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Some applications of $D\alpha$-closed sets in topological spaces

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ABSTRACT

In this paper, a new kind of sets called $D\alpha$-open sets are introduced and studied in a topological spaces. The class of all $D\alpha$-open sets is strictly between the class of all α-open sets and g-open sets. Also, as applications we introduce and study $D\alpha$-continuous, $D\alpha$-open, and $D\alpha$-closed functions between topological spaces. Finally, some properties of $D\alpha$-closed and strongly $D\alpha$-closed graphs are investigated.

1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology, and they are now the research topics of many topologies worldwide. Indeed a significant theme in General Topology and Real Analysis is the study of variously modified forms of continuity, separation axioms, etc. by utilizing generalized open sets. One of the most well-known notions and also inspiration source are the notion of α-open [1] sets introduced by Njåstad in 1965 and generalized closed (g-closed) subset of a topological space [2] introduced by Levine in 1970.

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Since then, many mathematicians turned their attention to the generalization of various concepts in General Topology by considering α-open sets [3–10] and generalized closed sets [11–13]. In 1982 Dunham [14] used the generalized closed sets to define a new closure operator, and thus a new topology τ^*, on the space, and examined some of the properties of this new topology. Throughout the present paper $(X, \tau), (Y, \sigma)$ and (Z, ν) denote topological spaces (briefly X, Y and Z) and no separation axioms are assumed on the spaces unless explicitly stated. For a subset A of a space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. Since we require the following known definitions, notations, and some properties, we recall in this section.

Definition 1.1. Let (X, τ) be a topological space and $A \subseteq X$. Then

(i) A is α-open [1] if $A \subseteq Int(Cl(Int(A))$ and α-closed [1] if $Cl(Int(Cl(A)) \subseteq A$.
(ii) A is generalized closed (briefly g-closed) [2] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(iii) A is generalized open(briefly g-open) [2] if $X \backslash A$ is g-closed.

The α-closure of a subset A of X [3] is the intersection of all α-closed sets containing A and is denoted by $Cl_{\alpha}(A)$. The α-interior of a subset A of X [3] is the union of all α-open sets contained in A and is denoted by $Int_{\alpha}(A)$. The intersection of all g-closed sets containing A [14] is called the g-closure of A and denoted by $Cl^*(A)$, and the g-interior of A [15] is the union of all g-open sets contained in A and is denoted by $Int^*(A)$.

We need the following notations:

- $\alpha O(X)$ (resp. $\alpha C(X)$) denotes the family of all α-open sets (resp. α-closed sets) in (X, τ).
- $GO(X)$ (resp. $GC(X)$) denotes the family of all generalized open sets (resp. generalized closed sets) in (X, τ).
- $\alpha O(X, x) = \{U \mid x \in U \in \alpha O(X, \tau)\}$, $O(X, x) = \{U \mid x \in U \in \tau\}$ and $\alpha C(X, x) = \{U \mid x \in U \in \alpha C(X, \tau)\}$.

Definition 1.2. A function $f : X \to Y$ is said to be:

(i) α-continuous [16] (resp. g-continuous [17]) if the inverse image of each open set in Y is α-open (resp. g-open) in X.
(ii) α-open [16] (resp. α-closed [16]) if the image of each open (resp. closed) set in X is α-open (resp. α-closed) in Y.
(iii) g-open [18] (resp. g-closed [18]) if the image of each open (resp. closed) set in X is g-open (resp. g-closed) in Y.

Definition 1.3. Let $f : X \to Y$ be a function:

(i) The subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the graph of f [19] and is usually denoted by $G(f)$.
(ii) a closed graph [19] if its graph $G(f)$ is closed sets in the product space $X \times Y$.
(iii) a strongly closed graph [20] if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $(U \times Cl(V)) \cap G(f) = \phi$.
(iv) a strongly α-closed graph [21] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \alpha O(X, x)$ and $V \in O(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \phi$.

Definition 1.4. A topological space (X, τ) is said to be:

(i) α-T_1 [9] (resp. g-T_1 [22]) if for any distinct pair of points x and y in X, there exist α-open (resp. g-open) set U in X containing x but not y and an α-open (resp. g-open) set V in X containing y but not x.
(ii) α-T_2 [8] (resp. g-T_2 [22]) if for any distinct pair of points x and y in X, there exist α-open (resp. g-open) sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Lemma 1.5. Let $A \subseteq X$, then

(i) $X \setminus Cl^*(A) = Int^*(X \setminus A)$.
(ii) $X \setminus Int^*(A) = Cl^*(X \setminus A)$.

Lemma 1.6. A function $f : (X, \tau) \to (Y, \sigma)$ has a closed graph [19] if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in O(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap V = \phi$.

Lemma 1.7. The graph $G(f)$ is strongly closed [23] if and only if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap Cl(V) = \phi$.

2. $D\alpha$-closed sets

In this section we introduce $D\alpha$-closed sets and investigate some of their basic properties.

Definition 2.1. A subset A of a space X is called $D\alpha$-closed if $Cl^*(Int(Cl^*(A))) \subseteq A$.

The collection of all $D\alpha$-closed sets in X is denoted by $D\alpha C(X)$.

Lemma 2.2. If there exists an g-closed set F such that $Cl^*(Int(F)) \subseteq A \subseteq F$, then A is $D\alpha$-closed.

Proof. Since F is g-closed, $Cl^*(F) = F$. Therefore, $Cl^*(Int(Cl^*(A))) \subseteq Cl^*(Int(Cl^*(F))) = Cl^*(Int(F)) \subseteq A$. Hence A is $D\alpha$-closed.

Remark 2.3. The converse of above lemma is not true as shown in the following example.

Example 2.4. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $F_X = \{\phi, \{c\}, \{b, c\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $D\alpha C(X) = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Therefore $\{c\} \in D\alpha C(X)$ and $\{a, c\} \in GC(X)$ but $Cl^*(Int\{a, c\}) = \{a, c\} \not\subset \{c\} \subset \{a, c\}$.

Theorem 2.5. Let (X, τ) be a topological space. Then

(i) Every α-closed subset of (X, τ) is $D\alpha$-closed.
(ii) Every g-closed subset of (X, τ) is $D\alpha$-closed.

Proof. (i) Since closed set is g-closed, $Cl^*(A) \subseteq Cl(A)$ [14]. Now, suppose A is α-closed in X, then $Cl(Int(Cl(A))) \subseteq A$. Therefore, $Cl^*(Int(Cl^*(A))) \subseteq Cl(Int(Cl(A))) \subseteq A$. Hence A is $D\alpha$-closed in X.

(ii) Suppose A is g-closed. Then $Cl^*(A) = A$ [14]. Therefore, $Int(Cl^*(A)) \subseteq Cl^*(A)$. Then $Cl^*(Int(Cl^*(A))) \subseteq Cl^*(Cl^*(A)) \subseteq Cl^*(A) = A$ [14]. Hence A is $D\alpha$-closed.

Remark 2.6. The converse of above theorem is not true as shown in the following example.

(i) $D\alpha$-closed set need not be α-closed. (see Example 2.7 below)
(ii) $D\alpha$-closed set need not be g-closed. (see Example 2.7 below)

Example 2.7. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a, b\}, X\}$. Then $F_X = \alpha C(X) = \{\phi, \{c\}, X\}$, $\alpha O(X) = \{\phi, \{a, b\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $D\alpha C(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Therefore $\{a\} \in D\alpha C(X)$, but $\{a\} \notin \alpha C(X)$ and $\{a\} \notin GC(X)$.

From the above discussions we have the following diagram in which the converses of implications need not be true.

α-closed set $\rightarrow$ $D\alpha$-closed set $\leftarrow$ g-closed set

Theorem 2.8. Arbitrary intersection of $D\alpha$-closed sets is $D\alpha$-closed.

Proof. Let $\{F_i : i \in \Lambda\}$ be a collection of $D\alpha$-closed sets in X. Then $Cl^*(Int(Cl^*(F_i))) \subseteq F_i$ for each i. Since $\cap F_i \subseteq F_i$ for each i, $Cl^*(\cap F_i) \subseteq Cl^*(F_i)$ for each i. Hence $Cl^*(\cap F_i) \subseteq \cap Cl^*(F_i), i \in \Lambda$. Therefore $Cl^*(Int(Cl^*(\cap F_i))) \subseteq Cl^*(Int(\cap Cl^*(F_i))) \subseteq Cl^*(\cap Int(Cl^*(F_i))) \subseteq \cap Cl^*(Int(Cl^*(F_i))) \subseteq \cap F_i$. Hence $\cap F_i$ is $D\alpha$-closed.

Remark 2.9. The union of two $D\alpha$-closed sets need not to be $D\alpha$-closed as shown in Example 2.7, where both $\{a\}$ and $\{b\}$ are $D\alpha$-closed sets but $\{a\} \cup \{b\} = \{a, b\}$ is not $D\alpha$-closed.

Corollary 2.10. If a subset A is $D\alpha$-closed and B is α-closed, then $A \cap B$ is $D\alpha$-closed.

Proof. Follows from Theorem 2.5 (i) and Theorem 2.8.

Corollary 2.11. If a subset A is $D\alpha$-closed and F is g-closed, then $A \cap F$ is $D\alpha$-closed.

Proof. Follows from Theorem 2.5 (ii) and Theorem 2.8.

Definition 2.12. Let A be a subset of a space X. The $D\alpha$-closure of A, denoted by $Cl^D_\alpha(A)$, is the intersection of all $D\alpha$-closed sets in X containing A. That is $Cl^D_\alpha(A) = \cap\{F : A \subseteq F$ and $F \in D\alpha C(X)\}$.

Theorem 2.13. Let A be a subset of X. Then A is $D\alpha$-closed set in X if and only if $Cl^D_\alpha(A) = A$.

Proof. Suppose A is $D\alpha$-closed set in X. By Definition 2.12, $Cl^D_\alpha(A) = A$. Conversely, suppose $Cl^D_\alpha(A) = A$. By Theorem 2.8 A is $D\alpha$-closed.

Theorem 2.14. Let A and B be subsets of X. Then the following results hold.

(i) $A \subseteq Cl^D_\alpha(A) \subseteq Cl_\alpha(A), Cl^D_\alpha(A) \subseteq Cl^*(A)$.
(ii) $Cl^D_\alpha(\phi) = \phi$ and $Cl^D_\alpha(X) = X$.
(iii) If $A \subseteq B$, Then $Cl^D_\alpha(A) \subseteq Cl^D_\alpha(B)$.
(iv) $Cl^D_\alpha(Cl^D_\alpha(A)) = Cl^D_\alpha(A)$.
(v) $Cl^D_\alpha(A) \cup Cl^D_\alpha(B) \subseteq Cl^D_\alpha(A \cup B)$.
(vi) $Cl^D_\alpha(A \cap B) \subseteq Cl^D_\alpha(A) \cap Cl^D_\alpha(B)$.

Proof. (i) Follows From Theorem 2.5 (i) and (ii), respectively.
(ii) and (iii) are obvious.
(iv) If $A \subseteq F$, $F \in D\alpha C(X)$, then from (iii) and Theorem 2.13, $Cl^D_\alpha(A) \subseteq Cl^D_\alpha(F) = F$. Again $Cl^D_\alpha(Cl^D_\alpha(A)) \subseteq Cl^D_\alpha(F) = F$. Therefore $Cl^D_\alpha(Cl^D_\alpha(A)) \subseteq \cap\{F : A \subseteq F, F \in D\alpha C(X)\} = Cl^D_\alpha(A)$.
(v) and (vi) follows from (iii).

Remark 2.15. The equality in the statements (v) of the above theorem need not be true as seen from Example 2.7, where $A = \{a\}$, $B = \{b\}$, and $A \cup B = \{a, b\}$. Then one can have that, $Cl^D_\alpha(A) = \{a\}$; $Cl^D_\alpha(B) = \{b\}$; $Cl^D_\alpha(A \cup B) = X$; $Cl^D_\alpha(A) \cup Cl^D_\alpha(B) = \{a, b\}$. Further more the equality in the statements (iv) of the above theorem need not be true as shown in the following example.

Example 2.16. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Then $F_X = GC(X) = D\alpha C(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $GO(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Let $A = \{a\}$, $B = \{b\}$, and $A \cap B = \phi$. Then one can have that, $Cl^D_\alpha(A) = \{a\}$; $Cl^D_\alpha(B) = \{a, b\}$; $Cl^D_\alpha(A \cap B) = \phi$; $Cl^D_\alpha(A) \cap Cl^D_\alpha(B) = \{a\}$.

3. $D\alpha$-open sets

In this section we introduce $D\alpha$-open sets and investigate some of their basic properties.

Definition 3.1. A subset A of a space X is called an $D\alpha$-open if $X \setminus A$ is $D\alpha$-closed. Let $D\alpha O(X)$ denote the collection of all an $D\alpha$-open sets in X.

Lemma 3.2. Let $A \subseteq X$, then

(i) $X \setminus Cl^*(X \setminus A) = Int^*(A)$.
(ii) $X \setminus Int^*(X \setminus A) = Cl^*(A)$.

Proof. Obvious.

Theorem 3.3. A subset A of a space X is $D\alpha$-open if and only if $A \subseteq Int^*(Cl(Int^*(A)))$.

Proof. Let A be $D\alpha$-open set. Then $X \setminus A$ is $D\alpha$-closed and $Cl^*(Int(Cl^*(X \setminus A))) \subseteq X \setminus A$. By Lemma 3.2 $A \subseteq Int^*(Cl(Int^*(A)))$. Conversely, suppose $A \subseteq Int^*(Cl(Int^*(A)))$. Then $X \setminus Int^*(Cl(Int^*(A))) \subseteq X \setminus A$. Hence $(Int^*(Cl(Int^*(X \setminus A)))) \subseteq X \setminus A$. This shows that $X \setminus A$ is $D\alpha$-closed. Thus A is $D\alpha$-open.

Lemma 3.4. If there exists g-open set V such that $V \subseteq A \subseteq Int^*(Cl(V))$, then A is $D\alpha$-open.

Proof. Since V is g-open, $X \setminus V$ is g-closed and $X \setminus Int^*(Cl(V)) \subseteq X \setminus A \subseteq X \setminus V$. Therefore From Lemma 3.2 $Cl^*(Int(X \setminus V)) \subseteq X \setminus A \subseteq X \setminus V$. From Lemma 2.2 we have $X \setminus A$ is $D\alpha$-closed. Hence A is $D\alpha$-open.

Remark 3.5. The converse of Lemma 3.4 need not to be true as seen from Example 2.4, where $\{a, b\} \in D\alpha O(X)$ and $\{b\} \in GO(X)$ but $\{b\} \subset \{a, b\} \not\subset \{b\}$.

Theorem 3.6. Let (X, τ) be a topological space. Then

(i) Every α-open subset of (X, τ) is $D\alpha$-open.
(ii) Every g-open subset of (X, τ) is $D\alpha$-open.

Proof. From Theorem 2.5, the proof is obvious.

Remark 3.7. The converse of the above theorem is not true as seen from Example 2.7, where $\{b, c\} \in D\alpha O(X)$ but $\{b, c\} \notin \alpha O(X)$ and $\{b, c\} \notin GO(X)$.

From the above discussions we have the following diagram in which the converses of implications need not be true.

α-open set $\rightarrow$ $D\alpha$-open set $\leftarrow$ g-open set

Theorem 3.8. Arbitrary union of $D\alpha$-open set is $D\alpha$-open.

Proof. Follows from Theorem 2.8.

Remark 3.9. The intersection of two $D\alpha$-open sets need not be $D\alpha$-open as seen from Example 2.7, where both $\{b, c\}$ and $\{a, c\}$ are $D\alpha$-open sets but $\{b, c\} \cap \{a, c\} = \{c\}$ is not $D\alpha$-open.

Corollary 3.10. If a subset A is $D\alpha$-open and B is α-open, then $A \cup B$ is $D\alpha$-open.

Proof. Follows from Theorem 3.6 (i) and Theorem 3.8.

Corollary 3.11. If a subset A is $D\alpha$-open and U is g-open, then $A \cup U$ is $D\alpha$-open.

Proof. Follows from Theorem 3.6 (ii) and Theorem 3.8.

Definition 3.12. Let A be a subset of a space X. The $D\alpha$-interior of A is denoted by $Int_\alpha^D(A)$, is the union of all an $D\alpha$-open sets in X contained in A. That is $Int_\alpha^D(A) = \cup\{U : U \subseteq A, U \in D\alpha O(X)\}$.

Lemma 3.13. If A is a subset of X, then

(i) $X \setminus Cl_\alpha^D(A) = Int_\alpha^D(X \setminus A)$.
(ii) $X \setminus Int_\alpha^D(A) = Cl_\alpha^D(X \setminus A)$.

Proof. Obvious.

Theorem 3.14. Let A be a subset of X. Then A is $D\alpha$-open if and only if $Int_\alpha^D(A) = A$.

Proof. Follows from Theorem 2.13 and Lemma 3.13.

Theorem 3.15. Let A and B be subsets of X. Then the following results hold.

(i) $Int_\alpha(A) \subseteq Int_\alpha^D(A) \subseteq A$, $Int^*(A) \subseteq Int_\alpha^D(A)$.
(ii) $Int_\alpha^D(\phi) = \phi$ and $Int_\alpha^D(X) = X$.
(iii) If $A \subseteq B$, then $Int_\alpha^D(A) \subseteq Int_\alpha^D(B)$.
(iv) $Int_\alpha^D(Int_\alpha^D(A)) = Int_\alpha^D(A)$.
(v) $Int_\alpha^D(A) \cup Int_\alpha^D(B) \subseteq Int_\alpha^D(A \cup B)$.
(vi) $Int_\alpha^D(A \cap B) \subseteq Int_\alpha^D(A) \cap Int_\alpha^D(B)$.

Proof. Obvious.

Remark 3.16. The equality in the statements (v) of Theorem 3.15 need not be true as seen from Example 2.7, where $A = \{b, c\}$, $B = \{a, c\}$, and $A \cup B = X$. Then one can have that, $Int_\alpha^D(A) = \{b, c\}$; $Int_\alpha^D(B) = \{c\}$; $Int_\alpha^D(A) \cup Int_\alpha^D(B) = \{b, c\}$; $Int_\alpha^D(A \cup B) = X$. Furthermore the equality in the statements (iv) of the above theorem need not be true as seen from Example 2.7, where $A = \{b, c\}$, $B = \{a, c\}$, and $A \cap B = \{c\}$. Then one can have that, $Int_\alpha^D(A) = \{b, c\}$; $Int_\alpha^D(B) = \{a, c\}$; $Int_\alpha^D(A \cap B) = \phi$; $Int_\alpha^D(A) \cap Int_\alpha^D(B) = \{c\}$.

Theorem 3.17. Let $x \in X$. Then $x \in Cl_\alpha^D(A)$ if and only if $U \cap A \neq \phi$ for every $D\alpha$-open set U containing x.

Proof. Let $x \in Cl_\alpha^D(A)$ and there exists $D\alpha$-open set U containing x such that $U \cap A = \phi$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is $D\alpha$-closed. Therefore $Cl_\alpha^D(A) \subseteq Cl_\alpha^D(X \setminus U) = X \setminus U$. This implies $x \notin Cl_\alpha^D(A)$, which is a contradiction. Conversely, assume that $U \cap A \neq \phi$ for every $D\alpha$-open set U containing x and $x \notin Cl_\alpha^D(A)$. Then there exists $D\alpha$-closed subset F containing A such that $x \notin F$. Hence $x \in X \setminus F$ and $X \setminus F$ is $D\alpha$-open. Therefore $A \subseteq F$, $(X \setminus F) \cap A = \phi$ This is a contradiction to our assumption.

Lemma 3.18. Let A be any subset of (X, τ). Then

(i) $A \cap Int^*(Cl(Int^*(A)))$ is $D\alpha$-open;
(ii) $A \cup Cl^*(Int(Cl^*(A)))$ is $D\alpha$-closed.

Proof.

(i) $Int^*(Cl(Int^*(A \cap Int^*(Cl(Int^*(A)))))) = Int^*(Cl(Int^*(A) \cap Int^*(Cl(Int^*(A))))) = Int^*(Cl(Int^*(A)))$. This implies that $A \cap Int^*(Cl(Int^*(A))) = A \cap Int^*(Cl(Int^*(A \cap Int^*(Cl(Int^*(A)))))) \subseteq Int^*(Cl(Int^*(A \cap Int^*(Cl(Int^*(A))))))$. Therefore $A \cap Int^*(Cl(Int^*(A)))$ is $D\alpha$-open.
(ii) From (i) we have $X \setminus (A \cup Cl^*(Int(Cl^*(A))) = (X \setminus A) \cap Int^*(Cl(Int^*(X \setminus A)))$ is $D\alpha$-open that further implies $A \cup Cl^*(Int(Cl^*(A)))$ is $D\alpha$-closed.

Theorem 3.19. If A is a subset of a topological space X,

(i) $Int_\alpha^D(A) = A \cap Int^*(Cl(Int^*(A)))$.
(ii) $Cl_\alpha^D(A) = A \cup Cl^*(Int(Cl^*(A)))$.

Proof.

(i) Let $B = Int_\alpha^D(A)$. Clearly B is $D\alpha$-open and $B \subseteq A$. Since B is $D\alpha$-open, $B \subseteq Int^*(Cl(Int^*(B))) \subseteq Int^*(Cl(Int^*(A)))$. This proves that $B \subseteq A \cap Int^*(Cl(Int^*(A)))$. By Lemma 3.18,

$A \cap Int^*(Cl(Int^*(A)))$ is $D\alpha$-open. By the definition of $Int_\alpha^D(A)$, $A \cap Int^*(Cl(Int^*(A))) \subseteq B$. Then it follows that $B = A \cap Int^*(Cl(Int^*(A)))$. Therefore $Int_\alpha^D(A) = A \cap Int^*(Cl(Int^*(A)))$.

(ii) By Lemma 3.13 we have $Cl_\alpha^D(A) = X \setminus Int_\alpha^D(X \setminus A)$,
$= X \setminus ((X \setminus A) \cap Int^*(Cl(Int^*(X \setminus A))))$, using (i)
$= X \setminus (X \setminus A) \cup (X \setminus Int^*(Cl(Int^*(X \setminus A)))$
$= A \cup Cl^*(Int(Cl^*(A)))$.

4. $D\alpha$-continuous functions

In this section we introduce $D\alpha$-continuous functions and investigate some of their basic properties.

Definition 4.1. A function $f : X \to Y$ is called $D\alpha$-continuous if the inverse image of each open set in Y is $D\alpha$-open in X.

Theorem 4.2.

(i) Every α-continuous function is $D\alpha$-continuous.
(ii) Every g-continuous function is $D\alpha$-continuous.

Proof. It is obvious from Theorem 3.6.

Remark 4.3.

(i) $D\alpha$-continuous function need not be α-continuous. (see Example 4.4 (i) below)
(ii) $D\alpha$-continuous function need not g-continuous. (see Example 4.4 (ii) below)

Example 4.4. (i) Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\phi, \{a\}, X\}$ and $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\phi, \{x, y\}, \{z\}, Y\}$. Let $f : X \to Y$ be a function defined by $f(a) = f(b) = x$, $f(c) = z$. One can have that $F_X = \{\phi, \{b, c\}, X\}$, $GC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, $\alpha O(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $D\alpha C(X) = D\alpha O(X) = P(X)$. Since $\{z\}$ is open in Y, $f^{-1}(\{z\}) = \{c\} \in D\alpha O(X)$, but $\{c\} \notin \alpha O(X)$. Therefore f is $D\alpha$-continuous but not α-continuous.

(ii) Let (X,τ) and (Y,σ) be the topological spaces in (i) and $f : X \to Y$ be a function defined by $f(a) = x$, $f(b) = f(c) = z$. Since $\{z\}$ is open in Y, $f^{-1}(\{z\}) = \{b, c\} \in D\alpha O(X)$, but $\{b, c\} \notin GO(X)$. Therefore f is $D\alpha$-continuous but not g-continuous.

From the above discussions we have the following diagram in which the converses of implications need not be true.

α-continuity $\to$ $D\alpha$-continuity $\leftarrow$ g-continuity

Theorem 4.5. Let $f : X \to Y$ be a function. Then the following are equivalent:

(i) f is $D\alpha$-continuous.
(ii) For each $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists $D\alpha$-open set $W \subset X$ containing x such that $f(W) \subset V$.
(iii) The inverse image of each closed set in Y is $D\alpha$-closed in X.
(iv) $f(Cl_\alpha^D(A)) \subseteq Cl(f(A))$ for every subset A of X.
(v) $Cl_\alpha^D(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every subset B of Y.
(vi) $f^{-1}(Int(B)) \subseteq Int_\alpha^D(f^{-1}(B))$ for every subset B of Y.

Proof. (i)⇒(ii) Since $V \subset Y$ containing $f(x)$ is open, then $f^{-1}(V) \in D\alpha O(X)$. Set $W = f^{-1}(V)$ which contains x, therefore $f(W) \subset V$.

(ii)⇒(i) Let $V \subset Y$ be open, and let $x \in f^{-1}(V)$, then $f(x) \in V$ and thus there exists $W_x \in D\alpha O(X)$ such that $x \in W_x$ and $f(W_x) \subset V$. Then $x \in W_x \subset f^{-1}(V)$, and so $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} W_x$ but $\bigcup_{x \in f^{-1}(V)} W_x \in D\alpha O(X)$ by Theorem 3.8. Hence $f^{-1}(V) \in D\alpha O(X)$, and therefore f is $D\alpha$-continuous.

(i)⇒(iii) Let $F \subset Y$ be closed. Then $Y \setminus F$ is open and $f^{-1}(Y \setminus F) \in D\alpha O(X)$, i.e. $X - f^{-1}(F) \in D\alpha O(X)$. Then $f^{-1}(F)$ is $D\alpha$-closed of X.

(iii)⇒(iv) Let $A \subseteq X$ and F be a closed set in Y containing $f(A)$. Then by (iii), $f^{-1}(F)$ is $D\alpha$-closed set containing A. It follows that $Cl_\alpha^D(A) \subseteq Cl_\alpha^D(f^{-1}(F)) = f^{-1}(F)$ and hence $f(Cl_\alpha^D(A)) \subseteq F$. Therefore $f(Cl_\alpha^D(A)) \subseteq Cl(f(A))$.

(iv)⇒(v) Let $B \subseteq Y$ and $A = f^{-1}(B)$. Then by assumption, $f(Cl_\alpha^D(A)) \subseteq Cl(f(A)) \subseteq Cl(B)$. This implies that $Cl_\alpha^D(A) \subseteq f^{-1}(Cl(B))$. Hence $Cl_\alpha^D(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(v)⇒(vi) Let $B \subseteq Y$. By assumption, $Cl_\alpha^D(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B))$. This implies that, $Cl_\alpha^D(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B))$ and hence $X \setminus Int_\alpha^D(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B))$. By taking complement on both sides we get $f^{-1}(Int(B)) \subseteq Int_\alpha^D(f^{-1}(B))$.

(vi)⇒(i) Let U be any open set in Y. Then $Int(U) = U$. By assumption, $f^{-1}(Int(U)) \subseteq Int_\alpha^D(f^{-1}(U))$ and hence $f^{-1}(U) \subseteq Int_\alpha^D(f^{-1}(U))$. Then $f^{-1}(U) = Int_\alpha^D(f^{-1}(U))$. Therefore by Theorem 3.14, $f^{-1}(U)$ is $D\alpha$-open in X. Thus f is $D\alpha$-continuous.

Theorem 4.6. Let $f : X \to Y$ be $D\alpha$-continuous and let $g : Y \to Z$ be continuous. Then $gof : X \to Z$ is $D\alpha$-continuous.

Proof. Obvious.

Remark 4.7. Composition of two $D\alpha$-continuous functions need not be $D\alpha$-continuous as seen from the following example.

Example 4.8. Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\phi, \{b\}, \{a, b\}, X\}$, $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\phi, \{x\}, Y\}$ and $Z = \{p, q, r\}$ associated with the topology $\nu = \{\phi, \{r\}, Z\}$ and $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = y$, $f(b) = x$, $f(c) = z$. Define $g : (Y, \sigma) \to (Z, \nu)$ by $g(x) = g(y) = p$, $g(z) = r$. One can have that $F_X = \{\phi, \{c\}, \{a, c\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $D\alpha C(X) = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, and $F_Y = \{\phi, \{y, z\}, Y\}$, $GC(Y) = \{\phi, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, Y\}$, $GO(Y) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, Y\}$, $D\alpha C(Y) = D\alpha O(Y) = P(X)$. Clearly, f and g are $D\alpha$-continuous. $\{r\}$ is open in Z. But $(gof)^{-1}(\{r\}) = f^{-1}(g^{-1}(\{r\})) = f^{-1}(\{z\}) = \{c\}$, which is not $D\alpha$-open in X. Therefore gof is not $D\alpha$-continuous.

5. $D\alpha$-open functions and $D\alpha$-closed functions

In this section we introduce $D\alpha$-open functions and $D\alpha$-closed functions and investigate some of their basic properties.

Definition 5.1. A function $f: X \to Y$ is said to be $D\alpha$-open (resp. $D\alpha$-closed) if the image of each open (resp. closed) set in X is $D\alpha$-open (resp. $D\alpha$-closed) in Y.

Theorem 5.2.

(i) Every α-open function is $D\alpha$-open.
(ii) Every g-open function is $D\alpha$-open.

Proof. It is obvious from Theorem 3.6.

Remark 5.3.

(i) $D\alpha$-open function need not be α-open. (see Example 5.4 below)
(ii) $D\alpha$-open function set need not be g-open. (see Example 5.5 below)

Example 5.4. (i) Let $X = \{x,y,z\}$ associated with the topology $\tau = \{\phi, \{x\}, X\}$ and $Y = \{a,b,c\}$ associated with the topology $\sigma = \{\phi, \{a, b\}, \{c\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = a$, $f(y) = b$ and $f(z) = c$. One can have that $F_Y = \alpha O(Y) = \{\phi, \{a, b\}, \{c\}, Y\}$, $GC(Y) = GO(Y) = D\alpha C(Y) = D\alpha O(Y) = P(X)$. Since $\{x\}$ is open in X, $f(\{x\}) = \{a\} \in D\alpha O(Y)$, but $\{a\} \notin \alpha O(Y)$. Therefore f is $D\alpha$-open function but not α-open.

Example 5.5. (ii) Let $X = \{x,y,z\}$ associated with the topology $\tau = \{\phi, \{y\}, \{x, y\}, X\}$ and $Y = \{a, b, c\}$ associated with the topology $\sigma = \{\phi, \{a\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = b$, $f(y) = c$ and $f(z) = a$. One can have that $F_Y = \{\phi, \{b, c\}, Y\}$, $GC(Y) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$, $GO(Y) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$, $D\alpha C(Y) = D\alpha O(Y) = P(X)$. Since $\{x,y\}$ is open in X, $f(\{x, y\}) = \{b, c\} \in D\alpha O(Y)$, but $\{b, c\} \notin GO(Y)$. Therefore f is $D\alpha$-open function but not g-open.

From the above discussions we have the following diagram in which the converses of implications need not be true.

α-open function $\to$ $D\alpha$-open function $\leftarrow$ g-open function

Theorem 5.6. Let $f: X \to Y$ be a function. The following statements are equivalent.

(i) f is $D\alpha$-open.
(ii) For each $x \in X$ and each neighborhood U of x, there exists $D\alpha$-open set $W \subseteq Y$ containing $f(x)$ such that $W \subseteq f(U)$.

Proof. (i)⇒(ii) Let $x \in X$ and U is a neighborhood of x, then there exists an open set $V \subseteq X$ such that $x \in V \subseteq U$. Set $W = f(V)$. Since f is $D\alpha$-open, $f(V) = W \in D\alpha O(Y)$ and so $f(x) \in W \subseteq f(U)$.

(ii)⇒(i) Obvious.

Theorem 5.7. Let $f: X \to Y$ be $D\alpha$-open (resp. $D\alpha$-closed) function and $W \subseteq Y$. If $A \subseteq X$ is a closed (resp. open) set containing $f^{-1}(W)$, then there exists $D\alpha$-closed (resp. $D\alpha$-open) set $H \subseteq Y$ containing W such that $f^{-1}(H) \subseteq A$.

Proof. Let $H = Y \setminus f(X \setminus A)$. Since $f^{-1}(W) \subseteq A$, we have $f(X \setminus A) \subseteq Y \setminus W$. Since f is $D\alpha$-open (resp. $D\alpha$-closed), then H is $D\alpha$-closed (resp. $D\alpha$-open) set and $f^{-1}(H) = X \setminus f^{-1}(f(X \setminus A)) \subset X \setminus (X \setminus A) = A$.

Corollary 5.8. If $f: X \to Y$ is $D\alpha$-open, then $f^{-1}(Cl^D_\alpha(B)) \subseteq Cl(f^{-1}(B))$ for each set $B \subset Y$.

Proof. Since $Cl(f^{-1}(B))$ is closed in X containing $f^{-1}(B)$ for a set $B \subseteq Y$. By Theorem 5.7, there exists $D\alpha$-closed set $H \subseteq Y$, $B \subseteq H$ such that $f^{-1}(H) \subseteq Cl(f^{-1}(B))$. Thus, $f^{-1}(Cl^D_\alpha(B)) \subseteq f^{-1}(Cl^D_\alpha(H)) \subseteq f^{-1}(H) \subseteq Cl(f^{-1}(B))$.

Theorem 5.9. A function $f: X \to Y$ is $D\alpha$-open if and only if $f(Int(A)) \subseteq Int^D_\alpha(f(A))$ for every subset A of X.

Proof. Suppose $f: X \to Y$ is $D\alpha$-open function and $A \subseteq X$. Then $Int(A)$ is open set in X and $f(Int(A))$ is $D\alpha$-open set contained in $f(A)$. Therefore $f(Int(A)) \subseteq Int^D_\alpha(f(A))$. Conversely, let be $f(Int(A)) \subseteq Int^D_\alpha(f(A))$ for every subset A of X and U is open set in X. Then $Int(U) = U$, $f(U) \subseteq Int^D_\alpha(f(U))$. Hence $f(U) = Int^D_\alpha(f(U))$. By Theorem 3.14 $f(U)$ is $D\alpha$-open.

Theorem 5.10. For any bijective function $f: (X, \tau) \to (Y, \sigma)$ the following statements are equivalent.

(i) f^{-1} is $D\alpha$-continuous function.
(ii) f is $D\alpha$-open function.
(iii) f is $D\alpha$-closed function.

Proof. (i)⇒(ii) Let U be an open set in X. Then $X \setminus U$ is closed in X. Since f^{-1} is $D\alpha$-continuous, $(f^{-1})^{-1}(X \setminus U)$ is $D\alpha$-closed in Y. That is $f(X \setminus U) = Y \setminus f(U)$ is $D\alpha$-closed in Y. This implies $f(U)$ is $D\alpha$-open in Y. Hence f is $D\alpha$-open function.

(ii)⇒ (iii) Let F be a closed set in X. Then $X \setminus F$ is open in X. Since f is $D\alpha$-open, $f(X \setminus F)$ is $D\alpha$-open in Y. That is $f(X \setminus F) = Y \setminus f(F)$ is $D\alpha$-open in Y. This implies $f(F)$ is $D\alpha$-closed in Y. Hence f is $D\alpha$-closed function.

(iii)⇒ (i) Let F be closed set in X. Since f is $D\alpha$-closed function, $f(F)$ is $D\alpha$-closed in Y. That is $(f^{-1})^{-1}(F)$ is $D\alpha$-closed in Y. Hence f^{-1} is $D\alpha$-continuous function.

Remark 5.11. Composition of two $D\alpha$-open functions need not be $D\alpha$-open as seen from the following example.

Example 5.12. Let $X = \{x, y, z\}$ associated with the topology $\tau = \{\phi, \{x, y\}, \{z\}, X\}$, $Y = \{p, q, r\}$ associated with the topology $\sigma = \{\phi, \{p\}, Y\}$ and $Z = \{a,b,c\}$ associated with the topology, $\nu = \{\phi, \{b\}, \{a, b\}, Z\}$. Define $f: (X, \tau) \to (Y, \sigma)$ by $f(x) = p$, $f(y) = q$, $f(z) = r$ and $g: (Y, \sigma) \to (Z, \nu)$ by $g(p) = b$, $g(q) = a$, $g(r) = c$. One can have that; $F_Y = \{\phi, \{q, r\}, Y\}$, $GC(Y) = \{\phi, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, Y\}$, $GO(Y) = \{\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, Y\}$, $D\alpha C(Y) = D\alpha O(Y) = P(X)$ and $F_Z = \{\phi, \{c\}, \{a, c\}, Z\}$, $GC(Z) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, Z\}$, $GO(Z) = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$, $D\alpha C(Z) = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, Z\}$, $D\alpha O(Z) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Z\}$. Clearly, f and g are $D\alpha$-open function. $\{z\}$ is open in X. But $gof(\{z\}) = g(f(\{z\})) = g(\{r\}) = \{c\}$ which is not $D\alpha$-open in Z. Therefore gof is not $D\alpha$-open function.

6. $D\alpha$-closed graph and strongly $D\alpha$-closed

In this section we introduce $D\alpha$-closed graph and strongly $D\alpha$-closed and investigate some of their basic properties.

Definition 6.1. A function $f: X \to Y$ has $D\alpha$-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $(U \times Cl^*(V)) \cap G(f) = \phi$.

Remark 6.2. Evidently every closed graph is $D\alpha$-closed. That the converse is not true is seen from the following example.

Example 6.3. Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\phi, \{a, b\}, X\}$ and $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\phi, \{x\}, \{x, y\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = f(c) = x$, $f(b) = y$. One can have that $F_X = \{\phi, \{c\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $F_Y = \{\phi, \{z\}, \{y, z\}, Y\}$, $GC(Y) = \{\phi, \{z\}, \{x, z\}, \{y, z\}, Y\}$, $GO(Y) = \{\phi, \{x\}, \{y\}, \{x, y\}, Y\}$. Since $\{a, c\} \in D\alpha O(X, c)$ and $\{y\} \in GO(Y, y)$ but $\{a, c\} \notin O(X)$ and $\{y\} \notin O(Y)$. Therefore $G(f)$ is $D\alpha$-closed but not closed.

Theorem 6.4. Let $f: (X, \tau) \to (Y, \sigma)$ be a function and

(i) f is $D\alpha$-closed graph;
(ii) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $f(U) \cap Cl^*(V) = \phi$.
(iii) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in D\alpha O(Y, y)$ such that $(U \times Cl_\alpha^D(V)) \cap G(f) = \phi$.
(iv) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in D\alpha O(Y, y)$ such that $f(U) \cap Cl_\alpha^D(V) = \phi$. Then
(1) (i)$\Leftrightarrow$ (ii)
(2) (i)$\Rightarrow$ (iii)
(3) (ii)$\Rightarrow$ (iv)
(4) (i)$\Rightarrow$ (iv)

Proof. (i)$\Rightarrow$(ii) Suppose f is $D\alpha$-closed graph. Then for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $(U \times Cl^*(V)) \cap G(f) = \phi$. This implies that for each $f(x) \in f(U)$ and $y \in Cl^*(V)$. Since $y \neq f(x)$, $f(U) \cap Cl^*(V) = \phi$.

(ii)$\Rightarrow$(i) Let $(x, y) \in (X \times Y) \setminus G(f)$. Then there exists $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $f(U) \cap Cl = {}^*(V) = \phi$. This implies that $f(x) \neq y$ for each $x \in U$ and $y \in Cl^*(V)$. Therefore $(U \times Cl^*(V)) \cap G(f) = \phi$.

(i)$\Rightarrow$(iii) Suppose f is $D\alpha$-closed graph. Then for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $(U \times Cl^*(V)) \cap G(f) = \phi$. Since g-open set is $D\alpha$-open, $Cl_\alpha^D(V) \subseteq Cl^*(V)$. Therefore $(U \times Cl_\alpha^D(V)) \cap G(f) = \phi$.

(ii)$\Rightarrow$(iv) Let $(x, y) \in (X \times Y) \setminus G(f)$. Then there exists $U \in D\alpha O(X, x)$ and $V \in GO(Y, y)$ such that $f(U) \cap Cl^*(V) = \phi$. Since $Cl_\alpha^D(V) \subseteq Cl^*(V)$, $f(U) \cap Cl_\alpha^D(V) \subseteq f(U) \cap Cl^*(V) = \phi$.

(i)$\Rightarrow$(iv) From (ii).

Definition 6.5. A topological space (X, τ) is said to be $D\alpha$-T_1 if for any distinct pair of points x and y in X, there exist $D\alpha$-open U in X containing x but not y and an $D\alpha$-open V in X containing y but not x.

Theorem 6.6.

(i) Every α-T_1 space is $D\alpha$-T_1.
(ii) Every g-T_1 space is $D\alpha$-T_1.

Proof. It is obvious from Theorem 3.6.

Remark 6.7. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.8. Let $f: X \to Y$ be any surjection with $G(f)$ $D\alpha$-closed. Then Y is g-T_1.

Proof. Let $y_1, y_2 (y_1 \neq y_2) \in Y$. The subjectivity of f gives the existence of an element $x_o \in X$ such that $f(x_o) = y_2$. Now $(x_o, y_1) \in (X \times Y) \setminus G(f)$. The $D\alpha$-closeness of $G(f)$ provides $U_1 \in D\alpha O(X, x_o)$, $V_1 \in GO(Y, y_1)$ such that $f(U_1) \cap Cl^*(V_1) = \phi$. Now $x_o \in U_1 \Rightarrow f(x_o) = y_2 \in f(U_1)$. This and the fact that $f(U_1) \cap Cl^*(V_1) = \phi$ guarantee that $y_2 \notin V_1$. Again from the subjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) \setminus G(f)$ and the $D\alpha$-closedness of $G(f)$ provides $U_2 \in D\alpha O(X, x_1)$, $V_2 \in GO(Y, y_2)$ such that $f(U_2) \cap Cl^*(V_2) = \phi$. Now $x_1 \in U_2 \Rightarrow f(x_1) = y_1 \in f(U_2)$ so that $y_1 \notin V_2$. Thus we obtain sets $V_1, V_2 \in GO(Y)$ such that $y_1 \in V_1$ but $y_2 \notin V_1$ while $y_2 \in V_2$ but $y_1 \notin V_2$. Hence Y is g-T_1.

Corollary 6.9. Let $f: X \to Y$ be any surjection with $G(f)$ $D\alpha$-closed. Then Y is $D\alpha$-T_1.

Proof. Follows From Theorems 6.6 (i) and 6.8.

Theorem 6.10. Let $f: X \to Y$ be any injective with $G(f)$ $D\alpha$-closed. Then X is $D\alpha$-T_1.

Proof. Let $x_1, x_2 (x_1 \neq x_2) \in X$. The injectivity of f implies $f(x_1) \neq f(x_2)$ whence one obtains that $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. The $D\alpha$-closedness of $G(f)$ provides $U_1 \in D\alpha O(X, x_1)$, $V_1 \in GO(Y, f(x_2))$ such that $f(U_1) \cap Cl^*(V_1) = \phi$. Therefore $f(x_2) \notin f(U_1)$ and a fortiori $x_2 \notin U_1$. Again $(x_2, f(x_1)) \in (X \times Y) \setminus G(f)$ and $D\alpha$-closedness of $G(f)$ as before gives $U_2 \in D\alpha O(X, x_2)$, $V_2 \in GO(Y, f(x_1))$ with $f(U_2) \cap Cl^*(V_2) = \phi$, which guarantees that $f(x_1) \notin f(U_2)$ and so $x_1 \notin U_2$. Therefore, we obtain sets U_1 and $U_2 \in D\alpha O(X)$ such that $x_1 \in U_1$ but $x_2 \notin U_1$ while $x_2 \in U_2$ but $x_1 \notin U_2$. Hence X is $D\alpha$-T_1.

Corollary 6.11. Let $f: X \to Y$ be any bijection with $G(f)$ $D\alpha$-closed. Then both X and Y are $D\alpha$-T_1.

Proof. It readily follows from Corollary 6.9 and Theorem 6.10.

Definition 6.12. A topological space (X, τ) is said to be $D\alpha$-T_2 if for any distinct pair of points x and y in X, there exist $D\alpha$-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.

Theorem 6.13.

(i) Every α-T_2 space is $D\alpha$-T_2.
(ii) Every g-T_2 space is $D\alpha$-T_2.

Proof. Obvious.

Remark 6.14. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.15. Let $f: X \to Y$ be any surjection with $G(f)$ $D\alpha$-closed. Then Y is g-T_2.

Proof. Let $y_1, y_2 (y_1 \neq y_2) \in Y$. The subjectivity of f gives a $x_1 \in X$ such that $f(x_1) = y_1$. Now $(x_1, y_2) \in (X \times Y) \setminus G(f)$. The $D\alpha$-closedness of $G(f)$ provides $U \in D\alpha O(X, x_1)$, $V \in GO(Y, y_2)$ such that $f(U) \cap Cl^*(V) = \phi$. Now $x_1 \in U \Rightarrow f(x_1) = y_1 \in f(U)$. This and the fact that $f(U) \cap Cl^*(V) = \phi$ guarantee that $y_1 \notin Cl^*(V)$. This mean that there exists $W \in GO(Y, y_1)$ such that $W \cap V = \phi$. Hence Y is g-T_2.

Corollary 6.16. Let $f : X \to Y$ be any surjection with $G(f)$ $D\alpha$-closed. Then Y is $D\alpha$-T_2.

Proof. Follows from Theorems 6.13 (ii) and 6.15.

Definition 6.17. A function $f : X \to Y$ has a strongly $D\alpha$-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in O(Y, y)$ such that $(U \times Cl(V)) \cap G(f) = \phi$.

Corollary 6.18. A strongly $D\alpha$-closed graph is $D\alpha$-closed. That the converse is not true is seen from Example 6.3, where $\{y\} \in GO(Y, y)$ but $\{y\} \notin O(Y)$. Therefore $G(f)$ is $D\alpha$-closed but not strongly $D\alpha$-closed.

Remark 6.19. Evidently every strongly α-closed graph (resp. strongly closed graph) is strongly $D\alpha$-closed graph. That the converse is not true is seen from the following example.

Example 6.20. Let $X = \{a, b, c\}$ associated with the topology $\tau = \{\phi, \{a, b\}, X\}$ and $Y = \{x, y, z\}$ associated with the topology $\sigma = \{\phi, \{x, y\}, \{z\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = f(c) = x$, $f(b) = y$. One can have that $F_X = \{\phi, \{c\}, X\}$, $GC(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $GO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\alpha O(X) = \{\phi, \{a, b\}, X\}$, $D\alpha O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Since $\{a, c\} \in D\alpha O(X, c)$ and $\{z\} \in O(Y, z)$ but $\{a, c\} \notin \alpha O(X)$ (resp. $\{a, c\} \notin O(X)$). Therefore $G(f)$ is strongly $D\alpha$-closed but not strongly α-closed (resp. strongly closed).

Theorem 6.21. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:

(i) f has strongly $D\alpha$-closed graph.
(ii) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in O(Y, y)$ such that $f(U) \cap Cl(V) = \phi$.
(iii) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in \alpha O(Y, y)$ such that $(U \times Cl_\alpha(V)) \cap G(f) = \phi$.
(iv) For each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in D\alpha O(X, x)$ and $V \in \alpha O(Y, y)$ such that $f(U) \cap Cl_\alpha(V) = \phi$.

Proof. Similar to the proof of Theorem 6.4.

Theorem 6.22. If $f : X \to Y$ is a function with a strongly $D\alpha$-closed graph, then for each $x \in X$, $f(x) = \cap\{Cl_\alpha(f(U)) : U \in D\alpha O(X, x)\}$.

Proof. Suppose the theorem is false. Then there exists a $y \neq f(x)$ such that $y \in \cap\{Cl_\alpha(f(U)) : U \in D\alpha O(X, x)\}$. This implies that $y \in Cl_\alpha(f(U))$ for every $U \in D\alpha O(X, x)$. So $V \cap f(U) \neq \phi$ for every $V \in \alpha O(Y, y)$. This, in its turn, indicates that $Cl_\alpha(V) \cap f(U) \supset V \supset f(U) \neq \phi$, which contradicts the hypothesis that f is a function with $D\alpha$-closed graph. Hence the theorem holds.

Theorem 6.23. If $f : X \to Y$ is $D\alpha$-continuous function and Y is T_2. Then $G(f)$ is strongly $D\alpha$-closed.

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Since Y is T_2, there exists a set $V \in O(Y, y)$ such that $f(x) \notin Cl(V)$. But $Cl(V)$ is closed. Now $Y \setminus Cl(V) \in O(Y, f(x))$. By Theorem 4.5 there exists $U \in D\alpha O(X, x)$ such that $f(U) \subseteq Y \setminus Cl(V)$. Consequently, $f(U) \cap Cl(V) = \phi$ and therefore $G(f)$ is strongly $D\alpha$-closed.

Theorem 6.24. Let $f : X \to Y$ be any surjection with $G(f)$ strongly $D\alpha$-closed. Then Y is T_1 and α-T_1.

Proof. Similar to the proof of Theorem 6.8 and T_1-ness always guarantees α-T_1-ness. Hence Y is α-T_1.

Corollary 6.25. Let $f : X \to Y$ be any surjection with $G(f)$ strongly $D\alpha$-closed. Then Y is $D\alpha$-T_1.

Proof. Follows From Theorems 6.6 (i) and 6.24.

Theorem 6.26. Let $f : X \to Y$ be any injective with $G(f)$ strongly $D\alpha$-closed. Then X is $D\alpha$-T_1.

Proof. Similar to the proof of Theorem 6.10.

Corollary 6.27. Let $f : X \to Y$ be any bijection with $G(f)$ strongly $D\alpha$-closed. Then both X and Y are $D\alpha$-T_1.

Proof. It readily follows from Corollary 6.25 and Theorem 6.26.

Theorem 6.28. Let $f : X \to Y$ be any surjection with $G(f)$ strongly $D\alpha$-closed. Then Y is T_2 and α-T_2.

Proof. Similar to the proof Theorem 6.15 and T_2-ness always guarantees α-T_2-ness. Hence Y is α-T_2.

Corollary 6.29. Let $f : X \to Y$ be any surjection with $G(f)$ strongly $D\alpha$-closed. Then Y is $D\alpha$-T_2.

Proof. Follows From Theorems 6.13 (i) and 6.28.

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