SOFT TOPOLOGY AND SOFT PROXIMITY AS FUZZY PREDICATES BY FORMULAE OF ŁUKASIEWICZ LOGIC

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This paper is dedicated to Professor L. A. Zadeh on the occasion of his 95th birthday and the 50th year of the birth of fuzzy logic

Abstract. In this paper, based in the Łukasiewicz logic, the definition of fuzzifying soft neighborhood structure and fuzzifying soft continuity are introduced. Also, the fuzzifying soft proximity spaces which are a generalizations of the classical soft proximity spaces are given. Several theorems on classical soft proximities are special cases of the theorems we prove in this paper.

1. Introduction

Many disciplines, including engineering, medicine, economics, and sociology, are highly dependent on the task of modeling uncertain data. When the uncertainty is highly complicated and difficult to characterize, classical mathematical approaches are often insufficient to derive effective or useful models. Testifying to the importance of uncertainties that cannot be defined by classical means, researchers are introducing alternative theories every day. In addition to classical probability theory, some of the most important results on this topic are fuzzy sets [23], intuitionistic fuzzy sets [3, 4], vague sets [6], interval mathematics [4, 7], and rough sets [12]. However, all of these new theories have inherent difficulties which are pointed out in [11]. A possible reason is that these theories possess inadequate parameterizations tools [10-11]. Molodtsov [11] introduced soft sets as a mathematical tool for dealing with uncertainties which is free from the above difficulties. Soft set theory has rich potential for practical applications in several domains, a few of which are indicated by Molodtsov in his pioneer work [11]. Maji et al. [9] described an application of soft set theory to a decision-making problem. Pei and Miao [13] investigated the relationships between soft sets and information systems. In 2001, Maji et al. [8] expanded the soft set to fuzzy soft set theory. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [1] presented some more properties of fuzzy soft sets. Yang et al.[20] combined the interval-valued fuzzy set and soft set models and introduced the concept of interval-valued fuzzy soft set. Algebraic structures of soft sets and fuzzy soft sets have been studied increasingly in recent years.In[2] introduced fuzzy soft groups by using a t-norm. Feng [5] defined soft semirings and investigated several related properties. Varol et al. [19] studied fuzzy soft rings. Topological structures of soft set and fuzzy soft set have been studied

Invited Paper: September 2016
Key words and phrases: Soft set, Soft topology, Fuzzifying soft topology, Fuzzifying soft proximity.

by some authors in recent years. Shabir and Naz [15] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. In [24], Zorlutuna et al. introduced some other new concepts in soft topological spaces. The soft topological structures of soft set theories dealing with uncertainties were studied by Tanay and Kandemir [16]. In [17] Fuzzy soft topology was studied. As a different approach to soft topology Varol et al. [19] interpreted categories related to categories of topological spaces as special categories of soft sets. In [18] the authors considered the soft interpretation of topological spaces. They defined soft topology and L-fuzzy soft topology, which are mappings from the parameter set E to 2^{2^X} and from E to L^{L^X} respectively (where L is a fuzzy lattice). Based on paper [24], in paper [22], the authors introduced the definitions of L-fuzzifying soft topological spaces and L-fuzzifying soft interior spaces .
In 1952, Rosser and Turquette [14] proposed emphatically the following problem: If there are many-valued theories beyond the level of predicates calculus, then what are the detail of such theories ? As an attempt to give a partial answer to this problem in the case of point set soft topology, we use a semantical method of continuous-valued logic to develop systematically fuzzifying soft topology. Briefly speaking, a fuzzifying soft topology on a set X assigns each crisp soft subset of X to a certain degree of being soft open, other than being definitely soft open or not. Roughly speaking, the semantical analysis approach transforms formal statements of interest, which are usually expressed as implication formulas in logical language, into some inequalities in the truth value set by truth valuation rules, and then these inequalities are demonstrated in an algebraic way and the semantic validity of conclusions is thus established. In this paper, based in the Łukasiewicz logic, the definition of fuzzifying soft neighborhood structure and fuzzifying soft continuity are introduced. Also, the fuzzifying soft proximity spaces which are a generalizations of the classical soft proximity spaces are given. Several theorems on classical soft proximities are special cases of the theorems we prove in this paper.

2. **Preliminaries**

First, we display the Łukasiewicz logic and corresponding set theoretical notations used in this paper in the following definition (see [21]).

Definition 2.1. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) (a) $[\alpha] = \alpha(\alpha \in [0,1])$;
 (b) $[\varphi \wedge \psi] = \min([\varphi], [\psi])$;
 (c) $[\varphi \to \psi] = \min(1, 1 - [\varphi] + [\psi])$;

(2) If $\widetilde{A} \in \Im(X), [x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition the following derived formulae are given,

(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$;

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi], [\psi])$;
(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;
(5) $[\exists x\varphi(x)] := [\neg\forall x\neg\varphi(x)] := \sup_{x\in X} [\varphi(x)]$;
(6) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then
(a) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \inf_{x\in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$,
(b) $[\widetilde{A} \equiv \widetilde{B}] := [\widetilde{A} \subseteq \widetilde{B}] \wedge [\widetilde{B} \subseteq \widetilde{A}]$.
where $\mathcal{F}(\mathcal{X})$ is the family of all fuzzy sets in X.

Often we do not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying did.

Second we give some basic concepts related to soft sets, soft topology, L-fuzzifying soft topology.

Definition 2.2. [11] (1) A soft set on an universe X is a pair (M, E) (here E is a nonempty parameter set), and $M : E \to 2^X$ (the set of all subset of X) is a mapping. The set of all soft sets on X is denoted by $\mathbf{S}(\mathbf{X}, \mathbf{E})$.
(2) For two given subsets (M, E), $(N, E) \in \mathbf{S}(\mathbf{X}, \mathbf{E})$, we say that (M, E) is a soft subset of (N, E), denoted by $(M, E) \sqsubseteq (N, E)$, if for all $e \in E$, $M(e) \subseteq N(e)$.
If $(M, E) \sqsubseteq (N, E)$ and $(M, E) \sqsupseteq (N, E)$, we say (M, E) and (N, E) be soft equal. We denote it by $(M, E) = (N, E)$.

Definition 2.3. [10] The union of two soft sets $(F, A)$and (G, B) on X is the soft set (H, C), where

$$C = A \cup B \text{ and } H(e) = \begin{cases} F(e) & e \in A\backslash B \\ G(e) & e \in B\backslash A \\ F(e) \cup G(e) & e \in A \cap B \end{cases} (\forall e \in C)$$

We write $(F, A) \sqcup (G, B) = (H, C)$.

Definition 2.4. [15] The intersection of two soft sets $(F, A)$and (G, B) on X is the soft set (H, C), where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ $(\forall e \in C)$. We write $(F, A) \sqcap (G, B) = (H, C)$.

Definition 2.5. [15] (1) For each $A \in 2^X$, $(\widetilde{A}, E) \in \mathbf{S}(\mathbf{X}, \mathbf{E})$ is defined by $\widetilde{A}(e) = A$ for each $e \in E$;we identify $\widetilde{\{x\}}$ with $\widetilde{x}$ for each $x \in X$. For each $(M, E) \in \mathbf{S}(\mathbf{X}, \mathbf{E})$, $(M^c, E) \in \mathbf{S}(\mathbf{X}, \mathbf{E})$ is defined by $M^c(e) = X\backslash M(e)(\forall e \in E)$; sometimes we use $(M, E)^c$ (resp. $\widetilde{A}$) to replace (M^c, E) (resp. $(\widetilde{A}, E)$).
(2) For a given subset $\{(H_\lambda, E)\}_{\lambda\in\Lambda} \subseteq \mathbf{S}(\mathbf{X}, \mathbf{E})$, we call members $(M, E) = \sqcup_{\lambda\in\Lambda}(H_\lambda, E)$ and $(N, E) = \sqcap_{\lambda\in\Lambda}(H_\lambda, E)$ of $\mathbf{S}(\mathbf{X}, \mathbf{E})$ union and intersection of the family $\{(H_\lambda, E)\}_{\lambda\in\Lambda}$, respectively, which are defined by $M(e) = \cup_{\lambda\in\Lambda}H_\lambda(e)$ $(\forall e \in E)$ and $N(e) = \cap_{\lambda\in\Lambda}H_\lambda(e)(\forall e \in E)$.
(3) For two given subsets $(M, E), (N, E) \in \mathbf{S}(\mathbf{X}, \mathbf{E})$, then
(i) $((M, E) \sqcup (N, E))^c = (M, E)^c \sqcap (N, E)^c$.;
(ii) $((M, E) \sqcap (N, E))^c = (M, E)^c \sqcup (N, E)^c$.

Definition 2.6. [19] Defined soft function $(f, g) : \mathbf{S}(\mathbf{X}, \mathbf{E}) \to \mathbf{S}(\mathbf{Y}, \mathbf{F})$ by

$$(f, g)(M, E) = (\overrightarrow{g}(M), f(E))$$

for each $(M,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$ and

$$(f,g)^{-1}(N,F) = (\overleftarrow{g} \circ N \circ f, f^{-1}(F))$$

for each $(N,F) \in \mathbf{S}(\mathbf{Y},\mathbf{F})$, where for every $\alpha \in f(E)$ and for every $e \in f^{-1}(F)$ we have

$$\overrightarrow{g}(M)(\alpha) = \bigcup_{f(e)=\alpha} g(M(e)), (\overleftarrow{g} \circ N \circ f)(e) = \overleftarrow{g}(N(f(e)))$$

$f(E)$ is the image of E in the category **SET**, $f^{-1}(F)$ is the preimage of F in the category **SET.** $\overrightarrow{g}(M)$ is defined by the Zadeh extension principle, $\overleftarrow{g}(M)$ is the backward operator induced by the mapping $g : X \to Y$.

Definition 2.7. [24] (1) The soft set $(M,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$ is called a soft point in X, denoted by e_M, if for the element $e \in E, M(e) \neq \phi$ and $M(e_\circ) = \phi$ for all $e_\circ \in E \backslash \{e\}$.
(2) The soft point e_M is said to be in the soft set (N,E), for each $e \in E$, denoted by $e_M \widetilde{\in} (N,E)$,we have $M(e) \subseteq N(e)$.
(3) Let $e_M \in \mathbf{SP}(\mathbf{X})$and $(N,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$. If $e_M \widetilde{\in} (N,E)$, then $e_M \widetilde{\notin} (N,E)^c$.

More Knowledge about the soft point can be founded in [24]. $\mathbf{SP}(\mathbf{X})$ denoted the set of all soft points in X. Obviously, if $e_M \widetilde{\in} \mathbf{SP}(\mathbf{X})$, then $(id_E, g)(e_M) \widetilde{\in} \mathbf{SP}(\mathbf{Y})$.

3. Fuzzifying Soft Topologies

In this section, we use the semantics of fuzzy logic to investigate soft topology, and to propose a soft topology whose logical fundament is fuzzy. Also, we discuss the soft neighborhood structures of a soft point and soft continuity in this framework.

Definition 3.1. Let X be a universe of discourse, $\widetilde{\tau} \in \Im(\mathbf{S}(\mathbf{X},\mathbf{E}))$ satisfy the following conditions:
(1) $\models \widetilde{\phi} \in \widetilde{\tau}$ and $\models \widetilde{X} \in \widetilde{\tau}$;
(2) $\forall (M,E), (N,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$, $\models (M,E) \in \widetilde{\tau} \wedge (N,E) \in \widetilde{\tau} \to (M,E) \sqcap (N,E) \in \widetilde{\tau}$;
(3) $\forall \{(F_\lambda,E)\}_{\lambda \in \Lambda} \subseteq \mathbf{S}(\mathbf{X},\mathbf{E})$, $\models \forall \lambda (\lambda \in \Lambda \to (F_\lambda,E) \in \widetilde{\tau}) \to \bigsqcup_{\lambda \in \Lambda} (F_\lambda,E) \in \widetilde{\tau}$.

Then $\widetilde{\tau}$ is called a soft fuzzifying topology and the triple $(X,\widetilde{\tau},E)$ is called a fuzzifying soft topological space. $\widetilde{\tau}(M,E)$ can be interpreted as the degree to which (M,E) is a soft open set. If (X,τ) is a soft topological space over X, define $\chi_\tau \in \Im(\mathbf{S}(\mathbf{X},\mathbf{E}))$ as follows: $\chi_\tau(M,E) = 1$, if $(M,E) \in \tau$; if not, $\chi_\tau(M,E) = 0$. Obviously, χ_τ is a special fuzzifying soft topology.

Remark 3.2. The conditions in Definition 3.1, may be rewritten respectively as follows:
(1) $\widetilde{\tau}(\widetilde{\phi}) = \widetilde{\tau}(\widetilde{X}) = 1$;
(2) $\forall (M,E), (N,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$, $\widetilde{\tau}((M,E) \sqcap (N,E)) \geq \widetilde{\tau}(M,E) \wedge \widetilde{\tau}(N,E))$;
(3) $\forall \{(F_\lambda,E)\}_{\lambda \in \Lambda} \subseteq \mathbf{S}(\mathbf{X},\mathbf{E})$, $\widetilde{\tau}(\bigsqcup_{\lambda \in \Lambda} (F_\lambda,E)) \geq \bigwedge_{\lambda \in \Lambda} \widetilde{\tau}(F_\lambda,E)$.

Definition 3.3. The family of fuzzifying soft closed sets is denoted by $\widetilde{F} \in \Im(\mathbf{S}(\mathbf{X},\mathbf{E}))$, and defined as $(M,E) \in \widetilde{F} := (M,E)^c \in \widetilde{\tau}$, i.e., $\widetilde{F}(M,E) = \widetilde{\tau}(M,E)^c$.

Theorem 3.4. (1) $\models \widetilde{\phi} \in \widetilde{\mathcal{F}}$ and $\models \widetilde{X} \in \widetilde{\mathcal{F}}$;
(2) $\forall (M,E),(N,E) \in \mathbf{S}(\mathbf{X},\mathbf{E})$, $\models (M,E) \in \widetilde{\mathcal{F}} \wedge (N,E) \in \widetilde{\mathcal{F}} \to (M,E) \sqcup (N,E) \in \widetilde{\mathcal{F}}$;
(3) $\forall \{(F_\lambda,E)\}_{\lambda\in\Lambda} \subseteq \mathbf{S}(\mathbf{X},\mathbf{E})$, $\models \forall\lambda(\lambda \in \Lambda \to (F_\lambda,E) \in \widetilde{\tau}) \to \underset{\lambda\in\Lambda}{\sqcap}(F_\lambda,E) \in \widetilde{\mathcal{F}}$.

Definition 3.5. Let $e_M \in \mathbf{SP}(\mathbf{X})$. The fuzzifying soft neighborhood system of e_M is denoted by $SN_{e_M} \in \Im(\mathbf{S}(\mathbf{X},\mathbf{E}))$ and defined as $SN_{e_M}(F,E) = \bigvee_{e_M \widetilde{\in}(G,E)\sqsubseteq(F,E)} \widetilde{\tau}(G,E)$.

Lemma 3.6. $\bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{e_M \widetilde{\in}(F,E)\sqsubseteq(G,E)} \widetilde{\tau}(G,E) = \widetilde{\tau}(F,E)$.

Proof. First, we have $\bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{e_M \widetilde{\in}(F,E)\sqsubseteq(G,E)} \widetilde{\tau}(G,E) \geq \widetilde{\tau}(F,E)$.
In the other hand, let $\beta_{e_M} = \{(G,E) : e_M \widetilde{\in}(F,E) \sqsubseteq (G,E)\}$. Then, for any $f \in \prod_{e_M \widetilde{\in}(F,E)} \beta_{e_M}$, we have $\bigsqcup_{e_M \widetilde{\in}(F,E)} f(e_M) = (F,E)$. Furthermore

$$\bigwedge_{e_M \widetilde{\in}(F,E)} \widetilde{\tau}(f(e_M)) \leq \widetilde{\tau}\left(\bigsqcup_{e_M \widetilde{\in}(F,E)} f(e_M)\right) = \widetilde{\tau}(F,E)$$

$$\widetilde{\tau}(F,E) \geq \bigvee_{f \in \prod_{e_M \widetilde{\in}(F,E)} \beta_{e_M}} \bigwedge_{e_M \widetilde{\in}(F,E)} \widetilde{\tau}(f(e_M)) = \bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{e_M \widetilde{\in}(F,E)\sqsubseteq(G,E)} \widetilde{\tau}(G,E)$$

□

Theorem 3.7. For any $e_M,(F,E)$,
$\models (F,E) \in \widetilde{\tau} \leftrightarrow \forall e_M (e_M \widetilde{\in}(F,E) \to \exists (G,E)\ (((G,E) \in SN_{e_M}) \wedge ((G,E) \sqsubseteq (F,E)))$.

Proof.

$$\begin{aligned}
[\forall e_M (e_M \widetilde{\in}(F,E) &\to \exists (G,E)\,(((G,E) \in SN_{e_M}) \wedge ((G,E) \sqsubseteq (F,E)))] \\
&= \bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{(F,E)\sqsubseteq(G,E)} SN_{e_M}(G,E) \\
&= \bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{(F,E)\sqsubseteq(G,E)} \bigvee_{e_M \widetilde{\in}(H,E)\sqsubseteq(G,E)} \widetilde{\tau}(H,E) \\
&= \bigwedge_{e_M \widetilde{\in}(F,E)} \bigvee_{e_M \widetilde{\in}(H,E)\sqsubseteq(F,E)} \widetilde{\tau}(H,E) \\
&= \widetilde{\tau}(F,E).
\end{aligned}$$

□

Theorem 3.8. The mapping $SN : \mathbf{SP}(\mathbf{X}) \to \Im^N(\mathbf{S}(\mathbf{X},\mathbf{E}))$, $e_M \mapsto SN_{e_M}$, where $\Im^N(\mathbf{S}(\mathbf{X},\mathbf{E}))$ is the set of all normal fuzzy soft subset of $\mathbf{S}(\mathbf{X},\mathbf{E})$, has the following properties:
(1) for any $e_M,(F,E)$,

$$\models (F,E) \in SN_{e_M} \to e_M \widetilde{\in}(F,E)$$

(2) for any $e_M,(F,E),(G,E)$,

$$\models ((F,E) \sqsubseteq (G,E)) \to ((F,E) \in SN_{e_M} \to (G,E) \in SN_{e_M})$$

(3) for any $e_M, (F,E), (G,E)$,

$$\models ((F,E) \in SN_{e_M}) \wedge ((G,E) \in SN_{e_M}) \to (F,E) \sqcap (G,E) \in SN_{e_M}$$

(4) for any $e_M, (F,E)$,

$$\models ((F,E) \in SN_{e_M}) \to \exists (G,E)((G,E) \in SN_{e_M}) \wedge ((G,E) \sqsubseteq (F,E)) \wedge \forall e_N (e_N \widetilde{\in} (G,E) \to (G,E) \in SN_{e_N}))$$

Conversely, if a mapping SN satisfies (2), (3), then $\widetilde{\tau}$ is a fuzzifying soft topology which is defined as

$$(F,E) \in \widetilde{\tau} := \forall e_M (e_M \widetilde{\in} (F,E) \to (F,E) \in SN_{e_M})$$

Specially, if it satisfies (1), (4) also, then for any $e_M \in \mathbf{SP}(\mathbf{X})$, SN_{e_M} is the fuzzifying soft neighborhood system of e_M with respect to $\widetilde{\tau}$.

Proof. (A) (1) If $[(F,E) \in SN_{e_M}] = \bigvee_{e_M \widetilde{\in} (G,E) \sqsubseteq (F,E)} \widetilde{\tau}(G,E) > 0$, then there exists $(G_\circ, E)$ such that $e_M \widetilde{\in} (G_\circ, E) \sqsubseteq (F,E)$. Now, we have $[e_M \widetilde{\in} (F,E)] = 1$. Therefore $[(F,E) \in SN_{e_M}] \leq [e_M \widetilde{\in} (F,E)]$ holds always.

(2) If $[(F,E) \sqsubseteq (G,E)] = 0$, then the result holds. Now, suppose that $[(F,E) \sqsubseteq (G,E)] = 1$. Therefore

$$[(F,E) \in SN_{e_M}] = \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)} \widetilde{\tau}(H,E) \leq \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (G,E)} \widetilde{\tau}(H,E) = [(G,E) \in SN_{e_M}]$$

(3)

$$\begin{aligned}
[(F,E) \quad \sqcap \quad (G,E) \in SN_{e_M}] &= \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (F,E) \sqcap (G,E)} \widetilde{\tau}(H,E) \\
&= \bigvee_{e_M \widetilde{\in} (H_1,E) \sqsubseteq (F,E), e_M \widetilde{\in} (H_2,E) \sqsubseteq (G,E)} \widetilde{\tau}((H_1,E) \sqcap (H_2,E)) \\
&\geq \bigvee_{e_M \widetilde{\in} (H_1,E \sqsubseteq (F,E), e_M \widetilde{\in} (H_2,E) \sqsubseteq (G,E)} (\widetilde{\tau}(H_1,E) \wedge \widetilde{\tau}(H_2,E)) \\
&= \left(\bigvee_{e_M \widetilde{\in} (H_1,E) \sqsubseteq (F,E)} \widetilde{\tau}(H_1,E) \right) \wedge \left(\bigvee_{e_M \widetilde{\in} (H_2,E) \sqsubseteq (G,E)} \widetilde{\tau}(H_2,E) \right) \\
&= [((F,E) \in SN_{e_M}) \wedge ((G,E) \in SN_{e_M})].
\end{aligned}$$

(4) From Lemma 3.6, we have

$$\bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) = \bigwedge_{e_N \widetilde{\in} (G,E)} \bigvee_{e_N \widetilde{\in} (H,E) \sqsubseteq (G,E)} \widetilde{\tau}(H,E) = \widetilde{\tau}(G,E)$$

Therefore

$$\begin{aligned}
[\exists (G,E) \quad (\quad & (G,E) \in SN_{e_M} \wedge (G,E) \sqsubseteq (F,E)) \wedge \forall e_N (e_N \widetilde{\in} (G,E) \to (G,E) \in SN_{e_N}))] \\
&= \bigvee_{(G,E) \sqsubseteq (F,E)} \left(SN_{e_M}(G,E) \wedge \bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) \right) \\
&= \bigvee_{(G,E) \sqsubseteq (F,E)} (SN_{e_M}(G,E) \wedge \widetilde{\tau}(G,E)) \\
&= \bigvee_{(G,E) \sqsubseteq (F,E)} \widetilde{\tau}(G,E) \geq \bigvee_{e_M \widetilde{\in} (G,E) \sqsubseteq (F,E)} \widetilde{\tau}(G,E) \\
&= SN_{e_M}((F,E)).
\end{aligned}$$

(B) Conversely, $\widetilde{\tau}(F,E) = \bigwedge_{e_M \widetilde{\in} (F,E)} SN_{e_M}(F,E)$.

(1) Clearly, $\widetilde{\tau}\left(\widetilde{\phi}\right) = 1$. For any $e_M \widetilde{\in} \mathbf{SP}(\mathbf{X})$, there exists $(F_\circ, E)$ such that $SN_{e_M}(F_\circ, E) = 1$ because SN_{e_M} is soft normal. From the second condition, we have $SN_{e_M}(\widetilde{X}) = 1$. Hence, $\widetilde{\tau}(\widetilde{X}) = \bigwedge_{e_M \widetilde{\in} \widetilde{X}} SN_{e_M}(\widetilde{X}) = 1$.

(2)

$$\begin{aligned}
\widetilde{\tau}((F,E) \sqcap (G,E)) &= \bigwedge_{e_M \widetilde{\in} (F,E) \sqcap (G,E)} SN_{e_M}((F,E) \sqcap (G,E)) \\
&\geq \bigwedge_{e_M \widetilde{\in} (F,E) \sqcap (G,E)} (SN_{e_M}(F,E) \wedge SN_{e_M}(G,E)) \\
&= \bigwedge_{e_M \widetilde{\in} (F,E) \sqcap (G,E)} SN_{e_M}(F,E) \wedge \bigwedge_{e_M \widetilde{\in} (F,E) \sqcap (G,E)} SN_{e_M}(G,E) \\
&\geq \bigwedge_{e_M \widetilde{\in} (F,E)} SN_{e_M}(F,E) \wedge \bigwedge_{e_M \widetilde{\in} ((G,E)} SN_{e_M}(G,E) \\
&= \widetilde{\tau}(F,E) \wedge \widetilde{\tau}(G,E).
\end{aligned}$$

(3)

$$\begin{aligned}
\widetilde{\tau}\left(\bigsqcup_{\lambda \in \Lambda} (F_\lambda, E)\right) &= \bigwedge_{e_M \widetilde{\in} \bigsqcup_{\lambda \in \Lambda} (F_\lambda, E)} SN_{e_M}\left(\bigsqcup_{\lambda \in \Lambda} (F_\lambda, E)\right) \\
&= \bigwedge_{\lambda \in \Lambda} \bigwedge_{e_M \widetilde{\in} (F_\lambda, E)} SN_{e_M}\left(\bigsqcup_{\lambda \in \Lambda} (F_\lambda, E)\right) \\
&\geq \bigwedge_{\lambda \in \Lambda} \bigwedge_{e_M \widetilde{\in} (F_\lambda, E)} SN_{e_M}(F_\lambda, E) \\
&= \bigwedge_{\lambda \in \Lambda} \widetilde{\tau}(F_\lambda, E).
\end{aligned}$$

(4) From the fourth condition, we have

$$SN_{e_M}(F,E) \leq \bigvee_{(G,E) \sqsubseteq (F,E)} \left(SN_{e_M}(G,E) \wedge \bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) \right),$$

and from the first condition, we have $SN_{e_M}(G,E) = 0$, for any $e_N \widetilde{\notin} (G,E)$. Consequently,

$$\begin{aligned}
SN_{e_M}(F,E) &\leq \bigvee_{e_M \widetilde{\in} (G,E) \sqsubseteq (F,E)} \left(SN_{e_M}(G,E) \wedge \bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) \right) \\
&\leq \bigvee_{e_M \widetilde{\in} (G,E) \sqsubseteq (F,E)} \bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) \\
&= \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)} \widetilde{\tau}(H,E).
\end{aligned}$$

On the other hand, if $e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)$, then

$$\bigwedge_{e_N \widetilde{\in} (H,E)} SN_{e_N}(H,E) \leq SN_{e_N}(H,E) \leq SN_{e_N}(F,E)$$

Now, we know that

$$\bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)} \widetilde{\tau}(H,E) = \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)} \bigwedge_{e_N \widetilde{\in} (H,E)} SN_{e_N}(H,E) \leq SN_{e_M}(F,E)$$

□

Definition 3.9. Let $(X,\widetilde{\tau},E),(Y,\widetilde{\sigma},F)$ be two fuzzifying soft topological spaces and $f:(X,\widetilde{\tau},E)\to(Y,\widetilde{\sigma},F)$ be a mapping. For any $e_M \in \mathbf{SP}(\mathbf{X})$,
(1) f is said to be a fuzzifying soft continuous at e_M, if

$$\models (G,E) \in SN^{\widetilde{\sigma}}_{f(e_M)} \to f^{-1}(G,E) \in SN^{\widetilde{\tau}}_{e_M}$$

(2) f is called fuzzifying soft continuous if it is fuzzifying soft continuous at every point $e_M \in \mathbf{SP}(\mathbf{X})$.

Theorem 3.10. A mapping $f:(X,\widetilde{\tau},E)\to(Y,\widetilde{\sigma},F)$ is fuzzifying soft continuous if and only if $\models (G,E) \in \widetilde{\sigma} \to f^{-1}(G,E) \in \widetilde{\tau}$.

Proof. If f is a fuzzifying soft continuous, then

$$\begin{aligned}\widetilde{\tau}\left(f^{-1}(G,E)\right) &= \bigwedge_{e_M \widetilde{\in} f^{-1}(G,E)} SN_{e_M}(f^{-1}(G,E)) \geq \bigwedge_{e_M \widetilde{\in} f^{-1}(G,E)} SN_{f(e_M)}(G,E) \\ &\geq \bigwedge_{e_N \widetilde{\in} (G,E)} SN_{e_N}(G,E) = \widetilde{\sigma}(G,E)\end{aligned}$$

Conversely, assume that the condition is satisfied. Then

$$\begin{aligned}SN_{e_M}(f^{-1}(G,E)) &= \bigvee_{e_M \widetilde{\in} (H,E) \sqsubseteq f^{-1}(G,E)} \widetilde{\tau}(H,E) \geq \bigvee_{f(e_M) \widetilde{\in} (F,E) \sqsubseteq (G,E)} \widetilde{\tau}(f^{-1}(F,E)) \\ &\geq \bigvee_{f(e_M) \widetilde{\in} (F,E) \sqsubseteq (G,E)} \widetilde{\sigma}(F,E) = SN_{f(e_M)}(G,E)\end{aligned}$$

which completes the proof. □

As a direct consequence of the definitions, we have the following

Theorem 3.11. Let $(X,\widetilde{\tau},E),(Y,\widetilde{\sigma},F)$, $(Z,\widetilde{\theta},G)$ be three fuzzifying soft topological spaces and $f:(X,\widetilde{\tau},E)\to(Y,\widetilde{\sigma},F)$ and $g:(Y,\widetilde{\sigma},F)\to(Z,\widetilde{\theta},G)$ be a mapping. For any $e_M \in \mathbf{SP}(\mathbf{X})$,
(1)If f is fuzzifying soft continuous at e_M and g is fuzzifying soft continuous at $f(e_M)$, then the composition $h = g \circ f$ is fuzzifying soft continuous at e_M.
(2) If f, g are fuzzifying soft continuous, then $h = g \circ f$ is fuzzifying soft continuous.

4. **Fuzzifying Soft Proximities**

Definition 4.1. Let X be a universe of discourse, $\widetilde{\delta} \in \Im(\mathbf{S}(\mathbf{X},\mathbf{E}) \times \mathbf{S}(\mathbf{X},\mathbf{E}))$, i.e., $\widetilde{\delta}:\mathbf{S}(\mathbf{X},\mathbf{E}) \times \mathbf{S}(\mathbf{X},\mathbf{E}) \to I$ satisfies the following conditions:
$(\widetilde{\delta}1) \models (F,E) \sqcap (G,E) \neq \widetilde{\phi} \to ((F,E),(G,E)) \in \widetilde{\delta}$;
$(\widetilde{\delta}2) \models \neg\left((F,E),\widetilde{\phi}\right) \in \widetilde{\delta}, \models \neg\left(\widetilde{\phi},(F,E)\right) \in \widetilde{\delta}$;
$(\widetilde{\delta}3) \models ((F,E) \sqcup (G,E),(H,E)) \in \widetilde{\delta} \to ((F,E),(H,E)) \in \widetilde{\delta} \vee ((G,E),(H,E)) \in \widetilde{\delta}$
and

$\models ((F,E),(G,E)\sqcup(H,E)) \in \widetilde{\delta} \to ((F,E),(G,E)) \in \widetilde{\delta} \vee ((F,E),(H,E)) \in \widetilde{\delta}$.

($\widetilde{\delta}4$) For every $(F,E),(G,E) \in \mathbf{S(X,E)}$ there exists $(H,E) \in \mathbf{S(X,E)}$,

$$\models ((F,E),(G,E)) \in \widetilde{\delta} \longleftrightarrow \Big(((F,E),(H,E)) \in \widetilde{\delta} \vee ((H,E)^c,(G,E)) \in \widetilde{\delta}\Big)$$

Then $\widetilde{\delta}$ is called a fuzzifying soft quasi-proximity on X and $(X,\widetilde{\delta})$ is called a fuzzifying soft quasi-proximity space.

A fuzzifying soft proximity is a fuzzifying soft quasi-proximity $\widetilde{\delta}$ which satisfies also

$$(\widetilde{\delta}5) \models ((F,E),(G,E)) \in \widetilde{\delta} \to ((G,E),(F,E)) \in \widetilde{\delta}$$

Theorem 4.2. Let $\widetilde{\delta}$ be a fuzzifying soft quasi-proximity on a set X and define $\widetilde{\tau}_{\widetilde{\delta}} \in \Im(\mathbf{S(X,E)})$ as follows:

$(F,E)) \in \widetilde{\tau}_{\widetilde{\delta}} := \forall e_M ((\{e_M\},(F^c,E)) \in \widetilde{\delta} \to e_M \widetilde{\notin} (F,E))$.

Then:

(1) $\widetilde{\tau}_{\widetilde{\delta}}$ is a fuzzifying soft topology on X.

(2) $SN^{\widetilde{\tau}_{\widetilde{\delta}}}_{e_M}((F,E)) = 1 - \widetilde{\delta}(\{e_M\},(F^c,E))$.

Proof. (1) Clearly, $\widetilde{\tau}_{\widetilde{\delta}}(\widetilde{\phi}) = \widetilde{\tau}_{\widetilde{\delta}}(\widetilde{X}) = 1$. Also, $\forall (M,E),\ (N,E) \in \mathbf{S(X,E)}$,

$$\begin{aligned}
\widetilde{\tau}_{\widetilde{\delta}}((M,E)\sqcap(N,E)) &= \bigwedge_{e_M \widetilde{\in} (M,E)\sqcap(N,E)} \Big(1 - \widetilde{\delta}(\{e_M\},(M^c,E)\sqcup(N^c,E))\Big) \\
&\geq \bigwedge_{e_M \widetilde{\in} (M,E)\sqcap(N,E)} \Big(\Big(1 - \widetilde{\delta}(\{e_M\},(M^c,E))\Big) \wedge \Big(1 - \widetilde{\delta}(\{e_M\},(N^c,E))\Big)\Big) \\
&\geq \widetilde{\tau}_{\widetilde{\delta}}(M,E) \wedge \widetilde{\tau}_{\widetilde{\delta}}(N,E)
\end{aligned}$$

Analogously we have that $\forall \{(F_\lambda,E)\}_{\lambda\in\Lambda} \subseteq \mathbf{S(X,E)}$, $\widetilde{\tau}_{\widetilde{\delta}}(\bigsqcup_{\lambda\in\Lambda}(F_\lambda,E)) \geq \bigwedge_{\lambda\in\Lambda} \widetilde{\tau}_{\widetilde{\delta}}(F_\lambda,E)$.

(2) If $e_M \widetilde{\in} (G,E) \sqsubseteq (F,E)$, then $\widetilde{\tau}_{\widetilde{\delta}}(G,E) \leq 1 - \widetilde{\delta}(\{e_M\},(G^c,E)) \leq 1 - \widetilde{\delta}(\{e_M\},(F^c,E))$ and hence $SN^{\widetilde{\tau}_{\widetilde{\delta}}}_{e_M}((F,E)) \leq 1 - \widetilde{\delta}(\{e_M\},(F^c,E))$. On the other hand, let $1 - \widetilde{\delta}(\{e_M\},(F^c,E)) > \epsilon > 0$, i.e., $\widetilde{\delta}(\{e_M\},(F^c,E)) < 1-\epsilon$. Let $(H,E) = \{e_N : \widetilde{\delta}(\{e_N\},(F^c,E)) < 1-\epsilon\}$. Then $e_M \widetilde{\in} (H,E) \sqsubseteq (F,E)$. For each $e_H \widetilde{\in} (H,E)$, there exists $(H_\circ,E) \in \mathbf{S(X,E)}$ such that $\widetilde{\delta}(\{e_H\},(H^c_\circ,E)) \vee \widetilde{\delta}((H_\circ,E),(F^c,E)) < 1-\epsilon$. Clearly, $(H_\circ,E) \sqsubseteq (H,E)$. Also, for each $e_H \in \mathbf{SP(X)}$, we have $e_H \widetilde{\in} (H_\circ,E)$ since $\widetilde{\delta}(\{e_H\},(H_\circ,E)) < 1$. Thus $(H,E) = \bigsqcup_{e_H \widetilde{\in} (H,E)} (H_\circ,E)$. For $e_H \widetilde{\in} (H,E)$, we have $\widetilde{\delta}(\{e_H\},(H^c,E)) \leq \widetilde{\delta}(\{e_H\},(H^c_\circ,E)) < 1-\epsilon$ and hence

$$\widetilde{\tau}_{\widetilde{\delta}}(H,E) = \bigwedge_{e_H \widetilde{\in} (H,E)} \Big(1 - \widetilde{\delta}(\{e_H\},(H^c,E))\Big) \geq \epsilon$$

which implies that $SN^{\widetilde{\tau}_{\widetilde{\delta}}}_{e_M}((F,E)) \geq \widetilde{\tau}_{\widetilde{\delta}}(H,E) \geq \epsilon$. This proves that $SN^{\widetilde{\tau}_{\widetilde{\delta}}}_{e_M}((F,E)) \geq 1 - \widetilde{\delta}(\{e_M\},(F^c,E))$ and hence (2) follows. □

If $\widetilde{\delta}$ is a fuzzifying soft quasi-proximity on a set X, we will refer to $\widetilde{\tau}_{\widetilde{\delta}}$ as the fuzzifying soft topology corresponding to $\widetilde{\delta}$.

Definition 4.3. A fuzzifying soft quasi-proximity $\widetilde{\delta}_1$ on a set X is said to be finer than another one $\widetilde{\delta}_2$ if $\models ((F,E),(G,E)) \in \widetilde{\delta}_1 \to ((F,E),(G,E)) \in \widetilde{\delta}_2$ for all $(F,E),(G,E) \in \mathbf{S(X,E)}$.

Theorem 4.4. Let $\widetilde{\delta}_1, \widetilde{\delta}_2$ be two fuzzifying soft quasi-proximities on a set X. If $\widetilde{\delta}_1$ is finer than $\widetilde{\delta}_2$, then $\widetilde{\tau}_{\widetilde{\delta}_1}$ is finer than $\widetilde{\tau}_{\widetilde{\delta}_2}$.

Proof. It is direct consequence of (2) in Theorem 4.2. □

Theorem 4.5. Let $\widetilde{\tau}$ be a fuzzifying soft topology on X and define the mapping $\widetilde{\delta} : \mathbf{S(X,E)} \times \mathbf{S(X,E)} \to I$ as follows:

$$\widetilde{\delta}((F,E),(G,E)) = 1 - \left(\bigvee_{(F,E)\sqsubseteq(H,E)\sqsubseteq(G,E)^c} \widetilde{\tau}((H,E)) \right),$$

Then $\widetilde{\delta}$ is a fuzzifying soft quasi-proximity on X and $\models \widetilde{\tau}_{\widetilde{\delta}} \equiv \widetilde{\tau}$.

Proof. It is clear that $\widetilde{\delta}((F,E),(G,E)) = 1$ if $(F,E) \sqcap (G,E) \neq \widetilde{\phi}$ and that $\widetilde{\delta}(\widetilde{\phi},(G,E)) = \widetilde{\delta}((F,E),\widetilde{\phi}) = 0$. Let $(F,E) = (F_1,E) \sqcup (F_2,E)$. Clearly, for $i = 1,2$, we have $\widetilde{\delta}((F_i,E),(G,E)) \leq \widetilde{\delta}((F,E),(G,E))$. On the other hand, suppose that $\widetilde{\delta}((F_i,E),(G,E)) < \theta < 1$. There are $(H_1,E),(H_2,E)$, $(F_i,E) \sqsubseteq (H_i,E) \sqsubseteq (G,E)^c$ and $1 - \widetilde{\tau}((H_i,E)) < \theta$ for $i = 1,2$. If $(H,E) = (H_1,E) \sqcup (H_2,E)$, then $1 - \widetilde{\tau}((H,E)) < \theta$ and $(F,E) \sqsubseteq (H,E) \sqsubseteq (G,E)^c$ which implies that $\widetilde{\delta}((F,E),(G,E)) < \theta$. Thus

$$\widetilde{\delta}((F_1,E) \sqcup (F_2,E),(G,E)) = \widetilde{\delta}((F_1,E),(G,E)) \vee \widetilde{\delta}((F_2,E),(G,E)).$$

In a similar way we prove that

$$\widetilde{\delta}((F,E),(G_1,E) \sqcup (G_2,E)) = \widetilde{\delta}((F,E),(G_1,E)) \vee \widetilde{\delta}((F,E),(G_2,E)).$$

Finally, let $\widetilde{\delta}((F,E),(G,E)) < \theta < 1$. There exists $(H,E) \in 2^{\mathbf{S(X,E)}}$, $(F,E) \sqsubseteq (H,E) \sqsubseteq (G,E)^c$, $1 - \widetilde{\tau}((H,E)) < \theta$. Now $\widetilde{\delta}((F,E),(H,E)^c) < 1 - \widetilde{\tau}((H,E)) < \theta$ and $\widetilde{\delta}((H,E),(G,E)) < 1 - \widetilde{\tau}((H,E)) < \theta$, which proves that $\widetilde{\delta}$ satisfies also($\widetilde{\delta}4$). Therefore $\widetilde{\delta}$ is a fuzzifying soft quasi-proximity. If $\models \widetilde{\tau}_1 \equiv \widetilde{\tau}_{\widetilde{\delta}}$, then (by Theorem 4.2) we have $SN^{\widetilde{\tau}_1}_{e_M}((F,E)) = 1 - \widetilde{\delta}(\{e_M\},(F^c,E))$. Since

$$\widetilde{\delta}(\{e_M\},(F^c,E)) = 1 - \left(\bigvee_{e_M \widetilde{\in} (H,E)\sqsubseteq(F,E)} \widetilde{\tau}((H,E)) \right) = 1 - SN^{\widetilde{\tau}}_{e_M}((F,E)),$$

it follows that $SN^{\widetilde{\tau}}_{e_M}((F,E)) = SN^{\widetilde{\tau}_1}_{e_M}((F,E))$ and hence $\models \widetilde{\tau} \equiv \widetilde{\tau}_1$. This completes the proof. □

Definition 4.6. Let $(X,\widetilde{\delta}_1)$, $(Y,\widetilde{\delta}_2)$ be fuzzifying soft quasi-proximity spaces. A mapping $f : (X,\widetilde{\delta}_1) \to (Y,\widetilde{\delta}_2)$ is said to be fuzzifying soft quasi-proximally continuous if $\widetilde{\delta}_1((F,E),(G,E)) \leq \widetilde{\delta}_2(f(F,E),f(G,E))$, for any $(F,E),(G,E) \in 2^{\mathbf{S(X,E)}}$.

Theorem 4.7. Let $(X,\widetilde{\delta}_1)$ and $(Y,\widetilde{\delta}_2)$ be two fuzzifying soft quasi-proximity spaces and let $f : (X,\widetilde{\delta}_1) \to (Y,\widetilde{\delta}_2)$ be a fuzzifying soft quasi-proximally continuous. Then f is fuzzifying soft $(\widetilde{\tau}_{\widetilde{\delta}_1},\widetilde{\tau}_{\widetilde{\delta}_2})$-continuous.

Proof. Let $e_M \in \mathbf{SP}(\mathbf{X})$,$(G,E) \in 2^{\mathbf{S}(\mathbf{Y},\mathbf{E})}$. In view of Theorem 4.2, we have that

$$SN_{e_M}^{\widetilde{\tau}_{\widetilde{\delta}_1}}(f^{-1}(G,E)) = 1 - \widetilde{\delta}_1(\{e_M\}, \left(f^{-1}(G,E)\right)^c) = 1 - \widetilde{\delta}_1(\{e_M\}, f^{-1}(G^c,E)).$$

If $(H,E) = f\left(f^{-1}(G^c,E)\right)$, $(G,E) \sqsubseteq (H,E)^c$. Thus

$$\widetilde{\delta}_1(\{e_M\}, \left(f^{-1}(G,E)\right)^c) \leq \widetilde{\delta}_2(f(\{e_M\}),(H,E)) = 1 - SN_{f(e_M)}^{\widetilde{\tau}_{\widetilde{\delta}_2}}((H,E)^c) \leq 1 - SN_{f(e_M)}^{\widetilde{\tau}_{\widetilde{\delta}_2}}((G,E))$$

and hence $SN_{e_M}^{\widetilde{\tau}_{\widetilde{\delta}_1}}(f^{-1}(H,E)) \geq SN_{f(e_M)}^{\widetilde{\tau}_{\widetilde{\delta}_2}}((G,E))$, which proves that f is a fuzzifying soft $(\widetilde{\tau}_{\widetilde{\delta}_1}, \widetilde{\tau}_{\widetilde{\delta}_2})$-continuous. □

Theorem 4.8. Let $f : X \to Y$ be a mapping and $\widetilde{\delta}$ be a fuzzifying soft quasi-proximity on Y. Then $\widetilde{\delta}' \in \mathbf{S}(\mathbf{X},\mathbf{E}) \times \mathbf{S}(\mathbf{X},\mathbf{E}) \to I$, $\widetilde{\delta}'((F,E),(G,E)) = \widetilde{\delta}(f(F,E), f(G,E))$ is a fuzzifying soft quasi-proximity on X.

Proof. Since the rest follow easily, we will only prove that $\widetilde{\delta}'$ satisfies $(\widetilde{\delta}4)$. If $\widetilde{\delta}'((F,E),(H,E)) \vee \widetilde{\delta}'((H,E)^c,(G,E)) < \theta < 1$, then $(F,E) \sqsubseteq (H,E)^c$ and so $\widetilde{\delta}'((F,E),(G,E)) \leq \widetilde{\delta}'((H,E)^c,(G,E)) < \theta$. On the other hand, let $\widetilde{\delta}'((F,E),(G,E)) < \theta < 1$. There exists $(K,E) \in 2^{\mathbf{S}(\mathbf{Y},\mathbf{E})}$ such that $\widetilde{\delta}(f(F,E),(K,E)) \vee \widetilde{\delta}((K,E)^c, f(G,E)) < \theta$. If $(H,E) = f^{-1}(K,E)$, $f(H,E) \sqsubseteq (K,E)$ and $f((H,E)^c) \sqsubseteq (K,E)^c$. Thus

$$\widetilde{\delta}'((F,E),(H,E)) = \widetilde{\delta}(f(F,E), f(H,E)) < \theta \textit{ and } \widetilde{\delta}'((H,E)^c,(G,E)) = \widetilde{\delta}'(f((H,E)^c), f(G,E)) < \theta$$

which implies that $\widetilde{\delta}'$ satisfies $(\widetilde{\delta}4)$. □

We will denote $\widetilde{\delta}'$ by $f^{-1}(\widetilde{\delta})$.

Theorem 4.9. Let $f : X \to Y$ be a function and $\widetilde{\delta}$ be a fuzzifying soft quasi-proximity on Y. Then:
(a) $f^{-1}(\widetilde{\delta})$ is the weakest fuzzifying soft quasi-proximity on X for which f is a fuzzifying soft proximally continuous.
(b) If $\widetilde{\delta}$ is a fuzzifying soft proximity, so is $f^{-1}(\widetilde{\delta})$.
(c) If $(Z, \widetilde{\delta}_\circ)$ is a fuzzifying soft quasi-proximity space and $g : (Z, \widetilde{\delta}_\circ) \to (X, f^{-1}(\widetilde{\delta}))$ is a function, then g is a fuzzifying soft proximally continuous if and only if the composition $f \circ g$ is a fuzzifying soft proximally continuous.
(d) $\models \widetilde{\tau}_{f^{-1}(\widetilde{\delta})} \equiv f^{-1}(\widetilde{\tau}_{\widetilde{\delta}})$.

Proof. (a) and (b) are direct consequences of the definitions.
(c) Necessity: It is clear that the composition of two fuzzifying soft proximally continuous is a fuzzifying soft continuous.
Sufficiency:

$$\widetilde{\delta}_\circ((F,E),(G,E)) \leq \widetilde{\delta}(f \circ g(F,E), f \circ g(G,E)) = f^{-1}(\widetilde{\delta})(g(F,E), g(G,E))$$

and thus g is fuzzifying soft proximally continuous.

(d) Let $\models \widetilde{\tau}_1 \equiv \widetilde{\tau}_{f^{-1}(\widetilde{\delta})}$. Then

$$SN_{e_M}^{\widetilde{\tau}_1}((F,E)) = 1 - f^{-1}(\widetilde{\delta})(\{e_M\},(F,E)^c) = 1 - \widetilde{\delta}(f(e_M), f(F^c,E))$$

Also,

$$SN_{e_M}^{f^{-1}(\widetilde{\tau}_{\widetilde{\delta}})}((F,E)) = SN_{f(e_M)}^{\widetilde{\tau}_{\widetilde{\delta}}}(f(F^c,E))^c = 1 - \widetilde{\delta}(f(e_M), f(F^c,E)) = SN_{e_M}^{\widetilde{\tau}_1}((F,E))$$

and thus $\models f^{-1}(\widetilde{\tau}_{\widetilde{\delta}}) \equiv \widetilde{\tau}_1$. This completes the proof. □

Theorem 4.10. Let $(X_\lambda, \widetilde{\delta}_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying soft quasi-proximity space, X a set and for each $\lambda \in \Lambda, f_\lambda : X \to X_\lambda$ be a mapping. Define $\widetilde{\delta} : \mathbf{S}(\mathbf{X},\mathbf{E}) \times \mathbf{S}(\mathbf{X},\mathbf{E}) \to I$ by

$$\widetilde{\delta}((F,E),(G,E)) = \bigwedge \left\{ \bigvee_{i,j} \bigwedge_{\lambda\in\Lambda} \widetilde{\delta}_\lambda \left((F_i,E),(G_j,E)\right) \right\},$$

where the infimum is taken over the family of all finite collections $(F_i,E),(G_j,E)$ of soft subsets of X with $(F,E) = \sqcup(F_i,E)$, $(G,E) = \sqcup(G_j,E)$. Then:

(1) $\widetilde{\delta}$ is a fuzzifying soft quasi-proximity on X.

(2) Every f_λ is a fuzzifying soft $\left(\widetilde{\delta}, \widetilde{\delta}_\lambda\right)$-proximally continuous and $\widetilde{\delta}$ is the weakest of all fuzzifying soft quasi-proximities $\widetilde{\delta}'$ on X for which each f_λ is a fuzzifying soft $\left(\widetilde{\delta}', \widetilde{\delta}_\lambda\right)$-proximally continuous.

(3) A mapping f, from a fuzzifying soft quasi-proximity space $(Y, \widetilde{\delta}_1)$ to $(X, \widetilde{\delta})$, is a fuzzifying soft proximally continuous if and only if each composition $f_\lambda \circ f$ is a fuzzifying soft proximally continuous.

(4) $\widetilde{\tau}_{\widetilde{\delta}}$ coincides with the weakest fuzzifying soft topology $\widetilde{\tau}$ on X for which each f_λ is a fuzzifying soft $\left(\widetilde{\tau}, \widetilde{\tau}_{\widetilde{\delta}_\lambda}\right)$- continuous.

(5) If each $\widetilde{\delta}_\lambda$ is a fuzzifying soft proximity, so is $\widetilde{\delta}$.

Proof. (1) It is easy to see that $\widetilde{\delta}((F,E),(G,E)) = 1$ when $(F,E) \sqcap (G,E) \neq \widetilde{\phi}$ and that $\widetilde{\delta}(\widetilde{\phi},(G,E)) = \widetilde{\delta}((F,E),\widetilde{\phi}) = 0$. Assume now that $\widetilde{\delta}((F,E),(G,E)) < \theta < 1$. There are sets $(F_1,E),...,(F_n,E)$, $(G_1,E),...,(G_m,E)$ with $(F,E) = \sqcup(F_i,E)$, $(G,E) = \sqcup(G_j,E)$, and for each pair (i, j) there exists $\lambda = \lambda(i,j) \in \Lambda$ with $\widetilde{\delta}_\lambda\left(f_\lambda(F_i,E), f_\lambda(G_j,E)\right) < \theta$. Let $(F',E) \sqsubseteq (F,E)$ and $(G',E) \sqsubseteq (G,E)$. Set

$$(F'_i,E) = (F_i,E) \sqcap (F',E), (G'_j,E) = (G_j,E) \sqcap (G',E).$$

Then $(F',E) = \sqcup(F'_i,E)$, $(G',E) = \sqcup(G'_j,E)$, and for each pair (i, j) there exists $\lambda = \lambda(i,j) \in \Lambda$ such that

$$\widetilde{\delta}_\lambda\left(f_\lambda(F'_i,E), f_\lambda(G'_j,E)\right) \leq \widetilde{\delta}_\lambda\left(f_\lambda(F_i,E), f_\lambda(G_j,E)\right) \leq \theta$$

and so $\widetilde{\delta}((F,E),(G,E)) \leq \theta$. This proves that $\widetilde{\delta}((F',E),(G',E)) \leq \widetilde{\delta}((F,E),(G,E))$. In analogous way, we prove that

$$\widetilde{\delta}((F_1,E) \sqcup (F_2,E),(G,E)) \leq \widetilde{\delta}((F_1,E),(G,E)) \vee \widetilde{\delta}((F_2,E),(G,E))$$

and

$$\widetilde{\delta}((F,E),(G_1,E) \sqcup (G_2,E)) \leq \widetilde{\delta}((F,E),(G_1,E)) \vee \widetilde{\delta}((F,E),(G_2,E))$$

Finally we prove that $\widetilde{\delta}$ satisfies $(\widetilde{\delta}4)$. So, assume that $\widetilde{\delta}((F,E),(G,E)) < \theta < 1$. There are sets $(F_1,E),...,(F_n,E)$, $(G_1,E),...,(G_m,E)$, $(F,E) = \sqcup(F_i,E)$, $(G,E) = \sqcup(G_j,E)$, such that , for each pair (i,j) there exists $\lambda = \lambda(i,j)$ with $\widetilde{\delta}_\lambda\left(f_\lambda(F_i,E), f_\lambda(G_j,E)\right) < \theta$. Now, given the pair (i, j) and $\lambda = \lambda(i,j)$, there exists $(H_{ij},E) \in 2^{\mathbf{S}(\mathbf{X}_\lambda,\mathbf{E})}$ such that $\widetilde{\delta}_\lambda\left(f_\lambda(F_i,E),(H_{ij},E)\right) < \theta$ and $\widetilde{\delta}_\lambda\left((H_{ij},E)^c, f_\lambda(G_j,E)\right) < \theta$. Let $(K_{ij},E) =$

$f_\lambda^{-1}(H_{ij}, E)$. Set $(H,E) = \sqcup_{j=1}^{m} \sqcap_{k=1}^{n} (K_{kj}, E)$. For each pair (i, j) and $\lambda = \lambda(i,j)$, we have that

$$\widetilde{\delta}_\lambda \left(f_\lambda(F_i,E), f_\lambda(\sqcap_{k=1}^{n}(K_{kj},E))\right) \leq \widetilde{\delta}_\lambda \left(f_\lambda(F_i,E), f_\lambda((H_{ij},E))\right) < \theta$$

and hence $\widetilde{\delta}((F,E),(H,E))) < \theta$. Also, $(H,E)^c = \sqcap_{j=1}^{m} \sqcup_{k=1}^{n} (K_{kj},E)^c$. Let $(M_j,E) = \sqcup_{k=1}^{n}(K_{kj},E)^c$. We have $\widetilde{\delta}((H,E)^c,(G_j,E)) \leq \widetilde{\delta}((M_j,E),(G_j,E))$. For each pair (i, j) and $\lambda = \lambda(i,j)$, we have
$\widetilde{\delta}(f_\lambda(K_{kj},E)^c, f_\lambda(G_j,E)) < \theta$ and hence $\widetilde{\delta}((M_j,E),(G_j,E)) \leq \theta$. Thus $\widetilde{\delta}((H,E)^c,(G_j,E)) \leq \theta$. for all j, which implies that $\widetilde{\delta}((H,E)^c,(G,E)) \leq \bigvee \widetilde{\delta}((H,E)^c,(G_j,E)) \leq \theta$. Thus $\widetilde{\delta}$ satisfies $(\widetilde{\delta}4)$ and therefore it is a fuzzifying quasi-proximity. In case each $\widetilde{\delta}_\lambda$ is a fuzzifying proximity so is $\widetilde{\delta}$.

(2) Since $\widetilde{\delta}((F,E),(G,E)) \leq \widetilde{\delta}_\lambda(f_\lambda(F,E), f_\lambda(G,E))$, each f_λ is a fuzzifying soft $\left(\widetilde{\delta}, \widetilde{\delta}_\lambda\right)$-proximally continuous. On the other hand, let $\widetilde{\delta}'$ be a fuzzifying soft quasi-proximity on X such that each f_λ is a fuzzifying soft $\left(\widetilde{\delta}', \widetilde{\delta}_\lambda\right)$-proximally continuous and assume that $\widetilde{\delta}((F,E),(G,E)) < \theta < 1$. There are sets $(F_1,E),...,(F_n,E)$, $(G_1,E),...,(G_m,E)$, $(F,E) = \sqcup(F_i,E)$, $(G,E) = \sqcup(G_j,E)$, and for each pair (i, j) there exists $\lambda = \lambda(i,j)$ such that $\widetilde{\delta}_\lambda \left(f_\lambda(F_i,E), f_\lambda(G_j,E)\right) < \theta$. Now $\widetilde{\delta}'((F,E),(G,E)) \leq \max_{i,j} \widetilde{\delta}'\left((F_i,E),(G_j,E)\right)$. For each pair (i, j) and $\lambda = \lambda(i,j)$, we have

$$\widetilde{\delta}'\left((F_i,E),(G_j,E)\right) \leq \widetilde{\delta}\left(f_\lambda(F_i,E), f_\lambda(G_j,E)\right) < \theta$$

So $\widetilde{\delta}'((F,E),(G,E)) \leq \theta$. This proves that $\widetilde{\delta}'((F,E),(G,E)) \leq \widetilde{\delta}((F,E),(G,E))$. Therefore $\widetilde{\delta}'$ is finer than $\widetilde{\delta}$.

(3) The condition is necessary since a composition of two fuzzifying proximally continuous mappings is a fuzzifying proximally continuous. Conversely suppose that the condition is satisfied. We need to show that $\widetilde{\delta}_1((F,E),(G,E)) \leq \widetilde{\delta}(f(F,E), f(G,E))$ for all $(F,E),(G,E) \in 2^{\mathbf{S(Y,E)}}$. So, assume that $\widetilde{\delta}(f(F,E), f(G,E)) < \theta < 1$. There are sets $(H_1,E),...,(H_n,E)$, $(K_1,E),...,(K_m,E)$, $f(F,E) = \sqcup(H_i,E)$, $f(G,E) = \sqcup(K_j,E)$, such that, for each pair (i, j) there exists $\lambda = \lambda(i,j)$ in Λ with $\widetilde{\delta}_\lambda \left(f_\lambda(H_i,E), f_\lambda(K_j,E)\right) < \theta$. Let $(F_i,E) = f^{-1}(H_i,E)$, $(G_j,E) = f^{-1}(K_j,E)$. Since $(F,E) \sqsubseteq \sqcup(F_i,E)$, $(G,E) \sqsubseteq \sqcup(G_j,E)$, we have that

$$\widetilde{\delta}_1((F,E),(G,E)) \leq \widetilde{\delta}_1(\sqcup(F_i,E), \sqcup(G_j,E)) \leq \max_{i,j} \widetilde{\delta}_1((F_i,E),(G_j,E))$$

For each pair (i, j) there exists $\lambda = \lambda(i,j)$, we have

$$\widetilde{\delta}_1((F_i,E),(G_j,E)) \leq \widetilde{\delta}_\lambda(f_\lambda \circ f(F_i,E), f_\lambda \circ f(G_j,E)) \leq \widetilde{\delta}_\lambda \left(f_\lambda(H_i,E), f_\lambda(K_j,E)\right) < \theta$$

and hence $\widetilde{\delta}_1((F,E),(G,E)) < \theta$. This proves that $\widetilde{\delta}_1((F,E),(G,E)) \leq \widetilde{\delta}(f(F,E), f(G,E))$. Therefore f is a fuzzifying soft $(\widetilde{\delta}_1, \widetilde{\delta})$-proximally continuous.

(4) Let $\models \widetilde{\tau} \equiv \widetilde{\tau}_{\widetilde{\delta}}$ and $\models \widetilde{\tau}_\lambda \equiv \widetilde{\tau}_{\widetilde{\delta}_\lambda}$. Since f_λ is a fuzzifying soft $(\widetilde{\delta}, \widetilde{\delta}_\lambda)$-proximally continuous, it follows that f_λ is soft $(\widetilde{\tau}, \widetilde{\tau}_\lambda)$- continuous and so $f^{-1}(\widetilde{\tau}_\lambda) \leq \widetilde{\tau}$.Thus, for $\widetilde{\tau}' = \bigvee_{\lambda \in \Lambda} f_\lambda^{-1}(\widetilde{\tau}_\lambda)$, we $\widetilde{\tau}' \leq \widetilde{\tau}$. To prove that $\widetilde{\tau} \leq \widetilde{\tau}'$, it suffices to show that $SN_{e_M}^{\widetilde{\tau}}((F,E)) \leq SN_{e_M}^{\widetilde{\tau}'}((F,E))$ for all $e_M \in \mathbf{SP(X)}$ and $(F,E) \in 2^{\mathbf{S(X,E)}}$. So, assume that $SN_{e_M}^{\widetilde{\tau}}((F,E)) > \theta > 0$. By Theorem 4.2, we have that $\widetilde{\delta}(\{e_M\},(F^c,E)) <$

$1-\theta$. Thus there are $(G_1,E),...,(G_m,E)\in 2^{\mathbf{S(X,E)}}$, with $(F,E)^c=\sqcup(G_j,E)$, such that, for each j there exists $\lambda=\lambda(j)$ in Λ with $\widetilde{\delta}_\lambda\left(f_\lambda(\{e_M\}),f_\lambda(G_j,E)\right)<1-\theta$ and so $SN_{f_\lambda(e_M)}(f_\lambda(G_j^c,E))>\theta$. Now,

$$(K_j,E)=f_\lambda^{-1}(f_\lambda(G_j^c,E))=\left(f_\lambda^{-1}(f_\lambda(G_j,E))\right)^c\sqsubseteq(G_j^c,E)$$

Since $\widetilde{\tau}'\geq f_\lambda^{-1}(\widetilde{\tau}_\lambda)$, we have that $SN_{e_M}^{\widetilde{\tau}'}((K_j,E))\geq SN_{e_M}^{f^{-1}(\widetilde{\tau}_\lambda)}((K_j,E))>\theta$. Now, $\sqcap_j(K_j,E)\sqsubseteq\sqcap_j(G_j^c,E)=(F,E)$ and hence

$$SN_{e_M}^{\widetilde{\tau}'}((F,E))\geq SN_{e_M}^{\widetilde{\tau}'}(\sqcap_j(K_j,E))\geq\bigwedge_j SN_{e_M}^{\widetilde{\tau}'}((K_j,E))>\theta$$

This proves that $SN_{e_M}^{\widetilde{\tau}'}((F,E))\geq SN_{e_M}^{\widetilde{\tau}}((F,E))$ and hence $\widetilde{\tau}'$ is finer than $\widetilde{\tau}$, which completes the proof. □

As a corollary we have the following theorem

Theorem 4.11. Let $(X_\lambda,\widetilde{\delta}_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying soft quasi-proximity spaces on X. Define $\widetilde{\delta}:\mathbf{S(X,E)}\times\mathbf{S(X,E)}\to I$ by

$$\widetilde{\delta}((F,E),(G,E))=\bigwedge\left\{\bigvee_{i,j}\bigwedge_{\lambda\in\Lambda}\widetilde{\delta}_\lambda\left((F_i,E),(G_j,E)\right)\right\},$$

where the infimum is taken over the family of all finite collections $(F_i,E),(G_j,E)$ of soft subsets of X with $(F,E)=\sqcup(F_i,E)$, $(G,E)=\sqcup(G_j,E)$. Then:
(1) $\widetilde{\delta}$ is a fuzzifying soft quasi-proximity on X. If each $\widetilde{\delta}_\lambda$ is a fuzzifying soft proximity, so is $\widetilde{\delta}$.
(2) $\widetilde{\delta}$ is the weakest of all fuzzifying soft quasi-proximities on X which are finer than each $\widetilde{\delta}_\lambda$.
(3) A mapping f, from a fuzzifying soft quasi-proximity space $(Y,\widetilde{\delta}_1)$ to $(X,\widetilde{\delta})$, is a fuzzifying soft proximally continuous if and only if f is a fuzzifying soft $(\widetilde{\delta}_1,\widetilde{\delta}_\lambda)-$proximally continuous for all $\lambda\in\Lambda$.
(4) $\widetilde{\tau}$ coincides with the weakest fuzzifying soft topology on X finer than each $\widetilde{\tau}_{\widetilde{\delta}_\lambda}$.

Definition 4.12. Let $(X_\lambda,\widetilde{\delta}_\lambda)_{\lambda\in\Lambda}$ be a family of a fuzzifying soft quasi-proximity spaces, $X=\Pi X_\lambda$ and for each $\lambda\in\Lambda$, $p_\lambda:X\to X_\lambda$ the λth projection. The product fuzzifying soft quasi-proximity space on X is the weakest fuzzifying soft quasi-proximity space on X for which p_λ is a fuzzifying proximally continuous.
We denote this by $\Pi\widetilde{\delta}_\lambda$

In view of Theorem 4.10, we have the following

Theorem 4.13. Let Let $(X_\lambda,\widetilde{\delta}_\lambda)_{\lambda\in\Lambda}$ and X be as above and let $\widetilde{\delta}=\Pi\widetilde{\delta}_\lambda$. Then:
(1) $\widetilde{\delta}((F,E),(G,E))=\bigwedge\left\{\bigvee_{i,j}\bigwedge_{\lambda\in\Lambda}\widetilde{\delta}_\lambda\left(\Pi\lambda(F_i,E),\Pi\lambda(G_j,E)\right)\right\}$, where the infimum is taken over the family of all finite families $(F_i,E),(G_j,E)$ of soft subsets of X with $(F,E)=\sqcup(F_i,E)$, $(G,E)=\sqcup(G_j,E)$.

(2) A mapping f, from a fuzzifying soft quasi-proximity space $(Y, \widetilde{\delta}_1)$ to $(X, \widetilde{\delta})$, is a fuzzifying soft proximally continuous if and only if each composition $\Pi_\lambda \circ f$ is a fuzzifying soft proximally continuous.
(3) If each $\widetilde{\delta}_\lambda$ is a fuzzifying soft proximity, so is $\widetilde{\delta}$.
(4) $\models \widetilde{\tau}_{\widetilde{\delta}} \equiv \prod \widetilde{\tau}_{\widetilde{\delta}_\lambda}$.

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