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Local β compactness as fuzzy predicates defined in Łukasiewicz logic

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Abstract.
In this paper, some characterizations of fuzzifying β-compactness are given, including characterizations in terms of nets and β-subbases. Several characterizations of locally β-compactness in the framework of fuzzifying topology are introduced and the mapping theorems are obtained.

Keywords: Łukasiewicz logic, semantics, fuzzifying topology, fuzzifying compactness, β-compactness, fuzzifying locally compactness, locally β-compactness.

1. Introduction

In 1952, Rosser and Turquette [8] proposed the following unsolved problem: If there are many-valued theories beyond the level of predicate calculus, then what are the details of such theories ? As an attempt to give a partial answer to this problem in the case of point set topology, Ying in 1991-1993 [13-15] used a semantical method of continuous-valued logic to develop systematically fuzzifying topology. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. So far, there has been significant research on fuzzifying topologies [3, 9- 16]. For example, Ying [16] introduced the concepts of compactness and established a generalization of Tychonoff's theorem in the framework of fuzzifying topology. In [12] the concept of local compactness in fuzzifying topology is introduced and some of its properties are established. The notion of β-open sets [1] was introduced by Abd El-Monsef, El-Deeb and Mahmoud in 1983 which was studied in Andrijevic [5] under the name semi-preopen sets. Dontchev and Przemiski [6] replaced the term semi-preopen by the term pre-semiopen. The concept of β-compact topological spaces was studied in [2, 4]. In [3] the concepts of fuzzifying β-open sets and fuzzifying β-continuity were introduced and studied. Also, Sayed in [10] introduced some concepts of fuzzifying β-separation axioms and clarified the relations of these axioms with each other as well as the relations with other fuzzifying separation axioms. Furthermore, Sayed and Abd-Allah [11] characterized the concepts of fuzzifying β-irresolute functions and used the finite intersection property to give a characterization of fuzzifying β-compact spaces. In this paper, the concepts of β-base and β-subbase of fuzzifying β-topology are introduced. Other characterizations of fuzzifying β-compactness are given, includ-

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ing characterizations in terms of nets and β-subbase. Several characterizations of locally β-compactness in the framework of fuzzifying topology are introduced and the mapping theorems are obtained. Thus we fill a gap in the existing literature on fuzzifying topology. We use the terminologies and notations in [3, 9- 16] without any explanation. We note that the set of truth values is the unit interval and we do often not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. We will use the symbol $\otimes$ instead of the second "AND" operation $\dot{\wedge}$ as dot is hardly visible. This mean that $[\alpha] \leq [\varphi \rightarrow \psi] \Leftrightarrow [\alpha] \otimes [\varphi] \leq [\psi]$. All of the contributions in General Topology in this paper which are not referenced may be original.

2. Preliminaries

We now give some definitions and results which are useful in the rest of the present paper.

Definition 2.1 [13] Let X be a set and $\tau \in \Im(P(X))$ is called a fuzzifying topology if it satisfies the following conditions

(T1) $\models X \in \tau$;

(T2) for any $A, B \in P(X)$, $\models (A \in \tau) \wedge (A \in \tau) \rightarrow ((A \cap B \in \tau)$;

(T3) for any $\{A_\lambda \mid \lambda \in \Lambda\}$, $\models \forall \lambda(\lambda \in \Lambda \rightarrow A_\lambda \in \tau) \rightarrow \bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$.

The pair (X, τ) is called a fuzzifying topological space.

(1) The fuzzifying neighborhood system of a point $x \in X$ is defined as $N_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$.

(2) The fuzzifying closure of a set $A \subseteq X$ is defined as $Cl(A)(x) = 1 - N_x(X - A)$.

(3) The fuzzifying interior of a set $\mu \in \Im(X)$ is defined as $Int(\mu) = \bigcup\{U \in P(X) \mid 1_U \leq \mu\}$, where 1_U is a characteristic function.

Definition 2.2 [3] The family of all fuzzifying β-open sets, denoted by $\tau_\beta \in \Im(P(X))$, is defined as

$A \in \tau_\beta := \forall x(x \in A \rightarrow x \in Cl(Int(Cl(A))))$, i. e., $\tau_\beta(A) = \bigwedge_{x \in A} Cl(Int(Cl(A)))(x))$.

(1) The family of all fuzzifying β-closed sets, denoted by $\Gamma_\beta \in \Im(P(X))$, is defined as $A \in \Gamma_\beta := X - A \in \tau_\beta$.

(2) The fuzzifying β-neighborhood system of a point $x \in X$ is denoted by $N_x^{\beta^X}$ (or N_x^β) $\in \Im(P(X))$ and defined as $N_x^\beta(A) = \bigvee_{x \in B \subseteq A} \tau_\beta(B)$.

(3) The fuzzifying β-closure of a set $A \subseteq X$, denoted by $Cl_\beta \in \Im(X)$, is defined as $Cl_\beta(A)(x) = 1 - N_x^\beta(X - A)$.

(4) ([9]) If $N(X)$ is the class of all nets in X, then the binary fuzzy predicates $\rhd^\beta, \propto^\beta \in \Im(N(X) \times X)$ are defined as $S \rhd^\beta x := \forall A(A \in N_x^\beta \rightarrow S \tilde{\subset} A)$, $S \propto^\beta x := \forall A(A \in N_x^\beta \rightarrow S \tilde{\precsim} A)$, where "$S \rhd^\beta x$" , "$S \propto^\beta x$" stand for "$S$ β-converges to x" , "x is an β-accumulation point of S", respectively; and "$\tilde{\subset}$", "$\tilde{\precsim}$" are the binary crisp predicates "almost in ","often in", respectively. The degree to which x is an β-adherence point of S is $adh_\beta S(x) = [S \propto^\beta x]$.

(5) ([3, 11]) If (X, τ) and (Y, σ) are two fuzzifying topological spaces and $f \in Y^X$, the unary fuzzy predicates $C_\beta, I_\beta \in \Im(Y^X)$, called fuzzifying β-continuity , fuzzifying β-irresoluteness, are given as $C_\beta(f) := \forall B(B \in \sigma \rightarrow f^{-1}(B) \in \tau_\beta)$, $I_\beta(f) := \forall B(B \in \sigma_\beta \rightarrow f^{-1}(B) \in \tau_\beta)$, respectively.

Definition 2.3 Let Ω be the class of all fuzzifying topological spaces.

(1) ([10]) A unary fuzzy predicate $T_2^\beta \in \Im(\Omega)$, called fuzzifying β-Hausdorffness, is given

$$T_2^\beta(X, \tau) = \forall x \forall y((x \in X \wedge y \in X \wedge x \neq y) \rightarrow \exists B \exists C(B \in N_x^\beta \wedge C \in N_y^\beta \wedge B \cap C \equiv \phi)).$$

(2) ([16]) A unary fuzzy predicate $\Gamma \in \Im(\Omega)$, called fuzzifying compactness , is given as $\Gamma(X, \tau) := (\forall \Re)(K_\circ(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, A) \otimes FF(\wp)))$ and if $A \subseteq X$, then $\Gamma(A) := \Gamma(A, \tau/A)$.

(3) ([16]) A unary fuzzy predicate $fI \in \Im(\Im(P(X)))$, called fuzzy finite intersection property, is given as $fI(\Re) := \forall \wp((\wp \leq \Re) \wedge FF(\wp) \rightarrow \exists x \forall B(B \in \wp \rightarrow x \in B))$.

(4) ([11]) A fuzzifying topological space (X, τ) is said to be fuzzifying β-topological space [11] if $\tau_\beta(A \cap B) \geq \tau_\beta(A) \wedge \tau_\beta(B)$.

(5) ([11]) A binary fuzzy predicate $K_\beta \in \Im(\Im(P(X)) \times P(X))$, called fuzzifying β-open covering, is given as $K_\beta(\Re, A) := K(\Re, A) \otimes (\Re \subseteq \tau_\beta)$.

(6) ([11]) A unary fuzzy predicate $\Gamma_\beta \in \Im(\Omega)$, called fuzzifying $\beta-$compactness, is given as $(X, \tau) \in \Gamma_\beta := (\forall \Re)(K_\beta(\Re, X) \longrightarrow (\exists \wp)((\wp \leq \Re) \wedge K(\wp, X) \otimes FF(\wp)))$ and if $A \subseteq X$, then $\Gamma_\beta(A) := \Gamma_\beta(A, \tau/A)$.

(7) ([12]) A unary fuzzy predicate $LC \in \Im(\Omega)$, called fuzzifying locally compactness, is given as $LC(X, \tau) := (\forall x)(\exists B)((x \in Int(B) \otimes \Gamma(B, \tau/B))$.

3. Fuzzifying β-base and β-subbase

Definition 3.1 Let (X,τ) be a fuzzifying topological space and $ß_\beta \subset \tau_\beta$. Then $ß_\beta$ is called β-base of τ_β if $ß_\beta$ fulfils the condition: $\models U \in N_x^\beta \to \exists V((V \in ß_\beta) \wedge (x \in V \subseteq U))$.

Remark 3.2 In above definition, we can obtain that $\tau_r = \{U \in P(X) \mid \tau(U) \geq r\}$ is a classical topology for each $r \in [0,1]$, similarly, $(ß_\beta)_r, (\tau_\beta)_r$. Then $(ß_\beta)_r$ is a β-base of $(\tau_\beta)_r$ if for each $U \in (N_x^\beta)_r$, there exists $V \in (ß_\beta)_r$ such that $x \in V \subseteq U$.

Example 3.3 Let $X = \{x,y,z\}$ and we define a fuzzifying topology $\tau \in \Im(P(X))$ as follows:

$$\begin{aligned}&\tau(X)=\tau(\emptyset)=1, \tau(\{x\})=0.8,\\&\tau(\{y\})=0.6, \tau(\{z\})=0.4,\\&\tau(\{x,y\})=0.6, \tau(\{y,z\})=0.6, \tau(\{z,x\})=0.4.\end{aligned}$$

Since $N_x(A) = \bigvee_{x\in B\subseteq A} \tau(B)$, we have

$$\begin{aligned}&N_x(X)=1, \forall x \in X,\\&N_x(\{x\})=N_x(\{x,y\})=N_x(\{x,z\})=0.8,\\&N_y(\{y\})=N_y(\{y,z\})=0.6, N_y(\{x,y\})=0.8,\\&N_z(\{z\})=N_z(\{x,z\})=0.4, N_z(\{y,z\})=0.6.\end{aligned}$$

Since $Cl(A)(x) = 1 - N_x(X-A)$,

$$\begin{aligned}&Cl(\{x\})(x)=1-N_x(\{y,z\})=1,\\&Cl(\{x\})(y)=1-N_y(\{y,z\})=0.4,\\&Cl(\{x\})(z)=1-N_z(\{y,z\})=0.4,\\&Cl(\{y\})=(0.2,1,0.6), Cl(\{z\})=(0.2,0.2,1),\\&Cl(\{x,y\})=(1,1,0.6), Cl(\{x,z\})=(1,0.4,1),\\&Cl(\{y,z\})=(0.2,1,1), Cl(X)=(1,1,1),\\&Cl(\emptyset)=(0,0,0).\end{aligned}$$

Then $Int(Cl(A)) = A$ and $Cl(Int(Cl(A))) = Cl(A)$ for all $A \in P(X)$. From Definition 2.2, we obtain a fuzzifying topology $\tau_\beta \in \Im(P(X))$ as follows:

$$\begin{aligned}&\tau_\beta(X)=\tau_\beta(\emptyset)=1, \tau_\beta(\{x\})=0.4,\\&\tau_\beta(\{y\})=0.2, \tau_\beta(\{z\})=0.2\\&\tau_\beta(\{x,y\})=0.6, \tau_\beta(\{x,z\})=0.4, \tau_\beta(\{y,z\})=0.2.\end{aligned}$$

(1) We define $ß_\beta \in \Im(P(X))$ as follows:

$$\begin{aligned}&ß_\beta(X)=ß_\beta(\emptyset)=1, ß_\beta(\{x\})=0.4,\\&ß_\beta(\{y\})=0.2, ß_\beta(\{z\})=0.2,\\&ß_\beta(\{x,y\})=0.6, ß_\beta(\{x,z\})=ß_\beta(\{y,z\})=0.\end{aligned}$$

Since $ß_\beta \subset \tau_\beta$ and $N_x^\beta(U) \leq \bigvee_{x\in V\subseteq U} ß_\beta(V)$ from Definition 3.1, then $ß_\beta$ is a β-base of τ_β.

(2) We define $c_\beta \in \Im(P(X))$ as follows:

$$\begin{aligned}&c_\beta(X)=c_\beta(\emptyset)=1, c_\beta(\{x\})=0.2,\\&c_\beta(\{y\})=0.2, c_\beta(\{z\})=0.2\\&c_\beta(\{x,y\})=0.6, c_\beta(\{x,z\})=c_\beta(\{y,z\})=0.\end{aligned}$$

We have $c_\beta \subset \tau_\beta$ and $N_x^\beta(\{x,z\}) = \tau_\beta(\{x,z\}) = 0.4 \nleq \bigvee_{x\in V\subseteq\{x,z\}} c_\beta(V) = 0.2$. Hence c_β is not a β-base of τ_β. Moreover, $(c_\beta)_{0.3} = \{X, \emptyset, \{x,y\}\}$ is not a β-base of $(\tau_\beta)_{0.3} = \{X,\emptyset,\{x\},\{x,y\},\{x,z\}\}$ in Remark 3.2.

Theorem 3.4 *$ß_\beta$ is an β-base of τ_β if and only if $\tau_\beta = ß_\beta^{(\cup)}$, where $ß_\beta^{(\cup)}(U) = \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$.*

Proof. Suppose that $ß_\beta$ is an β-base of τ_β. If $\bigcup_{\lambda\in\Lambda} V_\lambda = U$, then from Theorem 3.1 (1) (b) in [3], $\tau_\beta(U) = \tau_\beta\left(\bigcup_{\lambda\in\Lambda} V_\lambda\right) \geq \bigwedge_{\lambda\in\Lambda}\tau_\beta(V_\lambda) \geq \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$. Consequently, $\tau_\beta(U) \geq \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$.

To prove that $\tau_\beta(U) \leq \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$, we first prove $\tau_\beta(U) = \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U}\tau_\beta(V)$. (Indeed, assume $\delta_x = \{V : x \in V \subseteq U\}$. Then for any $f \in \prod_{x\in U}\delta_x$, $\bigcup_{x\in U} f(x) = U$, and furthermore

$$\begin{aligned}\tau_\beta(U) &= \tau_\beta\left(\bigcup_{x\in U} f(x)\right) \geq \bigwedge_{x\in U}\tau_\beta(f(x))\\&\geq \bigvee_{f\in\prod_{x\in U}\delta_x}\bigwedge_{x\in U}\tau_\beta(f(x)) = \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U}\tau_\beta(V).\end{aligned}$$

Also $\tau_\beta(U) \leq \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U}\tau_\beta(V)$. Therefore

$\tau_\beta(U) = \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U}\tau_\beta(V)$.

Now, since $N_x^\beta(U) \leq \bigvee_{x\in V\subseteq U} ß_\beta(V)$,

$$\begin{aligned}\tau_\beta(U) &= \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U}\tau_\beta(V) = \bigwedge_{x\in U} N_x^\beta(U)\\&\leq \bigwedge_{x\in U}\bigvee_{x\in V\subseteq U} ß_\beta(V) = \bigvee_{f\in\prod_{x\in U}\delta_x}\bigwedge_{x\in U\beta} ß_\beta(f(x)).\end{aligned}$$

Then $\tau_\beta(U) \leq \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$. Therefore $\tau_\beta(U) = \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$

Conversely, we assume $\tau_\beta(U) = \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda)$ and we will show that $ß_\beta$ is an β-base of τ_β, i.e., for any $U \subseteq X$, $N_x^\beta(U) \leq \bigvee_{x\in V\subseteq U} ß_\beta(V)$. Indeed, if $x \in V \subseteq U$, $\bigcup_{\lambda\in\Lambda} V_\lambda = V$, then there exists $\lambda_\circ \in \Lambda$ such that $x \in V_{\lambda_\circ}$ and $\bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda) \leq ß_\beta(V_{\lambda_\circ}) \leq \bigvee_{x\in V\subseteq U} ß_\beta(V)$. Therefore

$$\begin{aligned} N_x^\beta(U) &= \bigvee_{x\in V\subseteq U} \tau_\beta(V) \\ &= \bigvee_{x\in V\subseteq U} \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda) \\ &\leq \bigvee_{x\in V\subseteq U} ß_\beta(V). \end{aligned}$$

Theorem 3.5 *Let $ß_\beta \in \Im(P(X))$. Then $ß_\beta$ is an β-base for some fuzzifying β-topology τ_β if and only if it has the following properties:*
(1) $ß_\beta^{(\cup)}(X) = 1$;
(2) $\models (U \in ß_\beta) \wedge (V \in ß_\beta) \wedge (x \in U \cap V) \rightarrow \exists W((W \in ß_\beta) \wedge (x \in W \subseteq U \cap V)$.

Proof. If $ß_\beta$ is an β-base for some fuzzifying β-topology τ_β, then $\tau_\beta(X) = ß_\beta^{(\cup)}(X)$. Clearly, $ß_\beta^{(\cup)}(X) = 1$. In addition, if $x \in U \cap V$, then

$$\begin{aligned} &ß_\beta(U) \wedge ß_\beta(V) \leq \tau_\beta(U) \wedge \tau_\beta(V) \leq \tau_\beta(U \cap V) \\ &\leq N_x^\beta(U \cap V) \leq \bigvee_{x\in W\subseteq U\cap V} ß_\beta(W). \end{aligned}$$

Conversely, if $ß_\beta$ satisfies (1) and (2), then we have τ_β is a fuzzifying β-topology. In fact, $\tau_\beta(X) = 1$. For any $\{U_\lambda : \lambda \in \Lambda\} \subseteq P(X)$, we set

$$\delta_\lambda = \left\{ \{V_{\Phi_\lambda} : \Phi_\lambda \in \Lambda_\lambda\} : \bigcup_{\Phi_\lambda\in\Lambda_\lambda} V_{\Phi_\lambda} = U_\lambda \right\}.$$

Then for any $f \in \prod_{\lambda\in\Lambda} \delta_\lambda$, $\bigcup_{\lambda\in\Lambda} \bigcup_{V_{\Phi_\lambda}\in f(\lambda)} V_{\Phi_\lambda} = \bigcup_{\lambda\in\Lambda} U_\lambda$. Therefore

$$\begin{aligned} &\tau_\beta\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigvee_{\bigcup_{\Phi\in\Lambda} V_\Phi = \bigcup_{\lambda\in\Lambda} U_\lambda} \bigwedge_{\Phi\in\Lambda} ß_\beta(V_\Phi) \\ &\geq \bigvee_{f\in\prod_{\lambda\in\Lambda}\delta_\lambda} \bigwedge_{\lambda\in\Lambda} \bigwedge_{V_{\Phi_\lambda}\in f(\lambda)} ß_\beta(V_{\Phi_\lambda}) \\ &\geq \bigwedge_{\lambda\in\Lambda} \bigvee_{\{V_{\Phi_\lambda}:\Phi_\lambda\in\Lambda_\lambda\}\in\delta_\lambda} \bigwedge_{\Phi_\lambda\in\Lambda_\lambda} ß_\beta(V_{\Phi_\lambda}) = \bigwedge_{\lambda\in\Lambda} \tau_\beta(U_\lambda). \end{aligned}$$

Finally, we need to prove that $\tau_\beta(U \cap V) \geq \tau_\beta(U) \wedge \tau_\beta(V)$. If $\tau_\beta(U) > t, \tau_\beta(V) > t$, then there exists $\{V_{\lambda_1} : \lambda_1 \in \Lambda_1\}, \{V_{\lambda_2} : \lambda_2 \in \Lambda_2\}$ such that $\bigcup_{\lambda_1\in\Lambda_1} V_{\lambda_1} = U$, $\bigcup_{\lambda_2\in\Lambda_2} V_{\lambda_2} = V$ and for any $\lambda_1 \in \Lambda_1$, $ß_\beta(V_{\lambda_1}) > t$, for any $\lambda_2 \in \Lambda_2$, $ß_\beta(V_{\lambda_2}) > t$. Now, for any $x \in U \cap V$, there exists $\lambda_{1x} \in \Lambda_1, \lambda_{2x} \in \Lambda_2$ such that $x \in V_{\lambda_{1x}} \cap V_{\lambda_{2x}}$. From the assumption, we know that $t < ß_\beta(V_{\lambda_{1x}}) \wedge ß_\beta(V_{\lambda_{2x}}) \leq \bigvee_{x\in W\subseteq V_{\lambda_{1x}}\cap V_{\lambda_{2x}}} ß_\beta(W)$ and furthermore, there exists W_x such that $x \in W_x \subseteq V_{\lambda_{1x}} \cap V_{\lambda_{2x}} \subseteq U \cap V$, $ß_\beta(W_x) > t$. Since $\bigcup_{x\in U\cap V} W_x = U \cap V$, we have $t \leq \bigwedge_{x\in U\cap V} ß_\beta(W_x) \leq \bigvee_{\bigcup_{\lambda\in\Lambda} V_\lambda = U\cap V} \bigwedge_{\lambda\in\Lambda} ß_\beta(V_\lambda) = \tau_\beta(U \cap V)$. Now, let $\tau_\beta(U) \wedge \tau_\beta(V) = k$. For any natural number *n*, we have $\tau_\beta(U) > k - \frac{1}{n}$, $\tau_\beta(V) > k - \frac{1}{n}$ and so $\tau_\beta(U \cap V) \geq k - \frac{1}{n}$. Therefore $\tau_\beta(U \cap V) \geq k = \tau_\beta(U) \wedge \tau_\beta(V)$.

Definition 3.6 $\varphi_\beta \in \Im(P(X))$ is called an β-subbase of τ_β if $\varphi_\beta^{\cap}$ is an β-base of τ_β, where $\varphi_\beta^{\cap}(\bigcap_{\lambda\in\Lambda} V_\lambda) = \bigvee_{\bigcap_{\lambda\in\Lambda} V_\lambda = U} \bigwedge_{\lambda\in\Lambda} \varphi_\beta(V_\lambda)$, $\{V_\lambda : \lambda \in \Lambda\} \subset_f P(X)$, with " $\subset_f$ " standing for "a finite subset of".

Remark 3.7 From Remark 3.2 and Definition 3.6, since (X, τ_r) is a classical topology for each $r \in [0, 1]$ and $(\tau_\beta)_r$ is the collection of all β-open sets in X, then a β-subbase of $(\tau_\beta)_r$ is a collection $(\varphi_\beta)_r$ of β-open sets such that every β-open set of $(\tau_\beta)_r$ is the union of sets that are finite intersections of $\{V_\lambda : \lambda \in \Lambda\} \subset (\varphi_\beta)_r$ with a finite index Λ.

Theorem 3.8 $\varphi_\beta \in \Im(P(X))$ *is an β-subbase of some fuzzifying β-topology if and only if* $\varphi_\beta^{(\cup)}(X) = 1$.

Proof. We only demonstrate that $\varphi_\beta^{\cap}$ satisfies the second condition of Theorem 3.5, and others are obvious. In fact

$\varphi^{\sqcap}_{\beta}(U)\wedge\varphi^{\sqcap}_{\beta}(V)=\left(\bigvee_{\bigcap_{\lambda_1\in\Lambda_1}V_{\lambda_1}=U}\bigwedge_{\lambda_1\in\Lambda_1}\varphi_\beta(V_{\lambda_1})\right)\wedge$
$\left(\bigvee_{\bigcap_{\lambda_2\in\Lambda_2}V_{\lambda_2}=V}\bigwedge_{\lambda_2\in\Lambda_2}\varphi_\beta(V_{\lambda_2})\right)$
$=\bigvee_{\bigcap_{\lambda_1\in\Lambda_1}V_{\lambda_1}=U}\bigvee_{\bigcap_{\lambda_2\in\Lambda_2}V_{\lambda_2}=V}\left(\bigwedge_{\lambda_1\in\Lambda_1}\varphi_\beta(V_{\lambda_1})\right)$
$\wedge\left(\bigwedge_{\lambda_2\in\Lambda_2}\varphi_\beta(V_{\lambda_2})\right)$
$\leq\bigvee_{\bigcap_{\lambda\in\Lambda}V_\lambda=U\cap V}\left(\bigwedge_{\lambda\in\Lambda}\varphi_\beta(V_\lambda)\right)=\varphi^{\sqcap}_{\beta}(U\cap V).$
Therefore if $x\in U\cap V$, then $\varphi^{\sqcap}_{\beta}(U)\wedge\varphi^{\sqcap}_{\beta}(V)\leq\varphi^{\sqcap}_{\beta}(U\cap V)\leq\bigvee_{x\in W\subseteq U\cap V}\varphi^{\sqcap}_{\beta}(W)$.

4. Fuzzifying β-compact spaces

Theorem 4.1 *Let (X,τ) be a fuzzifying topological space, φ_β be an β-subbase of τ_β, and*
$\beta_1:=(\forall\Re)(K_{\varphi_\beta}(\Re,X)\rightarrow\exists\wp((\wp\leq\Re)\wedge K(\wp,X)\otimes FF(\wp)))$,
where $K_{\varphi_\beta}(\Re,X):=K(\Re,X)\otimes(\Re\subseteq\varphi_\beta)$;
$\beta_2:=(\forall S)((S$ *is a universal net in* $X)\rightarrow\exists x((x\in X)\wedge(S\rhd^\beta x))$;
$\beta_3:=(\forall S)((S\in N(X)\rightarrow(\exists T)(\exists x)((T<S)\wedge(x\in X)\wedge(T\rhd^\beta x))$,
where "$T<S$" stands for "T is a subnet of S";
$\beta_4:=(\forall S)((S\in N(X)\rightarrow\neg(adh_\beta S\equiv\phi))$;
$\beta_5:=(\forall\Re)(\Re\in\Im(P(X))\wedge\Re\subseteq\Gamma_\beta\otimes fI(\Re)\rightarrow\exists x\forall A(A\in\Re\rightarrow x\in A))$.
Then $\models(X,\tau)\in\Gamma_\beta\leftrightarrow\beta_i$, $i=1,2,...,5$.

Proof. (1) Since $\varphi_\beta\subseteq\tau_\beta$, $[\Re\subseteq\varphi_\beta]\leq[\Re\subseteq\tau_\beta]$ for any $\Re\in\Im(P(X))$. Then $[K_{\varphi_\beta}(\Re,X)]\leq[K_\beta(\Re,X)]$. Therefore $\Gamma_\beta(X,\tau)\leq[\beta_1]$.
(2) $[\beta_2]=\bigwedge\left\{\bigvee_{x\in X}[S\rhd^\beta x]:S\text{ is a universal net in }X\right\}$.
(2.1) Assume X is finite. We set $X=\{x_1,...,x_m\}$. For any universal net S in X, there exists $i_\circ\in\{1,...,m\}$ with $S\tilde{\subset}\{x_{i_\circ}\}$. In fact, if not, then for any $i\in\{1,...,m\}$, $S\tilde{\not\subset}\{x_{i_\circ}\}$, $S\tilde{\subset}X-\{x_{i_\circ}\}$ and $S\tilde{\subset}\bigcap_{i=1}^{m}(X-\{x_i\})=\phi$, a contradiction. Therefore $x_{i_\circ}\notin A$ and $N^\beta_{x_{i_\circ}}(A)=0$ (see[3],Theorem 4.2 (1)) provided $S\tilde{\not\subset}A$, and furthermore $[S\rhd^\beta x_{i_\circ}]=\bigwedge_{S\tilde{\not\subset}A}\left(1-N^\beta_{x_{i_\circ}}(A)\right)=1$. Therefore $[\beta_2]=1\geq[\beta_1]$.
(2.2) In general, to prove that $[\beta_1]\leq[\beta_2]$ we prove that for any $\lambda\in[0,1]$, if $[\beta_2]<\lambda$, then $[\beta_1]<\lambda$. Assume for any $\lambda\in[0,1]$, $[\beta_2]<\lambda$. Then there exists a universal net S in X such that $\bigvee_{x\in X}[S\rhd^\beta x]<\lambda$ and for any $x\in X$, $[S\rhd^\beta x]=\bigwedge_{S\tilde{\not\subset}A}\left(1-N^\beta_x(A)\right)<\lambda$, i.e., there exists $A\subseteq X$ with $S\tilde{\not\subset}A$ and $N^\beta_x(A)>1-\lambda$. Since φ_β is an β-subbase of $\tau_\beta,\varphi^{\sqcap}_{\beta}$ is an β-base of τ_β and from Definition 3.1, we have $\bigvee_{x\in B\subseteq A}\varphi^{\sqcap}_{\beta}(B)\geq N^\beta_x(A)>1-\lambda$, i.e., there exists $B\subseteq A$ such that $x\in B\subseteq A$ and

$$\bigvee\left\{\min_{\lambda\in\Lambda}\varphi_\beta(B_\lambda):\bigcap_{\lambda\in\Lambda}B_\lambda=B,B_\lambda\subseteq X,\lambda\in\Lambda\right\}=\varphi^{\sqcap}_{\beta}(B)>1-\lambda,$$

where Λ is finite. Therefore there exists a finite set Λ and $B_\lambda\subseteq X(\lambda\in\Lambda)$ such that $\bigcap_{\lambda\in\Lambda}B_\lambda=B$ and for any $\lambda\in\Lambda,\varphi_\beta(B_\lambda)>1-\lambda$. Since $S\tilde{\not\subset}A$ and Λ is finite, there exists $\lambda(x)\in\Lambda$ such that $S\tilde{\not\subset}B_{\lambda(x)}$. We set $\Re_\circ(B_{\lambda(x)})=\bigvee_{x\in X}\varphi_\beta(B_{\lambda(x)})$.If $\wp\leq\Re_\circ$, then for any $\delta>0,\wp_\delta\subseteq\{B_{\lambda(x)}:x\in X\}$. Consequently, for any $B\in\wp_\delta,S\tilde{\not\subset}B$ and $S\tilde{\subset}B^c$ because S is a universal net. If $[FF(\wp)]=1-\inf\{\delta\in[0,1]:F(\wp_\delta)\}=t$, then for any $n\in w$ (the non-negative integer), $\inf\{\delta\in[0,1]:F(\wp_\delta)\}<1-t+\frac{1}{n}$, and there exists $\delta_\circ<1-t+\frac{1}{n}$ such that $F(\wp_{\delta\circ})$. If $\delta_\circ=0$,then $P(X)=\wp_{\delta\circ}$ is finite and it is proved in (2.1). If $\delta_\circ>0$, then for any $B\in\wp_{\delta\circ},S\tilde{\subset}B^c$. Since $F(\wp_{\delta\circ})$, we have $S\tilde{\subset}\bigcap\{B^c:B\in\wp_{\delta\circ}\}\neq\phi$. i.e., $\bigcup\wp_{\delta\circ}\neq X$ and there exist $x_\circ\in X$ such that for any $B\in\wp_{\delta\circ},x_\circ\notin B$. Therefore, if $x_\circ\in B$, then $B\notin\wp_{\delta\circ}$, i.e., $\wp(B)<\delta\circ$, $K(\wp,X)=\bigwedge_{x\in X}\bigvee_{x\in B}\wp(B)\leq\bigvee_{x_\circ\in B}\wp(B)\leq\delta\circ<1-t+\frac{1}{n}$. Let $n\rightarrow\infty$. We obtain $K(\wp,X)\leq1-t$ and $[K(\wp,X)\otimes FF(\wp)]=0$. In addition, $[K_{\varphi_\beta}(\Re_\circ,X)]\geq1-\lambda$. In fact, $[\Re_\circ\subseteq\varphi_\beta]=1$ and $[K(\Re_\circ,X)]=\bigwedge_{x\in X}\bigvee_{x\in B}\Re_\circ(B)\geq\bigwedge_{x\in X}\Re_\circ(B_{\lambda(x)})\geq\bigwedge_{x\in X}\varphi_\beta(B_{\lambda(x)})\geq1-\lambda$ because $x\in B_{\lambda(x)}$. Now, we have
$[\beta_1]=(\forall\Re)(K_{\varphi_\beta}(\Re,X)\rightarrow\exists\wp((\wp\leq\Re)\wedge K(\wp,X)\otimes FF(\wp)))$
$\leq K_{\varphi_\beta}(\Re_\circ,X)\rightarrow\exists\wp((\wp\leq\Re_\circ)\wedge K(\wp,X)\otimes FF(\wp))$

$= \min(1, 1 - K_{\varphi_\beta}(\Re_\circ, X) + \bigvee_{\wp \le \Re_\circ} [K(\wp, X) \otimes FF(\wp))] \le \lambda$.

By noticing that λ is arbitrary, we have $[\beta_1] \le [\beta_2]$.

(3) It is immediate that $[\beta_2] \le [\beta_3]$.

(4) To prove that $[\beta_3] \le [\beta_4]$, first we prove that $[\exists T\ ((T < S) \wedge (T \rhd^\beta x))] \le [S \propto^\beta x]$, where $[\exists T\ ((T < S) \wedge (T \rhd^\beta x))] = \bigvee_{T<S} \bigwedge_{T \widetilde{\not\subset} A} \left(1 - N_x^\beta(A)\right)$ and $[S \propto^\beta x] = \bigwedge_{S \widetilde{\not\prec} A} \left(1 - N_x^\beta(A)\right)$. Indeed, for any $T < S$ one can deduce $\{A : S \widetilde{\not\prec} A\} \subseteq \{A : T \widetilde{\not\subset} A\}$ as follows. Suppose $T = S \circ K$. If $S \widetilde{\not\prec} A$, then there exists $m \in D$ such that $S(n) \notin A$ when $n \ge m$, where $\ge$ directs the domain D of S. Now, we will show that $T \widetilde{\not\subset} A$. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \ge p$, where $\ge$ directs the domain E of T. Moreover, there exists $n_1 \in E$ such that $K(n_1) \ge m$ because $T < S$, and there exists $n_2 \in E$ such that $n_2 \ge n_1, p$ because $(E, \ge)$ is directed. So, $K(n_2) \ge K(n_1) \ge m$, $S \circ K(n_2) \notin A$ and $S \circ K(n_2) = T(n_2) \in A$. They are contrary. Hence $\{A : S \widetilde{\not\prec} A\} \subseteq \{A : T \widetilde{\not\subset} A\}$. Therefore $[\exists T\ ((T < S) \wedge (T \rhd^\beta x))] = \bigvee_{T<S} \bigwedge_{T \widetilde{\not\subset} A} \left(1 - N_x^\beta(A)\right) = \bigvee_{T<S} \bigwedge_{\{A : T \widetilde{\not\subset} A\}} \left(1 - N_x^\beta(A)\right) \le \bigwedge_{\{A : S \widetilde{\not\prec} A\}} \left(1 - N_x^\beta(A)\right) = \bigwedge_{S \widetilde{\not\prec} A} \left(1 - N_x^\beta(A)\right) = [S \propto^\beta x]$. Therefore for any $x \in X$ and $S \in N(X)$ we have

$$
\begin{aligned}
[\beta_3] &= \bigwedge_{S \in N(X)} \bigvee_{x \in X} [\exists T\ ((T < S) \wedge (T \rhd^\beta x))] \\
&\le \bigwedge_{S \in N(X)} \bigvee_{x \in X} [S \propto^\beta x] \\
&= \bigwedge_{S \in N(X)} \neg \left(\bigwedge_{x \in X} \left(1 - [S \propto^\beta x]\right) \right) \\
&= \bigwedge_{S \in N(X)} [\neg(adh_\beta S \equiv \phi)] = [\beta_4].
\end{aligned}
$$

(5) We want to show that $[\beta_4] \le [\beta_5]$. For any $\Re \in \Im(P(X))$, assume $[fI(\Re)] = \lambda$. Then for any $\delta > 1 - \lambda$, if $A_1, ..., A_n \in \Re_\delta$, $A_1 \cap A_2 \cap ... \cap A_n \ne \phi$. In fact, we set $\wp(A_i) = \bigvee_{i=1}^n \Re(A_i)$. Then $\wp \le \Re$ and $FF(\wp) = 1$. By putting $\varepsilon = \lambda + \delta - 1 > 0$, we obtain $\lambda - \varepsilon < \lambda \le [FF(\wp) \to (\exists x)(\forall B)(B \in \wp \to x \in B)] = \bigvee_{x \in X} \bigwedge_{x \notin B} (1 - \wp(B))$. There exists $x_\circ \in X$ such that $\lambda - \varepsilon < \bigwedge_{x_\circ \notin B} (1 - \wp(B))$, $x_\circ \notin B$ implies $\wp(B) < 1 - \lambda + \varepsilon = \delta$ and $x_\circ \in \cap \wp_\delta = A_1 \cap A_2 \cap ... \cap A_n$. Now, we set $\vartheta_\delta = \{A_1 \cap A_2 \cap ... \cap A_n : n \in N, A_1, ..., A_n \in \Re_\delta\}$ and $S : \vartheta_\delta \to X, B \mapsto x_B \in B, B \in \vartheta_\delta$ and know that $(\vartheta_\delta, \subseteq)$ is a directed set and S is a net in X. Therefore $[\beta_4] \le [\neg(adh_\beta S \equiv \phi)] = \bigvee_{x \in X} \bigwedge_{S \widetilde{\not\prec} A} (1 - N_x^\beta(A))$. Assume $[\Re \subseteq \Gamma_\beta] = \mu$. Then for any $B \in P(X), \Re(B) \le 1 + \Gamma_\beta(B) - \mu$, and $[\Re \subseteq \Gamma_\beta \otimes fI(\Re) \to (\exists x)(\forall A)((A \in \Re) \to x \in A)] = \min(1, 2 - \mu - \lambda + \bigvee_{x \in X} \bigwedge_{x \notin A} (1 - \Re(A)))$. Therefore it suffices to show that for any $x \in X$, $\bigwedge_{S \widetilde{\not\prec} A} (1 - N_x^\beta(A)) \le 2 - \mu - \lambda + \bigwedge_{x \notin A} (1 - \Re(A))$, i.e., $\bigvee_{x \notin A} \Re(A) \le 2 - \mu - \lambda + \bigvee_{S \widetilde{\not\prec} A} N_x^\beta(A)$ for some $\delta > 1 - \lambda$. For any $t \in [0, 1]$, if $\bigvee_{x \notin A} \Re(A) > t$, then there exists $A_\circ$ such that $x_\circ \notin A_\circ$ and $\Re(A_\circ) > t$.

Case 1. $t \le 1 - \lambda$, then $t \le 2 - \mu - \lambda + \bigvee_{S \widetilde{\not\prec} A} N_x^\beta(A)$.

Case 2. $t > 1 - \lambda$. Here we set $\delta = \frac{1}{2}(t + 1 - \lambda)$ and have $A_\circ \in \Re_\delta, A_\circ \in \vartheta_\delta$. In addition, $t < \Re(A_\circ) \le 1 + \Gamma_\beta(A_\circ) - \mu$, $t + \mu - 1 \le \Gamma_\beta(A_\circ) = \tau_\beta(A_\circ^c)$. Since $A_\circ \in \vartheta_\delta$, we know that $S_B \in A_\circ$, *i.e.*, $S_B \notin A_\circ^c$ when $B \subseteq A_\circ$ and $S \widetilde{\not\prec} A_\circ^c$. Therefore $2 - \mu - \lambda + \bigvee_{S \widetilde{\not\prec} A} N_x^\beta(A) \ge 2 - \mu - \lambda + N_x^\beta(A_\circ^c) \ge 2 - \mu - \lambda + \tau_\beta(A_\circ^c) \ge t + (1 - \lambda) \ge t$. By noticing that t is arbitrary, we have completed the proof.

(6) To prove that $[\beta_5] = [(X, \tau) \in \Gamma_\beta]$ see [10, Theorem 4.3].

As a sense in Remarks 3.2 and 3.7, the above theorem is a generalization of the following corollary.

Corollary 4.2 *The following are equivalent for a topological space (X, τ).*

(a) X is an β-compact space.

(b) Every cover of X by members of an β-subbase of τ_β has a finite subcover.

(c) Every universal net in X β-converges to a point in X.

(d) Each net in X has a subnet that β-converges to some point in X.

(e) Each net in X has an β-adherent point.

(f) Each family of β-closed sets in X that has the finite intersection property has a non-void intersection.

Definition 4.3 Let $\{(X_s, \tau_s) : s \in S\}$ be a family of fuzzifying topological spaces, $\prod_{s \in S} X_s$ be the cartesian product of $\{X_s : s \in S\}$ and $\varphi = \{p_s^{-1}(U_s) : s \in$

$S, U_s \in p(X_s)\}$, where $p_t : \prod_{s\in S} X_s \to X_t (t \in S)$ is a projection. For $\Phi \subseteq \varphi$, $S(\Phi)$ stands for the set of indices of elements in Φ. The β-base $ß_\beta \in \Im(\prod_{s\in S} X_s)$ of $\prod_{s\in S} (\tau_\beta)_s$ is defined as

$V \in ß_\beta := (\exists\Phi)(\Phi \subset_f \varphi \wedge (\bigcap \Phi = V)) \to \forall s(s \in S(\Phi) \to V_s \in (\tau_\beta)_s)$, i.e.,

$ß_\beta(V) = \bigvee_{\Phi\subset_f\varphi, \bigcap\Phi=V} \bigwedge_{s\in S(\Phi)} (\tau_\beta)_s(V_s).$

Definition 4.4 Let (X,τ), (Y,σ) be two fuzzifying topological space. A unary fuzzy predicate $O_\beta \in \Im(Y^X)$, is called fuzzifying β-openness, is given as: $O_\beta(f) := \forall U(U \in \tau_\beta \to f(U) \in \sigma_\beta)$. Intuitively, the degree to which f is β-open is $[O_\beta(f)] = \bigwedge_{B\subseteq X} \min(1, 1 - \tau_\beta(U) + \sigma_\beta(f(U)))$.

Lemma 4.5 *Let (X,τ) and (Y,σ) be two fuzzifying topological space. For any $f \in Y^X$,*
$O_\beta(f) := \forall B(B \in ß^X_\beta \to f(B) \in \sigma_\beta)$, where $ß^X_\beta$ is an β-base of τ_β.

Proof. Clearly, $[O_\beta(f)] \le [\forall U(U \in ß^X_\beta \to f(U) \in \sigma_\beta)]$. Conversely, for any $U \subseteq X$, we are going to prove $\min(1, 1 - \tau_\beta(U) + \sigma_\beta(f(U))) \ge [\forall V(V \in ß^X_\beta \to f(V) \in \sigma_\beta)]$. If $\tau_\beta(U) \le \sigma_\beta(f(U))$, it is hold clearly. Now assume $\tau_\beta(U) > \sigma_\beta(f(U))$.

If $\mathcal{C} \subseteq P(X)$ with $\bigcup\mathcal{C} = U$, then $\bigcup_{V\in\mathcal{C}} f(V) = f(\bigcup\mathcal{C}) = f(U)$. Therefore

$$\begin{aligned}
&\tau_\beta(U) - \sigma_\beta(f(U))\\
&= \bigvee_{\mathcal{C}\subseteq P(X), \bigcup\mathcal{C}=U} \bigwedge_{V\in\mathcal{C}} ß^X_\beta(V)\\
&- \bigvee_{\mathcal{K}\subseteq P(Y), \bigcup\mathcal{K}=f(U)} \bigwedge_{W\in\mathcal{K}} \sigma_\beta(W)\\
&\le \bigvee_{\mathcal{C}\subseteq P(X), \bigcup\mathcal{C}=U} \bigwedge_{V\in\mathcal{C}} ß^X_\beta(V)\\
&- \bigvee_{\mathcal{C}\subseteq P(X), \bigcup\mathcal{C}=U} \bigwedge_{V\in\mathcal{C}} \sigma_\beta(f(V))\\
&\le \bigvee_{\mathcal{C}\subseteq P(X), \bigcup\mathcal{C}=U} \bigwedge_{V\in\mathcal{C}} \left(ß^X_\beta(V) - \sigma_\beta(f(V))\right)
\end{aligned}$$

$$\begin{aligned}
&\min(1, 1 - \tau_\beta(U) + \sigma_\beta(f(U)))\\
&\ge \bigvee_{\mathcal{C}\subseteq P(X), \bigcup\mathcal{C}=U} \bigwedge_{V\in\mathcal{C}} \min(1, 1 - ß^X_\beta(V) + \sigma_\beta(f(V)))\\
&\ge [\forall V(V \in ß^X_\beta \to f(V) \in \sigma_\beta)].
\end{aligned}$$

Lemma 4.6 *For any family $\{(X_s,\tau_s) : s \in S\}$ of fuzzifying topological spaces.*
(1) $\models (\forall s)(s \in S \to p_s \in O_\beta)$;
(2) $\models (\forall s)(s \in S \to p_s \in C_\beta)$.

Proof. (1) For any $t \in S$, we have $O_\beta(p_t) = \bigwedge_{U\in P(\prod_{s\in S} X_s)} \min(1, 1 - \left(\prod_{s\in S} (\tau_\beta)_s\right)(U) + (\tau_\beta)_t(p_t(U)))$.
Then it suffices to show that for any $U \in P(\prod_{s\in S} X_s)$, we have $(\tau_\beta)_t(p_t(U)) \ge \left(\prod_{s\in S} (\tau_\beta)_s\right)(U)$.
Assume
$\left(\prod_{s\in S} (\tau_\beta)_s\right)(U) = \bigvee_{\bigcup_{\lambda\in\Lambda} B_\lambda=U} \bigwedge_{\lambda\in\Lambda} \bigvee_{\Phi_\lambda\subset_f\varphi, \cap\Phi_\lambda=B_\lambda} \bigwedge_{s\in S(\Phi_\lambda)} (\tau_\beta)_s(V_s) > \mu$,
where $\Phi_\lambda = \{p_s^{-1}(V_s) : s \in S(\Phi_\lambda)\}(\lambda \in \Lambda)$.
Hence there exists $\{B_\lambda : \lambda \in \Lambda\} \subseteq P(\prod_{s\in S} X_s)$ such that $\bigcup_{\lambda\in\Lambda} B_\lambda = U$ and furthermore, for any $\lambda \in \Lambda$, there exists $\Phi_\lambda \subset_f \varphi$ such that $\cap\Phi_\lambda = B_\lambda$ and $\bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_\lambda$, where for any $s \in S(\Phi_\lambda)$ we have $(\tau_\beta)_s(V_s) > \mu$. Thus $p_t(U) = p_t(\bigcup_{\lambda\in\Lambda} \bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s))$.
(1) If for any $\lambda \in \Lambda$, $\bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s) = \phi$, then $U = \phi, p_t(U) = \phi$ and $(\tau_\beta)_t(p_t(U)) = 1$. Therefore $(\tau_\beta)_t(p_t(U)) \ge \left(\prod_{s\in S} (\tau_\beta)_s\right)(U)$.
(2) If there exists $\lambda_\circ \in \Lambda$, such that $\phi \ne \bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_{\lambda_\circ}$,
(i) If $t \notin S(\Phi_{\lambda_\circ})$, i.e., $t \in S - S(\Phi_{\lambda_\circ})$, $p_t(B_{\lambda_\circ}) = X_t$. Therefore $(\tau_\beta)_t(p_t(B_{\lambda_\circ})) = (\tau_\beta)_t(X_t) = 1$.
(ii) If $t \in S(\Phi_{\lambda_\circ})$, then $p_t(B_{\lambda_\circ}) = V_t \subseteq X_t$. Thus $p_t(U) = p_t((\bigcup_{t\in S(\Phi_{\lambda_\circ})} B_{\lambda_\circ}) \cup (\bigcup_{t\notin S(\Phi_{\lambda_\circ})} B_{\lambda_\circ})) = (\bigcup_{t\in S(\Phi_{\lambda_\circ})} p_t(B_{\lambda_\circ})) \cup (\bigcup_{t\notin S(\Phi_{\lambda_\circ})} p_t(B_{\lambda_\circ})) = V_t \cup X_t = X_t$.
Hence $(\tau_\beta)_t(p_t(U)) = (\tau_\beta)_t(X_t) = 1$ or $(\tau_\beta)_t(p_t(U)) = (\tau_\beta)_t(V_t) > \lambda$.
Therefore $(\tau_\beta)_t(p_t(U)) \ge \left(\prod_{s\in S} (\tau_\beta)_s\right)(U)$. Thus $O_\beta(p_t) = 1$.
(2) From Lemma 3.1 in [15] we have $\models (\forall s)(s \in S \to p_s \in C)$. Furthermore, for any two fuzzifying topological spaces (X,τ) and (Y,σ) and $f \in Y^X$, we

have $C(f) \leq C_\beta(f)$ (Theorem 3.3 in [3]). Therefore $\models (\forall s)(s \in S \to p_s \in C_\beta)$.

Theorem 4.7 *Let $\{(X_s, \tau_s) : s \in S\}$ be the family of fuzzifying topological spaces, then*
$\models \exists U(U \subseteq \prod_{s\in S} X_s \wedge \Gamma_\beta(U, \tau/U) \wedge \exists x(x \in Int_\beta(U)) \to \exists T(T \subset_f S \wedge \forall t(t \in S - T \wedge \Gamma_\beta(X_t, \tau_t)))$.

Proof. It suffices to show that

$$\bigvee_{U\in P(\prod_{s\in S} X_s)} \left(\Gamma_\beta(U, \tau/U) \wedge \bigvee_{x\in \prod_{s\in S} X_s} N_x^\beta(U) \right) \leq \bigvee_{T\subset_f S} \bigwedge_{t\in S-T} \Gamma_\beta(X_t, \tau_t).$$

Indeed, if

$$\bigvee_{U\in P(\prod_{s\in S} X_s)} \left(\Gamma_\beta(U, \tau/U) \wedge \bigvee_{x\in \prod_{s\in S} X_s} N_x^\beta(U) \right) > \mu > 0,$$

then there exists $U \in P(\prod_{s\in S} X_s)$ such that $\Gamma_\beta(U, \tau/U) > \mu$ and $\bigvee_{x\in \prod_{s\in S} X_s} N_x^\beta(U) > \mu$, where $N_x^\beta(U) = \bigvee_{x\in V\subseteq U} \left(\prod_{s\in S} (\tau_\beta)_s\right)(V)$. Furthermore, there exists V such that $x \in V \subseteq U$ and $\left(\prod_{s\in S} (\tau_\beta)_s\right)(V) > \mu$. Since *ß*$_\beta$ is an β-base of $\prod_{s\in S} (\tau_\beta)_s$,

$$\left(\prod_{s\in S} (\tau_\beta)_s\right)(V) = \bigvee_{\bigcup_{\lambda\in\Lambda} B_\lambda = V} \bigwedge_{\lambda\in\Lambda} ß_\beta(B_\lambda)$$
$$= \bigvee_{\bigcup_{\lambda\in\Lambda} B_\lambda = V} \bigwedge_{\lambda\in\Lambda} \bigvee_{\Phi_\lambda\subset_f \varphi, \cap\Phi_\lambda = B_\lambda} \bigwedge_{s\in S(\Phi_\lambda)} (\tau_\beta)_s(V_s) > \mu,$$

where $\Phi_\lambda = \{p_s^{-1}(V_s) : s \in S(\Phi_\lambda)\}(\lambda \in \Lambda)$.
Hence there exists $\{B_\lambda : \lambda \in \Lambda\} \subseteq P(\prod_{s\in S} X_s)$ such that $\bigcup_{\lambda\in\Lambda} B_\lambda = V$. Furthermore, for any $\lambda \in \Lambda$, there exists $\Phi_\lambda \subset_f \varphi$ such that $\cap\Phi_\lambda = B_\lambda$ and for any $s \in S(\Phi_\lambda)$, we have $(\tau_\beta)_s(V_s) > \mu$. Since $x \in V$, there exists B_{λ_x} such that $x \in B_{\lambda_x} \subseteq V \subseteq U$. Hence there exists $\Phi_{\lambda_x} \subset_f \varphi$ such that $\cap\Phi_{\lambda_x} = B_{\lambda_x}$ and $\bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_{\lambda_x} \subseteq \prod_{s\in S} X_s$ and for any $s \in S(\Phi_\lambda)$, we have $(\tau_\beta)_s(V_s) > 1 - \mu$. By $\bigcap_{s\in S(\Phi_\lambda)} p_s^{-1}(V_s) = B_{\lambda_x}$, we have $p_\delta(B_{\lambda_x}) = V_\delta \subseteq X_\delta$,if $\delta \in S(\Phi_{\lambda_x})$; $p_\delta(B_{\lambda_x}) = X_\delta$,if $\delta \in S - S(\Phi_{\lambda_x})$. Since $B_{\lambda_x} \subseteq U$, for any $\delta \in S - S(\Phi_{\lambda_x})$, we have $p_\delta(U) \supseteq p_\delta(B_{\lambda_x}) = X_\delta$ and $p_\delta(U) = X_\delta$. On the other hand, since for any $s \in S$ and $U_s \in P(X_s)$, $\left(\prod_{t\in S} (\tau_\beta)_t\right)\left(p_s^{-1}(U_s)\right) \geq (\tau_\beta)_s(U_s)$, we have , for any $s \in S$, $I_\beta(p_s) = \bigwedge_{U_s\in P(X_s)} \min\left(1, 1 - (\tau_\beta)_s(U_s) + \prod_{t\in S} (\tau_\beta)_t \left(p_s^{-1}(U_s)\right)\right) = 1$. Furthermore, since by Theorem 5.3 in [11], we have $\models \Gamma_\beta(X, \tau) \otimes I_\beta(f) \to \Gamma_\beta(f(X))$, then $\Gamma_\beta(U, \tau/U) = \Gamma_\beta(U, \tau/U) \otimes I_\beta(p_s) \leq \Gamma_\beta(p_\delta(U), \tau_\delta) = \Gamma_\beta(X_\delta, \tau_\delta)$. Therefore $\bigvee_{T\subset_f S} \bigwedge_{t\in S-T} \Gamma_\beta(X_t, \tau_t) \geq \bigwedge_{\delta\in S-S(\Phi_\lambda)} \Gamma_\beta(X_\delta, \tau_\delta) \geq \Gamma_\beta(U, \tau/U) > \mu$.

The above theorem is a generalization of the following corollary.

Corollary 4.8 *If there exists a coordinate β-neighborhood β-compact subset U of some point $x \in X$ of the product space, then all except a finite number of coordinate spaces are β-compact.*

Lemma 4.9 *For any fuzzifying topological space (X, τ), $A \subseteq X$,*
$\models T_2^\beta(X, \tau) \to T_2^\beta(A, \tau/A)$.

Proof.

$$[T_2^\beta(X, \tau)] = \bigwedge_{x,y\in X, x\neq y} \bigvee_{U,V\in P(X), U\cap V=\phi} (N_x^\beta(U), N_y^\beta(V))$$
$$\leq \bigwedge_{x,y\in X, x\neq y} \bigvee_{(U\cap A)\cap(V\cap A)=\phi} (N_x^{\beta^A}(U\cap A), N_y^{\beta^A}(V\cap A))$$
$$\leq \bigwedge_{x,y\in A, x\neq y} \bigvee_{U'\cap V'=\phi, U',V'\in P(A)} (N_x^{\beta^A}(U'), N_y^{\beta^A}(V'))$$
$$= T_2^\beta(A, \tau/A),$$

where $N_x^{\beta^A}(U) = \bigvee_{x\in C\subseteq U} \tau_\beta/A(C)$ and $\tau_\beta/A(B) = \bigvee_{B=V\cap A} \tau_\beta(V)$.

Lemma 4.10 *For any fuzzifying $\beta-$topological space (X, τ),*
$\models T_2^\beta(X, \tau) \otimes \Gamma_\beta(X, \tau) \to T_4^\beta(X, \tau)$.
For the definition of $T_4^\beta(X, \tau)$ see [10, Definition 2.1].

Proof. If $[T_2^\beta(X, \tau) \otimes \Gamma_\beta(X, \tau)] = 0$, then the result holds. Now, suppose that $[T_2^\beta(X, \tau) \otimes \Gamma_\beta(X, \tau)] >$

$\lambda > 0$. Then $T_2^\beta(X,\tau) + \Gamma_\beta(X,\tau) - 1 > \lambda > 0$. Therefore from Theorem 5.4 in [11]
$T_2^\beta(X,\tau) \otimes (\Gamma_\beta(A) \wedge \Gamma_\beta(B)) \wedge (A \cap B = \phi) \models^{ws} T_2^\beta(X,\tau) \to (\exists U)(\exists V)((U \in \tau_\beta) \wedge (V \in \tau_\beta) \wedge (A \subseteq U) \wedge (B \subseteq V) \wedge (A \cap B = \phi))$. Then for any $A, B \subseteq X, A \cap B = \phi$,
$T_2^\beta(X,\tau) \otimes (\Gamma_\beta(A) \wedge \Gamma_\beta(B))$
$\leq \bigvee_{U \cap V = \phi, A \subseteq U, B \subseteq V} \min(\tau_\beta(U), \tau_\beta(V))$
or equivalently $T_2^\beta(X,\tau) \leq \Gamma_\beta(A) \wedge \Gamma_\beta(B) \to \bigvee_{U \cap V = \phi, A \subseteq U, B \subseteq V} \min(\tau_\beta(U), \tau_\beta(V))$
Hence for any $A, B \subseteq X, A \cap B = \phi$,
$1 - [\Gamma_\beta(A) \wedge \Gamma_\beta(B)] + \bigvee_{U \cap V = \phi, A \subseteq U, B \subseteq V} \min(\tau_\beta(U), \tau_\beta(V)) + \Gamma_\beta(X,\tau) - 1 > \lambda$.

From Theorem 5.1 in [11] we have $\models \Gamma_\beta(X,\tau) \otimes A \in \Gamma_\beta \to \Gamma_\beta(A)$. Then $\Gamma_\beta(X,\tau) + [\tau_\beta(A^c) \wedge \tau_\beta(B^c)] - 1 = (\Gamma_\beta(X,\tau) + \tau_\beta(A^c) - 1) \wedge (\Gamma_\beta(X,\tau) + \tau_\beta(B^c) - 1) \leq (\Gamma_\beta(X,\tau) \otimes \tau_\beta(A^c)) \wedge (\Gamma_\beta(X,\tau) \otimes \tau_\beta(B^c)) \leq [\Gamma_\beta(A) \wedge \Gamma_\beta(B)]$. Thus $\Gamma_\beta(X,\tau) - [\Gamma_\beta(A) \wedge \Gamma_\beta(B)] - 1 \leq -[\tau_\beta(A^c) \wedge \tau_\beta(B^c)]$. So, $1 - [\tau_\beta(A^c) \wedge \tau_\beta(B^c)] + \bigvee_{U \cap V = \phi, A \subseteq U, B \subseteq V} \min(\tau_\beta(U), \tau_\beta(V)) > \lambda$, i.e.,

$$T_4^\beta(X,\tau) = \bigwedge_{A \cap B = \phi} \min(1, 1 - [\tau_\beta(A^c) \wedge \tau_\beta(B^c)] + \bigvee_{U \cap V = \phi, A \subseteq U, B \subseteq V} \min(\tau_\beta(U), \tau_\beta(V))) > \lambda.$$

The above lemma is a generalization of the following corollary.

Corollary 4.11 *Every β-compact β-Hausdorff topological space is β-normal.*

Lemma 4.12 *For any fuzzifying $\beta-$topological space (X,τ),*
$\models T_2^\beta(X,\tau) \otimes \Gamma_\beta(X,\tau) \to T_3^\beta(X,\tau)$. *For the definition of*
$T_3^\beta(X,\tau)$ *see [10, Definition 2.1].*

Proof. Immediate, set $A = \{x\}$ in the above lemma.

The above lemma is a generalization of the following corollary.

Corollary 4.13 *Every β-compact β-Hausdorff topological space is β-regular.*

Theorem 4.14 *For any fuzzifying topological space (X,τ) and $A \subseteq X$,*
$\models T_2^\beta(X,\tau) \otimes \Gamma_\beta(A) \to A \in \Gamma_\beta$.

Proof. For any $\{x\} \subset A^c$, we have $\{x\} \cap A = \phi$ and $\Gamma_\beta(\{x\}) = 1$. By Theorem 5.4 in [11]
$[T_2^\beta(X,\tau) \otimes (\Gamma_\beta(A) \wedge \Gamma_\beta(\{x\}))]$
$\leq \bigvee_{G \cap H_x = \phi, A \subseteq G, x \in H_x} \min(\tau_\beta(G), \tau_\beta(H_x)))$. Assume $\gamma_x = \{H_x : A \cap H_x = \phi, x \in H_x\}$, $\bigcup_{x \in A^c} f(x) \supseteq A^c$ and $\bigcup_{x \in A^c} f(x) \cap A = \bigcup_{x \in A^c} (f(x) \cap A) = \phi$. So, $\bigcup_{x \in A^c} f(x) = A^c$.
Therefore $[T_2^\beta(X,\tau) \otimes \Gamma_\beta(A)] \leq \bigvee_{G \cap H_x = \phi, A \subseteq G, x \in H_x} \tau_\beta(H_x)$
$\leq \bigwedge_{x \in A^c} \bigvee_{A \cap H_x = \phi, x \in H_x} \tau_\beta(H_x)$
$= \bigvee_{f \in \prod_{x \in A^c} \gamma_x} \bigwedge_{x \in A^c} \tau_\beta(f(x)) \leq \bigvee_{f \in \prod_{x \in A^c} \gamma_x} \tau_\beta(\bigcup_{x \in A^c} f(x)) = \bigvee_{f \in \prod_{x \in A^c} \gamma_x} \tau_\beta(A^c) = \Gamma_\beta(A)$.

The above theorem is a generalization of the following corollary.

Corollary 4.15 *β-compact subspace of β-Hausdorff topological space is β-closed.*

5. Fuzzifying locally β-compactness

Definition 5.1 Let Ω be a class of fuzzifying topological spaces. A unary fuzzy predicate $L_\beta C \in \Im(\Omega)$, called fuzzifying locally β-compactness, is given as follows:
$(X,\tau) \in L_\beta C := (\forall x)(\exists B)((x \in Int_\beta(B) \otimes \Gamma_\beta(B, \tau/B))$. Since $[x \in Int_\beta(X)] = N_x^\beta(X) = 1$, then $L_\beta C(X,\tau) \geq \Gamma_\beta(X,\tau)$. Therefore, $\models (X,\tau) \in \Gamma_\beta \to (X,\tau) \in L_\beta C$.
Also, since $\models (X,\tau) \in \Gamma \to (X,\tau) \in LC$ [12]and $\models (X,\tau) \in \Gamma_\beta \to (X,\tau) \in \Gamma$ [11], $\models (X,\tau) \in \Gamma_\beta \to (X,\tau) \in LC$.

Theorem 5.2 *For any fuzzifying topological space (X,τ) and $A \subseteq X$,*
$\models (X,\tau) \in L_\beta C \otimes A \in \Gamma_\beta \to (A, \tau/A) \in L_\beta C$.

Proof. We have

$$L_\beta C(X,\tau) = \bigwedge_{x\in X}\bigvee_{B\subseteq X}\max(0, N_x^{\beta^X}(B)+\Gamma_\beta(B,\tau/B)-1)$$
$$L_\beta C(A,\tau/A) = \bigwedge_{x\in A}\bigvee_{G\subseteq A}\max(0, N_x^{\beta^A}(G)+\Gamma_\beta(G,(\tau/A)/G)-1).$$

Now, suppose that $[(X,\tau)\in L_\beta C\otimes A\in\Gamma_\beta]>\lambda>0$. Then for any $x\in A$, there exists $B\subseteq X$ such that $N_x^{\beta^X}(B)+\Gamma_\beta(B,\tau/B)+\tau_\beta(X-A)-2>\lambda$. (*)
Set $E=A\cap B\in P(A)$. Then $N_x^{\beta^A}(E)=\bigvee_{E=C\cap B}N_x^{\beta^X}(C)\geq N_x^{\beta^X}(B)$ and for any $U\in P(E)$, we have

$$(\tau_\beta/A)_\beta/E(U)=\bigvee_{U=C\cap E}\tau_\beta/A(C)=\bigvee_{U=C\cap E}\bigvee_{C=D\cap A}\tau_\beta(D)=\bigvee_{U=D\cap A\cap E}\tau_\beta(D)=\bigvee_{U=D\cap E}\tau_\beta(D).$$

Similarly, $(\tau_\beta/B)_\beta/E(U)=\bigvee_{U=D\cap E}\tau_\beta(D)$. Thus, $(\tau_\beta/B)_\beta/E=(\tau_\beta/A)_\beta/E$ and $\Gamma_\beta(E,(\tau/A)/E)=\Gamma_\beta(E,(\tau/B)/E)$. Furthermore, $[E\in\Gamma_\beta/B]=\tau_\beta/B(B-E)=\tau_\beta/B(B\cap E^c)=\bigvee_{B\cap E^c=B\cap D}\tau_\beta(D)\geq\tau_\beta(X-A)=\Gamma_\beta(A)$. Since $\models(X,\tau)\in\Gamma_\beta\otimes A\in\Gamma_\beta\to(A,\tau/A)\in\Gamma_\beta$, from (*) we have for any $x\in A$ that
$$\bigvee_{G\subseteq A}\max(0,N_x^{\beta^A}(G)+\Gamma_\beta(G,(\tau/A)/G)-1)$$
$$\geq N_x^{\beta^A}(E)+\Gamma_\beta(E,(\tau/A)/E)-1$$
$$=N_x^{\beta^A}(E)+\Gamma_\beta(E,(\tau/B)/E)-1$$
$$\geq N_x^{\beta^X}(B)+[\Gamma_\beta(B,\tau/B)\otimes E\in\Gamma_\beta/B]-1$$
$$\geq N_x^{\beta^X}(B)+\Gamma_\beta(B,\tau/B)+[E\in\Gamma_\beta/B]-2$$
$$\geq N_x^{\beta^X}(B)+\Gamma_\beta(B,\tau/B)+[A\in\Gamma_\beta]-2>\lambda.$$
Therefore $L_\beta C(A,\tau/A)=\bigwedge_{x\in A}\bigvee_{G\subseteq A}\max(0,N_x^{\beta^A}(G)+\Gamma_\beta(G,(\tau/A)/G)-1)>\lambda$.
Hence $[(X,\tau)\in L_\beta C\otimes A\in\Gamma_\beta]\leq L_\beta C(A,\tau/A)$.

As a crisp result of the above theorem we have the following corollary.

Corollary 5.3 *Let A be an β-closed subset of locally β-compact space (X,τ). Then A with the relative topology τ/A is locally β-compact.*

The following theorem is a generalization of the statement "If X is an β-Hausdorff topological space and A is an β-dense locally β-compact subspace, then A is β-open", where A is an β-dense in a topological space X if and only if the β-closure of A is X.

Theorem 5.4 *For any fuzzifying β-topological space (X,τ) and $A\subseteq X$,*
$\models T_2^\beta(X,\tau)\otimes L_\beta C(A,\tau/A)\otimes(Cl_\beta(A)\equiv X)\to A\in\tau_\beta$.

Proof. Assume $[T_2^\beta(X,\tau)\otimes L_\beta C(A,\tau/A)\otimes(Cl_\beta(A)\equiv X)]>\lambda>0$. Then $L_\beta C(A,\tau/A)>\lambda-[T_2^\beta(X,\tau)\otimes(Cl_\beta(A)\equiv X)]+1=\lambda'>\lambda$,i. e., $\bigwedge_{x\in A}\bigvee_{B\subseteq A}\max(0,N_x^{\beta^A}(B)+\Gamma_\beta(B,(\tau/A)/B)-1)>\lambda'$. Thus for any $x\in A$, there exists $B_x\subseteq A$ such that $N_x^{\beta^A}(B_x)+\Gamma_\beta(B_x,(\tau/A)/B_x)-1>\lambda'$. i.e., $\bigvee_{H\cap A=B_x}\bigvee_{x\in K\subseteq H}\tau_\beta(K)+\Gamma_\beta(B_x,(\tau/A)/B_x)-1>\lambda'$. Hence there exists K_x such that $K_x\cap A=B_x$, $\tau_\beta(K_x)+\Gamma_\beta(B_x,(\tau/A)/B_x)-1>\lambda'$. Therefore $\tau_\beta(K_x)>\lambda'$.
(1) If for any $x\in A$ there exists K_x such that $x\in K_x\subseteq B_x\subseteq A$, then $\bigcup_{x\in A}K_x=A$ and $\tau_\beta(A)=\tau_\beta(\bigcup_{x\in A}K_x)\geq\bigwedge_{x\in A}\tau_\beta(K_x)\geq\lambda'>\lambda$.
(2) If there exists $x_\circ\in A$ such that $K_{x_\circ}\cap(B_{x_\circ}^c)\neq\phi$, $\tau_\beta(K_{x_\circ})+\Gamma_\beta(B_{x_\circ},(\tau/A)/B_{x_\circ})-1>\lambda'$. From the hypothesis $[T_2^\beta(X,\tau)\otimes L_\beta C(A,\tau/A)\otimes(Cl_\beta(A)\equiv X)]>\lambda>0$, we have $[T_2^\beta(X,\tau)\otimes(Cl_\beta(A)\equiv X)]\neq 0$. So $\tau_\beta(K_{x_\circ})+\Gamma_\beta(B_{x_\circ},(\tau/A)/B_{x_\circ})-1+[T_2^\beta(X,\tau)\otimes(Cl_\beta(A)\equiv X)]-1>0$. Therefore $\tau_\beta(K_{x_\circ})+\Gamma_\beta(B_{x_\circ},(\tau/A)/B_{x_\circ})-1+T_2^\beta(X,\tau)+[(Cl_\beta(A)\equiv X)]-1-1>\lambda$. Since $(\tau_\beta/A)_\beta/B_{x_\circ}(U)=\bigvee_{U=C\cap B_{x_\circ}}\tau_\beta/A(C)=\bigvee_{U=C\cap B_{x_\circ}}\bigvee_{C=D\cap A}\tau_\beta(D)=\bigvee_{U=D\cap B_{x_\circ}}\tau_\beta(D)=\tau_\beta/B_{x_\circ}(U),\Gamma_\beta(B_{x_\circ},(\tau/A)/B_{x_\circ})=\Gamma_\beta(B_{x_\circ},\tau/B_{x_\circ})$.
From Theorem 4.3 we have $\tau_\beta(B_{x_\circ}^c)\geq T_2^\beta(X,\tau)\otimes\Gamma_\beta(B_{x_\circ},\tau/B_{x_\circ})\geq T_2^\beta(X,\tau)+\Gamma_\beta(B_{x_\circ},\tau/B_{x_\circ})-1$.
Hence $\tau_\beta(K_{x_\circ})+\tau_\beta(B_{x_\circ}^c)+[Cl_\beta(A)\equiv X]-2>\lambda$.
Now, for any $y\in A^c$ we have $[Cl_\beta(A)\equiv X]=\bigwedge_{x\in X}(1-N_x^{\beta^X}(A^c))\leq 1-N_y^{\beta^X}(A^c)$. Since (X,τ) is a fuzzifying β-topological space, $\tau_\beta(K_{x_\circ})+\tau_\beta(B_{x_\circ}^c)-1\leq\tau_\beta(K_{x_\circ})\otimes\tau_\beta(B_{x_\circ}^c)\leq\tau_\beta(K_{x_\circ})\wedge\tau_\beta(B_{x_\circ}^c)\leq\tau_\beta(K_{x_\circ}\cap B_{x_\circ}^c)\leq N_y^{\beta^X}(K_{x_\circ}\cap B_{x_\circ}^c)\leq N_y^{\beta^X}(A^c)$, where $y\in K_{x_\circ}\cap B_{x_\circ}^c\subseteq H_{x_\circ}\cap(H_{x_\circ}\cap A)^c=H_{x_\circ}\cap(H_{x_\circ}^c\cup A^c)=H_{x_\circ}\cap A^c\subseteq A^c$. Therefore

$0<\lambda<\tau_\beta(K_{x_\circ})+\tau_\beta(B^c_{x_\circ})+[Cl_\beta(A)\equiv X]-2=\tau_\beta(K_{x_\circ})+\tau_\beta(B^c_{x_\circ})-1+[Cl_\beta(A)\equiv X]-1\leq N^{\beta^X}_y(A^c)+1-N^{\beta^X}_y(A^c)-1=0$, a contradiction. So, case (2) does not hold. We complete the proof.

Theorem 5.5 *For any fuzzifying β-topological space (X,τ),*
$\models T^\beta_2(X,\tau)\otimes(L_\beta C(X,\tau))^2\to\forall x\forall U(U\in N^{\beta^X}_x\to\exists V(V\in N^{\beta^X}_x\wedge Cl_\beta(V)\subseteq U\wedge\Gamma_\beta(V))),$
where $(L_\beta C(X,\tau))^2:=L_\beta C(X,\tau)\otimes L_\beta C(X,\tau)$.

Proof. We need to show that for any x and U, $x\in U$,
$T^\beta_2(X,\tau)\otimes(L_\beta C(X,\tau))^2\otimes N^{\beta^X}_x(U)\leq\bigvee_{V\subseteq X}(N^{\beta^X}_x(V)\wedge\bigwedge_{y\in U^c}N^{\beta^X}_x(V^c)\wedge\Gamma_\beta(V,\tau/V))$. Assume that $T^\beta_2(X,\tau)\otimes(L_\beta C(X,\tau))^2\otimes N^{\beta^X}_x(U)>\lambda>0$. Then for any $x\in X$ there exists C such that
$T^\beta_2(X,\tau)+N^{\beta^X}_x(C)+(L_\beta C(X,\tau))^2+N^{\beta^X}_x(U)-3>\lambda.$ (*)
Since (X,τ) is fuzzifying β-topological space, $N^{\beta^X}_x(C)+N^{\beta^X}_x(U)-1\leq N^{\beta^X}_x(C)\otimes N^{\beta^X}_x(U)\leq N^{\beta^X}_x(C)\wedge N^{\beta^X}_x(U)\leq N^{\beta^X}_x(C\cap U)=\bigvee_{x\in W\subseteq C\cap U}\tau_\beta(W)$. Therefore there exists W such that $x\in W\subseteq C\cap U$, and $T^\beta_2(X,\tau)+(L_\beta C(X,\tau))^2+\tau_\beta(W)-2>\lambda$. By Lemmas 4.3 and 4.5 we have $T^\beta_2(X,\tau)\leq T^\beta_2(C,\tau/C)$ and $T^\beta_2(C,\tau/C)+\Gamma_\beta(C,\tau/C)-1\leq T^\beta_2(C,\tau/C)\otimes\Gamma_\beta(C,\tau/C)\leq T^\beta_3(C,\tau/C)$.Thus $T^\beta_3(X,\tau)+\Gamma_\beta(C,\tau/C)+\tau_\beta(W)-2>\lambda$. Since for any $x\in W\subseteq U$, we have $T^\beta_3(C,\tau/C)\leq 1-\tau_\beta/C(W)+\bigvee_{G\subseteq C}\left((N^{\beta^C}_x(G)\wedge\bigwedge_{y\in C-W}N^{\beta^C}_y(C-G))\right)$, so there exists $G, x\in G\subseteq W$ such that

$$\begin{aligned}&\left((N^{\beta^C}_x(G)\wedge\bigwedge_{y\in C-W}N^{\beta^C}_y(C-G))\right)\\&\geq T^\beta_3(C,\tau/C)+\tau_\beta/C(W)-1\\&\geq T^\beta_3(C,\tau/C)+\tau_\beta(W)-1,\\&\left((N^{\beta^C}_x(G)\wedge\bigwedge_{y\in C-W}N^{\beta^C}_y(C-G))\right)\\&+\Gamma_\beta(C,\tau/C)-1>\lambda.\end{aligned}$$

Thus $N^{\beta^C}_x(G)=\bigvee_{D\cap C=G}N^{\beta^X}_x(D)=N^{\beta^X}_x(G\cup C^c)>\lambda'=\lambda+1-\Gamma_\beta(C,\tau/C)\geq\lambda$. Furthermore, for any $y\in C-W$, $N^{\beta^C}_y(C-G)=\bigvee_{D\cap C=C\cap G^c}N^{\beta^X}_y(G^c\cup C^c)=N^{\beta^X}_y(G^c)>\lambda'$ and $N^{\beta^X}_x(G)=N^{\beta^X}_x((G\cup C^c)\cap C)\geq N^{\beta^X}_x(G\cup C^c)\wedge N^{\beta^X}_x(C)>\lambda'$. Since $N^{\beta^X}_y(G^c)=\bigvee_{x\in B^c\subseteq G^c}\tau_\beta(B^c)>\lambda'$, for any $y\in C-W$, there exists B^c_y such that $y\in B^c_y\subseteq G^c$ and $\tau_\beta(B^c_y)>\lambda'$. Set $B^c=\bigcup_{y\in C-W}B^c_y$. Then $C-W\subseteq B^c\subseteq G^c$ and $\tau_\beta(B^c)\geq\bigwedge_{y\in C-W}\tau_\beta(B^c_y)\geq\lambda'$. Again, set $V=B\cap C$, then $V\subseteq(C-W)^c\cap C=(C^c\cup W)\cap C=C\cap W=W\subseteq U\cap C$ and $V^c=B^c\cup C^c$. Since (X,τ) is fuzzifying β-topological space,

$$\begin{aligned}N^{\beta^X}_x(V)&=N^{\beta^X}_x(B\cap C)\geq N^{\beta^X}_x(B)\wedge N^{\beta^X}_x(C)\\&\geq N^{\beta^X}_x(G)\wedge N^{\beta^X}_x(C)>\lambda.\end{aligned}\quad(1)$$

By (*) and Theorem 4.3, $\tau_\beta(C^c)\geq T^\beta_2(X,\tau)\otimes\Gamma_\beta(C,\tau/C)\geq T^\beta_2(X,\tau)+\Gamma_\beta(C,\tau/C)-1\geq\lambda'$. So $\tau_\beta(V^c)=\tau_\beta(B^c\cup C^c)\geq\tau_\beta(B^c)\wedge\tau_\beta(C^c)\geq\lambda'$, i.e., $\tau_\beta(V^c)+\Gamma_\beta(C,\tau/C)-1\geq\lambda$ and

$$\begin{aligned}&\Gamma_\beta(V,\tau/V)=\Gamma_\beta(V,(\tau/C)/V)\\&\geq\tau_\beta/C(C-V)+\Gamma_\beta(C,\tau/C)-1\\&\geq\tau_\beta(V^c)+\Gamma_\beta(C,\tau/C)-1\geq\lambda.\end{aligned}\quad(2)$$

$$\bigwedge_{y\in U^c}N^{\beta^X}_y(V^c)\geq\bigwedge_{y\in V^c}N^{\beta^X}_y(V^c)=\tau_\beta(V^c)\geq\lambda.\quad(3)$$

Thus by (1), (2) and (3), for any $x\in U$, there exists $V\subseteq U$ such that $N^{\beta^X}_x(V)>\lambda$, $\bigwedge_{y\in U^c}N^{\beta^X}_y(V^c)\geq\lambda$ and $\Gamma_\beta(V,\tau/V)\geq\lambda$. So

$$\bigvee_{V\subseteq X}(N^{\beta^X}_x(V)\wedge\bigwedge_{y\in U^c}N^{\beta^X}_y(V^c)\wedge\Gamma_\beta(V,\tau/V))\geq\lambda.$$

Theorem 5.6 *For any fuzzifying β-topological space (X,τ),*
$\models T^\beta_2(X,\tau)\otimes(L_\beta C(X,\tau))^2\to T^\beta_3(X,\tau)$

Proof. By Theorem 5.5, for any $x\in U$, we have $\bigvee_{x\in V\subseteq U}(N^{\beta^X}_x(V)\wedge\bigwedge_{y\in U^c}N^{\beta^X}_y(V^c)\geq[T^\beta_2(X,\tau)\otimes(\Gamma_\beta(C,\tau/C))^2\otimes N^{\beta^X}_x(U)]$.

Thus $1-N_x^{\beta^X}(U)+\bigvee_{x\in V\subseteq U}(N_x^{\beta^X}(V)\wedge\bigwedge_{y\in U^c}N_y^{\beta^X}(V^c)\geq$ $[T_2^\beta(X,\tau)\otimes(\Gamma_\beta(C,\tau/C))^2]$,
i.e., $[T_3^\beta(X,\tau)]\geq[T_2^\beta(X,\tau)\otimes(\Gamma_\beta(C,\tau/C))^2]$.

Theorem 5.7 *For any fuzzifying β-topological space (X,τ),*
$\models T_3^\beta(X,\tau)\otimes L_\beta C(X,\tau)\rightarrow\forall A\forall U(U\in N_A^{\beta^X}\otimes\Gamma_\beta(A,\tau/A)\rightarrow$
$\exists V(V\subseteq U\wedge U\in N_A^{\beta^X}\wedge\tau_\beta(V^c)\wedge\Gamma_\beta(V,\tau/V)))$,
where $U\in N_A^{\beta^X}:=(\forall x)(x\in A\wedge U\in N_x^{\beta^X})$.

Proof. We only need to show that for any $A,U\in P(X)$,
$[T_3^\beta(X,\tau)\otimes L_\beta C(X,\tau)\otimes\Gamma_\beta(A,\tau/A)\otimes N_A^{\beta^X}(U)]\leq\bigvee_{V\subseteq U}(N_A^{\beta^X}(V)\wedge\tau_\beta(V^c)\wedge\Gamma_\beta(V,\tau/V))$.
Indeed, if $[T_3^\beta(X,\tau)\otimes L_\beta C(X,\tau)\otimes\Gamma_\beta(A,\tau/A)\otimes N_A^{\beta^X}(U)]>\lambda>0$, then for any $x\in A$, there exists $C\in P(X)$ such that $[T_3^\beta(X,\tau)\otimes N_x^{\beta^X}(C)\otimes\Gamma_\beta(C,\tau/C)\otimes\Gamma_\beta(A,\tau/A)\otimes N_A^{\beta^X}(U)]>\lambda$.
Since (X,τ) is fuzzifying β-topological space,

$$\begin{aligned}&\bigvee_{x\in W\subseteq C\cap U}\tau_\beta(W)\\&=N_x^{\beta^X}(C\cap U)\geq N_x^{\beta^X}(C)\wedge N_x^{\beta^X}(U)\\&\geq N_x^{\beta^X}(C)\wedge N_A^{\beta^X}(U)\geq N_x^{\beta^X}(C)\otimes N_A^{\beta^X}(U).\end{aligned}$$

Then there exists W such that $x\in W\subseteq C\cap U$, and $[T_3^\beta(X,\tau)\otimes\tau_\beta(W)\otimes\Gamma_\beta(C,\tau/C)\otimes\Gamma_\beta(A,\tau/A)]>\lambda$. Therefore
$[T_3^\beta(X,\tau)]+\tau_\beta(W)-1>\lambda+2-\Gamma_\beta(C,\tau/C)-\Gamma_\beta(A,\tau/A)]=\lambda'\geq\lambda$. (*)
Since for any $x\in W$, $[T_3^\beta(X,\tau)]\leq 1-\tau_\beta(W)+\bigvee_{B\subseteq W}(N_x^{\beta^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{\beta^X}(B^c))$, we have

$$\bigvee_{B\subseteq W}(N_x^{\beta^X}(B)\wedge\bigwedge_{y\in W^c}N_y^{\beta^X}(B^c))>\lambda'.$$

Thus there exists B_x such that $x\in B_x\subseteq W\subseteq C\cap U$ and for any $y\in W^c$, we have $N_y^{\beta^X}(B_x^c)>\lambda'$, $N_x^{\beta^X}(B_x)>\lambda'$. Since $N_y^{\beta^X}(B_x^c)=\bigvee_{x\in G^c\subseteq B_x^c}\tau_\beta(G^c)>\lambda'$, then for any $y\in W^c$, there exists G_{xy} such that $x\in G_{xy}^c\subseteq B_x^c$ and $\tau_\beta(G_{xy}^c)>\lambda'$. Set $G_x^c=\bigcup_{y\in W^c}G_{xy}^c$, then $W^c\subseteq G_{xy}^c\subseteq B_x^c$ and $\tau_\beta(G_x^c)\geq\bigwedge_{y\in W^c}\tau_\beta(G_{xy}^c)\geq\lambda'$. Since $G_x\supseteq B_x$, $N_x^{\beta^X}(G_x)\geq N_x^{\beta^X}(B_x)>\lambda'$, i.e., $\bigvee_{x\in H\subseteq G_x}\tau_\beta(H)>\lambda'$. Thus there exists H_x such that $x\in H_x\subseteq G_x$ and $\tau_\beta(H_x)>\lambda'$. Hence for any $x\in A$, there exists H_x and G_x such that $x\in H_x\subseteq G_x\subseteq U$, $\tau_\beta(H_x)>\lambda'$ and $W\supseteq\bigcup_{x\in A}G_x\supseteq\bigcup_{x\in A}H_x\supseteq A$. We define $\Re\in\Im(P(A))$ as follows:

$$\Re(D)=\begin{cases}\bigvee_{H_x\cap A=D}\tau_\beta(H_x), & \exists H_x \text{ with } H_x\cap A=D,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\Gamma_\beta(A,\tau/A)=\mu>\mu-\epsilon(\ \epsilon>0)$. Then $1-K_\beta(\Re,A)+\bigvee_{\wp\leq\Re}[K(\Re,A)\otimes FF(\wp)]>\mu-\epsilon$, where

$$\begin{aligned}&[K(\Re,A)]=\bigwedge_{x\in A}\bigvee_{x\in B}\Re(B)\\&=\bigwedge_{x\in A}\bigvee_{x\in D}\Re(D)\\&=\bigwedge_{x\in A}\bigvee_{x\in D}\bigvee_{H_{x'}\cap A=D}\tau_\beta(H_{x'})\geq\lambda'\end{aligned}$$

$$\begin{aligned}&[\Re\subseteq\tau_\beta\backslash A]\\&=\bigwedge_{B\subseteq X}\min(1,1-\Re(B)+\tau_\beta\backslash A(B))\\&=\bigwedge_{B\subseteq X}\min(1,1-\bigvee_{H_x\cap A=B}\tau_\beta(H_x)+\bigvee_{H\cap A=B}\tau_\beta(H))=1.\end{aligned}$$

So, $K_\beta(\Re,A)=[K(\Re,A)]\geq\lambda'$. By (*), $[K(\Re,A)\otimes FF(\wp)]>\mu-\epsilon-1+K_\beta(\Re,A)\geq\mu-\epsilon-1+\lambda'\geq\lambda-\epsilon$.
Thus $\bigwedge_{x\in A}\bigvee_{x\in E}\Re(E)+1-\bigwedge\{\delta:F(\wp_\delta)\}-1>\lambda-\epsilon$, and $\bigwedge_{x\in A}\bigvee_{x\in E}\Re(E)>\lambda-\epsilon+\bigwedge\{\delta:F(\wp_\delta)\}$.
Hence there exists $\alpha>0$ such that $F(\wp_\alpha)$ and $\bigwedge_{x\in A}\bigvee_{x\in D}\Re(D)>\lambda-\epsilon+\alpha$. Therefore for any $x\in A$, there exists $D_x\subseteq A$ such that $\wp(D_x)>\lambda-\epsilon+\alpha$ and $\bigcup_{x\in A}D_x\supseteq A$. Suitably choose ϵ such that $\lambda-\epsilon>0$, then $\wp(D_x)>\alpha>0$. Since $\Re(D_x)\geq\wp(D_x)>0$, $D_x=H_{x'}\cap A$, i.e., $H_{x'}\cap A\in\wp_\alpha$. By $F(\wp_\alpha)$, so there exists finite $H_{x'_1},H_{x'_2},...,H_{x'_n}$ such that $\bigcup_{i=1}^n H_{x'_i}\supseteq A$ and $\bigcup_{i=1}^n H_{x'_i}\subseteq\bigcup_{i=1}^n G_{x'_i}$.Set $V=\bigcup_{i=1}^n G_{x'_i}$, and $V^c=\bigcap_{i=1}^n G_{x'_i}^c$, $A\subseteq V\subseteq U$, and $\tau_\beta(V^c)\geq\bigwedge_{1\leq i\leq n}\tau_\beta(G_{x'_i}^c)\geq\lambda'>\lambda$. Since for any $x\in A$, $G_x\subseteq W\subseteq C\cap U\subseteq C$, we have $V=\bigcup_{i=1}^n G_{x'_i}\subseteq W\subseteq C$. Because $\tau_\beta\backslash C(C-V)=\bigvee_{D\cap C=C\cap V^c}\tau_\beta(D)\geq\tau_\beta(V^c)\geq\lambda'$. Thus by (*), $\tau_\beta\backslash C(C-V)+\Gamma_\beta(C,\tau/C)-1>\lambda$. By Theo-

rem 5.1 in [11], $\Gamma_\beta(V,\tau/V) = \Gamma_\beta(V,\tau/C/V) \geq [\Gamma_\beta(C,\tau/C)\otimes\tau_\beta\backslash C(C-V)] > \lambda$.
Finally, we have for any $x\in A$,
$N_x^{\beta^X}(V) = N_x^{\beta^X}(\bigcup_{i=1}^n G_{x'_i}) \geq N_x^{\beta^X}(\bigcup_{i=1}^n H_{x'_i}) \geq \tau_\beta(\bigcup_{i=1}^n H_{x'_i}) \geq \bigwedge_{1\leq i\leq n}\tau_\beta(H_{x'_i}) \geq \lambda' > \lambda$. So $N_A^{\beta^X}(V) = \bigwedge_{x\in A} N_x^{\beta^X}(V) \geq \lambda$. Therefore $N_A^{\beta^X}(V)\wedge \tau_\beta(V^c)\wedge\Gamma_\beta(V,\tau/V)\geq\lambda$.
Thus $\bigvee_{V\subseteq U}(N_A^{\beta^X}(V)\wedge\tau_\beta(V^c)\wedge\Gamma_\beta(V,\tau/V))\geq\lambda$.

Theorem 5.8 *Let (X,τ) and (Y,σ) be two fuzzifying topological space and $f\in Y^X$ be surjective. Then $\models L_\beta C(X,\tau)\otimes C_\beta(f)\otimes O(f)\to LC(Y,\sigma)$, where $O(f) := (\forall U)((U\in\tau)\to(f(U)\in\sigma)$.*

Proof. If $[L_\beta C(X,\tau)\otimes C_\beta(f)\otimes O(f)] > \lambda > 0$, then for any $x\in X$,there exists $U\subseteq X$, such that $[N_x^{\beta^X}(U)\otimes\Gamma_\beta(U,\tau/U)\otimes C_\beta(f)\otimes O(f)] > \lambda$. Since $N_x^{\beta^X}(U) = \bigvee_{x\in V\subseteq U}\tau_\beta(V)$, so there exists $V'\subseteq X$ such that $x\in V'\subseteq U$ and $[\tau_\beta(V')\otimes\Gamma_\beta(U,\tau/U)\otimes C_\beta(f)\otimes O(f)] > \lambda$. By Theorem 5.2 in [11] , $[\Gamma_\beta(U,\tau/U)\otimes C_\beta(f)]\leq[\Gamma(f(U),\sigma/f(U))]$ and

$$\begin{aligned}
&[\tau(V')\otimes O(f)] = \max(0,\tau(V')+O(f)-1)\\
&= \max(0,\tau(V')+\\
&\bigwedge_{V\subseteq X}\min(1,1-\tau(V')+\sigma(f(V)))-1)\\
&\leq \max(0,\tau(V')+1-\tau(V')+\sigma(f(V))-1)\\
&= \sigma(f(V)) \leq N_{f(x)}^{Y}(f(V')) \leq N_{f(x)}^{Y}(f(U)).
\end{aligned}$$

Since f is surjective,

$$\begin{aligned}
&LC(Y,\sigma) = LC(f(X),\sigma)\\
&= \bigwedge_{y\in f(x)\subseteq f(X)}\ \bigvee_{U'=f(U)\subseteq f(X)}[N_y^{Y}(U')\otimes[\Gamma(U',\sigma/U')]\\
&\geq \bigwedge_{y\in f(x)\subseteq f(X)}[N_{f(x)}^{Y}(f(U))\otimes[\Gamma(f(U),\sigma/f(U))]\\
&\geq \bigwedge_{y\in f(x)\subseteq f(X)}[\tau(V')\otimes O(f)\otimes\Gamma_\beta(U,\tau/U)\otimes C_\beta(f)]\\
&\geq \lambda.
\end{aligned}$$

Theorem 5.9 *Let (X,τ) and (Y,σ) be two fuzzifying topological space and $f\in Y^X$ be surjective. Then $\models L_\beta C(X,\tau)\otimes I_\beta(f)\otimes O_\beta(f)\to L_\beta C(Y,\sigma)$.*

Proof. By Theorem 5.3 in [11], the proof is similar to the proof of Theorem 5.8.

Theorems 5.8 and 5.9 are a generalization of the following corollary.

Corollary 5.10 *Let (X,τ) and (Y,σ) be two topological space and $f:(X,\tau)\to(Y,\sigma)$ be surjective mapping. If f is an β-continuous (resp. β-irresolute), open (resp. β-open) and X is locally β-compact, then Y is locally compact (resp. locally β-compact) space.*

Theorem 5.11 *Let $\{(X_s,\tau_s): s\in S\}$ be a family of fuzzifying topological spaces, then*
$\models L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s)\to\forall s(s\in S\wedge L_\beta C(X_s,(\tau_\beta)_s)\wedge \exists T(T\subset_f S\wedge\forall t(t\in S-T\wedge\Gamma_\beta(X_t,\tau_t)))$.

Proof. It suffices to show that
$L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s) \leq \bigwedge_{s\in S}[L_\beta C(X_s,(\tau_\beta)_s)\wedge \bigvee_{T\subset_f S}\bigwedge_{t\in S-T}\Gamma_\beta(X_t,\tau_t)]$.
From Theorem 5.8 and Lemma 4.5 we have for any $t\in S$,
$L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s) = [L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s)\otimes C_\beta(p_t)\otimes O_\beta(p_t)]\leq L_\beta C(X_t,\tau_t)$.
So, $\bigwedge_{t\in S-T}L_\beta C(X_t,\tau_t)\geq L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s)$.
By Theorem 4.7 we have

$$\begin{aligned}
&\bigvee_{T\subset_f S}\bigwedge_{t\in S-T}\Gamma_\beta(X_t,\tau_t)\\
&\geq [\bigvee_{U\subseteq\prod_{s\in S}X_s}\Gamma_\beta(U,\prod_{s\in S}(\tau_\beta)_s/U)\otimes\bigvee_{X\subseteq\prod_{s\in S}X_s}N_x^{\beta^X}(U))]\\
&\geq \bigvee_{U\subseteq\prod_{s\in S}X_s}\ \bigvee_{X\subseteq\prod_{s\in S}X_s}[\Gamma_\beta(U,\prod_{s\in S}(\tau_\beta)_s/U)\otimes N_x^{\beta^X}(U))]\\
&\geq \bigwedge_{X\subseteq\prod_{s\in S}X_s}\ \bigvee_{U\subseteq\prod_{s\in S}X_s}[\Gamma_\beta(U,\prod_{s\in S}(\tau_\beta)_s/U)\otimes N_x^{\beta^X}(U))]\\
&= L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s).
\end{aligned}$$

Therefore

$$\begin{aligned}
&L_\beta C(\prod_{s\in S}X_s,\prod_{s\in S}(\tau_\beta)_s)\\
&\leq [\bigwedge_{t\in S-T}L_\beta C(X_t,\tau_t)\wedge\bigvee_{T\subset_f S}\bigwedge_{t\in S-T}\Gamma_\beta(X_t,\tau_t)].
\end{aligned}$$

We can obtain the following corollary in crisp setting.

Corollary 5.12 *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of nonempty topological spaces. If $\prod_{\lambda\in\Lambda} X_\lambda$ is locally β-compact, then each X_λ is locally β-compact and all but finitely many X_λ are β-compact.*

6. Conclusion

The present paper investigates topological notions when these are planted into the framework of Ying's fuzzifying topological spaces (in semantic method of continuous valued-logic). The main contributions of the present paper are to give characterizations of fuzzifying β-compactness. Also, we define the concept of locally β-compactness of fuzzifying topological spaces and obtain some basic properties of such spaces. There are some open questions for further study:
(1) One obvious problem is: our results are derived in the Łukasiewicz continuous logic. It is possible to generalize them to more general logic setting, like residuated lattice-valued logic considered in [17-18].
(2) What is the justification for fuzzifying locally β-compactness in the setting of (2, L) topologies?
(3) What is the justification for fuzzifying locally strong compactness in (M, L)-topologies etc?

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Answer to jifs15-1335

Dear, Dr. Referees for jifs15-1335

I wish to express my sincere thanks to referees for their valuable suggestions.

The following statements are my answer for your comments.

(A) Answer for Review 1

I investigate topological notions in semantic Ying's method of Łukasiewicz continuous logic. I think it an important results.

I revise introduction and conclusion.

(B) Answer for Review 2

[1] As your direction, I revise a title as " Local β compactness as fuzzy predicates defined in Łukasiewicz logic"

[2] As your direction, I simplify Introduction and insert section 2 (Preliminaries and definitions)

[3,4] For readers, I insert Example 3.3, Remarks 3.2 and 3.7. Every corollaries are the corresponding results in classical topology.

Remark 3.2 In above definition, we can obtain that $\tau_r = \{U \in P(X) \mid \tau(U) \geq r\}$ is a classical topology for each $r \in [0,1]$, similarly, $(ß_\beta)_r, (\tau_\beta)_r$. Then $(ß_\beta)_r$ is a β-base of $(\tau_\beta)_r$ if for each $U \in (N_x^\beta)_r$, there exists $V \in (ß_\beta)_r$ such that $x \in V \subseteq U$.

Example 3.3 Let $X = \{x, y, z\}$ and we define a fuzzifying topology $\tau \in \Im(P(X))$ as follows:

$$\tau(X) = \tau(\emptyset) = 1, \tau(\{x\}) = 0.8,$$
$$\tau(\{y\}) = 0.6, \tau(\{z\}) = 0.4,$$
$$\tau(\{x,y\}) = 0.6, \tau(\{y,z\}) = 0.6, \tau(\{z,x\}) = 0.4.$$

Since $N_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$, we have

$$N_x(X) = 1, \forall x \in X,$$
$$N_x(\{x\}) = N_x(\{x,y\}) = N_x(\{x,z\}) = 0.8,$$
$$N_y(\{y\}) = N_y(\{y,z\}) = 0.6, N_y(\{x,y\}) = 0.8,$$
$$N_z(\{z\}) = N_z(\{x,z\}) = 0.4, N_z(\{y,z\}) = 0.6.$$

Since $Cl(A)(x) = 1 - N_x(X - A)$,

$$Cl(\{x\})(x) = 1 - N_x(\{y,z\}) = 1,$$
$$Cl(\{x\})(y) = 1 - N_y(\{y,z\}) = 0.4,$$
$$Cl(\{x\})(z) = 1 - N_z(\{y,z\}) = 0.4,$$
$$Cl(\{y\}) = (0.2, 1, 0.6), Cl(\{z\}) = (0.2, 0.2, 1),$$
$$Cl(\{x,y\}) = (1, 1, 0.6), Cl(\{x,z\}) = (1, 0.4, 1),$$
$$Cl(\{y,z\}) = (0.2, 1, 1), Cl(X) = (1, 1, 1),$$
$$Cl(\emptyset) = (0, 0, 0).$$

Then $Int(Cl(A)) = A$ and $Cl(Int(Cl(A))) = Cl(A)$ for all $A \in P(X)$. From Definition 2.2, we obtain a fuzzifying topology $\tau_\beta \in \Im(P(X))$ as follows:

$$\tau_\beta(X) = \tau_\beta(\emptyset) = 1, \tau_\beta(\{x\}) = 0.4,$$
$$\tau_\beta(\{y\}) = 0.2, \tau_\beta(\{z\}) = 0.2$$
$$\tau_\beta(\{x,y\}) = 0.6, \tau_\beta(\{x,z\}) = 0.4, \tau_\beta(\{y,z\}) = 0.2.$$

(1) We define $ß_\beta \in \Im(P(X))$ as follows:

$$ß_\beta(X) = ß_\beta(\emptyset) = 1, ß_\beta(\{x\}) = 0.4,$$
$$ß_\beta(\{y\}) = 0.2, ß_\beta(\{z\}) = 0.2,$$
$$ß_\beta(\{x,y\}) = 0.6, ß_\beta(\{x,z\}) = ß_\beta(\{y,z\}) = 0.$$

Since $ß_\beta \subset \tau_\beta$ and $N_x^\beta(U) \leq \bigvee_{x \in V \subseteq U} ß_\beta(V)$ from Definition 3.1, then $ß_\beta$ is a β-base of τ_β.

(2) We define $c_\beta \in \Im(P(X))$ as follows:

$$c_\beta(X) = c_\beta(\emptyset) = 1, c_\beta(\{x\}) = 0.2,$$
$$c_\beta(\{y\}) = 0.2, c_\beta(\{z\}) = 0.2$$
$$c_\beta(\{x,y\}) = 0.6, c_\beta(\{x,z\}) = c_\beta(\{y,z\}) = 0.$$

We have $c_\beta \subset \tau_\beta$ and $N_x^\beta(\{x,z\}) = \tau_\beta(\{x,z\}) = 0.4 \not\leq \bigvee_{x \in V \subseteq \{x,z\}} c_\beta(V) = 0.2$. Hence c_β is not a β-base of τ_β. Moreover, $(c_\beta)_{0.3} = \{X, \emptyset, \{x,y\}\}$ is not a β-base of $(\tau_\beta)_{0.3} = \{X, \emptyset, \{x\}, \{x,y\}, \{x,z\}\}$ in Remark 3.2.

Remark 3.7 From Remark 3.2 and Definition 3.6, since (X, τ_r) is a classical topology for each $r \in [0,1]$ and $(\tau_\beta)_r$ is the collection of all β-open sets in X, then a β-subbase of $(\tau_\beta)_r$ is a collection $(\varphi_\beta)_r$ of β-open sets such that every β-open set of $(\tau_\beta)_r$ is the union of sets that are finite intersections of $\{V_\lambda : \lambda \in \Lambda\} \subset (\varphi_\beta)_r$ with a finite index Λ.