

# ON  $L$ -FUZZY  $(K, E)$ -SOFT GRILLS

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## Abstract

The aim of this paper is to present and make preliminary study of a new  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft topology  ${\mathcal T}_{{\mathcal G}}$  from old  $(L, \odot, \sqcap)$ fuzzy  $(K, E)$ -soft topology  $T$  via L-fuzzy  $(K, E)$ -soft grill  $G$ induced by L-fuzzy soft operators  $\phi_G$ ,  $\psi_G$ .

## 1. Introduction and Preliminaries

In 1999, Molodtsov [21] introduced the theory of soft sets as a new

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mathematical tool for dealing with uncertainties. Also, he applied this theory to several directions (see, for example, [22-24]). The soft set theory has been applied to many different fields (see, for example, [1, 2, 5, 6, 8, 10, 17, 19, 25, 32, 35, 36]). Later, few researches (see, for example, [3, 7, 15, 16, 20, 26, 30, 33, 37]) introduced and studied the notion of soft topological spaces.

Roy and Samanta [31] redefined some definitions on fuzzy soft set in another form and defined a fuzzy soft topology. The concept of L-fuzzy soft sets can be seen as a generalization of fuzzy soft sets introduced by Cetkin et al. [9, 18, 27].

Aygünoglu et al. [4] defined fuzzy soft topology which will be compatible to the fuzzy soft theory and investigated some of its fundamental properties and introduced fuzzy soft cotopology and given the relations between fuzzy soft topology and fuzzy soft cotopology.

On the other hand, Hájek [11] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices.

In this paper, we present a kind of  $L$ -fuzzy soft operator, by using L-fuzzy  $(K, E)$ -soft grill, which eventually given rise to another L-fuzzy soft operator which satisfies Kuratowski's L-fuzzy soft closure axioms, thereby inducing a new  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topology. Some properties of the induced  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topology will be investigated.

Let  $(L, \leq, \vee, \wedge, 0, 1)$  be a completely distributive lattice with least element  $0_L$  and the greatest element  $1_L$  in L.

**Definition 1.1** [11, 12, 28]. A complete lattice  $(L, \leq, \odot)$  is called a strictly two-sided commutative quantale (stsc-quantale, for short) if and only if it satisfies the following properties:

(L1)  $(L, \odot)$  is a commutative semigroup.

(L2)  $x = x \odot 1$  for each  $x \in L$  and 1 is the universal upper bound.

(L3)  $\odot$  is distributive over arbitrary joins, i.e.,  $(\bigvee_{i \in \Gamma} x_i) \odot y =$  $\bigvee_{i \in \Gamma} (x_i \odot y).$ 

There exists a further binary operation  $\rightarrow$  (called the implication operator or residuated) satisfying the following condition:

$$
x \to y = \bigvee \{ z \in L : x \odot z \le y \}.
$$

Then it satisfies the Galois correspondence, i.e.,  $(x \odot z) \leq y$  if and only if  $z \leq (x \rightarrow y).$ 

**Definition 1.2** [13, 14, 29, 34]. (1) An stsc-quantale  $(L, \leq, \odot,')$  is called an *MV-algebra* if and only if  $(x \to 0) \to 0 = x$ .

(2) An MV-algebra  $(L, \le, \odot,')$  is called *complete* if and only if it satisfies the following property:

$$
(x \rightarrow y) \rightarrow y = x \vee y, \forall x, y \in L.
$$

We always assume that  $(L, \leq, \odot,')$  is an stsc-quantale with an order reversing involution ' which is defined by

$$
x \oplus y = (x' \odot y')', \quad x' = x \to 0.
$$

ing the following condition:<br>  $=\forall \{z \in L : x \odot z \leq y\}.$ <br>
interspondence, i.e.,  $(x \odot z) \leq y$  if and only if<br>
29, 34]. (1) An stsc-quantale  $(L, \leq, \odot, \prime)$  is<br>
aly if  $(x \rightarrow 0) \rightarrow 0 = x$ .<br>  $\leq$ ,  $\odot$ ,  $\prime$ ) is called *complete* i **Lemma 1.3** [12, 13, 29]. For each x, y, z,  $x_i$ ,  $y_i$ ,  $w \in L$ , we have the following properties:

- (1)  $1 \to x = x$ ,  $0 \odot x = 0$  and  $x \to 0 = x'$ , (2) if  $y \le z$ , then  $x \odot y \le x \odot z$ ,  $x \oplus y \le x \oplus z$ ,  $x \rightarrow y \le x \rightarrow z$ and  $z \to x \leq y \to x$ , (3)  $x \odot y \leq x \land y \leq x \lor y \leq x \oplus y$ ,  $(x \rightarrow y) \rightarrow y = x \vee y$ ,  $\forall x, y \in L$ .<br>
We always assume that  $(L, \le, \odot, ')$  is an stsc-quantale with an order<br>
rsing involution ' which is defined by<br>  $x \oplus y = (x' \odot y')'$ ,  $x' = x \rightarrow 0$ .<br> **Lemma 1.3** [12, 13, 29]. For each  $x, y, z, x_i, y_i,$
- $\Lambda_i y_i)' = \bigvee_i y_i', \left(\bigvee_i y_i\right)' = \Lambda_i y_i'$

On *L*-fuzzy 
$$
(K, E)
$$
-soft Grills  
\n(5)  $x \odot (\Lambda_i y_i) \le \Lambda_i (x \odot y_i)$ ,  
\n(6)  $x \oplus (\Lambda_i y_i) = \Lambda_i (x \oplus y_i)$ ,  $x \oplus (\Lambda_i y_i) = \Lambda_i (x \oplus y_i)$ ,  
\n(7)  $x \rightarrow (\Lambda_i y_i) = \Lambda_i (x \rightarrow y_i)$ ,  
\n(8)  $(\Lambda_i x_i) \rightarrow y = \Lambda_i (x_i \rightarrow y)$ ,  
\n(9)  $x \rightarrow (\Lambda_i y_i) \ge \Lambda_i (x_i \rightarrow y)$ ,  
\n(10)  $(\Lambda_i x_i) \rightarrow y \ge \Lambda_i (x_i \rightarrow y)$ ,  
\n(11)  $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ ,  
\n(12)  $x \odot (x \rightarrow y) \le y$  and  $x \rightarrow y \le (y \rightarrow z) \rightarrow (x \rightarrow z)$ ,  
\n(13)  $x \odot (x' \oplus y') \le y'$ ,  $x \odot y = (x \rightarrow y')'$  and  $x \oplus y = x' \rightarrow y$ ,  
\n(14)  $(x \rightarrow y) \odot (z \rightarrow w) \le (x \odot y) \rightarrow (y \odot w)$ ,  
\n(15)  $x \rightarrow y \le (x \odot z) \rightarrow (y \odot z)$  and  $(x \rightarrow y) \odot (y \rightarrow z) \le x \rightarrow z$ ,  
\n(16)  $(x \rightarrow y) \odot (z \rightarrow w) \le (x \oplus z) \rightarrow (y \oplus w)$ .  
\nThroughout this paper, *X* refers to an initial universe, *E* and *K* are the  
sets of all the parameters for *X*, and *L*<sup>*X*</sup> is the set of all *L*-fuzzy soft set on *X*, where *f* is  
a mapping from *E* into *L*<sup>*X*</sup>, i.e.,  $f_e := f(e)$  is an *L*-fuzzy soft set on *X*,  
\nfor each  $e \in E$ . The set of all fuzzy soft sets is denoted by  $(L^X)^E$ . Let  
\n $f, g \in (L^X)^E$ . Then:  
\n(1

Throughout this paper,  $X$  refers to an initial universe,  $E$  and  $K$  are the sets of all the parameters for X, and  $L^X$  is the set of all L-fuzzy sets on X.

**Definition 1.4** [9]. A map f is called an L-fuzzy soft set on X, where f is a mapping from E into  $L^X$ , i.e.,  $f_e := f(e)$  is an L-fuzzy soft set on X, for each  $e \in E$ . The set of all fuzzy soft sets is denoted by  $(L^X)^E$ . Let  $f, g \in (L^X)^E$ . Then:

(1) f is an L-fuzzy soft subset g and we write  $f \sqsubseteq g$  if  $f_e \leq g_e$ , for each  $e \in E$ . f and g are equal if  $f \sqsubseteq g$  and  $g \sqsubseteq f$ .

(2) The intersection of f and g is an L-fuzzy soft set  $h = f \cap g$ , where  $h_e = f_e \wedge g_e$ , for each  $e \in E$ .

(3) The union of f and g is an L-fuzzy soft set  $h = f \sqcup g$ , where  $h_e = f_e \vee g_e$ , for each  $e \in E$ .

(4) An *L*-fuzzy soft set  $h = f \odot g$  is defined as  $h_e = f_e \odot g_e$ , for each  $e \in E$ .

(5) An *L*-fuzzy soft set  $h = f \oplus g$  is defined as  $h_e = f_e \oplus g_e$ , for each  $e \in E$ .

(6) The complement of an L-fuzzy soft set on X is denoted by  $f'$ , where O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
e union of f and g is an L-fuzzy soft set  $h = f \sqcup g$ , where<br>  $g_e$ , for each  $e \in E$ .<br>
L-fuzzy soft set  $h = f \odot g$  is defined as  $h_e = f_e \odot g_e$ , for<br>  $\Sigma$ .<br>
L-fuzzy soft set  $h = f \oplus g$  67 (3) The union of f and g is an L-fuzzy soft set  $h = f \sqcup g$ , where<br>  $h_e = f_e \vee g_e$ , for each  $e \in E$ .<br>
(4) An L-fuzzy soft set  $h = f \odot g$  is defined as  $h_e = f_e \odot g_e$ , for each  $e \in E$ .<br>
(5) An L-fuzzy soft set  $h = f \oplus g$  is def  $f'_e = (f_e)'$ , for each  $e \in E$ .

(7) f is called a *null L-fuzzy soft set* and denoted by  $0<sub>X</sub>$  if  $f<sub>e</sub>(x) = 0$ , for each  $e \in E$ , and  $x \in X$ .

(8)  $f_A$  is called *absolute L-fuzzy soft set* and denoted by  $1_X$  if  $f_e(x) = 1$ , for each  $e \in E$ , and  $x \in X$ .

An *L*-fuzzy soft point is an *L*-fuzzy soft set f such that  $f_e := f(e)$  is an *L*-fuzzy point and  $f_a := f(a) = \overline{0}$  for all  $a \in E \setminus \{e\}$ . We denote this *L*fuzzy soft point by  $f = e_x^t$ . For  $f, g \in (L^X)^E$ , we write fqg to mean that f soft set  $h = f \odot g$  is defined as  $h_e = f_e \odot g_e$ , for<br>
soft set  $h = f \oplus g$  is defined as  $h_e = f_e \oplus g_e$ , for each<br>
nent of an *L*-fuzzy soft set on *X* is denoted by *f'*, where<br>
a mapping given by  $f'_e = (f_e)'$ , for each  $e \in E$ .<br> is soft quasi-coincident with g, i.e., there exist at least one  $x \in X$ ,  $e \in E$ such that  $f_e(x) \nleq g'_e(x)$ . Negation of such a statement is denoted as  $f \overline{q}g$ . *E.*<br>
(6) The complement of an *L*-fuzzy soft set on *X* is denoted by *f'*, where  $E \rightarrow (L^X)^E$  is a mapping given by  $f'_e = (f_e)^t$ , for each  $e \in E$ .<br>
(7) *f* is called a *null L-fuzzy soft set* and denoted by  $0_X$  if  $f_e(x) =$ oft set and denoted by  $1_X$  if<br>
set f such that  $f_e := f(e)$  is an<br>  $a \in E \setminus \{e\}$ . We denote this  $L$ -<br>  $f^Y$  $f^E$ , we write fag to mean that f<br>
exist at least one  $x \in X$ ,  $e \in E$ <br>
statement is denoted as  $f \overline{q}g$ .<br>  $f^E$  ( f set  $f$  such that  $f_e := f(e)$  is an<br>  $a \in E \setminus \{e\}$ . We denote this  $L^X$ <br>  $f^E$ , we write  $fgg$  to mean that  $f$ <br>  $f^X$  statement is denoted as  $f\overline{q}g$ .<br>  $f^E$  (where  $T_k := T(k) : (L^X)^E$ <br>  $f^E$  (where  $T_k := T(k) : (L^X)^E$ <br>  $f^E$ An *L*-fuzzy soft point is an *L*-fuzzy soft set *f* such that  $f_e := f(e)$  is an<br>zzy point and  $f_a := f(a) = \overline{0}$  for all  $a \in E \setminus \{e\}$ . We denote this *L*-<br>y soft point by  $f = e'_x$ . For  $f, g \in (L^X)^E$ , we write *fag* to mean th

**Definition 1.5.** A map  $\mathcal{T}: K \to L^{(L^X)^E}$  (where  $\mathcal{T}_k := \mathcal{T}(k): (L^X)^E$  $\rightarrow L$  is a mapping for each  $k \in K$ ) is called an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ soft topology on  $X$  if it satisfies the following conditions:

(LSO1)  $T_k(0_X) = T_k(1_X) = 1$ .

(LSO2)  $\mathcal{T}_k(f \odot g) \ge \mathcal{T}_k(f) \odot \mathcal{T}_k(g)$  for all  $f, g \in (L^X)^E$ . (LSO3)  $\mathcal{T}_k(f \sqcap g) \ge \mathcal{T}_k(f) \odot \mathcal{T}_k(g)$  for all  $f, g \in (L^X)^E$ .  $f_i \in (L^X)^E$ 

The pair  $(X, \mathcal{T})$  is called an  $(L, \odot, \Box)$ -fuzzy- $(K, E)$ -soft topological space.

**Definition 1.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft On *L*-fuzzy (*K*, *E*)-soft Grills 63<br>
The pair  $(X, \mathcal{T})$  is called an  $(L, \odot, \Box)$ -fuzzy-(*K*, *E*)-soft topological<br>
space.<br> **Definition 1.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>
topological space. Then f  $f \in (L^X)^E$ , an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft On *L*-fuzzy  $(K, E)$ -soft Grills<br>
The pair  $(X, \mathcal{T})$  is called an  $(L, \odot, \Box)$ -fuzzy- $(K, E)$ -soft topological<br>
space.<br> **Definition 1.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>
topological space. Then for  $f \in (L^X$ closure of f is a mapping  $C_T : K \times (L^X)^E \times L_0 \to (L^X)^E$  defined as: On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $(X, \mathcal{T})$  is called an  $(L, \odot, \Box)$ -fuzzy- $(K, E)$ -soft topological<br> **on 1.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>
space. Then for  $f \in (L^X)^E$ , an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft

$$
C_{\mathcal{T}}(k, f, r) = \bigwedge \{g \in (L^X)^E : f \sqsubseteq g, \mathcal{T}(g') \ge r\}.
$$

**Definition 1.7.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft On *L*-fuzzy  $(K, E)$ -soft Grills 63<br>
The pair  $(X, T)$  is called an  $(L, \odot, \top)$ -fuzzy- $(K, E)$ -soft topological<br>
space.<br> **Definition 1.6.** Let  $(X, T)$  be an  $(L, \odot, \top)$ -fuzzy  $(K, E)$ -soft<br>
topological space. Then for  $f \in (L^X)^E$ topological space,  $e \in E$  and  $x \in X$ . For  $k \in K$ ,  $e_x^t \in (L^X)^E$  is said to be On *L*-fuzzy  $(K, E)$ -soft Grills<br>
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The pair  $(X, \mathcal{T})$  is called an  $(L, \odot, \Box)$ -fuzzy- $(K, E)$ -soft topological<br>
space.<br> **Definition 1.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>
topological space. Then for  $f \in$ zy  $(K, E)$ -soft Grills<br>
an  $(L, \odot, \sqcap)$ -fuzzy- $(K, E)$ -soft topological<br>  $T$ ) be an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft<br>  $\in (L^X)^E$ , an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft<br>  $\cdot \times (L^X)^E \times L_0 \rightarrow (L^X)^E$  defined as:<br>  $\in (L^X)^E : f \sqsubseteq g, T(g') \ge r$  $\in$   $Q_{\mathcal{T}_k}(e_x^t, r)$ , we have *fqg*, where On *L*-fuzzy  $(K, E)$ -soft Grills<br>
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X, T) is called an  $(L, \bigcirc, \bigcap$  *fuzzy* - $(K, E)$ -soft topological<br>
1.6. Let  $(X, T)$  be an  $(L, \bigcirc, \bigcirc)$ -fuzzy  $(K, E)$ -soft<br>
ce. Then for  $f \in (L^X)^E$ , an  $(L, \bigcirc, \bigcirc)$ -fuzzy  $(K, E)$ -soft<br>
map The pair  $(X, T)$  is called an  $(L, \bigcirc, \bigcap$ -fuzzy  $(K, E)$ -soft *iopological*<br>
re.<br> **Definition 1.6.** Let  $(X, T)$  be an  $(L, \bigcirc, \bigcirc)$ -fuzzy  $(K, E)$ -soft<br>
logical space. Then for  $f \in (L^X)^E$ , an  $(L, \bigcirc, \bigcirc)$ -fuzzy  $(K, E)$ -soft<br>
ure

$$
Q_{\mathcal{T}_k}(e_x^t, r) = \{g \in (L^X)^E : e_x^t q g, \mathcal{T}_k(h) \ge r\}.
$$

**Definition 1.8.** A map  $\mathcal{B}: K \to L^{(L^X)^E}$  (where  $\mathcal{B}_k := \mathcal{B}(k) : (L^X)^E$  $\rightarrow L$  is a mapping for each  $k \in K$ ) is called an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ soft base on  $X$  if it satisfies the following conditions:

(LSB1)  $\mathcal{B}_k(0_X) = \mathcal{B}_X(1_X) = 1.$ (LSB2)  $\mathcal{B}_k(f \odot g) \geq \mathcal{B}_k(f) \odot \mathcal{B}_k(g)$  for all  $f, g \in (L^X)^E$ .  $f \subseteq g, T(g') \ge r$ .<br>  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft<br>  $k \in K, e_x^l \in (L^X)^E$  is said to be<br>
if for every  $g \in Q_{T_k}(e_x^l, r)$ , we<br>  $e_x^l g, T_k(h) \ge r$ .<br>  $\bigcup^E$  (where  $B_k := B(k) : (L^X)^E$ <br>
lled an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -<br>
diditions:<br>
for (LSB3)  $\mathcal{B}_k(f \sqcap g) \geq \mathcal{B}_k(f) \odot \mathcal{B}_k(g)$  for all  $f, g \in (L^X)^E$ . 1  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft<br>  $k \in K$ ,  $e_x^t \in (L^X)^E$  is said to be<br>
if for every  $g \in Q_{T_k}(e_x^t, r)$ , we<br>  $e_x^t g g$ ,  $T_k(h) \ge r$ }.<br>  $\bigg)^E$  (where  $B_k := B(k) : (L^X)^E$ <br>
lled an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -<br>
miditions:<br>
for all  $f$ **Definition 1.9.** An  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft neighborhood system an *L*-fuzzy soft closure point of  $f \in (L^X)^E$  if for every  $g \in Q_{T_k}(e'_x, r)$ , we<br>have *fag*, where<br> $Q_{T_k}(e'_x, r) = \{g \in (L^X)^E : e'_x qg, T_k(h) \ge r\}$ .<br>**Definition 1.8.** A map  $B : K \to L^{(L^Y)^E}$  (where  $B_k := B(k) : (L^X)^E$ <br> $\to L$  is a mapp bsure point of  $f \in (L^X)^E$  if for every  $g \in Q_{T_k}(e_x^t, r)$ , we<br>  $\partial_{T_k}(e_x^t, r) = \{g \in (L^X)^E : e_x^t q g, T_k(h) \ge r\}.$ <br> **3.** A map  $B : K \rightarrow L^{(L^X)^E}$  (where  $B_k := B(k) : (L^X)^E$ <br>
it satisfies the following conditions:<br>  $0x) = B_X(1_X) = 1$ .<br>  $f$  $Q_{T_k}(e'_x, r) = \{g \in (L^X)^E : e'_x g g, T_k(h) \ge r\}.$ <br>
Definition 1.8. A map  $B : K \rightarrow L^{(L^X)^E}$  (where  $B_k := B(k) : (L^X)^E$ <br>
L is a mapping for each  $k \in K$ ) is called an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -<br>
base on X if it satisfies the following con  $e'_x$ ,  $r$ ) = { $g \in (L^X)^E : e'_x g g$ ,  $T_k(h) \ge r$ }.<br>
A map  $B : K \to L^{(L^X)^E}$  (where  $B_k := B(k) : (L^X)^E$ <br>
for each  $k \in K$ ) is called an  $(L, \bigcirc, \sqcap)$ -fuzzy  $(K, E)$ -<br>
tisfies the following conditions:<br>  $= B_X(1_X) = 1$ .<br>  $g \ge B_k(f) \bigcirc B_k(g)$  fo **Definition 1.8.** A map  $B: K \rightarrow l^{(L^X)^E}$  (where  $B_k := B(k) : (L^X)^E$ <br> *U* is a mapping for each  $k \in K$ ) is called an  $(L, \odot, \odot)$ -fluzzy  $(K, E)$ -<br> *base* on X if it satisfies the following conditions:<br>
(LSB1)  $B_k(0_X) = B_X(1_X) = 1$ **ition 1.8.** A map  $B: K \to L^{(L^X)F}$  (where  $B_k := B(k) : (L^X)^F$ <br>
mapping for each  $k \in K$ ) is called an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -<br>  $\exists$   $D \in X$  if it satisfies the following conditions:<br>  $\exists B_k (0_X) = B_X(1_X) = 1$ .<br>  $\exists B_k (f \odot g) \ge B_k(f$ 

on X is a set  $N = \{N^x : x \in X\}$  of mappings  $N^x : K \to L^{(L^X)^E}$  such that for each  $k \in K$ :

(1) 
$$
N_k^x(1_X) = 1
$$
 and  $N_k^x(1_X) = 0$ .  
\n(2)  $N_k^x(f \odot g) \ge N_k^x(f) \odot N_k^x(g)$  for each  $f, g \in (L^X)^E$ .

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\n- \n O. R. Sayed, E. Elsanousy and Y. H. Ragh\n
\n- \n (3) 
$$
N_k^x(f \cap g) \geq N_k^x(f) \odot N_k^x(g)
$$
 for each  $f, g \in (L^X)^E$ .\n
\n- \n (4) If  $f \sqsubseteq g$ , then  $N_k^x(f) \leq N_k^x(g)$ .\n
\n- \n (5)  $N_k^x(f) \leq f_e(x)$ , where  $f \in (L^X)^E$  and  $e \in E$ .\n
\n- \n (6)  $N_k^x(f) \leq \sqrt{\{N_k^x(g) : g_e(y) \sqsubseteq N_k^y(g), \forall y \in X, e \in E\}}$ .\n
\n- \n**2. L-fuzzy** (*K, E*) **-soft Grill Space**
\n- \n**Definition 2.1.** A map  $g: K \to L^{(L^X)^E}$  (where  $\mathcal{G}_k := \mathcal{G}(k): (L^X)^E$  *L* is a mapping for each  $k \in K$ ) is called an *L-fuzzy* (*K, E*)-soft grid on *f* it satisfies the following conditions for each  $k \in K$ :\n
\n- \n (LSG1)  $\mathcal{G}_k(f \oplus g) \leq \mathcal{G}_k(f) \oplus \mathcal{G}_k(g)$  for all  $f, g \in (L^X)^E$ .\n
\n- \n (LSG3) If  $f \sqsubseteq g$ , then  $\mathcal{G}_k(f) \leq \mathcal{G}_k(g)$ . The pair  $(X, \mathcal{G})$  is called an *L-fuzzy* (*K, E*)-soft *grill space*. If  $\mathcal{G}^1$  and  $\mathcal{G}^1$  is a *L-fuzzy* (*K, E*)-soft *grill space*. If  $\mathcal{G}^1$  and  $\mathcal{G}_k(f) \leq \mathcal{G}_$

### 2. L-fuzzy  $(K, E)$ -soft Grill Space

**Definition 2.1.** A map  $G: K \to L^{(L^X)^E}$  (where  $G_k := G(k) : (L^X)^E$  $\rightarrow$  L is a mapping for each  $k \in K$ ) is called an L-fuzzy  $(K, E)$ -soft grill on X if it satisfies the following conditions for each  $k \in K$ :

(LSG1) 
$$
\mathcal{G}_k(0_X) = 0
$$
 and  $\mathcal{G}_k(1_X) = 1$ .  
\n(LSG2)  $\mathcal{G}_k(f \oplus g) \le \mathcal{G}_k(f) \oplus \mathcal{G}_k(g)$  for all  $f, g \in (L^X)^E$ .  
\n(LSG3) If  $f \sqsubseteq g$ , then  $\mathcal{G}_k(f) \le \mathcal{G}_k(g)$ .

The pair  $(X, \mathcal{G})$  is called an *L*-fuzzy  $(K, E)$ -soft grill space. If  $\mathcal{G}^1$  and  $\mathcal{G}^2$  are *L*-fuzzy  $(K, E)$ -soft grills on *X*, then we say that  $\mathcal{G}^1$  is *finer* than  $\mathcal{G}^2$  ( $\mathcal{G}^2$  is coarser than  $\mathcal{G}^1$ ) denoted by  $\mathcal{G}^2 \sqsubseteq \mathcal{G}^1$  if and only if 2. *L*-fuzzy  $(K, E)$ -soft Grill Space<br>
Definition 2.1. A map  $G : K \rightarrow L^{(L^X)^E}$  (where  $G_k := G(k) : (L^X)^E$ <br>  $\geq L$  is a mapping for each  $k \in K$ ) is called an *L*-fuzzy  $(K, E)$ -soft grill on<br>
if it satisfies the following conditio 2. *L*-fuzzy  $(K, E)$ -soft Grill Space<br>
Definition 2.1. A map  $G: K \rightarrow L^{(L^X)^E}$  (where  $G_k := G(k) : (L^X)^E$ <br>  $\rightarrow L$  is a mapping for each  $k \in K$ ) is called an *L*-fuzzy  $(K, E)$ -soft grill on<br>
Xif it satisfies the following conditio  $f \in (L^X)^E$ (LSG2)  $\mathcal{G}_k(f \oplus g) \leq \mathcal{G}_k(f) \oplus \mathcal{G}_k(g)$  for all  $f, g \in (L^X)^E$ .<br>
(LSG3) If  $f \subseteq g$ , then  $\mathcal{G}_k(f) \leq \mathcal{G}_k(g)$ .<br>
The pair  $(X, G)$  is called an  $L_f luzzy (K, E)$ -soft grill space. If  $\mathcal{G}^1$  and<br>  $\mathcal{G}^2$  are  $L$ -fuzz

**Remark 2.2.** Let G be an L-fuzzy  $(K, E)$ -soft grill on X. By Lemma 1.3(3), (LSG2) and (LSG3), we have

$$
\mathcal{G}_k(f \sqcup g) \le \mathcal{G}_k(f) \oplus \mathcal{G}_k(g) \text{ for all } k \in K.
$$

**Proposition 2.3.** Let  $\mathcal{G}^1$ ,  $\mathcal{G}^2$  be L-fuzzy  $(K, E)$ -soft grills on X. Then a

$$
\mathcal{G}_k = \mathcal{G}_k^1 \vee \mathcal{G}_k^2 \text{ for all } k \in K
$$

is an L-fuzzy  $(K, E)$ -soft grill on X.

**Proof.** (LSG1) For all  $k \in K$ , we have

$$
\mathcal{G}_k(1_X) = \mathcal{G}_k^1(1_X) \bigvee \mathcal{G}_k^1(1_X) = 1 \bigvee 1 = 1.
$$

Also,

On *L*-fuzzy 
$$
(K, E)
$$
-soft Grills  
\n
$$
\mathcal{G}_k = \mathcal{G}_k^1 \vee \mathcal{G}_k^2 \text{ for all } k \in K
$$
\n*E*)-soft *grill on X.*  
\n1) For all  $k \in K$ , we have  
\n
$$
\mathcal{G}_k(1_X) = \mathcal{G}_k^1(1_X) \vee \mathcal{G}_k^1(1_X) = 1 \vee 1 = 1.
$$
  
\n
$$
\mathcal{G}_k(0_X) = \mathcal{G}_k^1(0_X) \vee \mathcal{G}_k^2(0_X) = 0 \vee 0 = 0.
$$
  
\neach  $f, g \in (L^X)^E$  and  $k \in K$ , we have  
\n
$$
f) \oplus \mathcal{G}_k(g) = \vee \mathcal{G}_k^i(f) \oplus \vee \mathcal{G}_k^i(g)
$$

(LSG2) For each  $f, g \in (L^X)^E$  and  $k \in K$ , we have

On *L*-fuzzy (*K*, *E*)-soft Grills 65  
\n
$$
G_k = G_k^1 \vee G_k^2 \text{ for all } k \in K
$$
\n*L*-*fuzzy* (*K*, *E*)-*soß grill* on *X*.  
\n**Proof.** (LSG1) For all  $k \in K$ , we have  
\n
$$
G_k(1_X) = G_k^1(1_X) \vee G_k^1(1_X) = 1 \vee 1 = 1.
$$
\n*9*,  
\n
$$
G_k(0_X) = G_k^1(0_X) \vee G_k^2(0_X) = 0 \vee 0 = 0.
$$
\n(LSG2) For each  $f$ ,  $g \in (L^X)^E$  and  $k \in K$ , we have  
\n
$$
G_k(f) \oplus G_k(g) = \bigvee_{i \in \{1,2\}} G_k^i(f) \oplus \bigvee_{i \in \{1,2\}} G_k^i(g)
$$
\n
$$
= \bigvee_{i \in \{1,2\}} (G_k^i(f) \oplus G_k^i(g))
$$
\n
$$
\geq \bigvee_{i \in \{1,2\}} G_k^i(f \oplus g) = G_k(f \oplus g).
$$
\n(LSG3) If  $f \subseteq g$ , then we have  $G_k^i(f) \leq G_k^i(g)$  for all  $k \in K$  and  
\n{1, 2}. Therefore,  
\n
$$
G_k(f) = \bigvee_{i \in \{1,2\}} G_k^i(f) \leq \bigvee_{i \in \{1,2\}} G_k^i(g) = G_k(g).
$$
\nThe next example shows that the meet of two *L*-fuzzy (*K*, *E*)-soft grills  
\nis not an *L*-fuzzy (*K*, *E*)-soft grill on *X*, in general.

 $\mathcal{G}_{k}^{i}(f) \leq \mathcal{G}_{k}^{i}(g)$  for all  $k \in K$  and  $i \in \{1, 2\}$ . Therefore,

$$
\mathcal{G}_k(f) = \bigvee_{i \in \{1,2\}} \mathcal{G}_k^i(f) \le \bigvee_{i \in \{1,2\}} \mathcal{G}_k^i(g) = \mathcal{G}_k(g).
$$

The next example shows that the meet of two *L*-fuzzy  $(K, E)$ -soft grills on X is not an L-fuzzy  $(K, E)$ -soft grill on X, in general.

**Example 2.4.** Let  $X = \{h_1, h_2, h_3\}$  with  $h_i = \text{house}$  for  $i \in \{1, 2, 3\}$  and  $E = \{e, b\}$  with  $e =$  expensive,  $b =$  beautiful. Define a binary operation  $\odot$ on [0, 1] by

> $x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\},\$  $x \oplus y = \min\{1, x + y\}, \quad x' = 1 - x.$

Then  $([0, 1], 0, \to, 0, 1)$  is an stsc-quantale [11, 13, 29]. Let  $f_i \in$ 66 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
Then  $([0, 1], \odot, \rightarrow, 0, 1)$  is an stsc-quantale [11, 13, 29]. Let  $f_i \in ([0, 1]^X)^E$  for  $i \in \{1, 2, 3, 4\}$  as follows:<br>  $(f_1)_e = (0.0, 1.0, 0.0), (f_1)_b = (1.0, 1.0, 1.0),$ <br>  $(f_2)_e = (1.$  $([0, 1]^X)^E$  for  $i \in \{1, 2, 3, 4\}$  as follows:

66  
\n66  
\n66  
\n67. Sayed, E. Elsanousy and Y. H. Ragh  
\n7. A graph  
\n7. A graph  
\n7. B graph  
\n8. A graph  
\n9. Let 
$$
f_i \in
$$
  
\n10.  $1]^X$ <sup>E</sup> for  $i \in \{1, 2, 3, 4\}$  as follows:  
\n $(f_1)_e = (0.0, 1.0, 0.0), (f_1)_b = (1.0, 1.0, 1.0),$   
\n $(f_2)_e = (1.0, 1.0, 0.0), (f_2)_b = (1.0, 1.0, 1.0),$   
\n $(f_3)_e = (0.0, 1.0, 1.0), (f_3)_b = (1.0, 1.0, 1.0),$   
\n $(f_4)_e = (1.0, 0.0, 0.0), (f_4)_b = (1.0, 1.0, 1.0).$   
\n8. A graph  
\n9. A graph  
\n10.  $1^{10.11^X} = \{k_1, k_2\}$ , we define *L*-fuzzy  $(K, E)$ -soft grills  $g^1$ ,  $g^2 : K \rightarrow$   
\n10.  $1^{10.11^X} = \{k_1, k_2\}$ , we define *L*-fuzzy  $(K, E)$ -soft grills  $g^1$ ,  $g^2 : K \rightarrow$   
\n10.  $1^{10.11^X} = \{k_1, k_2\}$ , we define *L*-fuzzy  $(K, E) = \{k_1, g^1, g^2 : K \rightarrow$   
\n11. If  $g = 1_X$ ,  
\n12.  $g^1$ ,  $(g) = \{0.6, f, g = f_2, g^2, g = \{0.9, f, g = f_4, g = f_5\}$ 

For  $K = \{k_1, k_2\}$ , we define L-fuzzy  $(K, E)$ -soft grills  $\mathcal{G}^1, \mathcal{G}^2 : K \to$ 0, 1]<sup>([0,1]<sup>X</sup>)<sup>E</sup> as follows:</sup>

(10, 1], 0, →, 0, 1) is an stsc-quantale [11, 13, 29]. Let 
$$
f_i \in
$$
  
\n
$$
f(Y) = \text{ for } i \in \{1, 2, 3, 4\} \text{ as follows:}
$$
\n
$$
(f_1)_e = (0.0, 1.0, 0.0), (f_1)_b = (1.0, 1.0, 1.0), (f_2)_e = (1.0, 1.0, 1.0), (f_3)_e = (0.0, 1.0, 1.0), (f_4)_e = (1.0, 0.0, 0.0), (f_4)_b = (1.0, 1.0, 1.0).
$$
\n
$$
K = \{k_1, k_2\}, \text{ we define } L\text{-fuzzy } (K, E) \text{-soft grills } g^1, g^2 : K \rightarrow
$$
\n
$$
g^1_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.3 & \text{if } g = f_1, \\ 0.6 & \text{if } g = f_2, \\ 0.7 & \text{if } g = f_3, \\ 0.8 & \text{if } g = f_3, \\ 0.9 & \text{otherwise,} \\ 0.1 & \text{otherwise,} \\ 0.1 & \text{otherwise,} \end{cases}
$$
\n
$$
g^1_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.2 & \text{if } g = f_2, \\ 0.3 & \text{if } g = f_3, \\ 0.5 & \text{if } g = f_2, \\ 0.6 & \text{otherwise,} \\ 0.7 & \text{if } g = f_2, \\ 0.8 & \text{if } g = f_3, \\ 0.8 & \text{if } g = f_2, \\ 0.9 & \text{otherwise,} \\ 0.1 & \text{otherwise,} \end{cases}
$$
\n
$$
(f_2 \oplus f_3)_e = (1.0, 1.0, 1.0), (f_2 \oplus f_3)_b = (1.0, 1.0, 1.0), (f_3 \oplus f_3)_b = (1.0, 1.0, 1.0), \text{ we have}
$$
\n
$$
(g^1_{k_1} \wedge g^2_{k_1})(f_2 \oplus f_3) = (g^1_{k_1}
$$

Since  $(f_2 \oplus f_3)_e = (1.0, 1.0, 1.0), (f_2 \oplus f_3)_b = (1.0, 1.0, 1.0),$  we have

$$
\begin{aligned} (\mathcal{G}_{k_1}^1 \wedge \mathcal{G}_{k_1}^2)(f_2 \oplus f_3) &= (\mathcal{G}_{k_1}^1 \wedge \mathcal{G}_{k_1}^2)(1_X) \\ &= 1 \wedge 1 = 1 \\ &\le ((\mathcal{G}_{k_1}^1 \wedge \mathcal{G}_{k_1}^2)(f_2)) \oplus ((\mathcal{G}_{k_1}^1 \wedge \mathcal{G}_{k_1}^2)(f_3)) \\ &= (0.6 \wedge 0) \oplus (0.5 \wedge 0) = 0 \oplus 0 = 0, \end{aligned}
$$

On *L*-fuzzy 
$$
(K, E)
$$
-soft Grills\n
$$
(\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_2 \oplus f_3) = (\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(1_X)
$$
\n
$$
= 1 \wedge 1 = 1
$$
\n
$$
\leq ((\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_2)) \oplus ((\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_3))
$$
\n
$$
= (0.7 \wedge 0) \oplus (0.3 \wedge 0) = 0 \oplus 0 = 0.
$$
\n\ni.e.,  $\mathcal{G}_{k}^1 \wedge \mathcal{G}_{k}^2$  is not an *L*-fuzzy  $(K, E)$ -soft grid on *X*.  
\n**Definition 2.5.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft logical space and  $\mathcal{G}$  be an *L*-fuzzy  $(K, E)$ -soft grid on *X*. Then a  
\nring  $\phi_{\mathcal{G}} : K \times (L^X)^E \times L \rightarrow (L^X)^E$  is called an *L*-fuzzy soft operator  
\nriated with an *L*-fuzzy  $(K, E)$ -soft grid  $G$  and an  $(L, \odot, \Box)$ -fuzzy *E*)-soft topological *T*, and is defined by  
\n $\phi_{\mathcal{G}}(k, f, r) = \bigvee \{e_x^1 \in (L^X)^E : \mathcal{G}_k(f \odot g) \geq r, \forall g \in \mathcal{Q}_{\mathcal{T}_k}(e_x^f, r)\}.$   
\nProposition 2.6. Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft grids  
\nlogical space and  $\mathcal{G}^1$ ,  $\mathcal{G}^2$  be two *L*-fuzzy  $(K, E)$ -soft grids on *X*. Then

Hence,  $\mathcal{G}^1_k \wedge \mathcal{G}^2_k$  is not an *L*-fuzzy  $(K, E)$ -soft grill on *X*.

**Definition 2.5.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $G$  be an *L*-fuzzy  $(K, E)$ -soft grill on *X*. Then a On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $(\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_2 \oplus f_3) = (\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(1_X)$ <br>  $= 1 \wedge 1 = 1$ <br>  $\leq ((\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_2)) \oplus ((\mathcal{G}_{k_2}^1 \wedge \mathcal{G}_{k_2}^2)(f_3))$ <br>  $= (0.7 \wedge 0) \oplus (0.3 \wedge$ mapping  $\phi_{\mathcal{G}} : K \times (L^X)^E \times L \to (L^X)^E$  is called an *L*-fuzzy soft operator associated with an L-fuzzy  $(K, E)$ -soft grill G and an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological T, and is defined by  $\leq ((G_{k_2}^1 \wedge G_{k_2}^2)(f_2)) \oplus ((G_{k_2}^1 \wedge G_{k_2}^2)(f_3))$ <br>  $= (0.7 \wedge 0) \oplus (0.3 \wedge 0) = 0 \oplus 0 = 0.$ <br>
Hence,  $G_k^1 \wedge G_k^2$  is not an *L*-fluzzy  $(K, E)$ -soft grill on *X*.<br> **Definition 2.5.** Let  $(X, T)$  be an  $(L, \odot, \uparrow, \uparrow)$ -=  $(0.7 \wedge 0) \oplus (0.3 \wedge 0) = 0 \oplus 0 = 0$ .<br>
ce,  $g_k^1 \wedge g_k^2$  is not an *L*-fuzzy  $(K, E)$ -soft grill on *X*.<br> **Definition 2.5.** Let  $(X, T)$  be an  $(L, \bigcirc, \bigcap)$ -fuzzy  $(K, E)$ -soft<br>
dogical space and  $G$  be an *L*-fuzzy  $(K, E)$ -sof in paping  $\phi_{\mathcal{G}} : K \times (L^X)^E \times L \rightarrow (L^X)^E$  is called an *L*-fuzzy soft operator<br>sociated with an *L*-fuzzy  $(K, E)$ -soft grill *G* and an  $(L, \odot, \top)$ -fuzzy<br>*K*, *E*)-soft topological *T*, and is defined by<br> $\phi_{\mathcal{G}}(k, f, r)$ 1 2  $\phi_i : K \times (L^X)^E \times L \rightarrow (L^X)^E$  is called an  $L_2fuzzy$  soft operator<br>
suspeciated with an  $L$ -fuzzy  $(K, E)$ -soft grill G and an  $(L, \bigcirc, \bigcap)$ -fuzzy<br>  $(K, E)$ -soft topological T, and is defined by<br>  $\phi_G(k, f, r) = \sqrt{\{e'_X \in (L^X)^E : \mathcal{$ 

$$
\phi_{\mathcal{G}}(k, f, r) = \bigvee \{e_x^t \in (L^X)^E : \mathcal{G}_k(f \odot g) \ge r, \,\forall g \in Q_{\mathcal{T}_k}(e_x^t, r)\}.
$$

**Proposition 2.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $\mathcal{G}^1$ ,  $\mathcal{G}^2$  be two L-fuzzy  $(K, E)$ -soft grills on X. Then  $f \in (L^X)^E$ (*K*, *E*) -soft topological '*I*, and is defined by<br>  $\phi_{\mathcal{G}}(k, f, r) = \sqrt{\{e_x^f \in (L^X)^E : \mathcal{G}_k(f \odot g) \ge r, \forall g \in \mathcal{Q}_{T_k}(e_x^f, r)\}}.$ <br> **Proposition 2.6.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \mathcal{T})$  fuzzy  $(K, E)$  -soft<br>
topological spac  $\frac{1}{k} \in (L^{\lambda})^b : \mathcal{G}_k(f \odot g) \ge r, \forall g \in \mathcal{Q}_{T_k}(e_x^t, r)).$ <br>
Let  $(X, \mathcal{T})$  be an  $(L, \odot, \bigcap, \text{fixz}y$   $(K, E) \text{-soft}$ <br>  $\mathcal{G}^1, \mathcal{G}^2$  be two L-fuzzy  $(K, E) \text{-soft}$  grills on X. Then<br>
ach  $f \in (L^X)^E$ :<br>
f), then  $\phi_{\mathcal{G}^1}(k,$ 

(1) If 
$$
\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)
$$
, then  $\phi_{\mathcal{G}^1}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^2}(k, f, r)$ .  
\n(2)  $\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) = \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r)$ .

**Proof.** (1) Let  $\mathcal{G}^1$ ,  $\mathcal{G}^1$  be two *L*-fuzzy  $(K, E)$ -soft grills on *X* with  $\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)$  for all  $k \in K$  and  $f \in (L^X)^E$  such that

$$
\phi_{\mathcal{G}^1}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}^2}(k, f, r).
$$

Then there exists  $e_x^t \in (L^X)^E$  such that

$$
\phi_{\mathcal{G}^1}(k, f, r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}^2}(k, f, r).
$$

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It implies that  $\mathcal{G}_k^1(f \odot h) \ge r$  for all  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f) \le \mathcal{G}_k^2(f)$ ,  $\mathcal{G}_k^2(f \odot h) \ge r$  for every  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq \phi_{\math$ D. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $\mathcal{G}_k^1(f \odot h) \ge r$  for all  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f)$ <br>
∴ ⊙  $h$ ) ≥ r for every  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq$ <br>
s a contradiction. It implies that  $\mathcal{G}_k^1(f \odot h) \ge r$  for all  $h \in \mathcal{Q}_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f)$ d Y. H. Raghp<br>  $\in Q_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f)$ <br>  $\in Q_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq$ O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>plies that  $G_k^1(f \odot h) \ge r$  for all  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$ . Since  $G_k^1(f)$ <br>(*f*),  $G_k^2(f \odot h) \ge r$  for every  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq$ <br>k, *f*, *r*). It is a contradictio  $\mathcal{Q}_k^2(f)$ ,  $\mathcal{Q}_k^2(f \odot h) \ge r$  for every  $h \in \mathcal{Q}_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq$ 68 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
It implies that  $G_k^1(f \odot h) \ge r$  for all  $h \in Q_{T_k}(e_x^t, r)$ . Since  $G_k^1(f)$ <br>  $\leq G_k^2(f)$ ,  $G_k^2(f \odot h) \ge r$  for every  $h \in Q_{T_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq \phi_{G^2}(k, f, r)$ . It is a cont O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
that  $\mathcal{G}_k^1(f \odot h) \ge r$  for all  $h \in \mathcal{Q}_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f)$ <br>  $\mathcal{G}_k^2(f \odot h) \ge r$  for every  $h \in \mathcal{Q}_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \sqsubseteq$ <br>
). It is a contradict  $\phi_{\mathcal{G}^2}(k, f, r)$ . It is a contradiction. O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
mplies that  $\mathcal{G}_k^1(f \odot h) \ge r$  for all  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$ . Since  $\mathcal{G}_k^1(f)$ <br>  $\hat{\zeta}_k^2(f)$ ,  $\mathcal{G}_k^2(f \odot h) \ge r$  for every  $h \in Q_{\mathcal{T}_k}(e_x^t, r)$  and so  $e_x^t \subseteq$ <br>  $(k, f, r)$ 68 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
It implies that  $\mathcal{G}_k^1(f \circ h) \ge r$  for all  $h \in \mathcal{Q}_{T_k}(e'_x, r)$ . Since  $\mathcal{G}_k^1(f)$ <br>  $\leq \mathcal{G}_k^2(f)$ ,  $\mathcal{G}_k^2(f \circ h) \ge r$  for every  $h \in \mathcal{Q}_{T_k}(e'_x, r)$  and so  $e'_x \in$ <br>  $\phi$ b. R. Sayed, E. Eisanousy and Y. H. Ragnp<br>  $G_k^1(f \circ h) \ge r$  for all  $h \in Q_{T_k}(e_x^f, r)$ . Since  $G_k^1(f)$ <br>  $\circ h \ge r$  for every  $h \in Q_{T_k}(e_x^f, r)$  and so  $e_x^t \subseteq$ <br>
a contradiction.<br>  $x^X$   $y^E$  and  $k \in K$ . Then by (1), we have<br>  $f$ 

(2) Let 
$$
f \in (L^X)^E
$$
 and  $k \in K$ . Then by (1), we have  
\n
$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsupseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r).
$$
\n(2.1)

It suffices to show that  $\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r)$ . So, suppose that

$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r).
$$

 $t_x$   $\in$   $(L^X)^E$ ,  $k \in K$  such that

$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsupseteq e_x^t \sqsupseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r) \tag{2.2}
$$

 $\leq g_k^2(f)$ ,  $g_k^2(f \circ h) \geq r$  for every  $h \in Q_{T_k}(e_x^f, r)$  and so  $e_x^f \subseteq$ <br>  $\phi_0^2(k, f, r)$ . It is a contradiction.<br>
(2) Let  $f \in (L^X)^E$  and  $k \in K$ . Then by (1), we have<br>  $\phi_{g^1 \vee g^2}(k, f, r) \supseteq \phi_{g^1}(k, f, r) \vee \phi_{g^2}(k, f, r)$  $t'_x \sqsupset \phi_{\mathcal{G}^1}(k, f,$  $\begin{aligned}\n\mathbf{P} \geq r & \text{ for every } h \in Q_{T_k}(e_x^t, r) \text{ and so } e_x^t \subseteq \text{intraidiction.} \\
\text{and } k \in K. \text{ Then by (1), we have} \\
f, r) & \supseteq \phi_{\mathcal{G}}(k, f, r) \vee \phi_{\mathcal{G}}(k, f, r). \qquad (2.1) \\
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) & \supseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r). \text{ So,} \\
k, f, r$  $\phi_x^t \rightrightarrows \phi_{\mathcal{G}^2}(k, f, r)$ . Hence, there  $\Phi_{\mathcal{G}^2}(k, f, r)$ . It is a contradiction.<br>
(2) Let  $f \in (L^X)^E$  and  $k \in K$ . Then by (1), we have<br>  $\Phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \supseteq \Phi_{\mathcal{G}^1}(k, f, r) \vee \Phi_{\mathcal{G}^2}(k, f, r)$ . (2.1)<br>
It suffices to show that  $\Phi_{\mathcal{G}^1 \$  $g_1, g_2 \in Q_T(e_x^t, r)$  such that  $\mathcal{G}_k^1(f \odot g_1) = 0$  and  $\mathcal{G}_k^2(f \odot g_2) = 0$ tradiction.<br>
such  $k \in K$ . Then by (1), we have<br>  $\int f(r) \equiv \phi_{gl}(k, f, r) \sqrt{\phi_{gl}(k, f, r)}$  (2.1)<br>  $\int \phi_{gl} \sqrt{g^2(k, f, r)} \equiv \phi_{gl}(k, f, r) \sqrt{\phi_{gl}(k, f, r)}$ . So,<br>  $f, r \equiv \phi_{gl}(k, f, r) \sqrt{\phi_{gl}(k, f, r)}$ <br>  $\int f, r \equiv \phi_{gl}(k, f, r) \sqrt{\phi_{gl}(k, f, r)}$ <br>  $\int f, \int f \equiv$ f by (1), we have<br>
f, r)  $\sqrt{\phi_{g^2}(k, f, r)}$ . (2.1)<br>  $\equiv \phi_{g^1}(k, f, r) \sqrt{\phi_{g^2}(k, f, r)}$ . So,<br>
f, r)  $\sqrt{\phi_{g^2}(k, f, r)}$ .<br>  $\Rightarrow$  f r)  $\sqrt{\phi_{g^2}(k, f, r)}$ .<br>  $\Rightarrow$  (2.2)<br>  $\Rightarrow$  f  $\Rightarrow$  f  $\Rightarrow$   $\phi_{g^2}(k, f, r)$ . Hence, there<br>  $\Rightarrow$  f  $\Rightarrow$  g (2) Let  $f \in (L^X)^F$  and  $k \in K$ . Then by (1), we have<br>  $\phi_{Q^1 \setminus Q^2}(k, f, r) \supseteq \phi_{Q^1}(k, f, r) \vee \phi_{Q^2}(k, f, r)$ . (2.1)<br>
It suffices to show that  $\phi_{Q^1 \setminus Q^2}(k, f, r) \subseteq \phi_{Q^1}(k, f, r) \vee \phi_{Q^2}(k, f, r)$ . So,<br>
suppose that<br>  $\phi_{Q^$  $=(g_1 \odot g_2) \in Q_T(e_x^t, r)$  and  $\mathcal{G}_k^1(f \odot g) = 0$  and we have<br>  $\binom{1}{2}(k, f, r)$ . (2.1)<br>  $\binom{1}{2}(k, f, r)$ . (2.1)<br>  $\binom{1}{2}(k, f, r)$ . (2.2)<br>  $\binom{1}{2}(k, f, r)$ . (2.2)<br>  $\binom{1}{2}(k, f, r)$ . Hence, there<br>  $= 0$  and  $\binom{1}{2}(f \odot g_2) = 0$ <br>
and  $\binom{1}{2}(f \odot g) = 0$  and<br>  $g = 0$  proving tha (2) Let  $f \in (L^{\alpha})^{\mu}$  and  $k \in K$ . Then by (1), we have<br>  $\phi_{g^1 \vee g^2}(k, f, r) \supseteq \phi_{g^1}(k, f, r) \vee \phi_{g^2}(k, f, r).$  (2.1)<br>
uffices to show that  $\phi_{g^1 \vee g^2}(k, f, r) \in \phi_{g^1}(k, f, r) \vee \phi_{g^2}(k, f, r).$  So,<br>
pose that<br>  $\phi_{g^1 \vee g$  $\mathcal{G}_k^2(f \odot g) = 0$ . Consequently,  $(\mathcal{G}^1 \vee \mathcal{G}^2)_k(f \odot g) = 0$  proving that  $e_x^t \square$ 2) Let  $f \in (L^{\alpha})^{\beta}$  and  $k \in K$ . Then by (1), we have<br>  $\phi_{Q^1 \vee Q^2}(k, f, r) \supseteq \phi_{Q^1}(k, f, r) \vee \phi_{Q^2}(k, f, r)$ . (2.1)<br>
ffices to show that  $\phi_{Q^1 \vee Q^2}(k, f, r) \supseteq \phi_{Q^1}(k, f, r) \vee \phi_{Q^2}(k, f, r)$ . So,<br>
sose that<br>  $\phi_{Q^1 \ve$  $\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r)$ . It contradicts (2.2). Hence, which implies that  $e'_x \rightrightarrows e'_y$ <br>
which implies that  $e'_x \rightrightarrows e'_y$  (*k*, *f*, *r*) and  $e'_x \rightrightarrows e'_y$  (*k*, *f*, *r*). Hence, there<br>
exist  $g_1, g_2 \in Q_T(e'_x, r)$  such that  $\mathcal{G}_k^1(f \circ g_1) = 0$  and  $\mathcal{G}_k^2(f \circ g_2) = 0$ <br>
f

$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r). \tag{2.3}
$$

From  $(2.1)$  and  $(2.3)$ , we have

$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) = \phi_{\mathcal{G}^1}(k, f, r) \vee \phi_{\mathcal{G}^2}(k, f, r).
$$

**Proposition 2.7.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $G$  be an L-fuzzy  $(K, E)$ -soft grill on X. Then for all  $f, g \in (L^X)^E$ , the following statements hold:

(1) 
$$
f \, f \subseteq g
$$
, then  $\phi_{\mathcal{G}}(k, f, r) \subseteq \phi_{\mathcal{G}}(k, g, r)$ .  
\n(2)  $f \, \mathcal{G}_k(f) = 0$ , then  $\phi_{\mathcal{G}}(k, f, r) = 0_X$ .  
\n(3)  $\phi_{\mathcal{G}}(k, f \oplus g, r) = \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r)$ .  
\n(4)  $\phi_{\mathcal{G}}(k, f \sqcup g, r) = \phi_{\mathcal{G}}(k, f, r) \sqcup \phi_{\mathcal{G}}(k, g, r)$ .  
\n(5)  $\phi_{\mathcal{G}}(k, f \odot g, r) \subseteq \phi_{\mathcal{G}}(k, f, r) \bigcirc \phi_{\mathcal{G}}(k, g, r)$ .  
\n(6)  $\phi_{\mathcal{G}}(k, f \sqcap g, r) \subseteq \phi_{\mathcal{G}}(k, f, r) \sqcap \phi_{\mathcal{G}}(k, g, r)$ .  
\n(7)  $f \, \mathcal{G}_k(g) = 0$ , then we have  $\phi_{\mathcal{G}}(k, f \oplus g, r) = \phi_{\mathcal{G}}(k, f, r)$ .  
\n(8)  $\phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) \subseteq \phi_{\mathcal{G}}(k, f, r) = C_{T_k}(k, \phi_{\mathcal{G}}(k, f, r), r)$   
\n $\subseteq C_{T_k}(k, f, r)$ .  
\n(9)  $f \, \mathcal{T}_k(f') \ge r$ , then  $\phi_{\mathcal{G}}(k, f, r) \subseteq f$ .  
\n**Proof.** (1) Suppose that  $\phi_{\mathcal{G}}(k, f, r) \subseteq \phi_{\mathcal{G}}(k, g, r)$ . Then there exists  $e_x^l \in (L^X)^E$  such that  
\n $\phi_{\mathcal{G}}(k, f, r) \supseteq e_x^l \supseteq \phi_{\mathcal{G}}(k, g, r)$ .  
\nSo, we have  $\mathcal{G}_k(f \odot h) \ge r$  for all  $h$ 

**Proof.** (1) Suppose that  $\phi_{\mathcal{G}}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}}(k, g, r)$ . Then there exists  $e_x^t \in (L^X)^E$  such that

$$
\phi_{\mathcal{G}}(k, f, r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}}(k, g, r)
$$

 $\in$   $Q_{\mathcal{T}_k}(e^t, r)$ ,  $k \in K$ . Since  $f \sqsubseteq g$ , by Lemma 1.3(2) and Definition 2.1(LSG3), we have  $f \odot h \sqsubseteq g \odot h$ ,  $t<sub>x</sub>$ ,  $r$ ),  $k \in K$ . Hence,  $e_x^t \sqsubseteq \phi_G(k, g, r)$ . It is a contradiction, and hence  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$ . (c)  $\psi g(h, y, t, h, t) = \psi g(h, y, t) - C\tau_k(h, \psi g(h, y, t, h, t))$ <br>  $\subseteq C\tau_k(k, f, r)$ .<br>
(9) If  $T_k(f') \ge r$ , then  $\phi_g(k, f, r) \sqsubseteq f$ .<br> **Proof.** (1) Suppose that  $\phi_g(k, f, r) \sqsubseteq \phi_g(k, g, r)$ . Then there exists<br>  $\in (L^X)^E$  such that<br>  $\phi_g(k, f, r) \sqsupseteq e'_x \sq$  $\begin{aligned}\n\Psi_Q(x, y, t, k, t) &= \Psi_Q(x, y, t, t) - \nabla_{\tilde{R}}(x, \Psi_Q(x, y, t, t, t)) \\
(f') &\ge r, \text{ then } \phi_Q(k, f, r) \sqsubseteq f. \\
\end{aligned}$ Suppose that  $\phi_Q(k, f, r) \sqsubseteq \phi_Q(k, g, r)$ . Then there exists such that<br>  $\phi_Q(k, f, r) \sqsupseteq e'_x \sqsupseteq \phi_Q(k, g, r)$ .<br>  $\mathcal{G}_k(f \odot h) \ge r$   $\Box$  (9) If  $T_k(f') \ge r$ , then  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq f$ .<br> **Proof.** (1) Suppose that  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq f$ .<br> **Proof.** (1) Suppose that  $\phi_{\mathcal{G}}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}}(k, g, r)$ . Then there exists  $e_x^l \in (L^X)^E$  such that<br>  $\phi_{\math$ 

 $x^t \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$ . Then for all  $h \in$  $\mathcal{T}_k(e^t_x, r)$ ,  $\mathcal{G}_k(f \odot h) \ge r$ . But  $f \sqsubseteq 1$ <sub>X</sub> implies that  $f \odot h \sqsubseteq f$ . Hence,  $G_k(f) \ge r$ . It is a contradiction. Thus,  $\phi_G(k, f, r) = 0$ <sub>X</sub>.

(3) Since  $f \sqsubseteq f \oplus g$  and  $g \sqsubseteq f \oplus g$ , by (1),  $\phi_G(k, f, r) \sqsubseteq$  $\phi_{\mathcal{G}}(k, f \oplus g, r)$  and  $\phi_{\mathcal{G}}(k, g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$ . Hence, we have 70 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
(3) Since  $f \sqsubseteq f \oplus g$  and  $g \sqsubseteq f \oplus g$ , by (1),  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq$ <br>  $\phi_{\mathcal{G}}(k, f \oplus g, r)$  and  $\phi_{\mathcal{G}}(k, g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$ . Hence, we have<br>  $\phi_{\mathcal{G}}(k, f, r) \opl$ 

$$
\phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r). \tag{2.4}
$$

Conversely, suppose that

$$
\phi_{\mathcal{G}}(k, f \oplus g, r) \mathcal{L} \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r).
$$

Then there exists  $e_x^t \in (L^X)^E$  such that

$$
\phi_{\mathcal{G}}(k, f \oplus g, r) \sqsupseteq e_x^t \sqsupseteq \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r).
$$

O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $f \sqsubseteq f \oplus g$  and  $g \sqsubseteq f \oplus g$ , by (1),  $\phi_g(k, f, r) \sqsubseteq$ <br>
and  $\phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . Hence, we have<br>  $(k, f, r) \oplus \phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . (2.4)<br>
suppose that<br>  $\phi_g(k, f \oplus g, r) \sqsubseteq$ 70 **IO.** R. Sayed, E. Elsanousy and Y. H. Raghp<br>
(3) Since  $f \subseteq f \oplus g$  and  $g \subseteq f \oplus g$ , by (1),  $\phi_g(k, f, r) \subseteq$ <br>  $\phi_g(k, f \oplus g, r)$  and  $\phi_g(k, g, r) \subseteq \phi_g(k, f \oplus g, r)$ . Hence, we have<br>  $\phi_g(k, f, r) \oplus \phi_g(k, g, r) \subseteq \phi_g(k, f \oplus g, r)$ . (2.4)<br>
C  $x_1, h_2 \in Q_{\mathcal{T}_k}(e_x^t, r)$  such that  $\mathcal{G}_k(f \odot h_1) \not\geq r$ and  $G_k(g \odot h_2) \not\geq r$ , and hence  $G_k((f \odot h_1) \oplus (f \odot h_2)) \not\geq r$ . Let  $h = h_1 \odot h_2$ O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>
(3) Since  $f \sqsubseteq f \oplus g$  and  $g \sqsubseteq f \oplus g$ , by (1),  $\phi_g(k, f, r) \sqsubseteq$ <br>  $g(k, f \oplus g, r)$  and  $\phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . Hence, we have<br>  $\phi_g(k, f, r) \oplus \phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . (2.4)<br>
Conv  $\in \mathcal{Q}_{\mathcal{T}_k}(e_x^t, r)$  and  $(f \oplus g) \odot h \sqsubseteq ((f \odot h_1) \oplus (g \odot h_2))$ . Then  $\mathcal{G}_k((f \oplus g))$ (3) Since  $f \sqsubseteq f \oplus g$  and  $g \sqsubseteq f \oplus g$ , by (1),  $\phi_g(k, f, r) \sqsubseteq$ <br>  $\phi_g(k, f \oplus g, r)$  and  $\phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . Hence, we have<br>  $\phi_g(k, f, r) \oplus \phi_g(k, g, r) \sqsubseteq \phi_g(k, f \oplus g, r)$ . (2.4)<br>
Conversely, suppose that<br>  $\phi_g(k, f \oplus g, r) \sqsubseteq \phi$  $x \not\subseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$ . It is a contradiction. So, It implies that there exist  $h_1, h_2 \in Q_{T_k}(e'_x, r)$  such that  $\mathcal{G}_k(f \circ h_1) \not\ge r$ <br>
and  $\mathcal{G}_k(g \circ h_2) \not\ge r$ , and hence  $\mathcal{G}_k((f \circ h_1) \oplus (f \circ h_2)) \not\ge r$ . Let  $h = h_1 \circ h_2$ <br>  $\in Q_{T_k}(e'_x, r)$  and  $(f \oplus g) \circ h \sqsubseteq ((f \circ h_1) \oplus (g \$ d hence  $G_k((f \odot h_1) \oplus (f \odot h_2)) \ge r$ . Let  $h = h_1 \odot h_2$ <br>  $\oplus g) \odot h \sqsubseteq ((f \odot h_1) \oplus (g \odot h_2))$ . Then  $G_k((f \oplus g)$ <br>  $(k, f \oplus g, r)$ . It is a contradiction. So,<br>  $g, r$ ) $\sqsubseteq \phi_g(k, f, r) \oplus \phi_g(k, g, r)$ . (2.5)<br>  $k$ , the result follows.<br>

$$
\phi_{\mathcal{G}}(k, f \oplus g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r). \tag{2.5}
$$

From  $(2.4)$  and  $(2.5)$ , the result follows.

- (4) Similar to (3).
- (5) Obvious by (1).
- $(6)$  Similar to  $(5)$ .
- $(7)$  Follows from  $(2)$  and  $(3)$ .

(8) Suppose that  $\phi_{\mathcal{G}}(k, f, r) \not\sqsubseteq C_{\mathcal{T}_k}(k, f, r)$ . Then for  $k \in K$ , there exists  $e_x^t \in (L^X)^E$  such that  $\sum_{k=1}^{n} \sum_{k=1}^{n} \sum_{k=1}^{n}$ 

$$
\phi_{\mathcal{G}}(k, f, r) \sqsupseteq e_x^t \sqsupset C_{\mathcal{T}_k}(k, f, r).
$$

 $\in$   $Q_{\mathcal{T}_k}(e_x^t, r)$  such that  $f \overline{q}g$  for each  $e \in E$ ,  $x \in X$ , i.e.,  $f \odot g = 0$ <sub>X</sub> and hence  $\mathcal{G}_k(f \odot g) = 0$ . It implies that  $\phi_g(k, f, r)$   $\mathbb{Z}e_x^t$  for  $h \in K$ . It is a contradiction and hence,

$$
\phi_{\mathcal{G}}(k, f, r) \sqsubseteq C_{\mathcal{T}_k}(k, f, r). \tag{2.6}
$$

Suppose that  $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\sqsubseteq \phi_{\mathcal{G}}(k, f, r)$ . Then for  $k \in K$ , On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $\exists e_x^l$  for  $h \in K$ . It is a contradiction and hence,<br>  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq C_{\mathcal{T}_k}(k, f, r)$ . (2.6)<br>
Suppose that  $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\sqsubseteq \phi_{\mathcal{G}}(k, f, r)$ . Then for  $k \in K$ ,<br>
the there exists  $e_x^t \in (L^X)^E$  such that On *L*-fuzzy  $(K, E)$ -soft Grills<br>
71<br>
It is a contradiction and hence,<br>  $\phi_{\mathcal{G}}(k, f, r) \subseteq C_{T_k}(k, f, r)$ . (2.6)<br>  $\iota C_{T_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \subseteq \phi_{\mathcal{G}}(k, f, r)$ . Then for  $k \in K$ ,<br>  $(L^X)^E$  such that<br>  $C_{T_k}(k, \phi_{\mathcal{G}}(k, f, r$ 

$$
C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \supseteq e_x^t \supseteq \phi_{\mathcal{G}}(k, f, r).
$$

Then  $\phi_{\mathcal{G}}(k, f, r)$  *qg* and  $g \in Q_{\mathcal{T}_k}(e^t, r)$  for  $e \in X$ . It implies that On *L*-fuzzy  $(K, E)$ -soft Grills<br>
71<br>
a contradiction and hence,<br>  $\phi_{\mathcal{G}}(k, f, r) \subseteq C_{T_k}(k, f, r)$ . (2.6)<br>  $\phi_{k_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\subseteq \phi_{\mathcal{G}}(k, f, r)$ . Then for  $k \in K$ ,<br>  $\phi_{k_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsupseteq \phi_{k_k}(k, f, r)$ . T On *L*-fuzzy  $(K, E)$ -soft Grills 71<br>  $\mathbb{Z}e_x^f$  for  $h \in K$ . It is a contradiction and hence,<br>  $\phi_{\mathcal{G}}(k, f, r) \subseteq C_{\mathcal{T}_k}(k, f, r).$  (2.6)<br>
Suppose that  $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\subseteq \phi_{\mathcal{G}}(k, f, r)$ . Then for  $k \in K$ ,<br>  $x \subseteq \phi_G(k, f, r)$  and  $g \in Q_{\mathcal{T}_k}(e_x^t, r)$ . Hence,  $\mathcal{G}_k(f \odot g) \ge r$  and  $e_x^t \sqsubseteq$ On *L*-fuzzy  $(K, E)$ -soft Grills 71<br>
It is a contradiction and hence,<br>  $\phi_{\mathcal{G}}(k, f, r) \subseteq C_{T_k}(k, f, r).$  (2.6)<br>
tt  $C_{T_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\subseteq \phi_{\mathcal{G}}(k, f, r).$  Then for  $k \in K$ ,<br>  $((L^X)^F$  such that<br>  $C_{T_k}(k, \phi_{\mathcal{G}}(k, f, r$  $\phi_{\mathcal{G}}(k, f, r)$ . It is a contradiction and hence,

$$
C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r). \tag{2.7}
$$

Therefore,  $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r)$ . From (2.6) and (2.7), we have

$$
\phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsubseteq C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r) \sqsubseteq C_{\mathcal{T}_k}(k, f, r).
$$

(9) Obvious by using (8).

The next example shows that the reverse inclusion in Proposition 2.7(5) is not true, in general.

**Example 2.8.** Let  $X = \{h_1, h_2\}$  with  $h_i = \text{house}$  for  $i \in \{1, 2\}$  and  $E = \{e, b\}$  with  $e =$  expensive,  $b =$  beautiful. Let  $([0, 1], 0, \rightarrow, 0, 1)$  be a  $e'_x \sqsubseteq \phi_G(k, f, r)$  and  $g \in Q_{T_k}(e'_{x}, r)$ . Hence,  $G_k(f \circ g) \ge r$  and  $e'_x \sqsubseteq$ <br>  $\phi_G(k, f, r)$ . It is a contradiction and hence,<br>  $C_{T_k}(k, \phi_G(k, f, r), r) \sqsubseteq \phi_G(k, f, r)$ . (2.7)<br>
Therefore,  $C_{T_k}(k, \phi_G(k, f, r), r) \equiv \phi_G(k, f, r)$ . From (2.6) and  $f_i \in ([0, 1]^X)^E$  for  $i \in \{1, 2, 3, 4, 5\}$  as follows:

$$
(f_1)_e = (0.5, 0.4), (f_1)_b = (0.0, 0.0),
$$
  
\n $(f_2)_e = (0.3, 0.3), (f_2)_b = (0.0, 0.0),$   
\n $(f_3)_e = (0.2, 0.3), (f_3)_b = (0.0, 0.0),$ 

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$$
(f_4)_e = (0.7, 0.8), (f_4)_b = (0.0, 0.0),
$$
  
 $(f_5)_e = (0.2, 0.2), (f_5)_b = (0.0, 0.0).$ 

Then

$$
f_1 \odot 1_X = f_1, \quad f_4 \odot 1_X = f_4,
$$
  

$$
f_1 \odot f_4 = f_5, \quad f_5 \odot 1_X = f_5.
$$

Let  $K = \{k_1, k_2\}$  be given. Define a  $([0, 1], \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft 72 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $(f_4)_e = (0.7, 0.8), (f_4)_b = (0.0, 0.0),$ <br>  $(f_5)_e = (0.2, 0.2), (f_5)_b = (0.0, 0.0).$ <br>
Then<br>  $f_1 \odot 1_X = f_1, f_4 \odot 1_X = f_4,$ <br>  $f_1 \odot f_4 = f_5, f_5 \odot 1_X = f_5.$ <br>
Let  $K = \{k_1, k_2\}$  be given. D  $\mathcal{T}, \mathcal{G} : K \to [0, 1]^{([0, 1]^X)^E}$  as follows:

 0 otherwise, 0.8 if , 0.7 if , 1 if 1 or 0 , 0 otherwise, 0.6 if , 0.5 if , 1 if 1 or 0 , 2 1 2 1 <sup>1</sup> <sup>2</sup> g f g f g g g f g f g g X X k X X k 0.4 if 1 , 1 if 1 , 0.3 if 1 , 1 3 2 3 X X k k f f f f g g 1 if 1 , X X g g Since <sup>Q</sup> <sup>e</sup> <sup>r</sup> <sup>X</sup> <sup>k</sup> <sup>h</sup> and , , 0.3 <sup>1</sup> <sup>4</sup>

0 otherwise,

0 otherwise.

 $1_X \in Q_{\mathcal{T}_h}$  ( $e_{h}^{0.4}$ ,  $\in Q_{\mathcal{T}_{k_1}}(e_{h_2}^{0.4}, r)$  for all  $r \in [0, 1]$ , and  $\mathcal{G}_{k_1}(f_1) = \mathcal{G}_{k_1}(f_4)$  $f_1 \odot f_4 = f_5, \quad f_5 \odot f_1 x = f_5.$ <br>
Let  $K = \{k_1, k_2\}$  be given. Define a  $([0, 1], 0, \Box)$ -fuzzy  $(K, E)$ -soft<br>
(opology and  $[0, 1]$ -fuzzy  $(K, E)$ -soft grill  $T, G: K \rightarrow [0, 1]^{([0, 1]^X)^E}$  as<br>
follows:<br>  $\mathcal{T}_{k_1}(g) = \begin{cases} 1 & \text{if }$ 0.4  $e_{h_2}^{0.4} \sqsubseteq \phi_G(k_1, f_1, 0.3)$  and  $e_{h_2}^{0.4}$  $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_1, f_4, 0.3)$  but  $e_{h_2}^{0.4} \sqsubseteq$  $e_{h_2}^{0.4}$  $\phi_{\mathcal{G}}(k_1, f_1 \odot f_4, 0.3)$  because  $\mathcal{G}_{k_1}(f_5) = 0$ . Also, because  $1_X \in \mathcal{Q}_{\mathcal{T}_{k_2}}$ pology and [0, 1]-fuzzy  $(K, b)$ -soft grill  $T, G : K \to [0, 1]^{x_0, x_1}$  as<br>
bllows:<br>  $T_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X \text{ or } 0_X, \\ 0.5 & \text{if } g = f_1, \\ 0.6 & \text{if } g = f_2, \\ 0 & \text{otherwise,} \end{cases}$   $T_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X \text{ or } 0_X, \\ 0.8 & \text{if } g = f_2$  $\frac{0.4}{h_2}$ ,  $\mathcal{L}(e_{h_2}^{0,4}, r)$  for all  $r \in [0, 1]$ , and  $\mathcal{G}_{k_2}(f_1) = \mathcal{G}_{k_2}(f_4) = 0.4$ , then  $e_{h_2}^{0,4} \sqsubseteq$  $e_{h_2}^{0.4}$  $\phi_{\mathcal{G}}(k_2, f_1, 0.4)$  and  $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_2, f_4,$ if  $g = 1_X$  or  $0_X$ ,<br>
if  $g = f_1$ ,<br>
if  $g = f_2$ ,<br>
if  $g = f_2$ ,<br>
otherwise,<br>
if  $g = 1_X$ ,<br>
otherwise,<br>
if  $g = 1_X$ ,<br>
3 if  $f_3 \sqsubseteq f \sqsubset 1_X$ ,<br>
3 if  $f_3 \sqsubseteq f \sqsubset 1_X$ ,<br>
3 if  $f_3 \sqsubseteq f \sqsubset 1_X$ ,<br>
otherwise,<br>
otherwise,<br>  $h_1(e_{b_2}^{0,4}, r)$  $e_{h_2}^{0.4} \sqsubseteq \phi_G(k_2, f_4, 0.4)$  but  $e_{h_2}^{0.4}$  $\begin{aligned} \n\mathbf{g} &= \begin{cases} 1 & \text{if } \mathbf{g} = 1_X \text{ or } 0_X, \\ 0.7 & \text{if } \mathbf{g} = f_1, \\ 0.8 & \text{if } \mathbf{g} = f_2, \\ 0 & \text{otherwise,} \end{cases} \\ \n\mathbf{g} &= \begin{cases} 1 & \text{if } \mathbf{g} = 1_X, \\ 0.4 & \text{if } f_3 \subseteq f \sqsubset 1_X, \\ 0 & \text{otherwise.} \end{cases} \\ \n\begin{bmatrix} 0, 1 \end{bmatrix}, \quad$  $e_{h_2}^{0.4} \mathcal{L} \phi_{\mathcal{G}}(k_2, f_1 \odot f_2)$ because  $G_{k_2}(f_5) = 0$ . Hence,

$$
\phi_{\mathcal{G}}(k, f_1 \odot f_4, r) \mathbb{Z} \phi_{\mathcal{G}}(k, f_1, r) \odot \phi_{\mathcal{G}}(k, f_4, r) \text{ for all } k \in K.
$$

**Lemma 2.9.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological On *L*-fuzzy  $(K, E)$ -soft Grills<br> **Lemma 2.9.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological<br>
space and  $G$  be an *L*-fuzzy  $(K, E)$ -soft grill on *X*. Then for  $f, g \in (L^X)^E$ <br>
and for all  $k \in K$ ,<br>  $\phi_G(k, f, r) \$ space and G be an L-fuzzy  $(K, E)$ -soft grill on X. Then for  $f, g \in (L^X)^E$ On *L*-fuzzy (*K*, *E*)-soft Grills 73<br> **Lemma 2.9.** Let  $(X, T)$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological<br>
space and  $G$  be an *L*-fuzzy  $(K, E)$ -soft grill on *X*. Then for  $f, g \in (L^X)^E$ <br>
and for all  $k \in K$ ,<br>  $\phi_G(k, f$ On *L*-fuzzy  $(K, E)$ -soft Grills<br> **Lemma 2.9.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological<br>
re and  $\mathcal{G}$  be an *L*-fuzzy  $(K, E)$ -soft grill on *X*. Then for  $f, g \in (L^X)^E$ <br>
for all  $k \in K$ ,<br>  $\phi_{\mathcal{G}}(k,$ **ma 2.9.** Let  $(X, T)$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological<br>  $d \circ g$  be an  $L$ -fuzzy  $(K, E)$ -soft grill on  $X$ . Then for  $f, g \in (L^X)^E$ <br>  $\exists l \ k \in K$ ,<br>  $(k, f, r) \odot (\phi_g(k, g, r))' = \phi_g(k, f \odot g', r) \odot (\phi_g(k, g, r))'$ .<br> **f.** Let  $f, g \$ 

$$
\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))' = \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.
$$

**Proof.** Let  $f, g \in (L^X)^E$  and  $f = (f \odot g') \oplus (f \odot g)$ . Then by Proposition 2.7, we have

$$
u \ k \in \mathbb{A},
$$
  
\n
$$
(k, f, r) \odot (\phi_{G}(k, g, r))' = \phi_{G}(k, f \odot g', r) \odot (\phi_{G}(k, g, r))'.
$$
  
\n**f.** Let  $f, g \in (L^{X})^{E}$  and  $f = (f \odot g') \oplus (f \odot g)$ . Then by  
\non 2.7, we have  
\n
$$
\phi_{G}(k, f, r) = \phi_{G}(k, (f \odot g') \oplus (f \odot g), r)
$$
\n
$$
= \phi_{G}(k, f \odot g', r) \oplus \phi_{G}(k, f \odot g, r).
$$
\n
$$
\sqsubseteq \phi_{G}(k, f \odot g', r) \oplus \phi_{G}(k, g, r).
$$
\n
$$
((k, f, r) \odot (\phi_{G}(k, g, r))' \sqsubseteq \phi_{G}(k, f \odot g', r) \odot (\phi_{G}(k, g, r))').
$$
\n
$$
g(k, f \odot g', r) \sqsubseteq \phi_{G}(k, f, r).
$$
 Hence,  
\n
$$
((k, f \odot g', r) \odot (\phi_{G}(k, g, r))' \sqsubseteq \phi_{G}(k, f, r) \odot (\phi_{G}(k, g, r))').
$$
  
\ne,  
\n
$$
(k, f, r) \odot (\phi_{G}(k, g, r))' = \phi_{G}(k, f \odot g', r) \odot (\phi_{G}(k, g, r))'.
$$
  
\n**orem 2.10.** Let  $G$  be an *L*-fuzzy  $(K, E)$ -soft grid over an  
\n
$$
f \rightarrow f \rightarrow f
$$

Thus,

$$
\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r)) \subseteq \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.
$$

Again  $\phi_{\mathcal{G}}(k, f \odot g', r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r)$ . Hence,

$$
\phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r)) \subseteq \phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))'.
$$

Therefore,

$$
\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))^{'} = \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))^{'}.
$$

**Theorem 2.10.** Let  $G$  be an L-fuzzy  $(K, E)$ -soft grill over an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft topological space  $(X, \mathcal{T})$ . Then the following  $\Phi_{\mathcal{G}}(k, f \odot g', r) \oplus \phi_{\mathcal{G}}(k, f \odot g, r)$ <br>
Thus,<br>
Thus,<br>  $\Phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r)) \subseteq \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$ <br>
Again  $\phi_{\mathcal{G}}(k, f \odot g', r) \subseteq \phi_{\mathcal{G}}(k, f, r)$ . Hence,<br>  $\phi_{\mathcal{G}}(k, f \odot g', r) \od$ statements are equivalent for  $f, g \in (L^X)^E$  and for all  $k \in K$ : s,<br>  $\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r)) \subseteq \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$ <br>
in  $\phi_{\mathcal{G}}(k, f \odot g', r) \subseteq \phi_{\mathcal{G}}(k, f, r)$ . Hence,<br>  $\phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r)) \subseteq \phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$ <br>
re

(1) 
$$
f \odot \phi_{\mathcal{G}}(k, f, r) = 0_X \Rightarrow \phi_{\mathcal{G}}(k, f, r) = 0_X
$$
.

(2) 
$$
\phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r)), r) = 0_X
$$
.

(3)  $\phi_G(k, f \odot \phi_G(k, f, r), r) = \phi_G(k, f, r).$ 

**Proof.** (1)  $\Rightarrow$  (2) Let  $f \in (L^X)^E$  and  $k \in K$ . Then

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\n**Proof.** (1) 
$$
\Rightarrow
$$
 (2) Let  $f \in (L^X)^E$  and  $k \in K$ . Then  
\n
$$
(f \odot (\phi_g(k, f, r))') \odot \phi_g(k, f \odot (\phi_g(k, f, r))', r) = 0_X,
$$
\nand so,  $\phi_g(k, f \odot (\phi_g(k, f, r))', r) = 0_X$ .  
\n(2)  $\Rightarrow$  (3) Let  $f \in (L^X)^E$  and  
\n
$$
f = (f \odot (f \odot \phi_g(k, f, r))') \oplus (f \odot \phi_g(k, f, r)).
$$
\nThen for all  $k \in K$ ,

 $\phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r))', r) =$ 

(2) 
$$
\Rightarrow
$$
 (3) Let  $f \in (L^X)^E$  and

$$
f = (f \odot (f \odot \phi_{\mathcal{G}}(k, f, r))) \oplus (f \odot \phi_{\mathcal{G}}(k, f, r)).
$$

Then for all  $k \in K$ ,

74 O. R. Sayed, E. Elsanousy and Y. H. Ragh  
\n**Proof.** (1) 
$$
\Rightarrow
$$
 (2) Let  $f \in (L^X)^E$  and  $k \in K$ . Then  
\n
$$
(f \odot (\phi_G(k, f, r))') \odot \phi_G(k, f \odot (\phi_G(k, f, r))', r) = 0_X,
$$
\nand so,  $\phi_G(k, f \odot (\phi_G(k, f, r))', r) = 0_X$ .  
\n(2)  $\Rightarrow$  (3) Let  $f \in (L^X)^E$  and  
\n
$$
f = (f \odot (f \odot \phi_G(k, f, r))') \oplus (f \odot \phi_G(k, f, r)).
$$
\nThen for all  $k \in K$ ,  
\n $\phi_G(k, f, r) = \phi_G(k, f \odot (f \odot \phi_G(k, f, r))', r) \oplus \phi_G(k, f \odot \phi_G(k, f, r), r)$   
\n
$$
= \phi_G(k, f \odot (\phi_G(k, f, r))', r) \oplus \phi_G(k, f \odot \phi_G(k, f, r), r)
$$
  
\n
$$
= \phi_G(k, f \odot \phi_G(k, f, r), r).
$$
\n(3)  $\Rightarrow$  (1) Suppose that  $f \odot \phi_G(k, f, r) = 0_X$  for all  $k \in K$  and  
\n $f \in (L^X)^E$ . Then we have  
\n $\phi_G(k, f, r) = \phi_G(k, f \odot \phi_G(k, f, r), r) = \phi_G(k, 0_X, r) = 0_X$ .  
\n**Definition 2.11.** Let  $G$  be an *I*-fuzzy  $(K, E)$ -soft grid on  $X$ . We define  
\nan *I*-fuzzy soft operator  $\psi : K \times (L^X)^E \times L_0 \rightarrow (L^X)^E$  by  $\psi(k, f, r) =$   
\n $f \oplus \phi_G(k, f, r)$  for every  $f \in (L^X)^E$  and  $k \in K$ .  
\n**Theorem 2.12.** For  $f, g \in (L^X)^E$  and  $k \in K$ , the mapping  $\psi$   
\nsatisfies the following:  
\n(1)  $f \subseteq \psi(k, f$ 

(3)  $\Rightarrow$  (1) Suppose that  $f \odot \phi_g(k, f, r) = 0$ <sub>X</sub> for all  $k \in K$  and  $f \in (L^X)^E$ . Then we have

$$
\phi_{\mathcal{G}}(k, f, r) = \phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, 0_X, r) = 0_X.
$$

**Definition 2.11.** Let  $G$  be an *L*-fuzzy  $(K, E)$ -soft grill on *X*. We define  $f \oplus \phi_{\mathcal{G}}(k, f, r)$  for every  $f \in (L^X)^E$  and  $k \in K$ .

**Theorem 2.12.** For  $f, g \in (L^X)^E$  and  $k \in K$ , the mapping  $\psi$ satisfies the following:

(1)  $f \sqsubseteq \psi(k, f, r)$ .

(2) 
$$
\psi(k, 0_X, r) = 0_X
$$
.

(3)  $\psi(k, f \oplus g, r) = \psi(k, f, r) \oplus \psi(k, g, r).$ 

(4) 
$$
\psi(k, f \sqcup g, r) = \psi(k, f, r) \sqcup \psi(k, g, r)
$$
.

(5) 
$$
\psi(k, \psi(k, f, r), r) = \psi(k, f, r)
$$
.

On *L*-fuzzy  $(K, E)$ -soft Grills<br>
(4)  $\psi(k, f \sqcup g, r) = \psi(k, f, r) \sqcup \psi(k, g, r)$ .<br>
(5)  $\psi(k, \psi(k, f, r), r) = \psi(k, f, r)$ .<br> **Proof.** Let  $f, g \in (L^X)^E$ ,  $r \in L_0$ . Hence, in view of Proposition 2.7<br>
(3), (8) for all  $k \in K$ , we have:<br>
(1) Obv f,  $g \in (L^X)^E$ ,  $r \in L_0$ . Hence, in view of Proposition 2.7 (2), (3), (8) for all  $k \in K$ , we have:

(1) Obvious.

(2) Suppose that  $G_k(0_X) = 0$ . Then  $\phi_G(k, 0_X, r) = 0_X$ . It implies that  $\psi(k, 0_X, r) = 0_X.$ 

(3)

$$
\psi(k, f \oplus g, r) = (f \oplus g) \oplus \phi_{G}(k, f \oplus g, r)
$$
  
=  $(f \oplus g) \oplus (\phi_{G}(k, f, r) \oplus \phi_{G}(k, g, r))$   
=  $(f \oplus \phi_{G}(k, f, r)) \oplus (g \oplus \phi_{G}(k, g, r))$   
=  $\psi(k, f, r) \oplus \psi(k, g, r).$ 

(4) Similar to that of (3).

(5)

$$
\psi(k, \psi(k, f, r), r)
$$
  
=  $\psi(k, f \oplus \phi_{G}(k, f, r))$   
=  $(f \oplus \phi_{G}(k, f, r)) \oplus (\phi_{G}(k, f \oplus \phi_{G}(k, g, r), r))$   
=  $(f \oplus \phi_{G}(k, f, r)) \oplus (\phi_{G}(k, f, r) \oplus \phi_{G}(k, \phi_{G}(k, f, r), r))$   
=  $(f \oplus \phi_{G}(k, f, r)) \oplus \phi_{G}(k, \phi_{G}(k, f, r), r)$   
=  $f \oplus \phi_{G}(k, f, r) = \psi(k, f, r).$ 

**Definition 2.13.** For any L-fuzzy  $(K, E)$ -soft grill on an  $(L, \odot, \Box)$ fuzzy  $(K, E)$ -soft topological space  $(X, T)$ , there exists a unique

 $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft topology  $\mathcal{T}_{\mathcal{G}}$  on X given by  $(\mathcal{T}_{\mathcal{G}})_k(k)$  =  $\forall \{r : \psi(k, f', r) = f'\}$  such that  $\psi(k, f, r) = C_{T_{\mathcal{G}}}(k, f, r)$  for every O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $(\forall x \in \mathcal{X})$ .  $(\forall x \in \mathcal{X})$ . For every  $\exists x \in \mathcal{X}$ .  $(\forall x \in \mathcal{X})$ . For every<br>  $(\forall x \in \mathcal{X})$ .  $\forall x \in \mathcal{X}$ .<br>  $(\forall x \in \mathcal{X})$ . Theorem 2.14. Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ . Fuzz  $f \in (L^X)^E$  and  $k \in K$ .

**Theorem 2.14.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space. Then  $T_k(k) \le (T_{\mathcal{G}})_k(f)$  for any L-fuzzy  $(K, E)$ -soft 76 O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft topology  $T_g$  on X given by  $(T_g)_k(k) = \bigvee \{r : \psi(k, f', r) = f'\}$  such that  $\psi(k, f, r) = C_{T_g}(k, f, r)$  for every<br>  $f \in (L^X)^E$  and  $k \in K$ .<br> **Theorem 2.14.** L grill G on X,  $f \in (L^X)^E$  and  $k \in K$ . O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $\odot$ ,  $\Box$ ) -fuzzy (*K*, *E*) -soft topology  $T_G$  on *X* given by  $(T_G)_k(k) =$ <br>  $\cdots$   $\psi(k, f', r) = f'$ } such that  $\psi(k, f, r) = C_{T_G}(k, f, r)$  for every<br>  $\vdots (L^X)^E$  and  $k \in K$ .<br> **Theorem 2** 

**Proof.** Suppose that  $\mathcal{T}_k(f) \nleq (\mathcal{T}_G)_k(f)$  for all  $f \in (L^X)^E$  and  $k \in K$ . Then there exists  $r \in L$  such that

$$
\mathcal{T}_k(f) \ge r > \left(\mathcal{T}_\mathcal{G}\right)_k(f) \text{ for all } k \in K. \tag{2.8}
$$

Since  $\mathcal{T}_k(f) \ge r$  for all  $k \in K$ ,  $C_{\mathcal{T}}(k, f', r) = f'$  for all  $k \in K$ . By Proposition 2.7, it follows that  $\phi_G(k, f', r) \sqsubseteq f'$ . Hence,  $C_{T_G}(k, f', r)$  $= \psi(k, f', r) = f'$  and so  $(\mathcal{T}_{\mathcal{G}})_k(f) \ge r$ . It contradicts (2.8). Hence,  $\mathcal{T}_k(f) \leq (\mathcal{T}_\mathcal{G})_k(f).$ ogical space. Then  $T_k(k) \le (T_G)_k(f)$  for any L-fuzzy  $(K, E)$ -soft<br>  $G$  on  $X, f \in (L^X)^E$  and  $k \in K$ .<br> **roof.** Suppose that  $T_k(f) \le (T_G)_k(f)$  for all  $f \in (L^X)^E$  and  $k \in K$ .<br>
there exists  $r \in L$  such that<br>  $T_k(f) \ge r > (T_G)_k(f)$  for  $T_k(f) \ge r > (T_g)_k(f)$  for all  $k \in K$ . (2.8)<br>  $T_k(f) \ge r$  for all  $k \in K$ ,  $C_T(k, f', r) = f'$  for all  $k \in K$ . By<br>
sosition 2.7, it follows that  $\phi_G(k, f', r) \subseteq f'$ . Hence,  $C_{T_g}(k, f', r)$ <br>  $(k, f', r) = f'$  and so  $(T_g)_k(f) \ge r$ . It contradicts (2.8

**Theorem 2.15.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $G$  be an L-fuzzy  $(K, E)$ -soft grill on X. Then for each  $f \in (L^X)^E$  and for all  $k \in K$ , the following statements hold:

- (1)  $T_{\mathcal{G}}(f') \ge r$  if and only if  $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq f$ .
- (2) If  $\mathcal{G}(f) = 0$ , then  $\mathcal{T}_{\mathcal{G}}(f') \geq r$ .
- (3)  $T_G((\phi_G(k, f, r))') \geq r$ .

Proof. (1) By Definition 2.13, it is obvious.

(2) Let  $G(f') = 0$ . Then by Proposition 2.7(2), we have  $\phi_G(k, f, r)$  $= 0_X$  and so

$$
C_{\mathcal{T}_{\mathcal{G}}}(k, f, r) = \psi(k, f, r) = f \oplus \phi_{\mathcal{G}}(k, f, r) = f \text{ for all } k \in K.
$$

Hence,  $T_{\mathcal{G}}(f') \geq r$ .

On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $C_{T_G}(k, f, r) = \psi(k, f, r) = f \oplus \phi_G(k, f, r) = f$  for all  $k \in K$ .<br>
ce,  $T_G(f') \ge r$ .<br>
(3) Let  $f \in (L^X)^E$  and  $k \in K$ . Then from the definition of  $\psi_G$  and by<br>
oosition 2.7(8), we have<br>  $\psi(k, \phi_G(k, f, r), r)$  $f \in (L^X)^E$  and  $k \in K$ . Then from the definition of  $\psi_{\mathcal{G}}$  and by Proposition 2.7(8), we have

$$
\psi(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r).
$$

Consequently,  $T_{\mathcal{G}}((\phi_{\mathcal{G}}(k, f, r))') \geq r$ .

On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $C_{T_G}(k, f, r) = \psi(k, f, r) = f \oplus \phi_G(k, f, r) = f$  for all  $k \in K$ .<br>
Hence,  $T_G(f') \ge r$ .<br>
(3) Let  $f \in (L^X)^E$  and  $k \in K$ . Then from the definition of  $\psi_G$  and by<br>
Proposition 2.7(8), we have<br>  $\psi(k, \phi_G(k,$ **Theorem 2.16.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $G^1$ ,  $G^2$  be two L-fuzzy  $(K, E)$ -soft grills on X On *L*-fuzzy  $(K, E)$ -soft Grills<br>  $C_T_G(k, f, r) = \psi(k, f, r) = f \oplus \phi_G(k, f, r) = f$  for all  $k \in K$ .<br>
Hence,  $T_G(f') \ge r$ .<br>
(3) Let  $f \in (L^X)^E$  and  $k \in K$ . Then from the definition of  $\psi_G$  and by<br>
Proposition 2.7(8), we have<br>  $\psi(k, \phi_G(k, f,$ On *L*-fuzzy  $(K, E)$ -soft Grills<br>
77<br>  $(k, f, r) = \psi(k, f, r) = f \oplus \phi_G(k, f, r) = f$  for all  $k \in K$ .<br>  $g(f') \ge r$ .<br>  $f \in (L^X)^F$  and  $k \in K$ . Then from the definition of  $\psi_G$  and by<br>  $g(k, f, r), r \ge \phi_G(k, f, r) \oplus \phi_G(k, \phi_G(k, f, r), r) = \phi_G(k, f, r)$ .<br>  $h =$  $f \in (L^X)^E$ . Then the following statements hold:  $\epsilon K$ . Then from the definition of  $\psi_{\mathcal{G}}$  and by<br>  $f, r$ ) $\oplus \phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r)$ .<br>  $\psi$ ) $\circ$  2 r.<br>  $\tau$ <br>  $\tau$ ) be an  $(L, \odot, \Box)$  fuzzy  $(K, E)$  soft<br>  $\frac{1}{2}$  be two  $L$ -fuzzy  $(K, E)$ -soft g

(1)  $(\mathcal{T}_{\mathcal{G}^2})_k(f) \leq (\mathcal{T}_{\mathcal{G}^1})_k(f)$ . (2) If  $T_k = (T_{\mathcal{G}^1})_k$ , then  $(T_{\mathcal{G}^1})_k = (T_{\mathcal{G}^2})_k$ .

**Proof.** (1) Suppose that  $(\mathcal{T}_{\mathcal{G}^2})_k \leq (\mathcal{T}_{\mathcal{G}^1})_k$  for all  $f \in (L^X)^E$ . Then there exists  $r \in L$  such that

$$
(\mathcal{T}_{\mathcal{G}^2})_k(f) \ge r > (\mathcal{T}_{\mathcal{G}^1})_k(f) \text{ for all } k \in K. \tag{2.9}
$$

Since  $(\mathcal{T}_{\mathcal{G}^2})_k(f) \geq r$  for all  $k \in K$ ,

$$
C_{\mathcal{T}_{\mathcal{G}^2}}(k, f', r) = \psi_{\mathcal{G}^2}(k, f', r)
$$

$$
= f' \oplus \phi_{\mathcal{G}^2}(k, f', r) = f'
$$

$$
\Rightarrow \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.
$$

 $\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)$ , by Proposition 2.6, we have

$$
\phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.
$$

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Since  $g_k^1(f) \le g_k^2(f)$ , by Proposition 2.6, we have<br>  $\phi_{g^2}(k, f', r) \sqsubseteq \phi_{g^2}(k, f', r) \sqsubseteq f'$ .<br>
Thus,  $C_{\mathcal{T}_{c,1}}(k, f', r) = f'$ . It implies that  $(\mathcal{T}_{d^1})_k(f) \ge r$ . It contradicts Thus,  $C_{\mathcal{T}_{c^1}}(k, f', r) = f'$ . It implies that  $T_{\mathcal{G}^1}(k, f', r) = f'.$  It implies that  $(T_{\mathcal{G}^1})_k(f) \geq r.$  It contradicts (2.9). Hence,  $({\cal T}_{\mathcal{G}^2})_k(f) \le ({\cal T}_{\mathcal{G}^1})_k(f)$ .

(2) Follows from Theorem 2.14 and (1) of this theorem.

**Corollary 2.17.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space and  $\mathcal{G}^1$ ,  $\mathcal{G}^2$  be two L-fuzzy  $(K, E)$ -soft grills on X. 78 **O. R.** Sayed, E. Elsanousy and Y. H. Raghp<br>
Since  $\mathcal{G}_K^1(f) \leq \mathcal{G}_K^2(f)$ , by Proposition 2.6, we have<br>  $\phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.$ <br>
Thus,  $C_{\mathcal{T}_{\mathcal{G}}^1}(k, f', r) = f'.$  It implies that  $(\mathcal{T}_{\mathcal{G$  $f \in (L^X)^E$ ,  $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) =$  $({\cal T}_{\mathcal{G}^1})_k(f) \vee ({\cal T}_{\mathcal{G}^2})_k(f).$ 

**Proof.** By Theorem 2.16, we have  $(T_{g^1 \vee g^2})_k(f) \leq (T_{g^1})_k(f)$  $\bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f)$ . So, we only prove that  $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \geq (\mathcal{T}_{\mathcal{G}^1})_k(f) \vee$  $({\mathcal T}_{{\mathcal G}^2})_k(f)$ . Suppose that

$$
(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \not\geq (\mathcal{T}_{\mathcal{G}^1})_k(f) \vee (\mathcal{T}_{\mathcal{G}^2})_k(f).
$$

Then there exists  $r \in L$  such that

$$
(\mathcal{T}_{\mathcal{G}^1\vee\mathcal{G}^2})_k(f)
$$

Note that  $r \leq (T_{g1})_k(f) \vee (T_{g2})_k(f)$  implies that  $(T_{g1})_k(f) \geq r$  or  $(\mathcal{T}_{\mathcal{G}^2})_k(f) \ge r$ . By Theorem 2.15, we have

$$
\phi_{\mathcal{G}^1}(k, f', r) \sqsubseteq f', \quad \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.
$$

Also, by Proposition 2.6, we have

$$
\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f', r) = \phi_{\mathcal{G}^1}(k, f', r) \forall \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.
$$

Consequently,  $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \geq r$ . It is a contradiction.

**Definition 2.18.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft On *L*-fuzzy  $(K, E)$ -soft Grills 79<br> **Definition 2.18.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \sqcap)$ -fuzzy  $(K, E)$ -soft<br>
topological space. Define a mapping  $\mathcal{P}: K \to L^{(L^X)^E}$  by<br>  $\mathcal{P}_k^f(f) = \begin{cases} 1, & \text{if } fgq, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.$ On *L*-fuzzy  $(K, E)$ -soft Grills 79<br>
Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>
ine a mapping  $\mathcal{P}: K \to L^{(L^X)^E}$  by<br>  $(f) = \begin{cases} 1, & \text{if } f g q, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.} \end{cases}$ <br>  $f(K, E)$ -soft grill on *X*. We call this

$$
\mathcal{P}_k^f(f) = \begin{cases} 1, & \text{if } fgq, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.} \end{cases}
$$

Then  $P$  is an *L*-fuzzy  $(K, E)$ -soft grill on *X*. We call this *L*-fuzzy  $(K, E)$ soft grill the L-fuzzy  $(K, E)$ -soft principle grill generated by an L-fuzzy soft set f. **Definition 2.18.** Let  $(X, T)$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft<br>topological space. Define a mapping  $P : K \rightarrow L^{(L^X)^E}$  by<br> $P_k^f(f) = \begin{cases} 1, & \text{if } f g g, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.} \end{cases}$ <br>Then  $P$  is an *L*-fuzzy  $(K, E)$ -soft grill

**Lemma 2.19.** Let  $(X, \mathcal{T})$  be an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft  $f, g \in (L^X)^E$ . Then

On *L*-fuzzy (*K*, *E*)-soft Grills 79  
\n**Definition 2.18.** Let (*X*, *T*) be an (*L*, 
$$
\odot
$$
,  $\sqcap$ )-fuzzy (*K*, *E*)-soft  
\nopological space. Define a mapping  $\mathcal{P}: K \to L^{(L^X)E}$  by  
\n
$$
\mathcal{P}_k^f(f) = \begin{cases} 1, & \text{if } fgq, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.} \end{cases}
$$
\nThen  $\mathcal{P}$  is an *L*-fuzzy (*K*, *E*)-soft grid on *X*. We call this *L*-fuzzy (*K*, *E*)-  
\nsoft grid the *L*-fuzzy (*K*, *E*)-soft principle grid] generated by an *L*-fuzzy soft  
\n
$$
\mathcal{L} = \mathcal{L} \mathcal
$$

Proof. Obvious.

**Theorem 2.20.** Let G be an L-fuzzy  $(K, E)$ -soft grill on an  $(L, \odot, \Box)$ fuzzy  $(K, E)$ -soft topological space  $(X, T)$ . Define a mapping  $\mathcal{B}(\mathcal{G}, T)$ :

$$
(\mathcal{B}(\mathcal{G},\mathcal{T}))_k(f) = \bigvee \{r : f = g \odot h', \mathcal{T}_k(g) \ge r, \mathcal{G}_k(h) = 0\}.
$$

Then  $\mathcal{B}(\mathcal{G}, \mathcal{T})$  is an  $(L, \odot, \Box)$ -fuzzy- $(K, E)$ -soft base on X.

**Proof.** (LSB1) Since  $1_X = 1_X \odot 0_X'$  and  $0_X = 0_X \odot 0_X'$  with  $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$  and  $\mathcal{G}_k(0_X) = 0$  for  $k \in K$ , we have  $(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(0_X) = (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(1_X) = 1$  for all  $r \in L$ .

(LSB2) Suppose that

$$
(\mathcal{B}(\mathcal{G},\mathcal{T}))_k(f\odot g) \not\geq (\mathcal{B}(\mathcal{G},\mathcal{T}))_k(f)\odot (\mathcal{B}(\mathcal{G},\mathcal{T}))_k(g)
$$

for all  $f, g \in (L^X)^E$  and  $k \in K$ . Then there exists  $r \in L$  such that

$$
(\mathcal{B}(\mathcal{G},\,\mathcal{T}))_k(f\odot g)< r\leq (\mathcal{B}(\mathcal{G},\,\mathcal{T}))_k(f)\odot(\mathcal{B}(\mathcal{G},\,\mathcal{T}))_k(g).
$$

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(LSB2) Suppose that<br>  $(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (f \odot g) \geq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (g)$ <br>
for all  $f, g \in (L^X)^E$  and  $k \in K$ . Then there exists  $r \in L$  such that<br>  $(\mathcal$ Since  $r \leq (B(G, T))_k(f) \odot (B(G, T))_k(g)$ , there exist  $f_1, g_1, h_1, h_2$ O. R. Sayed, E. Elsanousy and Y. H. Raghp<br>  $(B(G, T))_k (f \odot g) \ge (B(G, T))_k (f) \odot (B(G, T))_k (g)$ <br>
rall  $f, g \in (L^X)^E$  and  $k \in K$ . Then there exists  $r \in L$  such that<br>  $(B(G, T))_k (f \odot g) < r \le (B(G, T))_k (f) \odot (B(G, T))_k (g)$ .<br>
note  $r \le (B(G, T))_k ($  $\in (L^X)^E$  such that  $\mathcal{T}_k(f_1) \ge r$ ,  $\mathcal{T}_k(g_1) \ge r$  and  $\mathcal{G}_k(h_1) = \mathcal{G}_k(h_2) = 0$ , where  $f = f_1 \odot h_1'$  and  $g = g_1 \odot h_2'$ . It implies that  $\mathcal{T}_k(f_1 \odot g_1) \ge r$  and  $\mathcal{G}_k(h_1 \oplus h_2) = 0$ . Since Sayed, E. Elsanousy and Y. H. Raghp<br>
at<br>
at<br>  $(f \odot g) \geq (B(G, T))_k(f) \odot (B(G, T))_k(g)$ <br>
and  $k \in K$ . Then there exists  $r \in L$  such that<br>  $g \odot g$   $\leq r \leq (B(G, T))_k(f) \odot (B(G, T))_k(g)$ .<br>  $k(f) \odot (B(G, T))_k(g)$ , there exist  $f_1, g_1, h_1, h_2$ ince  $r \le (B(G, T))_k(f) \circ (B(G, T))_k(g)$ , there exist  $f_1, g_1, h_1, h_2$ <br>  $((t^X)^E$  such that  $T_k(f_1) \ge r$ ,  $T_k(g_1) \ge r$  and  $\mathcal{G}_k(h_1) = \mathcal{G}_k(h_2) = 0$ ,<br>
there  $f = f_1 \circ h_1'$  and  $g = g_1 \circ h_2'$ . It implies that  $T_k(f_1 \circ g_1) \ge r$  and<br>

$$
f \odot g = (f_1 \odot h_1') \odot (g_1 \odot h_2')
$$
  
=  $(f_1 \odot g_1) \odot (h_1 \oplus h_2)'$ .

We have  $(B(G, T))_k$   $(f \odot g) \ge r$ . It is a contradiction. Hence,

 $(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (f \odot g) \geq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k (g).$ 

(LSB3) Similar to (LSB2).

**Corollary 2.21.** For any L-fuzzy  $(K, E)$ -soft grill  $G$  on an  $(L, \odot, \Box)$ -fuzzy  $(K, E)$ -soft topological space  $(X, T)$ , we have  $\mathcal{T}_k(f) \leq$  $(\mathcal{B}(\mathcal{G},\mathcal{T}))_k(f) \leq (T_{\mathcal{G}})_k(f)$  for each  $f \in (L^X)^E$ ,  $k \in$ 

Proof. Obvious.

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