)].$$

(2) First, for any x, y with $x\neq y$, if $\bigvee_{A\cap B=\emptyset}(N_x^R(A)\wedge N_y^R(B))<t$, then $N_x^R(A)<t$ or $N_y^R(B)<t$ provided $A\cap B=\emptyset$, i.e., $A\cap B\neq\emptyset$ when $A\in(N_x^R)_t$ and $B\in(N_y^R)_t$. Now, set a net $S^*:(N_x^R)_t\times$

$(N_y^R)_t \longrightarrow X$, $(A,B) \longmapsto x_{(A,B)} \in A \cap B$. Then for any $A \in (N_x^R)_t$, $B \in (N_y^R)_t$, we have $S^* \subsetneqq A$ and $S^* \subsetneqq B$. Therefore, if $S^* \not\subsetneqq A$ and $S^* \not\subsetneqq B$, then $A \notin (N_x^R)_t$, $B \notin (N_y^R)_t$, i.e., $N_x^R(A) \vee N_y^R(B)) < t$. Then

$$\bigvee_{S^* \not\subsetneqq A} \bigvee_{S^* \not\subsetneqq B} (N_x^R(A) \vee N_y^R(B)) \le t. \text{ Moreover, } \bigwedge_{S \subseteq X} \bigvee_{S \not\subsetneqq A} \bigvee_{S \not\subsetneqq B} (N_x^R(A) \vee N_y^R(B)) \le t.$$

Second, for any positive integer i, there exists x_i, y_i with $x_i \ne y_i$, and

$$\bigvee_{A \cap B = \emptyset} (N_{x_i}^R(A) \wedge N_{y_i}^R(B)) < [(X,\tau) \in T_2^R] + 1/i,$$

$$\text{and hence} \qquad \bigwedge_{S \subseteq X} \bigvee_{S \not\subsetneqq A} \bigvee_{S \not\subsetneqq B} (N_{x_i}^R(A) \vee N_{y_i}^R(B)) < [(X,\tau) \in T_2^R] + 1/i.$$

So we have

$$\begin{aligned} &[\forall S \forall x \forall y ((S \subseteq X) \wedge (x \in X) \wedge (y \in X) \wedge (S \rhd^R x) \wedge (S \rhd^R y) \longrightarrow x = y)] \\ &= \bigwedge_{x \ne y} \bigwedge_{S \subseteq X} \bigvee_{S \not\subsetneqq A} \bigvee_{S \not\subsetneqq B} (N_x^R(A) \vee N_y^R(B)) \le [(X,\tau) \in T_2^R]. \end{aligned}$$

□

Definition 2.12. The fuzzifying regular-local base β_x^R of x is a function from $P(X)$ into $I = [0,1]$ satisfying the following conditions:

(1) $\models \beta_x^R \subseteq N_x^R$, and
(2) $\models A \in N_x^R \longrightarrow \exists B (B \in \beta_x^R \wedge x \in B \subseteq A)$.

Lemma 2.13.

$$\models A \in N_x^R \longleftrightarrow \exists B (B \in \beta_x^R \wedge x \in B \subseteq A).$$

Proof. From condition (1) in Definition 2.12 and Theorem 2.23 (3) of [10] we have, $N_x^R(A) \ge N_x^R(B) \ge \beta_x^R(B)$ for each $B \in P(X)$ such that $x \in B \subseteq A$. So, $N_x^R(A) \ge \bigvee_{x \in B \subseteq A} \beta_x^R(B)$. From condition (2) in Definition 2.12 we have $N_x^R(A) \le \bigvee_{x \in B \subseteq A} \beta_x^R(B)$. Hence $N_x^R(A) = \bigvee_{x \in B \subseteq A} \beta_x^R(B)$.

□

Theorem 2.14. If β_x^R is a fuzzifying regular-local basis of x, then

$$\models (X,\tau) \in T_2^R \longleftrightarrow \forall x \forall y (x \in X \wedge y \in X \wedge x \ne y \longrightarrow \exists B (B \in \beta_x^R \wedge y \in \neg(Cl_R(B)))).$$

Proof. From Lemma 2.13 we have:

$$\begin{aligned} &[\forall x \forall y (x \in X \wedge y \in X \wedge x \ne y \longrightarrow \exists B (B \in \beta_x^R \wedge y \in \neg(Cl_R(B))))] \\ &= \bigwedge_{x \ne y} \bigvee_{B \in P(X)} \min(\beta_x^R(B), \neg(1 - N_y^R(X - B))) \\ &= \bigwedge_{x \ne y} \bigvee_{B \in P(X)} \min(\beta_x^R(B), N_y^R(X - B)) \\ &= \bigwedge_{x \ne y} \bigvee_{B \in P(X)} \bigvee_{y \in C \subseteq X - B} \min(\beta_x^R(B), \beta_y^R(C)) \\ &= \bigwedge_{x \ne y} \bigvee_{B \cap C = \emptyset} \bigvee_{x \in D \subseteq B,\, y \in E \subseteq C} \min(\beta_x^R(D), \beta_y^R(E)) \\ &= \bigwedge_{x \ne y} \bigvee_{B \cap C = \emptyset} \min(\bigvee_{x \in D \subseteq B} \beta_x^R(D), \bigvee_{y \in E \subseteq C} \beta_y^R(E)) \\ &= \bigwedge_{x \ne y} \bigvee_{B \cap C = \emptyset} \min(N_x^R(B), N_y^R(C)) = [(X,\tau) \in T_2^R]. \end{aligned}$$

□

Theorem 2.15. $\models (X,\tau)\in R_1^R \longrightarrow (X,\tau)\in R_0^R$.

Proof. From Lemma 2.5 (1), the proof is immediate.

□

Theorem 2.16. (1) $\models (X,\tau)\in T_1^R \longrightarrow (X,\tau)\in R_0^R$;
(2) $\models (X,\tau)\in T_1^R \longrightarrow (X,\tau)\in R_0^R \wedge (X,\tau)\in T_0^R$;
(3) If $T_0^R(X,\tau)=1$, then $\models (X,\tau)\in T_1^R \longleftrightarrow (X,\tau)\in R_0^R \wedge (X,\tau)\in T_0^R$.

Proof. (1) $T_1^R(X,\tau)=\bigwedge_{x\neq y}[H^R(x,y)] \le \bigwedge_{x\neq y}[K^R(x,y)\longrightarrow H^R(x,y)] = R_0^R(X,\tau)$.

(2) It is obtained from (1) and from Theorem 2.15 (1).

(3) Since $T_0^R(X,\tau)=1$, for every $x,y\in X$ such that $x\neq y$, then we have $[K^R(x,y)]=1$. Now, $[(X,\tau)\in R_0^R \wedge (X,\tau)\in T_0^R] = [(X,\tau)\in R_0^R] = \bigwedge_{x\neq y}\min(1, 1-[K^R(x,y)]+[H^R(x,y)])$
$=\bigwedge_{x\neq y}[H^R(x,y)] = T_1^R(X,\tau)$.

□

Theorem 2.17. (1) $\models (X,\tau)\in R_0^R \otimes (X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_1^R$, and
(2) If $T_0^R(X,\tau)=1$, then $\models (X,\tau)\in R_0^R \otimes (X,\tau)\in T_0^R \longleftrightarrow (X,\tau)\in T_1^R$.

Proof.

$$
\begin{aligned}
(1)\quad & [(X,\tau)\in R_0^R \otimes (X,\tau)\in T_0^R] \\
& = \max(0, R_0^R(X,\tau)+T_0^R(X,\tau)-1) \\
& = \max(0, \bigwedge_{x\neq y}\min(1,1-[K^R(x,y)]+[H^R(x,y)]) + \bigwedge_{x\neq y}[K^R(x,y)]-1) \\
& \le \max(0, \bigwedge_{x\neq y}\min(1,1-[K^R(x,y)]+[H^R(x,y)]) + [K^R(x,y)]-1) \\
& = \bigwedge_{x\neq y}[H^R(x,y)] = T_1^R(X,\tau).
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad [(X,\tau)\in R_0^R \otimes (X,\tau)\in T_0^R] & = [(X,\tau)\in R_0^R] \\
& = \bigwedge_{x\neq y}\min(1,1-[K^R(x,y)]+[H^R(x,y)]) \\
& = \bigwedge_{x\neq y}[H^R(x,y)] = T_1^R(X,\tau),
\end{aligned}
$$

because, $T_0^R(X,\tau)=1$, we have for each x,y such that $x\neq y$ $[K^R(x,y)]=1$.

□

Theorem 2.18. (1) $\models (X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_0^R \longrightarrow (X,\tau)\in T_1^R)$, and
(2) $\models (X,\tau)\in R_0^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_1^R)$.

Proof. From Theorem 2.16 (1) and Theorem 2.17 (1) we have

$$
\begin{aligned}
(1)\quad & [(X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_0^R \longrightarrow (X,\tau)\in T_1^R)] \\
& = \min(1, 1-[(X,\tau)\in T_0^R] + \min(1, 1-[(X,\tau)\in R_0^R] + [(X,\tau)\in T_1^R])) \\
& = \min(1, 1-[(X,\tau)\in T_0^R] + 1 - [(X,\tau)\in R_0^R] + [(X,\tau)\in T_1^R]) \\
& = \min(1, 1-([(X,\tau)\in T_0^R] + [(X,\tau)\in R_0^R] - 1) + [(X,\tau)\in T_1^R]) = 1.
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & [(X,\tau)\in R_0^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_1^R)] \\
& = \min(1, 1-([(X,\tau)\in T_0^R] + [(X,\tau)\in R_0^R] - 1) + [(X,\tau)\in T_1^R]) = 1.
\end{aligned}
$$

□

Theorem 2.19. (1) $\models (X,\tau)\in T_2^R \longrightarrow (X,\tau)\in R_1^R$;
(2) $\models (X,\tau)\in T_2^R \longrightarrow (X,\tau)\in R_1^R \wedge (X,\tau)\in T_0^R$;
(3) If $T_0^R(X,\tau)=1$, then $\models (X,\tau)\in T_2^R \longleftrightarrow (X,\tau)\in R_1^R \wedge (X,\tau)\in T_0^R$.

Proof. (1) The proof is similar to that of Theorem 2.16 (1).

(2) It is obtained from (1) above and Theorem 2.6 (3).

(3) The proof is similar to that of Theorem 2.16 (3).

□

Theorem 2.20. (1) $\models (X,\tau)\in R_1^R \otimes (X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_2^R$, and
(2) If $T_0^R(X,\tau)=1$, then $\models (X,\tau)\in R_1^R \otimes (X,\tau)\in T_0^R \longleftrightarrow (X,\tau)\in T_2^R$.

Proof. The proof is similar to that of Theorem 2.17 (1) and (2), respectively.

□

Theorem 2.21. (1) $\models (X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_1^R \longrightarrow (X,\tau)\in T_2^R)$, and
(2) $\models (X,\tau)\in R_1^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_2^R)$.

Proof. The proof is similar to that of Theorem 2.18 (1) and (2), respectively.

□

Theorem 2.22. If $T_0^R(X,\tau)=1$, then:
(1) $\models ((X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_0^R \longrightarrow (X,\tau)\in T_1^R)) \wedge ((X,\tau)\in T_1^R \longrightarrow \neg((X,\tau)\in T_0^R \longrightarrow \neg((X,\tau)\in R_0^R)))$;
(2) $\models ((X,\tau)\in R_0^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_1^R)) \wedge ((X,\tau)\in T_1^R \longrightarrow \neg((X,\tau)\in T_0^R \longrightarrow \neg((X,\tau)\in R_0^R)))$;
(3) $\models ((X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_0^R \longrightarrow (X,\tau)\in T_1^R)) \wedge ((X,\tau)\in T_1^R \longrightarrow \neg((X,\tau)\in R_0^R \longrightarrow \neg((X,\tau)\in T_0^R)))$;
(4) $\models ((X,\tau)\in R_0^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_1^R)) \wedge ((X,\tau)\in T_1^R \longrightarrow \neg((X,\tau)\in R_0^R \longrightarrow \neg((X,\tau)\in T_0^R)))$.

Proof. For simplicity we put, $T_0^R(X,\tau)=\alpha$, $R_0^R(X,\tau)=\beta$ and $T_1^R(X,\tau)=\gamma$. Now, applying Theorem 2.17 (2), the proof is obtained with some relations in fuzzy logic as follows:

(1)

$$\begin{aligned}1 &= (\alpha\otimes\beta \longleftrightarrow \gamma) = (\alpha\otimes\beta \longrightarrow \gamma)\wedge(\gamma \longrightarrow \alpha\otimes\beta)\\ &= \neg((\alpha\otimes\beta)\otimes\neg\gamma)\wedge\neg(\gamma\otimes\neg(\alpha\otimes\beta))\\ &= \neg(\alpha\otimes\neg(\neg(\beta\otimes\neg\gamma)))\wedge\neg(\gamma\otimes(\alpha\longrightarrow\neg\beta))\\ &= (\alpha\longrightarrow\neg(\beta\otimes\neg\gamma))\wedge(\gamma\longrightarrow\neg(\alpha\longrightarrow\neg\beta))\\ &= (\alpha\longrightarrow(\beta\longrightarrow\gamma)\wedge(\gamma\longrightarrow\neg(\alpha\longrightarrow\neg\beta))),\end{aligned}$$

since $\otimes$ is commutative one can have the proof of statements (2) - (4) in a similar way as (1).

By a similar procedure to Theorem 2.22 one can have the following theorem.

□

Theorem 2.23. If $T_0^R(X,\tau)=1$, then:
(1) $\models ((X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_1^R \longrightarrow (X,\tau)\in T_2^R)) \wedge ((X,\tau)\in T_2^R \longrightarrow \neg((X,\tau)\in T_0^R \longrightarrow \neg((X,\tau)\in R_1^R)))$;
(2) $\models ((X,\tau)\in R_1^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_2^R)) \wedge ((X,\tau)\in T_2^R \longrightarrow \neg((X,\tau)\in T_0^R \longrightarrow \neg((X,\tau)\in R_1^R)))$;

(3) $\models ((X,\tau)\in T_0^R \longrightarrow ((X,\tau)\in R_1^R \longrightarrow (X,\tau)\in T_2^R)) \wedge ((X,\tau)\in T_2^R \longrightarrow \neg((X,\tau)\in R_1^R \longrightarrow \neg((X,\tau)\in T_0^R)))$;

(4) $\models ((X,\tau)\in R_1^R \longrightarrow ((X,\tau)\in T_0^R \longrightarrow (X,\tau)\in T_2^R)) \wedge ((X,\tau)\in T_2^R \longrightarrow \neg((X,\tau)\in R_1^R \longrightarrow \neg((X,\tau)\in T_0^R)))$.

Lemma 2.24. (1) If $D\subseteq B$, then $\bigvee_{A\cap B=\emptyset} N_x^R(A) = \bigvee_{A\cap B=\emptyset,\, D\subseteq B} N_x^R(A)$,

(2) $\bigvee_{A\cap B=\emptyset} \bigwedge_{y\in D} N_y^R(X-A) = \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \tau_R(B)$.

Proof. (1) Since $D\subseteq B$, then:

$$\bigvee_{A\cap B=\emptyset} N_x^R(A) = \bigvee_{A\cap B=\emptyset} N_x^R(A)\wedge[D\subseteq B] = \bigvee_{A\cap B=\emptyset,\, D\subseteq B} N_x^R(A).$$

(2) Let $y\in D$ and $A\cap B=\emptyset$. Then,

$$\begin{aligned}\bigvee_{A\cap B=\emptyset,\, D\subseteq B}\tau_R(B) &= \bigvee_{A\cap B=\emptyset,\, D\subseteq B}\tau_R(B)\wedge[y\in D]\\ &= \bigvee_{y\in D\subseteq B\subseteq X-A}\tau_R(B) = \bigvee_{y\in B\subseteq X-A}\tau_R(B)\\ &= N_y^R(X-A) = \bigwedge_{y\in D} N_y^R(X-A)\\ &= \bigvee_{A\cap B=\emptyset}\ \bigwedge_{y\in D} N_y^R(X-A).\end{aligned}$$

□

Definition 2.25.

$$RT_3^{(1)}(X,\tau) := \forall x\forall D(x\in X\wedge D\in F\wedge x\notin D\longrightarrow \exists A(A\in N_x^R\wedge(D\subseteq X-Cl_R(A)))).$$

Theorem 2.26. $\models (X,\tau)\in T_3^R \longleftrightarrow (X,\tau)\in RT_3^{(1)}$.

Proof. Now, we have

$$\begin{aligned}RT_3^{(1)}(X,\tau) &= \bigwedge_{x\notin D}\min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)}\min(N_x^R(A), \bigwedge_{y\in D}(1-Cl_R(A)(y))))\\ &= \bigwedge_{x\notin D}\min(1, 1-\tau(X-D) + \bigvee_{A\in P(X)}\min(N_x^R(A), \bigwedge_{y\in D} N_y^R(X-A)))\end{aligned}$$

and $\quad T_3^R(X,\tau) = \bigwedge_{x\notin D}\min(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B}\min(N_x^R(A), \tau_R(B)))$.

So, the result holds if we prove that

$$\bigvee_{A\in P(X)}\min(N_x^R(A), \bigwedge_{y\in D} N_y^R(X-A)) = \bigvee_{A\cap B=\emptyset,\, D\subseteq B}\min(N_x^R(A), \tau_R(B)) \qquad (*)$$

It is clear that, on the left-hand side of $(*)$ in the case of $A\cap D\neq\emptyset$ there exists $y\in X$ such that $y\in D$ and $y\notin X-A$. So, $\bigwedge_{y\in D} N_y^R(X-A)=0$ and thus $(*)$ becomes

$$\bigvee_{A\in P(X),\, A\cap B=\emptyset}\min(N_x^R(A), \bigwedge_{y\in D} N_y^R(X-A)) = \bigvee_{A\cap B=\emptyset,\, D\subseteq B}\min(N_x^R(A), \tau_R(B)),$$

which is obtained from Lemma 2.24.

□

Definition 2.27.

$$RT_3^{(2)}(X,\tau) := \forall x \forall B (x \in B \wedge B \in \tau \longrightarrow \exists A (A \in N_x^R \wedge Cl_R(A) \subseteq B)).$$

Theorem 2.28. $\models (X,\tau) \in T_3^R \longleftrightarrow (X,\tau) \in RT_3^{(2)}$.

Proof. From Theorem 2.26, we have

$$T_3^R(X,\tau) = \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^R(A), \bigwedge_{y \in D} N_y^R(X - A))).$$

Now,

$$\begin{aligned} RT_3^{(2)}(X,\tau) &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^R(A), \bigwedge_{y \in X - B} (1 - Cl_R(A)(y)))) \\ &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^R(A), \bigwedge_{y \in X - B} (1 - (1 - N_y^R(X - A))))) \\ &= \bigwedge_{x \in B} \min(1, 1 - \tau(B) + \bigvee_{A \in P(X)} \min(N_x^R(A), \bigwedge_{y \in X - B} N_y^R(X - A))). \end{aligned}$$

Now, put $B = X - D$, we have

$$\begin{aligned} RT_3^{(2)}(X,\tau) &= \bigwedge_{x \notin D} \min(1, 1 - \tau(X - D) + \bigvee_{A \in P(X)} \min(N_x^R(A), \bigwedge_{y \in D} N_y^R(X - A))) \\ &= T_3^R(X,\tau). \end{aligned}$$

□

Definition 2.29. Let φ be a subbase of τ, then:

$$RT_3^{(3)}(X,\tau) := \forall x \forall D (x \in D \wedge D \in \varphi \longrightarrow \exists B (B \in N_x^R \wedge Cl_R(B) \subseteq D)).$$

Theorem 2.30. $\models (X,\tau) \in T_3^R \longleftrightarrow (X,\tau) \in RT_3^{(3)}$.

Proof. Since $[\varphi \subseteq \tau] = 1$, and with regard to Theorems 2.26 and 2.28 we have

$$RT_3^{(3)}(X,\tau) \geq RT_3^{(2)}(X,\tau) = T_3^R(X,\tau).$$

So, it suffices to prove that $RT_3^{(3)}(X,\tau) \leq RT_3^{(2)}(X,\tau)$ and this is obtained if we prove for any $x \in A$,

$$\min(1, 1 - \tau(A) + \bigvee_{B \in P(X)} \min(N_x^R(B), \bigwedge_{y \in X - A} N_y^R(X - B))) \geq RT_3^{(3)}(X,\tau).$$

Set $RT_3^{(3)}(X,\tau) = \delta$. then, for any $x \in X$ and any $D_{\lambda_i} \in P(X), \lambda_i \in I_\lambda$ (I_λ denotes a finite index set), $\lambda \in \Lambda$, $\bigcup_{\lambda \in \Lambda} \bigcap_{\lambda_i \in I_\lambda} D_{\lambda_i} = A$ we have,

$$1 - \varphi(D_{\lambda_i}) + \bigvee_{B \in P(X)} \min(N_x^R(B), \bigwedge_{y \in X - D_{\lambda_i}} N_y^R(X - B)) \geq \delta > \delta - \epsilon,$$

where ϵ is any positive number. Thus,

$$\bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^R(X-B)) > \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.$$

Set $\gamma_{\lambda_i} = \{B : B \subseteq D_{\lambda_i}\}$. Then, from the completely distributive law we have

$$\begin{aligned}
&\bigwedge_{\lambda_i\in I_\lambda} \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^R(X-B))\\
&= \bigvee_{f\in\Pi\{\gamma_{\lambda_i}:\lambda_i\in I_\lambda\}} \bigwedge_{\lambda_i\in I_\lambda} \min(N_x^R(f(\lambda_i)), \bigwedge_{y\in X-D_{\lambda_i}} N_y^R(X-f(\lambda_i)))\\
&= \bigvee_{f\in\Pi\{\gamma_{\lambda_i}:\lambda_i\in I_\lambda\}} \min(\bigwedge_{\lambda_i\in I_\lambda} N_x^R(f(\lambda_i)), \bigwedge_{\lambda_i\in I_\lambda} \bigwedge_{y\in X-D_{\lambda_i}} N_y^R(X-f(\lambda_i)))\\
&= \bigvee_{f\in\Pi\{\gamma_{\lambda_i}:\lambda_i\in I_\lambda\}} \min(\bigwedge_{\lambda_i\in I_\lambda} N_x^R(f(\lambda_i)), \bigwedge_{y\in \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-f(\lambda_i)))\\
&= \bigvee_{B\in P(X)} \min(\bigwedge_{\lambda_i\in I_\lambda} N_x^R(B), \bigwedge_{y\in \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-B))\\
&= \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-B)),
\end{aligned}$$

where $B = f(\lambda_i)$.
Similarly, we can prove

$$\begin{aligned}
&\bigwedge_{\lambda\in\Lambda} \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-B))\\
&= \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in \bigcup_{\lambda\in\Lambda} \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-B))\\
&\leq \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in \bigcap_{\lambda\in\Lambda} \bigcup_{\lambda_i\in I_\lambda} X-D_{\lambda_i}} N_y^R(X-B))\\
&\leq \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-A} N_y^R(X-B)),
\end{aligned}$$

so we have

$$\begin{aligned}
&\bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-A} N_y^R(X-B))\\
&\geq \bigwedge_{\lambda\in\Lambda} \bigwedge_{\lambda_i\in I_\lambda} \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-D_{\lambda_i}} N_y^R(X-B))\\
&\geq \bigwedge_{\lambda\in\Lambda} \bigwedge_{\lambda_i\in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon.
\end{aligned}$$

For any I_λ and Λ that satisfy $\bigcup_{\lambda\in\Lambda} \bigcap_{\lambda_i\in I_\lambda} D_{\lambda_i} = A$ the above inequality is true. So,

$$\begin{aligned}
&\bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-A} N_y^R(X-B))\\
&\geq \bigvee_{\cup_{\lambda\in\Lambda} D_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{\cap_{\lambda_i\in I_\lambda} D_{\lambda_i} = D_\lambda} \bigwedge_{\lambda_i\in I_\lambda} \varphi(D_{\lambda_i}) - 1 + \delta - \epsilon\\
&= \tau(A) - 1 + \delta - \epsilon.
\end{aligned}$$

$$\text{i.e.,}\ \min(1, 1-\tau(A) + \bigvee_{B\in P(X)} \min(N_x^R(B), \bigwedge_{y\in X-A} N_y^R(X-B))) \geq \delta - \epsilon.$$

Because ϵ is any arbitrary positive number, when $\epsilon \longrightarrow 0$ we have $RT_3^{(2)}(X,\tau) \geq \delta = RT_3^{(3)}(X,\tau)$. So, $\models (X,\tau) \in T_3^R \longleftrightarrow (X,\tau) \in RT_3^{(3)}$.

□

Definition 2.31. Let (X,τ) be any fuzzifying topological space.

(1) $RT_4^{(1)}(X,\tau) := \forall A \forall B (A \in \tau \wedge B \in F \wedge A \cap B \equiv \emptyset \longrightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge B \subseteq X - Cl_R(G)))$, and

(2) $RT_4^{(2)}(X,\tau) := \forall A \forall B (A \in F \wedge B \in \tau \wedge A \subseteq B \longrightarrow \exists G (G \in \tau \wedge A \subseteq G \wedge Cl_R(G) \subseteq B))$.

Theorem 2.32. $\models (X,\tau) \in T_4^R \longleftrightarrow (X,\tau) \in RT_4^{(i)}$, where $i = 1, 2$.

Proof. The proof is similar to that of Theorems 2.26 and 2.28.

□

3. Relation among fuzzifying separation axioms

Lemma 3.1. For any $\alpha, \beta \in I$, we have $(1 \wedge (1 - \alpha + \beta)) + \alpha \leq 1 + \beta$.

Theorem 3.2. $\models (X,\tau) \in T_3^R \otimes (X,\tau) \in T_1 \longrightarrow (X,\tau) \in T_2^R$.

Proof. From Theorem 2.2 [8] we have, $T_1(X,\tau) = \bigwedge_{y\in X} \tau(X - \{y\})$ and applying Lemma 3.1 we have,

$$\begin{aligned}
&T_3^R(X,\tau) + T_1(X,\tau)\\
&= \bigwedge_{x\notin D} \min\left(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\ D\subseteq B} \min(N_x^R(A), \tau_R(B))\right) + \bigwedge_{y\in X} \tau(X-\{y\})\\
&\leq \bigwedge_{x\in X,\ x\neq y} \bigwedge_{y\in X} \min\left(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^R(A), N_y^R(B))\right) + \bigwedge_{y\in X} \tau(X-\{y\})\\
&= \bigwedge_{x\in X,\ x\neq y} \left(\bigwedge_{y\in X} \min(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^R(A), N_y^R(B))) + \bigwedge_{y\in X} \tau(X-\{y\})\right)\\
&\leq \bigwedge_{x\in X,\ x\neq y} \bigwedge_{y\in X} \left(\min(1, 1-\tau(X-\{y\}) + \bigvee_{A\cap B=\emptyset} \min(N_x^R(A), N_y^R(B))) + \tau(X-\{y\})\right)\\
&\leq \bigwedge_{x\neq y} \left(1 + \bigvee_{A\cap B=\emptyset} \min(N_x^R(A), N_y^R(B))\right) = 1 + \bigwedge_{x\neq y} \bigvee_{A\cap B=\emptyset} \min(N_x^R(A), N_y^R(B)) = 1 + T_2^R(X,\tau),
\end{aligned}$$

namely, $T_2^R(X,\tau) \geq T_3^R(X,\tau) + T_1(X,\tau) - 1$. Thus, $T_2^R(X,\tau) \geq \max(0, T_3^R(X,\tau) + T_1(X,\tau) - 1)$.

□

Theorem 3.3. $\models (X,\tau) \in T_4^R \otimes (X,\tau) \in T_1 \longrightarrow (X,\tau) \in T_3^R$.

Proof. It is equivalent to prove that $T_3^R(X,\tau) \geq T_4^R(X,\tau) + T_1(X,\tau) - 1$. In fact,

$$\begin{aligned}
T_4^R(X,\tau) + T_1(X,\tau) &= \bigwedge_{E\cap D=\emptyset} \min\Bigg(1, 1 - \min(\tau(X-E), \tau(X-D))\\
&\qquad + \bigvee_{A\cap B=\emptyset,\ E\subseteq A,\ D\subseteq B} \min(\tau_R(A), \tau_R(B))\Bigg) + \bigwedge_{z\in X} \tau(X-\{z\})\\
&\leq \bigwedge_{x\notin D} \min\Bigg(1, 1 - \min(\tau(X-\{x\}), \tau(X-D))
\end{aligned}$$

$$+ \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + \bigwedge_{z\in X} \tau(X-\{z\})$$

$$
\begin{aligned}
&= \bigwedge_{x\notin D} \min\Big(1, \max\Big(1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B)), 1-\tau(X-\{x\}) \\
&\qquad + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
&= \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big), \min\Big(1, 1-\tau(X-\{x\}) \\
&\qquad + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big)\Big) + \bigwedge_{z\in X} \tau(X-\{z\}) \\
&\le \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + \tau(X-\{x\}), \\
&\qquad \min\Big(1, 1-\tau(X-\{x\}) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + \tau(X-\{x\})\Big) \\
&\le \bigwedge_{x\notin D} \max\Big(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + \tau(X-\{x\}), 1\Big) \\
&\le \bigwedge_{x\notin D} \Big(\min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + 1\Big) \\
&= \bigwedge_{x\notin D} \min\Big(1, 1-\tau(X-D) + \bigvee_{A\cap B=\emptyset,\, D\subseteq B} \min(N_x^R(A), \tau_R(B))\Big) + 1 \\
&= T_3^R(X,\tau) + 1.
\end{aligned}
$$

□

4. Conclusion

This paper considers fuzzifying topologies, a special case of I-fuzzy topologies (bifuzzy topologies) introduced by Ying [6]. It extend some fundamental results in general topology to fuzzifying topology.

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