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On special operations on Ω-closeness on topological spaces

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In this paper, the concepts of Ω^*-closed and Ω^*-continuous maps are introduced and several properties of them are investigated. These concepts are used to obtain several results concerning the preservation of Ω-closed sets. Moreover, we use Ω^*-closed and Ω^*-continuous maps to obtain a characterization of semi-$T_{\frac{1}{2}}$ spaces.

Keywords: Ω-closed set; Ω-open set; Ω-continuity.

AMS Subject Classification: 54A05, 54C08

1. Introduction

Some types of sets play an important role in the study of properties in topological spaces. Many authors introduced and studied various generalized properties and conditions containing some forms of sets in topological spaces. In 2005, Noiri and Sayed [14] introduced the class of Ω-closed sets. By the mean of these sets they introduced and studied Ω-continuous and Ω-irresolute maps. In 2008, Sayed [15] introduced Ω-open sets and studied some applications on them. In this paper, we obtain some new decompositions of Ω-continuity. Also, new forms of continuity (which we call Ω^*-closed and Ω^*-continuous) are introduced and several properties of them are investigated. We use these concepts to obtain some results concerning the preservation of Ω-closed sets. Furthermore, we characterize semi-$T_{\frac{1}{2}}$ spaces in terms of Ω-closed sets.

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2. Preliminaries

Throughout this paper, (X, τ), (Y, σ) and (Z, υ) (or simply, X, Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. The family of all closed subsets of X (respectively, Y) is denoted by F_X (respectively, F_Y). All sets are assumed to be subsets of topological spaces. The closure and the interior of a set A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a space X is said to be regular open (respectively, regular closed) [17] if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (respectively, $A = \mathrm{Cl}(\mathrm{Int}(A))$). A subset A of a space X is said to be semiopen [10] (respectively, preopen [13]) if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ (respectively, $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$). The complement of a semiopen set is said to be semiclosed [4]. The family of all semiopen (respectively, semiclosed) sets in a space X is denoted by $\mathrm{SO}(X)$ (respectively, $\mathrm{SC}(X)$). The semi-interior of A [6], denoted by $\mathrm{sInt}(A)$, is the union of all semiopen subsets of A while the semiclosure of A [4, 6], denoted by $\mathrm{sCl}(A)$, is the intersection of all semiclosed supersets of A. It is well known that $\mathrm{sInt}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(A))$ [1] and $\mathrm{sCl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(A))$ [9]. A subset A of a space X is said to be sg-closed [3] (respectively, Ω-closed [14]) if $\mathrm{sCl}(A) \subseteq U$ (respectively, $\mathrm{sCl}(A) \subseteq \mathrm{Int}(U)$) whenever $A \subseteq U$ and U is semiopen. The complement of an Ω-closed set is said to be Ω-open [15] or equivalently, a subset A of a space X is said to be Ω-open [15, Proposition 2.3(1)] if $\mathrm{Cl}(F) \subseteq \mathrm{sInt}(A)$ whenever $F \subseteq A$ and F is semiclosed. The family of all Ω-open (respectively, Ω-closed, sg-closed) sets in X will be denoted by $\Omega O(X)$ (respectively, $\Omega C(X)$, $\mathrm{SGC}(X)$). A map $f : X \to Y$ is said to be RC-continuous [2] (respectively, contra-semicontinuous [8]) if $f^{-1}(V)$ is regular closed (respectively, semiclosed) subset of X for every open subset V of Y. A map $f : X \to Y$ is said to be Ω-continuous [14] (respectively, sg-continuous [18]) if $f^{-1}(V)$ is Ω-closed (respectively, sg-closed) subset of X for every closed subset V of Y. A map $f : X \to Y$ is said to be irresolute [7] (respectively, Ω-irresolute [14]) if $f^{-1}(V)$ is semiopen (respectively, Ω-closed) subset of X for every semiopen (respectively, Ω-closed) subset V of Y. A map $f : X \to Y$ is said to be presemiopen [16] (respectively, pre-semiclosed [16], pre-Ω-closed [15]) if $f(F)$ is semiopen (respectively, semiclosed and Ω-closed) subset of Y whenever F is semiopen (respectively, semiclosed and Ω-closed) subset of X. A space X is said to be semi-$T_{\frac{1}{2}}$ [5] if and only if every sg-closed set is semiclosed.

3. Ω-Closed Sets and Ω-Continuity

Theorem 3.1. *An Ω-closed set is* sg-*closed.*

Proof. Let A be an Ω-closed set and $A \subseteq U$, where U is a semiopen set in X. Then $\mathrm{sCl}(A) \subseteq \mathrm{Int}(U) \subseteq U$. Hence A is sg-closed. □

Remark 3.1. (1) *sg*-closed need not be Ω-closed;
(2) Semiclosed and Ω-closed are independent;

(3) Preopen and Ω-closed are independent;
(4) Open and Ω-closed are independent.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The subset $\{b, c\}$ is semi-closed and hence sg-closed but it is not Ω-closed.

Example 3.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{c, d\}\}$. The subset $\{a, b, d\}$ of X is Ω-closed but it is not semiclosed.

Example 3.3. Let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$. The subset $\{b\}$ of X is Ω-closed but it is not open and hence not preopen.

Example 3.4. $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. The subset $\{a, b\}$ is open and hence pre-open but it is not Ω-closed.

Theorem 3.2. *A subset A of a space X is regular open if and only if A is semiopen and Ω-closed.*

Proof. Let A be a regular open subset of X. Then A is semiopen. To prove that A is Ω-closed, let $A \subseteq G$, where G is a semiopen subset of X. Then $\mathrm{sCl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(A)) = A = \mathrm{Int}(A) \subseteq \mathrm{Int}(G)$. Therefore A is Ω-closed.

Conversely, let A be a semiopen and an Ω-closed set. Since $A \subseteq A$, then $\mathrm{sCl}(A) \subseteq \mathrm{Int}(A)$. But $\mathrm{sCl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(A))$. Then $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Int}(A)$ and $A \subseteq \mathrm{Int}(A)$. Hence $\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(A) = A$. Therefore A is regular open. □

Corollary 3.1. *A subset A of a space X is regular closed if and only if A is semiclosed and Ω-open.*

Corollary 3.2. *A subset A of a space X is clopen if and only if A is semiopen, semiclosed, Ω-open and Ω-closed.*

From the above discussion we have Fig. 1.

Theorem 3.3. *If the map $f : X \to Y$ is Ω-continuous, then it is sg-continuous.*

Proof. From Theorem 3.1, the proof is straightforward. □

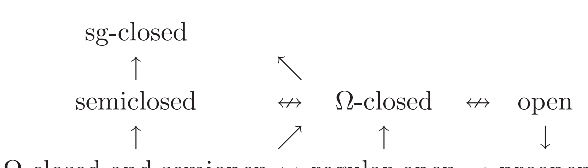

Fig. 1. (Where → means "tends to and not vice versa", ↔ means "are equivalent" and ↮ means "are independent").

The converse of the above theorems is not true as shown by the following example.

Example 3.5. $X = Y = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. The identity map $f : X \to Y$ is sg-continuous but not Ω-continuous.

Theorem 3.4. *A map $f : X \to Y$ is RC-continuous if and only if f is both Ω-continuous and contra-semicontinuous.*

Proof. From Theorem 3.2 the proof is straightforward. □

Proposition 3.1. *In a semi-$T_{\frac{1}{2}}$ space X, the classes $\Omega C(X)$ and $\mathrm{SC}(X)$ are a partition of $\mathrm{SGC}(X)$.*

Proposition 3.2. *Let X be a topological space. If $\tau = F_X$, then*

(1) $\mathrm{SO}(X) = \mathrm{SC}(X) = \tau$.
(2) $\Omega O(X) = \Omega C(X) = P(X)$.

4. Ω^*-Closed and Ω^*-Continuous Maps

In this section, we introduce a new type of maps called Ω^*-closed and Ω^*-continuous maps and obtain some of their properties and characterizations.

Definition 4.1. A map $f : X \to Y$ is said to be Ω^*-closed if $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$ whenever H is a semiclosed subset of X, O is an Ω-open subset of Y and $f(H) \subseteq O$.

Definition 4.2. A map $f : X \to Y$ is said to be Ω^*-continuous if $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(H))$ whenever H is a semiopen subset of Y, O is an Ω-closed subset of X and $O \subseteq f^{-1}(H)$.

Remark 4.1. The constant map $f : X \to Y$ defined by $f(x) = y$ for every $x \in X$ and $y \notin D[V]$ (the derived set of V), where V is a semiclosed subset of Y, is Ω^*-continuous.

The following example shows that Ω^*-continuous is not continuous, not Ω^*-closed, and not Ω-irresolute.

Example 4.1. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X, $Y = \{x, y, z\}$ and $\sigma = \{Y, \phi, \{x\}, \{y, z\}\}$ be a topology on Y. One can obtain that: $F_X = \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}, \mathrm{SO}(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\mathrm{SC}(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\Omega C(X) = \{X, \phi, \{a\}, \{b\}\}$, and $\Omega O(X) = \{X, \phi, \{a, c\}, \{b, c\}\}$. Furthermore, $F_Y = \mathrm{SO}(Y) = \mathrm{SC}(Y) = \sigma$ and $\Omega C(Y) = \Omega O(Y) = P(Y)$. Define the map $f : X \to Y$ by $f(a) = x, f(b) = y$ and $f(c) = z$. We have that f is Ω^*-continuous but not continuous, not Ω^*-closed, not Ω-irresolute, and not Ω-continuous.

The following example shows that Ω-continuous does not imply Ω^*-continuous.

Example 4.2. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ be a topology on X, $Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$ be a topology on Y. One can obtain that: $F_X = \mathrm{SO}(X) = \mathrm{SC}(X) = \tau$, and $\Omega C(X) = \Omega O(X) = P(X)$. Furthermore, $F_Y = \{Y, \phi, \{q\}\} = \mathrm{SC}(Y) = \Omega C(Y)$ and $\mathrm{SO}(Y) = \Omega O(Y) = \sigma$. Define the map $f : X \to Y$ by $f(a) = f(c) = p$ and $f(b) = q$. We have that f is Ω-continuous. Now, we prove that f is not Ω^*-continuous. Since $\{p\} \in \mathrm{SO}(Y)$ and $\{a, c\} \subseteq f^{-1}(\{p\})$, where $\{a, c\} \in \Omega C(X)$. But $\mathrm{sCl}(\{a, c\}) = X$ and $f^{-1}(\mathrm{Int}\{p\}) = f^{-1}(\{p\}) = \{a, c\}$. Hence $\mathrm{sCl}(\{a, c\}) \subsetneq f^{-1}(\mathrm{Int}\{p\})$. Therefore f is not Ω^*-continuous.

From the above two examples we note that:

(1) Ω-continuity and Ω^*-continuity are independent.
(2) Continuity and Ω^*-continuity are independent.

Theorem 4.1. *Let $f : X \to Y$ be a bijection. Then f is Ω^*-closed if and only if $\mathrm{sCl}(O) \subseteq f(\mathrm{Int}(H))$ whenever H is a semiopen subset of X, O is an Ω-closed subset of Y and $O \subseteq f(H)$.*

Proof. Necessity: Let $O \subseteq f(H)$, where H is a semiopen subset of X and O is an Ω-closed subset of Y. Since f is a bijection, then $f(X - H) = f(X) - f(H) = Y - f(H) \subseteq Y - O$. Again since f is Ω^*-closed, $f(\mathrm{Cl}(X - H)) \subseteq \mathrm{sInt}(Y - O)$. Hence $f(X - \mathrm{Int}(H)) \subseteq Y - \mathrm{sCl}(O)$ or $Y - f(\mathrm{Int}(H)) \subseteq Y - \mathrm{sCl}(O)$. Therefore $\mathrm{sCl}(O) \subseteq f(\mathrm{Int}(H))$.

Sufficiency: Let $f(H) \subseteq O$, where H is a semiclosed subset of X and O is an Ω-open subset of Y. Then $Y - O \subseteq Y - f(H) = f(X - H)$. Hence $\mathrm{sCl}(Y - O) \subseteq f(\mathrm{Int}(X-H))$ which implies $Y - \mathrm{sInt}(O) \subseteq f(X - \mathrm{Cl}(H)) = Y - f(\mathrm{Cl}(H))$. Therefore $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$ and f is Ω^*-closed map. □

Theorem 4.2. *Let $f : X \to Y$ be a map. Then f is Ω^*-continuous if and only if $f^{-1}(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$ whenever $f^{-1}(H) \subseteq O$, where H is a semiclosed subset of Y and O is an Ω-open subset of X.*

Proof. Necessity: Suppose that $f^{-1}(H) \subseteq O$, where H is a semiclosed subset of Y and O is an Ω-open subset of X. Then $X - O \subseteq X - f^{-1}(H) = f^{-1}(Y) - f^{-1}(H) = f^{-1}(Y - H)$. Since f is Ω^*-continuous, then $\mathrm{sCl}(X - O) \subseteq f^{-1}(\mathrm{Int}(Y - H))$. Hence $X - \mathrm{sInt}(O) \subseteq f^{-1}(Y - \mathrm{Cl}(H)) = X - f^{-1}(\mathrm{Cl}(H))$. Therefore $f^{-1}(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$.

Sufficiency: Let $O \subseteq f^{-1}(H)$, where O is an Ω-closed subset of X and H is a semiopen subset of Y. Then $X - f^{-1}(H) \subseteq X - O$ or $f^{-1}(Y - H) \subseteq X - O$. By hypothesis, $f^{-1}(\mathrm{Cl}(Y - H)) \subseteq \mathrm{sInt}(X - O)$. Hence $f^{-1}(Y - \mathrm{Int}(H)) \subseteq X - \mathrm{sCl}(O)$ which implies $X - f^{-1}(\mathrm{Int}(H)) \subseteq X - \mathrm{sCl}(O)$. Thus $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(H))$. Therefore f is Ω^*-continuous. □

Theorem 4.3. *A map $f : X \to Y$ is Ω^*-closed if and only if $\mathrm{Cl}(H) \subseteq f^{-1}(\mathrm{sInt}(O))$ whenever H is a semiclosed subset of X, O is an Ω-open subset of Y and $H \subseteq f^{-1}(O)$.*

Proof. Necessity: Let $H \subseteq f^{-1}(O)$, where H is a semiclosed subset of X and O is an Ω-open subset of Y. Then $f(H) \subseteq f(f^{-1}(O)) \subseteq O$. Since f is Ω^*-closed, $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$. Therefore $\mathrm{Cl}(H) \subseteq f^{-1}(f(\mathrm{Cl}(H))) \subseteq f^{-1}(\mathrm{sInt}(O))$.

Sufficiency: Let $f(H) \subseteq O$, where H is a semiclosed subset of X and O is an Ω-open subset of Y. Then $H \subseteq f^{-1}(f(H)) \subseteq f^{-1}(O)$. By hypothesis, $\mathrm{Cl}(H) \subseteq f^{-1}(\mathrm{sInt}(O))$. So $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$. Therefore f is Ω^*-closed map. □

Theorem 4.4. *A map $f : X \to Y$ is Ω^*-continuous if and only if $f(\mathrm{sCl}(O)) \subseteq \mathrm{Int}(H)$ whenever $f(O) \subseteq H$, where O is an Ω-closed subset of X and H is a semiopen subset of Y.*

Proof. Necessity: Suppose that $f(O) \subseteq H$, where O is an Ω-closed subset of X and H is a semiopen subset of Y. Then $O \subseteq f^{-1}(f(O)) \subseteq f^{-1}(H)$. Since f is Ω^*-continuous, then $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(H))$. Hence $f(\mathrm{sCl}(O)) \subseteq f(f^{-1}(\mathrm{Int}(H))) \subseteq \mathrm{Int}(H)$.

Sufficiency: Let $O \subseteq f^{-1}(H)$, where O is an Ω-closed subset of X and H is a semiopen subset of Y. Then $f(O) \subseteq f(f^{-1}(H)) \subseteq H$. By hypothesis, $f(\mathrm{sCl}(O)) \subseteq \mathrm{Int}(H)$. Thus $\mathrm{sCl}(O) \subseteq f^{-1}(f(\mathrm{sCl}(O))) \subseteq f^{-1}(\mathrm{Int}(H))$. Therefore f is Ω^*-continuous. □

Theorem 4.5. *Let $f : X \to Y$ be a bijection. Then f is Ω^*-closed if and only if f^{-1} is Ω^*-continuous.*

Proof. Let f be Ω^*-closed map and $O \subseteq (f^{-1})^{-1}(H) = f(H)$, where O is an Ω-closed subset of Y and H is a semiopen subset of X. From Theorem 4.1 we have $\mathrm{sCl}(O) \subseteq f(\mathrm{Int}(H)) = (f^{-1})^{-1}(\mathrm{Int}(H))$. Hence f^{-1} is Ω^*-continuous.

Conversely, let f^{-1} is Ω^*-continuous and $F \subseteq f(K)$ or $F \subseteq (f^{-1})^{-1}(K)$, where F is an Ω-closed subset of Y and K is a semiopen subset of X. Then $\mathrm{sCl}(F) \subseteq (f^{-1})^{-1}(\mathrm{Int}(K))$ or $\mathrm{sCl}(F) \subseteq f(\mathrm{Int}(K))$. Therefore, from Theorem 4.1 we have f is Ω^*-closed. □

Theorem 4.6. *Let $f : X \to Y$ be a function for which $f(H)$ is semiclosed subset of Y and $f(\mathrm{Cl}(H)) \subseteq \mathrm{Cl}(f(H))$ for every semiclosed subset H of X. Then f is Ω^*-closed.*

Proof. Suppose that $f(H) \subseteq O$, where H is a semiclosed subset of X and O is an Ω-open subset of Y. Since O is an Ω-open, then $\mathrm{Cl}(f(H)) \subseteq \mathrm{sInt}(O)$. Hence $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$. Therefore f is Ω^*-closed. □

Remark 4.2. The converse of the above theorem is not true by the following example.

Example 4.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ be a topology on X. One can obtain that: $F_X = \{X, \phi, \{c\}, \{b, c\}\}, \mathrm{SO}(X) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $\mathrm{SC}(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$, $\Omega C(X) = \{X, \phi, \{b, c\}\}$ and $\Omega O(X) = \{X, \phi, \{a\}\}$. Define the map $f : X \to X$ by $f(a) = a, f(b) = c$ and $f(c) = b$. We have that f is Ω^*-closed. Now, $f(\{b\}) = \{c\}$, where $\{b\}, \{c\} \in \mathrm{SC}(X)$. But $f(\mathrm{Cl}(\{b\})) \subsetneq \mathrm{Cl}(f(\{b\}))$.

Theorem 4.7. *Let $f : X \to Y$ be a function for which $f^{-1}(V)$ is semiopen in X and $\mathrm{Int}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Int}(V))$ for every semiopen subset V of Y. Then f is Ω^*-continuous.*

Proof. Suppose that $O \subseteq f^{-1}(V)$, where O is an Ω-closed subset of X and V is a semiopen subset of Y. Since O is an Ω-closed, then $\mathrm{sCl}(O) \subseteq \mathrm{Int}(f^{-1}(V))$. Hence $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(V))$. Therefore f is Ω^*-continuous. □

Remark 4.3. The converse of the above theorem is not true by the following example.

Example 4.4. In Example 4.3, f is Ω^*-continuous. But $f^{-1}(\{a, c\}) = \{a, b\}$, where $\{a, c\}, \{a, b\} \in \mathrm{SO}(X)$, and $\mathrm{Int}(f^{-1}(\{a, c\})) \subsetneq f^{-1}(\mathrm{Int}(\{a, c\}))$.

5. Preserving Ω-Closed Sets

In this section, the concepts of Ω^*-closed and Ω^*-continuous maps are used to study the preservation of Ω-closed set.

Theorem 5.1. *If $f : X \to Y$ is irresolute and Ω^*-closed map, then $f^{-1}(B)$ is Ω-closed (Ω-open) subset of X whenever B is Ω-closed (Ω-open) subset of Y.*

Proof. Assume that B is an Ω-closed subset of Y and $f^{-1}(B) \subseteq U$, where U is a semiopen subset of X. Then $X - U \subseteq X - f^{-1}(B) = f^{-1}(Y - B)$ or $f(X - U) \subseteq Y - B$. Since f is Ω^*-closed, then $f(\mathrm{Cl}(X - U)) \subseteq \mathrm{sInt}(Y - B) = Y - \mathrm{sCl}(B)$. Hence $\mathrm{Cl}(X - U) \subseteq f^{-1}(Y - \mathrm{sCl}(B)) = X - f^{-1}(\mathrm{sCl}(B))$. Thus $f^{-1}(\mathrm{sCl}(B)) \subseteq X - \mathrm{Cl}(X - U) = \mathrm{Int}(U)$. Since f is irresolute, then $\mathrm{sCl}(f^{-1}(B)) \subseteq \mathrm{sCl}(f^{-1}(\mathrm{sCl}(B))) = f^{-1}(\mathrm{sCl}(B)) \subseteq \mathrm{Int}(U)$. Therefore $f^{-1}(B)$ is an Ω-closed subset of X.

A similar argument shows that the inverse image of Ω-open sets is Ω-open. □

Remark 5.1. From the above theorem we note that if $f : X \to Y$ is irresolute and Ω^*-closed, then f is Ω-irresolute.

The converse of the above remark is not true as illustrated by the following example.

Example 5.1. Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}\}, Y = \{p, q\}$ and $\sigma = \{Y, \phi, \{p\}\}$. Define $f : X \to Y$ as follows: $f(a) = f(c) = p$ and $f(b) = q$. Then f is Ω-irresolute but not irresolute.

Theorem 5.2. *If $f : X \to Y$ is Ω^*-continuous and pre-semiclosed, then $f(A)$ is an Ω-closed subset of Y whenever A is an Ω-closed subset of X.*

Proof. Assume that A is an Ω-closed subset of X and $f(A) \subseteq V$, where V is a semiopen subset of Y. Then $A \subseteq f^{-1}(V)$. Since f is Ω^*-continuous, then $\mathrm{sCl}(A) \subseteq f^{-1}(\mathrm{Int}(V))$. Hence $f(\mathrm{sCl}(A)) \subseteq \mathrm{Int}(V)$. Since f is pre-semiclosed, $\mathrm{sCl}(f(A)) \subseteq \mathrm{sCl}(f(\mathrm{sCl}(A))) = f(\mathrm{sCl}(A)) \subseteq \mathrm{Int}(V)$. Therefore $f(A)$ is an Ω-closed subset of Y. □

Remark 5.2. From the above theorem we note that if $f : X \to Y$ is Ω^*-continuous and pre-semiclosed, then f is pre-Ω-closed.

The converse of the above remark is not true as illustrated by the following example.

Example 5.2. Let $X = \{x, y\}, \tau = \{X, \phi, \{x\}\}, Y = \{p, q, r\}$ and $\sigma = \{Y, \phi, \{p\}, \{q, r\}\}$. Define $f : X \to Y$ as follows: $f(x) = p$ and $f(y) = r$. Then f is pre-Ω-closed but not pre-semiclosed. Also, if we redefine $f : X \to Y$ to be $f(x) = f(y) = q$, then f is pre-Ω-closed but not Ω^*-continuous,

Theorem 5.3. *A space X is semi-$T_{\frac{1}{2}}$ if and only if every singleton $\{x\}, x \in X$ which is Ω-open is semiopen.*

Proof. ($\Rightarrow$) Since in any space X and $x \in X$, the singleton $\{x\}$ is semiclosed or Ω-open [14, Proposition 5.1], then suppose $\{x\}, x \in X$ is Ω-open and it is not semiclosed. Since X is semi-$T_{\frac{1}{2}}$, then $\{x\}$ is semiopen.

($\Leftarrow$) From [14, Proposition 5.1], $\{x\}$ is semiclosed or semiopen. Then X is semi-$T_{\frac{1}{2}}$ [14, Theorem 5.1]. □

Theorem 5.4. *For any space Y and $f : X \to Y$ is Ω^*-continuous, then the space X is semi-$T_{\frac{1}{2}}$.*

Proof. Let A be an Ω-closed subset of X and Y be the set X with the topology $\sigma = \{Y, A, \phi\}$. Finally, let $f : X \to Y$ be the identity mapping. By assumption, f is Ω^*-continuous. Since A is Ω-closed in X and open in Y, and $A \subseteq f^{-1}(A)$, then $\mathrm{sCl}(A) \subseteq f^{-1}(\mathrm{Int}(A)) = f^{-1}(A) = A$. Hence A is semiclosed in X. Therefore the space Y is semi-$T_{\frac{1}{2}}$ [14, Theorem 5.1]. □

Remark 5.3. The converse of the above theorem is not true as shown by the following example.

Example 5.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Now, $\Omega C(X) = \{X, \phi, \{a\}, \{b\}\}$, $\mathrm{SO}(X) = P(X) - \{c\}$, and $\mathrm{SC}(X) = P(X) - \{a, b\}$. Therefore the space X is semi-$T_{\frac{1}{2}}$. Define $f : X \to X$ as: $f(a) = b$ and $f(b) = f(c) = c$. Now $X \subseteq f^{-1}(\{b, c\})$ but $\mathrm{sCl}(X) \subsetneq f^{-1}(\mathrm{Int}(\{b, c\}))$. Hence f is not Ω^*-continuous.

Theorem 5.5. *For any space X and $f : X \to Y$ is Ω^*-closed, then the space Y is semi-$T_{\frac{1}{2}}$.*

Proof. Let B be an Ω-open subset of Y and X be the set Y with the topology $\tau = \{X, A, \phi\}$, where $A = X - B$. Finally, let $f : X \to Y$ be the identity mapping. By assumption, f is Ω^* -closed. Since B is Ω-open in Y and closed in X, and $f(B) \subseteq B$, it follows that $B = f(B) \subseteq f(\mathrm{Cl}(B)) \subseteq \mathrm{sInt}(B)$. Hence B is semiopen in Y. Therefore the space Y is semi-$T_{\frac{1}{2}}$ [14, Theorem 5.1]. □

Remark 5.4. The converse of the above theorem is not true as shown by the following example.

Example 5.4. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Now, $F_X = \{X, \phi, \{c\}, \{b, c\}\}$, $\mathrm{SO}(X) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, $\mathrm{SC}(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$, $\Omega C(X) = \{X, \phi, \{b, c\}\}$, and $\Omega O(X) = \{X, \phi, \{a\}\}$. Therefore the space X is semi-$T_{\frac{1}{2}}$. Define $f : X \to X$ as: $f(a) = b$, $f(b) = a$, and $f(c) = c$. Now, $f(\{b\}) \subseteq \{a\}$ but $f(\mathrm{Cl}(\{b\})) \subsetneq \mathrm{sInt}(\{a\})$. Hence f is not Ω^*-closed.

Theorem 5.6. *If the open and closed subsets of Y coincide, then the map $f : X \to Y$ is Ω^*-closed if and only if $f(H)$ is semiclosed subset of Y and $f(\mathrm{Cl}(H)) = f(H)$ for every semiclosed subset H of X.*

Proof. Let f be Ω^*-closed map and H be a semiclosed subset of X. Then by Proposition 3.2 $f(H)$ is both Ω-open and Ω-closed subset of Y. Since $f(H) \subseteq f(H)$, then $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(f(H)) \subseteq f(H)$. Hence $f(\mathrm{Cl}(H)) = \mathrm{sInt}(f(H)) = f(H)$. Therefore $f(H)$ is semiopen and hence semiclosed.

The converse can easily be shown. □

Corollary 5.1. *If the open and closed sets of Y coincide, then the map $f : X \to Y$ is Ω^*-closed if and only if f is pre-semiclosed and $f(\mathrm{Cl}(H)) = f(H)$ for every semiclosed subset H of X.*

Theorem 5.7. *If the open and closed sets of X coincide, then the map $f : X \to Y$ is Ω^*-continuous if and only if $f^{-1}(V)$ is a semiopen subset of X and $f^{-1}(V) = f^{-1}(\mathrm{Int}(V))$ for every semiopen subset V of Y.*

Proof. Let f be Ω^*-continuous and V be a semiopen subset of Y. Then by Proposition 3.2 $f^{-1}(V)$ is both Ω-open and Ω-closed subset of X. Since $f^{-1}(V) \subseteq f^{-1}(V)$, then $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Int}(V)) \subseteq f^{-1}(V)$. Hence $\mathrm{sCl}(f^{-1}(V)) = f^{-1}(\mathrm{Int}(V)) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is semiclosed and so semiopen.

The converse can easily be shown. □

Corollary 5.2. *If the open and closed sets of Y coincide, then the map $f : X \to Y$ is Ω^*-continuous if and only if f is irresolute and $f^{-1}(V) = f^{-1}(\mathrm{Int}(V))$ for every semiopen subset V of Y.*

Theorem 5.8. *Let $f : X \to Y$ be a map for which $f(H)$ is semiclosed subset of Y, $f(\mathrm{Cl}(H)) \subseteq \mathrm{Cl}(f(H))$ for every semiclosed subset H of X and $g : Y \to Z$ be Ω^*-closed. Then $g \circ f : X \to Z$ is Ω^*-closed.*

Proof. Let H be a semiclosed subset of X and O be an Ω-open subset of Z such that $g(f(H)) \subseteq O$. Then $g(\mathrm{Cl}(f(H))) \subseteq \mathrm{sInt}(O)$. Therefore $g(f(\mathrm{Cl}(H))) \subseteq g(\mathrm{Cl}(f(H))) \subseteq \mathrm{sInt}(O)$. Hence $g \circ f$ is Ω^*-closed. □

Theorem 5.9. *If $f : X \to Y$ is Ω^*-closed and $g : Y \to Z$ is Ω-irresolute and presemiopen, then $g \circ f : X \to Z$ is Ω^*-closed.*

Proof. Let H be a semiclosed subset of X and O be an Ω-open subset of Z such that $g(f(H)) \subseteq O$. Then $f(H) \subseteq g^{-1}(O)$. Since g is Ω-irresolute, then $g^{-1}(O)$ is Ω-open subset of Y. Since f is Ω^*-closed, then $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(g^{-1}(O))$. Hence $g(f(\mathrm{Cl}(H))) \subseteq g(\mathrm{sInt}(g^{-1}(O))) = \mathrm{sInt}(g(\mathrm{sInt}(g^{-1}(O)))) \subseteq \mathrm{sInt}(g(g^{-1}(O))) \subseteq \mathrm{sInt}(O)$. Therefore $g \circ f$ is Ω^*-closed. □

Theorem 5.10. *If $f : X \to Y$ is Ω^*-continuous and $g : Y \to Z$ is irresolute and $\mathrm{Int}(g^{-1}(H)) \subseteq g^{-1}(\mathrm{Int}(H))$, then $g \circ f : X \to Z$ is Ω^*-continuous.*

Proof. Let H be a semiopen subset of Z and O be an Ω-closed subset of X such that $O \subseteq (g \circ f)^{-1}(H)$. Then $O \subseteq f^{-1}(g^{-1}(H))$ and $g^{-1}(H)$ is semiopen in Y. Since f is Ω^*-continuous, then $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(g^{-1}(H))) \subseteq f^{-1}(g^{-1}\mathrm{Int}((H))) = (g \circ f)^{-1}(\mathrm{Int}(H))$. Therefore $g \circ f$ is Ω^*-continuous. □

The following example shows that the restrictions of Ω^*-closed and Ω^*-continuous maps can fail to be Ω^*-closed and Ω^*-continuous, respectively.

Example 5.5. Let X be an indiscrete space with a nonempty proper subset B. The identity mapping $f : X \to X$ is Ω^*-closed and hence by Theorem 4.5 is Ω^*-continuous.

First, we prove that $f|_B : B \to X$ is not Ω^*-closed. Observe that $f(B) = f|_B(B)$ is Ω-open in X (Proposition 3.2). Then $f|_B(B) \subseteq f(B)$, where $f(B)$ is Ω-open in X and B is a semi-closed in B. But $f|_B(\mathrm{Cl}_B(B)) = f|_B(B) = f(B) \subsetneq \mathrm{sInt}(f(B))$ (where $\mathrm{Cl}_B(B)$ is the closure of B in B). Hence $f|_B$ is not Ω^*-closed.

Second, we prove that $f|_B : B \to X$ is not Ω^*-continuous. Since $B \subseteq (f|_B)^{-1}(B)$, where B is semiopen in X (Proposition 3.2) and Ω-closed in B. But $(f|_B)^{-1}(\mathrm{Int}(B)) = f^{-1}(\mathrm{Int}(B)) \cap B \not\supseteq \mathrm{sCl}_B(B) = B$ (where $\mathrm{sCl}_B(B)$ is the semi-closure of B in B). Hence $f|_B$ is not Ω^*-continuous.

Now, we have the following two theorems.

Theorem 5.11. *If $f : X \to Y$ is Ω^*-closed and B is semiclosed subset of X, then $f|_B : B \to Y$ is Ω^*-closed.*

Proof. Suppose $f|_B(H) \subseteq O$, where O is Ω-open in Y and H is semiclosed in B. Then H is semiclosed in X (see [14, Theorem 2.6]) and $f|_B(H) = f(H)$. Therefore $f(H) \subseteq O$. Since f is Ω^*-closed, then $f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$. Hence $f|_B(\mathrm{Cl}_B(H)) \subseteq f|_B(\mathrm{Cl}(H)) = f(\mathrm{Cl}(H)) \subseteq \mathrm{sInt}(O)$. Therefore $f|_B$ is Ω^*-closed. □

Theorem 5.12. *If $f : X \to Y$ is Ω^*-continuous and B is open and Ω-closed subset of X, then $f|_B : B \to Y$ is Ω^*-continuous.*

Proof. Assume $O \subseteq (f|_B)^{-1}(H)$, where O is Ω-closed in B and H is semiopen in Y. Then $O \subseteq f^{-1}(H)$ and O is Ω-closed relative to X (see [15, Theorem 3.3]). Since f is Ω^*-continuous, then $\mathrm{sCl}(O) \subseteq f^{-1}(\mathrm{Int}(H))$. Hence $\mathrm{sCl}(O) \cap B \subseteq f^{-1}(\mathrm{Int}(H)) \cap B = (f|_B)^{-1}(\mathrm{Int}(H))$. Since B is open in X, then $\mathrm{sCl}(O) \cap B = \mathrm{sCl}_B(O)$ [11]. Therefore $\mathrm{sCl}_B(O) \subseteq (f|_B)^{-1}(\mathrm{Int}(H))$ and $f|_B : B \to Y$ is Ω^*-continuous. □

References

1. D. Andrijević, Semi-preopen sets, *Mat. Vesnik* **38** (1986) 24–32.
2. S. P. Arya, T. Nour and R. Gupta, On strongly continuous functions, *Kyungpook Math. J.* **14** (1974) 131–143.
3. P. Bhattacharyya and B. K. Lahiri, Semi-generalized closed sets in topology, *Indian J. Math.* **29**(3) (1987) 375–382.
4. N. Biswas, On characterization of semi-continuous functions, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **48**(8) (1970) 399–402.
5. S. E. Bohn and J. Lee, Semitopological groups, *Amer. Math. Monthly* **72** (1965) 996–998.
6. S. G. Crossely and S. K. Hildebrand, Semi-closure, *Texas J. Sci.* **22** (1971) 99–112.
7. S. G. Crossely and S. K. Hildebrand, Semi-topological properties, *Fund. Math.* **74** (1972) 233–254.
8. J. Dontchev and T. Noiri, Contra-semicontinuous functions, *Math. Pannon.* **10**(2) (1999) 159–168.
9. D. S. Janković and I. L. Relly, On semiseparation properties, *Indian J. Pure Appl. Math.* **16**(9) (1985) 957–964.
10. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly* **70** (1963) 36–41.
11. S. N. Maheshwari and R. Prasad, On *s*-regular spaces, *Glas. Math.* **10**(30) (1975) 347–350.
12. S. N. Maheshwari and R. Prasad, Some new separation axioms, *Ann. Soc. Sci. Bruxelles Ser. I* **89** (1975) 395–402.
13. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* **53** (1982) 47–53.
14. T. Noiri and O. R. Sayed, On Ω-closed sets and Ω*s*-closed sets in topological spaces, *Acta. Math. Hungar.* **107**(4) (2005) 315–326.
15. O. R. Sayed, On Ω-open sets and Ω*s*-open sets in topological spaces, *Int. J. Gen. Topol.* **1**(1) (2008) 11–19.
16. D. Sivaraj, Semihomeomorphisms, *Period. Math. Hungar.* **48** (1986) 139–145.
17. M. H. Stone, Application of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* **41** (1937) 375–381.
18. P. Sundaram, H. Maki and K. Balachandran, Semi-generalized continuous maps and semi-$T_{\frac{1}{2}}$ spaces, *Bull. Fukuoka Univ. Ed. Part III* **40** (1991) 33–40.