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# TWO TYPES OF FUZZY SEMI-CONTINUITY AND FOUR TYPES OF FUZZY IRRESOLUTNESS IN FUZZIFYING TOPOLOGICAL SPACES

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The concept of semi-open sets is extended in fuzzifying topology in two various types one of them is stronger than the other but the converse may not true. These concepts and two types of semi-continuity induced by them are introduced and studied in fuzzifyng topology. Furthermore, four types of irresolute functions are considered between fuzzifying topological spaces.

The concept of fuzzifying topology with the semantic method of continuous valued logic was introduced by M.S.Ying [8]. For the definitions and results in fuzzifying topology which we used in the present paper we refer to [8,9,10]. In general topology the concepts of semi-open sets and semi-continuity was introduced by N.L.Levine [6] and the concept of semi-closed sets was introduced by N.Biswas [2]. It is worth to mention that these concepts considered in fuzzy topology by K.K.Azad [1]. In S.G.Crossley and S.K.Hildbrand [4] the concept of irresolute functions are introduced and we note that these types of functions are considered by M.N.Mukherjee and S.B.Sinha [7] in fuzzy topology [3]. In the present paper we introduce and study two extentions of semi-open sets and semi-continuity in fuzzifying topology. Furthermore, depending on these types of semi-open sets, four types pf irresolute functions are introduced and studied in fuzzifying topology.

## 2. Preliminaries.

In the sequal note that for any formula $\varphi$, the symbol $[\varphi]$ means the truth value of $\varphi$, where the set of truth values is the unit interval. Also,

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a formula $\varphi$ is valid, we write $\models \varphi$, if and only if $[\varphi] = 1$ for every interpretation.

First, we present the fuzzy logical and corresponding set theoritcal notations [8,9] since we need the major of them in this paper.

(1) $[\alpha] := \alpha(\alpha = [0,1]); [\varphi \wedge \psi] := \min([\varphi], [\psi]);$
$[\varphi \to \psi] := \min(1, 1 - [\varphi] + [\psi]);$

(2) If $X$ is the universe of discourse, then
$[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)];$

(3) If $\tilde{A} \in \mathcal{F}(X)$, where $\mathcal{F}(X)$ is the family of all fuzzy sets of $X$, then
$[x \in \tilde{A}] := \tilde{A}(x)$.

In addition the following derived formula are given

(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi];$

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi], [\psi]);$

(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)];$

(4) $[\varphi \dot\wedge \psi] := [\neg(\varphi \to \neg\psi)] = \max(0, [\varphi] + [\psi] - 1);$

(5) $[\varphi \dot\vee \psi] := [\neg\varphi \to \psi] = [\neg(\neg\varphi \dot\wedge \neg\psi)] = \min(1, [\varphi] + [\psi];$

(6) $[\exists(x)\varphi(x)] : [\neg\forall x \neg\varphi(x)] = \sup_{x \in X} \varphi(x);$

(7) If $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$, then

(a) $\tilde{A} \subseteq \tilde{B} := \forall x(x \in \tilde{A} \to x \in \tilde{B}) = \inf_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x));$

(b) $\tilde{A} \equiv \tilde{B} := (\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A});$

(c) $\tilde{A} \dot\equiv \tilde{B} := (\tilde{A} \subseteq \tilde{B}) \dot\wedge (\tilde{B} \subseteq \tilde{A});$

We do often not distingush the connectives and their truth value functions and state strictly our results on formalization M.S.Ying do [8,9,10].

Secondly, we give the following definitions and results [8] in fuzzifying topology which are used in the sequal.

**Definition 2.1.** Let $X$ be a universe of discourse, $P(X)$ is the family of subsets of $X$ and $\tau \in \mathcal{F}(P(X))$ satisfy the following conditions:

(1) $\tau(X) = 1;$

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B);$

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau(\bigcup_{\lambda \in \Lambda} A_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then $\tau$ is called a fuzzifying topology and $(X, \tau)$ is a fuzzifying topological space.

**Definition 2.2.** The family of fuzzifying closed sets is denoted by $F \in \mathcal{F}(P(X))$ and defined as

$$A \in F := X \sim A \in \tau,$$

where $X \sim A$ is the complement of $A$.

**Definition 2.3.** Let $x \in X$. The neighborhood system of $x$ is denoted by $N_x \in \mathcal{F}(P(X))$ and defined as follows:

$$N_x(A) = \sup_{x \in B \subseteq A} \tau(B).$$

**Definition 2.4.** (Lemma 5.2 [8]). The closure $\bar{A}$ of $A$ is defined as

$$\bar{A}(x) = 1 - N_x(X \sim A).$$

In Theorem 5.3 [8] M.S. Ying proved that the closure $cl : P(X) \to \mathcal{F}(X)$, is a fuzzifying closure operator (see Definition 5.3 [8]) i.e., its extention $cl : \mathcal{F}(X) \to \mathcal{F}(X)$,

$$cl(\tilde{A}) = \bigcup_{\alpha \in [0,1]} \alpha \wedge \bar{\tilde{A}}_\alpha, \tilde{A} \in \mathcal{F}(X),$$

where $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}$ is the $\alpha$-cut of $\tilde{A}$, satisfies the following Kuratowski closure axims:

(1) $\models cl(\phi) \equiv \phi$;
(2) for any $\tilde{A} \in \mathcal{F}(X), \models \tilde{A} \subseteq cl(\tilde{A})$;
(3) for any $\tilde{A}, \tilde{B} \in \mathcal{F}(X), \models cl(\tilde{A} \cup \tilde{B}) \equiv cl(\tilde{A}) \cup cl(\tilde{B})$;
(4) for any $\tilde{A} \in \mathcal{F}(X), \models cl(cl(\tilde{A})) \subseteq cl(\tilde{A})$.

**Definition 2.5.** For any $A \in P(X)$, the interior $A^\circ$ of $A$ is defined as

$$A^\circ(x) = N_x(A).$$

**Corollary 2.1.**

$$\tau(A) = \inf_{x \in A} A^\circ(x)$$

**Definition 2.6 [10].** Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces. A unary fuzzy predicate $C \in \mathcal{F}(Y^X)$, called fuzzy continuity, is given as follows:

$$C(f) := \forall u(u \in U \to f^{-1}(u) \in \tau).$$

Now, we add the following concept:

**Definition 2.7.** For any $\tilde{A} \in \mathcal{F}(X)$,

$$\models (\tilde{A})^\circ \equiv X \sim \overline{(X \sim \tilde{A})}.$$

**Lemma 2.1.** If $[\tilde{A} \subseteq \tilde{B}] = 1$, then

(1) $\models \tilde{\tilde{A}} \subseteq \tilde{\tilde{B}}$;

(2) $\models (\tilde{A})^\circ \subseteq (\tilde{B})^\circ$.

**Lemma 2.2.** Let $(X, \tau)$ be a fuzzifying topological space. For any $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$,

(1) $\models X^\circ \equiv X$;

(2) $\models (\tilde{A})^\circ \subseteq \tilde{A}$;

(3) $\models (\tilde{A} \cap \tilde{B})^\circ \equiv (\tilde{A})^\circ \cap (\tilde{B})^\circ$;

(4) $\models (\tilde{A})^{\circ\circ} \supseteq (\tilde{A})^\circ$.

## 3. Fuzzifying semi-open sets.

**Definition 3.1.** Let $(X, \tau)$ be a fuzzifying topological space.

(1) The family of fuzzifying semi-open sets of type $i$ is denoted by $(S\tau)^i \in \mathcal{F}(P(X)), i = 1, 2$ and defined as follows:

(i) $A \in (S\tau)^1 := \exists B(B \in \tau \wedge B \subseteq A \wedge A \subseteq \bar{B})$;

(ii) $A \in (S\tau)^2 := \exists B(B \in \tau \wedge B^\circ \subseteq A \wedge A \subseteq \bar{B})$.

**Theorem 3.1.**

(1) $\models A \in (S\tau)^1 \leftrightarrow \exists B(B \in \tau \wedge B \subseteq A \wedge A \subseteq B^{\circ -})$;

(2) $\models A \in (S\tau)^2 \leftrightarrow \exists B(B \in \tau \wedge B^\circ \subseteq A \wedge A \subseteq B^{\circ -})$;

(3) $\models A \in (S\tau)^1 \rightarrow A \in (S\tau)^2$.

**Proof.** (1) We need to prove that for each $B \in P(X), [B \in \tau \wedge A \subseteq \bar{B}] = [B \in \tau \wedge A \subseteq B^{\circ -}]$. If $\inf_{y \in B} B^\circ(y) = 0$, then $\tau(B) = 0 = [B \in \tau \wedge A \subseteq \bar{B}] = [B \in \tau \wedge A \subseteq B^{\circ -}]$. Suppose $\inf_{y \in B} B^\circ(y) = t > 0$, then there exists $y. \in B$ such that $B^\circ_{B^\circ(y.)} = B$ (Indeed, suppose that for every $y \in B$ there exists $x \in B$ such that $x \notin B^\circ_{B^\circ(y)}$. Then for each $y \in B, B^\circ(x) < B^\circ(y)$ and so $B^\circ(x) \leq \inf_{y \in B} B^\circ(y)$. If $B^\circ(x) = \inf_{y \in B} B^\circ(y)$ then $B^\circ_{B^\circ(x)} = B$, a contradiction. If $B^\circ(x) < \inf_{y \in B} B^\circ(y)$, a contradiction also.). Let $x \in A$. Then $B^{\circ -}(x) = \overline{\bigcup_{y \in B} B^\circ(y) \wedge B^\circ_{B^\circ(y)}}(x) \geq B^\circ(y.) \wedge (\overline{B^\circ_{B^\circ(y.)}})(x) \geq t \wedge \bar{B}(x)$. So $\inf_{x \in A} B^{\circ -}(x) \geq \inf_{x \in A}(t \wedge \bar{B}(x)) = t \wedge \inf_{x \in A} \bar{B}(x)$. Thus $t \wedge \inf_{x \in A} B^{\circ -}(x) \geq t \wedge \inf_{x \in A} \bar{B}(x)$ and since $t \wedge \inf_{x \in A} B^{\circ -}(x) \geq t \wedge \inf_{x \in A} \bar{B}(x)$, then $[B \in \tau \wedge A \subseteq B^\circ] = [B \in \tau \wedge A \subseteq \bar{B}]$. Hence, $(S\tau)^1(A) = \sup_{B \subseteq A}(\tau(B) \wedge [A \subseteq B^{\circ -}])$.

(2) In (1) we proved that for each $B \in P(X), [B \in \tau \wedge \subseteq \bar{B}] = [B \in \tau \wedge A \subseteq B^{\circ -}]$. So, for each $B \in P(X), [B \in \tau] \wedge [B^\circ \subseteq A] \wedge [A \subseteq \bar{B}] = [B \in \tau] \wedge [B^\circ \subseteq A] \wedge A \subseteq B^{\circ -}]$. Thus,

$$(S\tau)^2(A) = \sup_{B \in P(X)} ([B \in \tau] \wedge [B^\circ \subseteq A] \wedge A \subseteq B^{\circ -}].$$

(3) Immediate.

**Remark 3.1.** In crisp setting i.e. if the underlying fuzzifying topology is an ordinary topology we have for each $A \in P(X), (S\tau)^1(A) = (S\tau)^2(A)$. But in fuzzifying setting in general, there exists $A \in P(X)$ such that $(S\tau)^2(A) \not\leq (S\tau)^1(A)$ as illustrated by the following counterxample.

**Counterexample 3.1.** Let $X = \{a, b, c\}$ and consider the fuzzifying topology $\tau$ defined on $X$ as follows:
$\tau(X) = \tau(\phi) = 1, \tau(\{a\}) = \tau(\{a,b\}) = \frac{1}{12}, \tau(\{b\}) = \frac{2}{12}, \tau(\{c\}) = \tau(\{a,c\}) = \frac{3}{12}, \tau(\{b,c\}) = \frac{4}{12}$. One can deduce that $(S\tau)^2(\{a,b\}) = \frac{4}{12} \leq / \frac{2}{12} = (S\tau)^1(\{a,b\})$.

**Theorem 3.2.** Let $(X, \tau)$ be a fuzzifying topological space. Then

(1) $\models A \in \tau \to A \in (S\tau)^1$;

(2) For any $\{A_\lambda : \lambda \in \wedge\} \subseteq P(X)$;

(a) $(S\tau)^1(\underset{\lambda\in\wedge}{\cup} A_\lambda) \geq \underset{\lambda\in\wedge}{\wedge} (S\tau)^1(A_\lambda)$;

(a) $(S\tau)^2(\underset{\lambda\in\wedge}{\cup} A_\lambda) \geq \underset{\lambda\in\wedge}{\wedge} (S\tau)^2(A_\lambda)$;

**Proof.** (1) Ummediate.

(2) (a) From Lemma 1.1[8], $[(\underset{\lambda\in\wedge}{\cup} A_\lambda) \subseteq \bar{B}] \geq \underset{\lambda\in\wedge}{\wedge} [A_\lambda \subseteq \bar{B}]$.
Then

$$\begin{aligned}(S\tau)^1(\underset{\lambda\in\wedge}{\cup} A_\lambda) &= \sup_{B \subseteq \underset{\lambda\in\wedge}{\cup} A_\lambda} (\tau(B) \wedge [\underset{\lambda\in\wedge}{\cup} A_\lambda \subseteq \bar{B}] \\ &\geq \sup_{B \subseteq \underset{\lambda\in\wedge}{\cup} A_\lambda} \underset{\lambda\in\wedge}{\wedge} (\tau(B) \wedge [A_\lambda \subseteq \bar{B}]).\end{aligned}$$

For each $\lambda \in \wedge$ put $M_\lambda = \{B : B \subseteq A_\lambda\}$. Then $\underset{\lambda\in\wedge}{\cup} M_\lambda \subseteq \{B : B \subseteq \underset{\lambda\in\wedge}{\cup} A_\lambda\}$.
So,

$$\begin{aligned}&\sup_{B\in P(X), B \subseteq \underset{\lambda\in\wedge}{\cup} A_\lambda} \underset{\lambda\in\wedge}{\wedge} (\tau(B) \wedge [A_\lambda \subseteq \bar{B}]) \\ &\geq \sup_{B\in P(X), B\in M_\lambda, \lambda\in\wedge} \underset{\lambda\in\wedge}{\wedge} (\tau(B) \wedge [A_\lambda \subseteq \bar{B}]) \\ &= \sup_{f\in \prod_{\lambda\in\wedge} M_\lambda} \underset{\lambda\in\wedge}{\wedge} (\tau(f(\lambda) \wedge [A_\lambda \subseteq \overline{f(\lambda)}]) \\ &= \underset{\lambda\in\wedge}{\wedge} \sup_{B \subseteq A_\lambda} (\tau(B) \wedge [A_\lambda \subseteq \bar{B}]) \\ &= \underset{\lambda\in\wedge}{\wedge} (S\tau)^1(A_\lambda).\end{aligned}$$

(2) From Lemma 1.1 [8], $[B^\circ \subseteq \underset{\lambda\in\wedge}{\cup} A_\lambda] \geq \underset{\lambda\in\wedge}{\wedge} [B^\circ \subseteq A_\lambda]$.

So

$$(S\tau)^2(\bigcup_{\lambda\in\Lambda} A_\lambda) = \sup_{B\in P(X)} (\tau(B) \wedge [B^\circ \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda] \wedge [\bigcup_{\lambda\in\Lambda} A_\lambda \subseteq \bar{B}])$$
$$\geq \sup_{B\in P(X)} (\tau(B) \wedge \bigwedge_{\lambda\in\Lambda} [B^\circ \subseteq A_\lambda] \wedge \bigwedge_{\lambda\in\Lambda} [A_\lambda \subseteq \bar{B}])$$
$$= \sup_{B\in P(X)} \bigwedge_{\lambda\in\Lambda} (\tau(B) \wedge [B^\circ \subseteq A_\lambda] \wedge [A_\lambda \subseteq \bar{B}]).$$

Put $M_\lambda = P(X)$ for each $\lambda \in \Lambda$. Then

$$\sup_{B\in P(X)} \bigwedge_{\lambda\in\Lambda} (\tau(B) \wedge [B^\circ \subseteq A_\lambda] \wedge [A_\lambda \subseteq \bar{B}])$$
$$= \sup_{f\in \prod_{\lambda\in\Lambda} M_\lambda} \bigwedge_{\lambda\in\Lambda} (\tau(f(\lambda)) \wedge [f(\lambda)^\circ \subseteq A_\lambda] \wedge [A_\lambda \subseteq \overline{f(\lambda)}])$$
$$= \bigwedge_{\lambda\in\Lambda} \sup_{B\in M_\lambda} (\tau(B) \wedge [B^\circ \subseteq A_\lambda] \wedge [A_\lambda \subseteq \bar{B}])$$
$$= \bigwedge_{\lambda\in\Lambda} \sup_{B\in P(X)} (\tau(B) \wedge [B^\circ \subseteq A_\lambda] \wedge [A_\lambda \subseteq \bar{B}])$$
$$= \bigwedge_{\lambda\in\Lambda} (S\tau)^2(A_\lambda).$$

**Corollary 3.1.**
(1) $\models A \in \tau \to A \in (S\tau)^2$;
(2) (a) $\models X \in (S\tau)^1$, (b) $\models \phi \in (S\tau)^1$,
(2) (a) $\models X \in (S\tau)^2$, (b) $\models \phi \in (S\tau)^2$,

**Proof.** (1) From Theorem 3.1 (3) and Theorem 3.2 (1).
(2) From Theorem 3.2 (1).
(3) From Theorem 3.1 (3) and from (2).

**Definition 3.2.** Let $(X, \tau)$ be a fuzzifying topological space. The family of all fuzzifing semi-closed sets of type $i$ will be denoted by $(SF)^i \in \mathcal{F}(P(X)), i = 1, 2$ and defined as follows:

$$(SF)^i(A) := (S\tau)^i(X \sim A).$$

**Theorem 3.3.** Let $(X, \tau)$ be a fuzzifying topological space. Then,
(1) $\models F \subseteq (SF)^i$;
(2) (a) $\models X \in (SF)^i$, (b) $\models \phi \in (SF)^i$;
(3) From any $\{A_\lambda : \lambda \in \Lambda\} \subseteq P(X), (SF)^i(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} (SF)^i(A_\lambda)$.

## 4. Fuzzifying $(S\tau)^i$-neighborhood structure of a point.

**Definition 4.1.** Let $x \in X$. The fuzzifying $(S\tau)^i$-neighborhood system of $x, i = 1,2$, is denoted by $N_x^{(S\tau)^i} \in \mathcal{F}(P(X))$ and defined as follows:

$$N_x^{(S\tau)^i}(A) = \sup_{x \in B \subseteq A} (S\tau)^i(B).$$

**Remark 4.1.** If there is no confusion we write $N_x^i$ for $N_x^{(S\tau)^i}$. A similar remark for the consequences concepts depend on $(S\tau)^i$ is considered without mention in the sequal of the paper, where $i = 1,2$.

**Theorem 4.1.**

(1) $\models A \in (S\tau)^i \leftrightarrow \forall x(x \in A \rightarrow \exists B(B \in (S\tau)^i \wedge x \in B \subseteq A))$;

(2) $\models A \in (S\tau)^i \leftrightarrow \forall x(x \in A \rightarrow \exists B(B \in N_x^i \wedge B \subseteq A))$;

**Proof.** (1) Note that $[\forall x(x \in A \rightarrow \exists B(B \in (S\tau)^i \wedge x \in B \subseteq A))] = \inf_{x \in A} \sup_{x \in B \subseteq A} (S\tau)^i(B)$. First it is clear that $\inf_{x \in A} \sup_{x \in B \subseteq A} (S\tau)^i(B) \geq (S\tau)^i(A)$. Second, let $M_x = \{B : x \in B \subseteq A\}$ for each $x \in A$. Then for any $f \in \prod_{x \in A} M$ we have $\bigcup_{x \in A} f(x) = A$ and so $(S\tau)^i(A) = (S\tau)^i(\bigcup_{x \in A} f(x)) \geq \bigwedge_{x \in A} (S\tau)^i(f(x))$. Thus, $(S\tau)^i(A) = \sup_{f \in \prod_{x \in A} M_x} \bigwedge_{x \in A} (S\tau)^i(f(x)) = \bigwedge_{x \in A} \sup_{B \in M_x} (S\tau)^i(f(x)) = \bigwedge_{x \in A} \sup_{x \in B \in A} (S\tau)^i(B)$.

(2) Applying (1) we have $[\forall x(x \in A \rightarrow \exists B(B \in N_x^i \wedge B \subseteq A))] = \inf_{x \in A} \sup_{B \subseteq A} \sup_{x \in C \subseteq B} (S\tau)^i(C) = \inf_{x \in A} \sup_{x \in C \subseteq A} (S\tau)^i(C) = (S\tau)^i(A)$.

**Corollary 4.1.**

$$\inf_{x \in A} N_x^i(A) = (S\tau)^i(A).$$

**Theorem 4.2.** The mapping $N^i : X \rightarrow \mathcal{F}^N(P(X)), x \longmapsto N_x^i$, where $\mathcal{F}^N(P(X))$ is the set of all normal fuzzy subset of $P(X)$ has the following properties:

(1) for any $x, A, \models A \in N_x^i \rightarrow x \in A$;

(2) for any $x, A, B, \models A \subseteq B \rightarrow (A \in N_x^i \rightarrow B \in N_x^i)$;

(3) for any $x, A, B, \models A \in N_x^i \rightarrow \exists H(H \in N_x^i \wedge H \subseteq A \wedge \forall y(y \in H \rightarrow H \in N_y^i))$.

**Proof.** Since for each $x \in X, N_x^i(X) = 1$, then each $N_x^i$ is normal.

(1) If $[A \in N_x^i] = 0$, then the result holds. If $[A \in N_x^i] = \sup_{x \in H \subseteq A} (S\tau)^i(H) > 0$, then there exists $H_o$ such that $x \in H_o \subseteq A$. Now, we have $[x \in A] = 1$. Therefore, $[A \in N_x^i] \leq [x \in A]$ holds always.

(2) Immediate.

(3) $[\exists H(H \in N_x^i \wedge H \subseteq A \wedge \forall y(y \in H \to H \in N_y^i))]$

$$= \sup_{H \subseteq A}(N_x^i(H) \wedge \inf_{y \in H} N_y^i(H) = \sup_{H \subseteq A}(N_x^i) \wedge (S\tau)^i(H))$$

$$= \sup_{H \subseteq A}(S\tau)^i(H) \geq \sup_{x \in H \subseteq A}(S\tau)^i(H) = [A \in N_x^i].$$

## 5. $(S\tau)^i$ closure and $(S\tau)^i$-interior.

**Definition 5.1.**

(1) The $(S\tau)^i$-closure of $A$ is denoted and defined as follows:

$$cl^i(A)(x) = \inf_{x \notin B \supseteq A}(1 - (SF)^i(B));$$

(2) The $(S\tau)^i$-interior of $A$ is denoted and defined as follows:

$$int^i(A)(x) = N_x^i(A).$$

**Theorem 5.1.** For any $x, A, B$

(a) $cl^i(A)(x) = 1 - N_x^i(X \sim A)$;
(b) $\models cl^i(\phi) \equiv \phi$;
(c) $\models A \subseteq cl^i(A)$;
(d) $\models x \in cl^i(A) \leftrightarrow \forall B(B \in N_x^i \to A \cap B \neq \phi)$;
(e) $\models A \equiv cl^i(A) \leftrightarrow A \in (SF)^i$;
(f) $\models B \equiv cl^i(A) \to B \in (SF)^i$.

**Proof.**

(a) $cl^i(A)(x) = \inf_{x \notin B \supseteq A}(1 - (SF)^i(B)) = \inf_{x \in X \sim B \subseteq X \sim A}(1 - (S\tau)^i(X \sim B))$

$$1 - \sup_{x \in X \sim B \subseteq X \sim A}(S\tau)^i(X \sim B) = 1 - N_x^i(X \sim A);$$

(b) From (a) we have, $cl^i(\phi)(x) = 1 - N_x^i(X \sim \phi) = 0$.

(c) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $N_x^i(A) = 0$. If $x \in A$, then $cl^i(A)(x) = 1 - N_x^i(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq cl^i(A)] = 1$.

(d) Applying (a) we have,

$$[\forall B(B \in N_x^i \to A \cap B \neq \phi)] = \inf_{B \subseteq X \sim A}(1 - N_x^i(B)) = 1 - N_x^i(X \sim A)$$

$$= [x \in cl^i(A)].$$

(e) From Corollary 4.1 and from (a) and (c) above we have,

$$\begin{aligned}[A \equiv cl^i(A)] &= \inf_{x\in X\sim A}(1 - cl^i(A)(x)) \\ &= \inf_{x\in X\sim A} N^i_x(X \sim A) \\ &= (S\tau)^i(X \sim A) \\ &= (SF)^i(A).\end{aligned}$$

(f) If $[A \subseteq B] = 0$, then $[B \dot{\equiv} cl^i(A)] = 0$ (indeed,

$$\begin{aligned}[B \equiv cl^i(A)] &= \max(0, [B \subseteq cl^i(A)] + [cl^i(A) \subseteq B] - 1) \\ &\leq \max(0, \inf_{x\in B} cl^i(A)(x) + [A \subseteq B] - 1) \\ &= \max(o, \inf_{x\in B} cl^i(A)(x) + 0 - 1) = 0.).\end{aligned}$$

Now we suppose $[A \subseteq B] = 1, [B \subseteq cl^i(A)] = 1 - \sup_{x\in B\sim A} N^i_x(X \sim A)$, $[cl^i(A) \subseteq B] = \inf_{x\in X\sim B} N^i_x(X \sim A)$. So, $[B \dot{\equiv} cl^i(A)] = \max(0, \inf_{x\in X\sim B} N^i_x(X \sim A) - \sup_{x\in B\sim A} N^i_x(X \sim A)$. If $[B \dot{\equiv} cl^i(A)] > t$, then $\inf_{x\in X\sim B} N^i_x(X \sim A) > t + \sup_{x\in B\sim A} N^i_x(X \sim A)$. For any $x \in X \sim B$, $\sup_{x\in C\subseteq X\sim A} (S\tau)^i(C) > t + \sup_{x\in B\sim A} N^i_x(X \sim A)$ and so there exists $C_x$ such that $x \in C_x \subseteq X \sim A$ and $(S\tau)^i(C_x) > t + \sup_{x\in B\sim A} SN_x(X \sim A)$. Now, we prove that $C_x \subseteq X \sim B$. If not, then there exists $x' \in B \sim A$ with $x' \in C_x$. Hence, $\sup_{x\in B\sim A} N^i_x(X \sim A) \geq N^i_{x'}(X \sim A) \geq (S\tau)^i(C_x) > t + \sup_{x\in B\sim A} SN_x(X \sim A)$, a contradiction. Therefore,

$$\begin{aligned}(SF)^i(B) = (S\tau)^i(X \sim B) = \inf_{x\in X\sim B} N^i_x(X \sim B) \geq \inf_{x\in X\sim B}(S\tau)^i(C_x) > t \\ + \sup_{x\in B\sim A} N^i_x(X \sim A) > t.\end{aligned}$$

Since $t$ is arbitrary, it holds that $[B \dot{\equiv} cl^i(A)] \leq [B \in (SF)^i]$.

**Theorem 5.2.** For any $x, A, B$,
(1)(a) $\models int^i(A) \equiv X \sim cl^i(X \sim A)$;
(b) $\models int^i(X) \equiv X$;
(c) $\models int^i(A) \subseteq A$;
(d) $\models B \dot{\equiv} int^i(A) \rightarrow B \in (S\tau)^i$;
(e) $\models B \in (S\tau)^i \wedge B \subseteq A \rightarrow B \subseteq int^i(A)$;
(f) $\models A \equiv int^i(A) \leftrightarrow A \in (S\tau)^i$.

**Proof.** (a) From Theorem 5.1 (a), $cl^i(A \sim A)(x) = 1 - N_x^i(A) = 1 - int^i(A)(X)$. Then, $[int^i(A) \equiv X \sim cl^i(X \sim A)] = 1$.

(b) and (c) are obtained from (a) above and from Theorem 5.1 (b) and (c).

(d) From (a) above and from Theorem 5.1 (f) we have

$$[B \doteq int^i(A)] = [X \sim B \doteq cl^i(X \sim A)] \le [X \sim B \in (SF)^i] \\ = [B \in (S\tau)^i].$$

(e) If $[B \subseteq A] = 0$, then the result holds. If $[B \subseteq A] = 1$, then

$$[B \subseteq int^i(A)] = \inf_{x \in B} int^i(A)(x) = \inf_{x \in B} N_x^i(A) > \inf_{x \in B} N_x^i(B) \\ = (S\tau)^i(B) = [B \in (S\tau)^i \wedge B \subseteq A].$$

(f) From Corollary 4.1 we have

$$[A \equiv int^i(A)] = \min(\inf_{x \in A} int^i(A)(x), \inf_{x \in X \sim A} (1 - int^i(A)(x))) \\ = \inf_{x \in A} int^i(A)(x) = \inf_{x \in A} N_x^i(A) = (S\tau)^i(A) = [A \in (S\tau)^i].$$

## 6. $(S\tau)^i$-Continuous functions.

**Definition 6.1.** Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces. A unary fuzzy predicate $S^iC \in \mathcal{F}(Y^X)$ called fuzzy semi-continuity of type $i$ is given as follows:

$$f \in S^iC := \forall u(u \in U \to f^{-1}(u) \in (S\tau)^i).$$

**Definition 6.2.** Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces. The unary fuzzy predicates $S^i\alpha_j \in \mathcal{F}(Y^X)$, where $j = 1, 2, ..., 5$, as follows:

(1) $S^i\alpha_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in (SF)^i_X)$,

where $F_Y$ is the family of all fuzzifying closed subsets of $Y$ and $(SF)^i_X$ is the family of all fuzzifying semi-closed subsets of type $i$ of $X$.

(2) $S^i\alpha_2(f) := \forall x \forall u(u \in N^Y_{f(x)} \to f^{-1}(u) \in N_x^{iX}$;

where $N^Y$ is the fuzzifying neighborhood system of $Y$ and $N^{iX}$ is the fuzzifying semi-neigborhood systems of type $i$ of $X$.

(3) $S^i\alpha_3(f) := \forall x \forall u(u \in N^Y_{f(x)} \to \exists \nu(f(\nu) \subseteq u \to \nu \in N_x^{iX}))$;

(4) $S^i\alpha_4(f) := \forall A(f(cl^i_X(A)) \subseteq cl_Y(f(A)))$;

(5) $S^i\alpha_5(f) := \forall B(cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))$.

**Theorem 6.1.**
$\models f \in S^iC \leftrightarrow f \in S^i\alpha_j$, where $j = 1, 2, 3, 4, 5$.

**Proof.** (a) We will prove that $\models f \in S^iC \leftrightarrow f \in S^i\alpha_1$.

$$\begin{aligned}[f \in S^i\alpha_1] &= \inf_{B \in P(Y)} \min(1, 1 - F_Y(B) + (SF)^i_X(f^{-1}(B))) \\ &= \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + (S\tau)^i(X \sim f^{-1}(B))) \\ &= \inf_{B \in P(Y)} \min(1, 1 - U(Y \sim B) + (S\tau)^i(f^{-1}(Y \sim B))) \\ &= \inf_{u \in P(Y)} \min(1, 1 - U(u) + (S\tau)^i(f^{-1}(u))) \\ &= [f \in S^iC].\end{aligned}$$

(b) We want to prove that $\models f \in S^iC \leftrightarrow f \in S^i\alpha_2$. First, we prove that $S^i\alpha_2(f) \geq S^iC(f)$. If $N^Y_{f(x)}(u) \leq N^{iX}_x(f^{-1}(u))$, then $\min(1, 1 - N^Y_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) = 1 \geq S^iC(f)$. Suppose $N^Y_{f(x)}(u) > N^{iX}_x(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$ than $x \in f^{-1}(A) \subseteq f^{-1}(u)$
Then,

$$\begin{aligned}N^Y_{f(x)}(u) - N^{iX}_x(f^{-1}(u)) &= \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} (S\tau)^i(B) \\ &\leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} (S\tau)^i(f^{-1}(A)) \\ &\leq \sup_{f(x) \in A \subseteq u} (U(A) - (S\tau)^i(f^{-1}(A))).\end{aligned}$$

So,

$$1 - N^Y_{f(x)}(u) + N^{iX}_x(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u} (1 - U(A) + (S\tau)^i(f^{-1}(A)))$$

and thus,

$$\begin{aligned}&\min(1, 1 - N^Y_{f(x)}(u) + N^{iX}_x(u))) \\ &\geq \inf_{f(x) \in A \subseteq u} \min(1, 1 - U(A) + (S\tau)^i(f^{-1}(A))) \\ &\geq \inf_{\nu \in P(Y)} \min(1, 1 - U(\nu) + (S\tau)^i(f^{-1}(\nu))) = S^iC(f).\end{aligned}$$

Hence,

$$\inf_{x \in X} \inf_{u \in P(X)} \min(1, 1 - N^Y_{f(x)} + N^{iX}_x(f^{-1}(u))) \geq [f \in S^iC].$$

Secondly, we prove that $S^iC(f) \geq S^i\alpha_2(f)$. From Corollary 4.1, we have

$$\begin{aligned} S^iC(f) &= \inf_{u\in P(Y)} \min(1, 1 - U(u) + (S\tau)^i(f^{-1}(u))) \\ &\geq \inf_{u\in P(Y)} \min(1, 1 - \inf_{f(x)\in u} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{iX}_x(f^{-1}(u))) \\ &= \inf_{u\in P(Y)} \min(1, 1 - \inf_{x\in f^{-1}(u)} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{iX}_x(f^{-1}(u))) \\ &\geq \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) = S^i\alpha_2(f). \end{aligned}$$

(c) We prove that $\models f \in S^i\alpha_2 \leftrightarrow f \in S^i\alpha_3$. From Theorem 4.2 (2) we have,

$$\begin{aligned} S^i\alpha_3(f) &= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + \sup_{\nu\in P(X) f(\nu)\subseteq u} N^{iX}_x(\nu)) \\ &= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) = S^i\alpha_2(f). \end{aligned}$$

(d) We want to prove that $\models f \in S^i\alpha_4 \leftrightarrow f \in S^i\alpha_5$. First, For any $B \in P(Y)$ one can deduce that $f^{-1}(f(cl^i_X(f^{-1}(B)))) \subseteq cl^i_X(f^{-1}(B))] = 1$ since for each fuzzy set $\tilde{A}$ we have $[f^{-1}(f(A)) \supseteq A] = 1$ and one can deduce that $[cl_Y(f(f^{-1}(B))) \subseteq cl_Y(B)] = 1$ since $[f(f^{-1}(B)) \subseteq B] = 1$. Then from Lemma 1.2(2) [10] we have,

$$\begin{aligned} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] &\geq [f^{-1}(f(cl^i_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] \\ &\geq [f^{-1}(f(cl^i_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \\ &\geq [f(cl^i_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))]. \end{aligned}$$

Therefore,

$$\begin{aligned} S^i\alpha_5 &= \inf_{B\in P(Y)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \\ &\geq \inf_{B\in P(Y)} [f(cl^i_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))] \\ &\geq \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq (cl_Y(f(A))] = S^i\alpha_4(f). \end{aligned}$$

Secondly, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$

and $f^{-1}(B) \supseteq A$. Hence from Lemma 1.2 (1) [10] we have,

$$\begin{aligned} S^i\alpha_4(f) &= \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq cl_Y(f(A))] \\ &\geq \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq f(f^{-1}(cl_Y(f(A))))] \\ &\geq \inf_{A\in P(X)} [cl^i_X(A) \subseteq f^{-1}(cl_Y(f(A)))]. \\ &\geq \inf_{B\in P(Y), B=f(A)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \\ &\geq \inf_{B\in P(X)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] = S^i\alpha_5(f). \end{aligned}$$

(e) We want to prove that $\models f \in S^i\alpha_5 \leftrightarrow f \in S^i\alpha_2$.

$$\begin{aligned} S^i\alpha_5(f) &= [\forall B(cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))] \\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-N^{iX}_x(X \sim f^{-1}(B))) + 1 - N^Y_{f(x)}(Y \sim B)) \\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-N^Y_{f(x)}(Y \sim B) + N^{iX}_x(f^{-1}(Y \sim B))) \\ &= \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1-N^Y_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) = S^i\alpha_2(f). \end{aligned}$$

**Theorem 6.2.** Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. For any $f \in Y^X, \models f \in C \to f \in S^iC$.

**Proof.** The proof is obtained from Theorem 3.2 (1) and Corollary 3.1 (1).

## 7. $(i,j)$-Irresolute functions.

**Definition 7.1.** Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. A unary fuzzy predicate $(i,j)-I \in \mathcal{F}(Y^X)$ called fuzzy $(i,j)$-irresolute of type $(i,j)$ is given as follows:

$$f \in (i,j)-I := \forall u(u \in (SU)^j \to f^{-1}(u) \in (S\tau)^i), \text{ where } i,j \in \{1,2\}.$$

**Theorem 7.1.** Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces and let $f \in Y^X$. Then,

(1) $\models f \in (i,j)-I \to f \in S^iC$;

(2) $\models f \in (i,2)-I \to f \in (i,1)-I$;

(3) $\models f \in (1,j)-I \to f \in (2,j)-1$.

**Proof.** (1) From Theorem 3.1(3) and Theorem 3.2(1) we have, $U(u) \leq (SU)^j(u)$ then the result holds.

From Theorem 3.1(3) we have for any fuzzifying topology $\tau$ that $[(S\tau)^1 \subseteq (S\tau)^2] = 1$, then (2) and (3) are obtained.

**Definition 7.2.** Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. We define the unary fuzzy predicates $(i,j)-I\alpha_k \in \mathcal{F}(Y^X)$, where $k = 1,2,3,4,5$ and $i,j \in \{1,2\}$, as follows:

(1) $f \in (i,j)-I\alpha_1 := \forall u(u \in (SF)^j_Y \to f^{-1}(u) \in (SF)^i_X$, where $(SF)^j_Y$ is the family of semi-closed subsets of $Y$ of type $j$ and $(SF)^i_X$ is the family of semi-closed subsets of $X$ of type $i$.

(2) $f \in (i,j)-I\alpha_2 := \forall x\forall u(u \in N^{jY}_{f(x)} \to f^{-1}(u) \in N^{iX}_x)$, where $N^{jY}$ is the semi-neighborhood system of type $j$ of $Y$ and $N^{iX}$ is the semi-neighborhood system of type $i$ of $X$;

(3) $f \in (i,j)-I\alpha_3 := \forall x\forall u(u \in N^{jY}_{f(x)} \to \exists\nu(f(\nu) \subseteq u \to \nu \in N^{iX}_x))$;

(4) $f \in (i,j)-I\alpha_4 := \forall A(f(cl^i_X(A)) \subseteq cl^j_Y(f(A)))$;

(5) $f \in (i,j)-I\alpha_5 := \forall B(cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B)))$.

**Theorem 7.2.**

$$\models f \in (i,j)-I \leftrightarrow f \in (i,j)-I\alpha_k, k = 1,2,3,4,5.$$

**Proof.** (a) We will prove that $\models f \in (i,j)-I \leftrightarrow f \in (i,j)-I\alpha_1$.

$$\begin{aligned}
[f \in (i,j)-I\alpha_1] &= \inf_{B\in P(Y)} \min(1, 1-(SF)^j_Y(B)+(SF)^i_X(f^{-1}(B))) \\
&= \inf_{B\in P(Y)} \min(1, 1-(SU)^j(Y\sim B)+(S\tau)^i(X\sim f^{-1}(B))) \\
&= \inf_{B\in P(Y)} \min(1, 1-(SU)^j(Y\sim B)+(S\tau)^i(f^{-1}(Y\sim B))) \\
&= \inf_{u\in P(Y)} \min(1, 1-(SU)^j(u)+(S\tau)^i(f^{-1}(u))) \\
&= [f \in (i,j)-I].
\end{aligned}$$

(b) We will prove that $\models f \in (i,j)-I \leftrightarrow f \in (i,j)-I\alpha_2$. First, we prove that $[f \in (i,j)-I\alpha_2] \geq [f \in (i,j)-I]$. If $N^{jY}_{f(x)}(u) \leq N^{iX}_x(f^{-1}(u)), \min(1, 1-N^{jY}_{f(x)}(u)+N^{iX}_x(f^{-1}(u))) = 1 \geq [f \in (i,j)-I]$. Suppose $N^{jY}_{f(x)}(u) > N^{iX}_x(f^{-1}(u))$. It is clear that, if $f(x) \in A \subseteq u$, then $x \in f^{-1}(A) \subseteq f^{-1}(u)$.

Then,

$$N^{jY}_{f(x)}(u) - N^{iX}_x(f^{-1}(u)) = \sup_{f(x)\in A\subseteq u} (SU)^j(A) - \sup_{x\in B\subseteq f^{-1}(u)} (S\tau)^i(B)$$
$$\leq \sup_{f(x)\in A\subseteq u} (SU)^j(A) - \sup_{f(x)\in A\subseteq u} (S\tau)^i(f^{-1}(A))$$
$$\leq \sup_{f(x)\in A\subseteq u} ((SU)^j(A) - (S\tau)^i(f^{-1}(A))).$$

So,

$$1 - N^{jY}_{f(x)}(u) + N^{iX}_x(f^{-1}(u)) \geq \inf_{f(x)\in A\subseteq u} (1 - (SU)^j(A) + (S\tau)^i(f^{-1}(A))).$$

and thus,

$$\min(1, 1 - N^{jY}_{f(x)}(u) + N^{iX}_x(f^{-1}(u)))$$
$$\geq \inf_{f(x)\in A\subseteq u} \min(1, 1 - (SU)^j(A) + (S\tau)^i(f^{-1}(A)))$$
$$\geq \inf_{\nu\in P(Y)} \min(1, 1 - (SU)^j(\nu) + (S\tau)^i(f^{-1}(\nu))) = [f \in (i,j) - I].$$

Hence,

$$\inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^{jY}_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) \geq [f \in (i,j) - I].$$

Secondly, we prove that $[f \in (i,j) - I] \geq [f \in (i,j) - I\alpha_2]$. From Corollary 4.1,

$$[f \in (i,j) - I] = \inf_{u\in P(Y)} \min(1, 1 - (SU)^j(u) + (S\tau)^i(f^{-1}(u)))$$
$$\geq \inf_{u\in P(Y)} \min(1, 1 - \inf_{f(x)\in u} N^{jY}_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{iX}_x(f^{-1}(u)))$$
$$\geq \inf_{u\in P(Y)} \min(1, 1 - \inf_{x\in f^{-1}(u)} N^{jY}_{f(x)}(u) + \inf_{x\in f^{-1}(u)} N^{iX}_x(f^{-1}(u)))$$
$$\geq \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^{jY}_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) = [f \in (i,j) - I\alpha_2].$$

(c) We ~~prof~~ prove that $\models f \in (i,j) - I\alpha_2 \leftrightarrow f \in (i,j) - I\alpha_3$. From Theorem 4.2(2) we have,

$$[f \in (i,j) - I\alpha_3] = \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^{jY}_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} N^{iX}_x(v))$$
$$= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^{iJ}_{f(x)}(u) + N^{iX}_x(f^{-1}(u)))$$
$$= [f \in (i,j) - I\alpha_2].$$

(d) We prove that $\models f \in (i,j)-I\alpha_4 \leftrightarrow f \in (i,j)-I\alpha_5$. First, for any $B \in P(Y)$ one can deduce that $f^{-1}(f(cl^i_X(f^{-1}(B)))) \supseteq cl^i_X(f^{-1}(B))] = 1$ since for any fuzzy set $\tilde{A}$ we have $[f^{-1}(f(\tilde{A})) \supseteq \tilde{A}] = 1$ and one can deduce that $[cl^j_Y(f(f^{-1}(B))) \subseteq cl^j_Y(B)] = 1$ since $[ff^{-1}(B) \subseteq B] = 1$. Then from Lemma 1.2(2) [10] we have,

$$\begin{aligned}
[cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B))] &\geq [f^{-1}(f(cl^i_X(f^{-1}(B)))) \subseteq f^{-1}(cl^j_Y(B))] \\
&\geq [f^{-1}(f(cl^i_X(f^{-1}(B)))) \subseteq f^{-1}(cl^j_Y(f(f^{-1}(B))))] \\
&\geq [f(cl^i_X(f^{-1}(B))) \subseteq cl^j_Y(f(f^{-1}(B)))]
\end{aligned}$$

Therefore,

$$\begin{aligned}
[f \in (i,j)-I\alpha_5] &= \inf_{B\in P(Y)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B))] \\
&\geq \inf_{B\in P(Y)} [f(cl^i_X(f^{-1}(B))) \subseteq cl^j_Y(f(f^{-1}(B)))] \\
&\geq \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq (cl^j_Y(f(A))] = [f \in (i,j)-I\alpha_4.
\end{aligned}$$

Secondly, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence from Lemma 1.2 (1) [10] we have,

$$\begin{aligned}
[f \in (i,j)-I\alpha_4] &= \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq cl^j_Y(f(A))] \\
&\geq \inf_{A\in P(X)} [f(cl^i_X(A)) \subseteq f(f^{-1}(cl^j_Y(f(A))))] \\
&\geq \inf_{A\in P(X)} [cl^i_X(A) \subseteq f^{-1}(cl^j_Y(f(A)))] \\
&\geq \inf_{B\in P(Y), B=f(A)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B))] \\
&\geq \inf_{B\in P(Y)} [cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B))] \\
&= [f \in (i,j)-I\alpha_5].
\end{aligned}$$

(e) We want to prove that $\models f \in (i,j)-I\alpha_2 \leftrightarrow f \in (i,j)-I\alpha_5$.

$$\begin{aligned}
&[f \in (i,j)-I\alpha_5] = [\forall B(cl^i_X(f^{-1}(B)) \subseteq f^{-1}(cl^j_Y(B)))] \\
&= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-N^{iX}_x(X \sim f^{-1}(B))) + 1 - N^{jY}_{f(x)}(Y \sim B)) \\
&= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-N^{jY}_{f(x)}(Y \sim B) + N^{iX}_x(f^{-1}(Y \sim B))) \\
&= \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1-N^{jY}_{f(x)}(u) + N^{iX}_x(f^{-1}(u))) \\
&= [f \in (i,j)-I\alpha_2].
\end{aligned}$$

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