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ON L-FUZZY (K, E)-SOFT GRILLS

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Abstract

The aim of this paper is to present and make preliminary study of a new $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topology $\mathcal{T}_{\mathcal{G}}$ from old $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topology $\mathcal{T}$ via L-fuzzy (K, E)-soft grill $\mathcal{G}$ induced by L-fuzzy soft operators $\phi_{\mathcal{G}}, \psi_{\mathcal{G}}$.

1. Introduction and Preliminaries

In 1999, Molodtsov [21] introduced the theory of soft sets as a new

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mathematical tool for dealing with uncertainties. Also, he applied this theory to several directions (see, for example, [22-24]). The soft set theory has been applied to many different fields (see, for example, [1, 2, 5, 6, 8, 10, 17, 19, 25, 32, 35, 36]). Later, few researches (see, for example, [3, 7, 15, 16, 20, 26, 30, 33, 37]) introduced and studied the notion of soft topological spaces.

Roy and Samanta [31] redefined some definitions on fuzzy soft set in another form and defined a fuzzy soft topology. The concept of L-fuzzy soft sets can be seen as a generalization of fuzzy soft sets introduced by Cetkin et al. [9, 18, 27].

Aygünoglu et al. [4] defined fuzzy soft topology which will be compatible to the fuzzy soft theory and investigated some of its fundamental properties and introduced fuzzy soft cotopology and given the relations between fuzzy soft topology and fuzzy soft cotopology.

On the other hand, Hájek [11] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices.

In this paper, we present a kind of L-fuzzy soft operator, by using L-fuzzy (K, E)-soft grill, which eventually given rise to another L-fuzzy soft operator which satisfies Kuratowski's L-fuzzy soft closure axioms, thereby inducing a new $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topology. Some properties of the induced $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topology will be investigated.

Let $(L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with least element 0_L and the greatest element 1_L in L.

Definition 1.1 [11, 12, 28]**.** A complete lattice $(L, \leq, \odot)$ is called a *strictly two-sided commutative quantale* (*stsc-quantale*, for short) if and only if it satisfies the following properties:

(L1) $(L, \odot)$ is a commutative semigroup.

(L2) $x = x \odot 1$ for each $x \in L$ and 1 is the universal upper bound.

(L3) $\odot$ is distributive over arbitrary joins, i.e., $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.

There exists a further binary operation $\to$ (called the implication operator or residuated) satisfying the following condition:

$$x \to y = \bigvee\{z \in L : x \odot z \leq y\}.$$

Then it satisfies the Galois correspondence, i.e., $(x \odot z) \leq y$ if and only if $z \leq (x \to y)$.

Definition 1.2 [13, 14, 29, 34]**.** (1) An stsc-quantale $(L, \leq, \odot, ')$ is called an *MV-algebra* if and only if $(x \to 0) \to 0 = x$.

(2) An MV-algebra $(L, \leq, \odot, ')$ is called *complete* if and only if it satisfies the following property:

$$(x \to y) \to y = x \vee y, \quad \forall x, y \in L.$$

We always assume that $(L, \leq, \odot, ')$ is an stsc-quantale with an order reversing involution $'$ which is defined by

$$x \oplus y = (x' \odot y')', \quad x' = x \to 0.$$

Lemma 1.3 [12, 13, 29]**.** *For each* $x, y, z, x_i, y_i, w \in L$, *we have the following properties*:

(1) $1 \to x = x, 0 \odot x = 0$ *and* $x \to 0 = x'$,

(2) *if* $y \leq z$, *then* $x \odot y \leq x \odot z, x \oplus y \leq x \oplus z, x \to y \leq x \to z$ *and* $z \to x \leq y \to x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)' = \bigvee_i y_i', (\bigvee_i y_i)' = \bigwedge_i y_i'$,

(5) $x \odot (\bigwedge_i y_i) \leq \bigwedge_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$, $x \oplus (\bigvee_i y_i) = \bigvee_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x_i \to y)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ *and* $x \to y \leq (y \to z) \to (x \to z)$,

(13) $x \odot (x' \oplus y') \leq y'$, $x \odot y = (x \to y')'$ *and* $x \oplus y = x' \to y$,

(14) $(x \to y) \odot (z \to w) \leq (x \odot y) \to (y \odot w)$,

(15) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $(x \to y) \odot (y \to z) \leq x \to z$,

(16) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

Throughout this paper, X refers to an initial universe, E and K are the sets of all the parameters for X, and L^X is the set of all L-fuzzy sets on X.

Definition 1.4 [9]**.** A map f is called an *L-fuzzy soft set* on X, where f is a mapping from E into L^X, i.e., $f_e := f(e)$ is an L-fuzzy soft set on X, for each $e \in E$. The set of all fuzzy soft sets is denoted by $(L^X)^E$. Let $f, g \in (L^X)^E$. Then:

(1) f is an L-fuzzy soft subset g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L-fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L-fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L-fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L-fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L-fuzzy soft set on X is denoted by f', where $f' : E \to (L^X)^E$ is a mapping given by $f'_e = (f_e)'$, for each $e \in E$.

(7) f is called a *null L-fuzzy soft set* and denoted by 0_X if $f_e(x) = 0$, for each $e \in E$, and $x \in X$.

(8) f_A is called *absolute L-fuzzy soft set* and denoted by 1_X if $f_e(x) = 1$, for each $e \in E$, and $x \in X$.

An L-fuzzy soft point is an L-fuzzy soft set f such that $f_e := f(e)$ is an L-fuzzy point and $f_a := f(a) = \overline{0}$ for all $a \in E \setminus \{e\}$. We denote this L-fuzzy soft point by $f = e_x^t$. For $f, g \in (L^X)^E$, we write fqg to mean that f is soft quasi-coincident with g, i.e., there exist at least one $x \in X$, $e \in E$ such that $f_e(x) \not\leq g'_e(x)$. Negation of such a statement is denoted as $f\overline{q}g$.

Definition 1.5. A map $\mathcal{T} : K \to L^{(L^X)^E}$ (where $\mathcal{T}_k := \mathcal{T}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topology* on X if it satisfies the following conditions:

(LSO1) $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$.

(LSO2) $\mathcal{T}_k(f \odot g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g)$ for all $f, g \in (L^X)^E$.

(LSO3) $\mathcal{T}_k(f \sqcap g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g)$ for all $f, g \in (L^X)^E$.

(LSO4) $\mathcal{T}_k(\bigsqcup_{i \in \Lambda} f_i) \geq \bigwedge_{i \in \Lambda} \mathcal{T}(f_i)$ for all $f_i \in (L^X)^E$.

The pair $(X, \mathcal{T})$ is called an $(L, \odot, \sqcap)$-*fuzzy*-(K, E)-*soft topological space*.

Definition 1.6. Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space. Then for $f \in (L^X)^E$, an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft closure of *f* is a mapping $C_{\mathcal{T}} : K \times (L^X)^E \times L_0 \to (L^X)^E$ defined as:

$$C_{\mathcal{T}}(k, f, r) = \bigwedge\{g \in (L^X)^E : f \sqsubseteq g, \mathcal{T}(g') \geq r\}.$$

Definition 1.7. Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space, $e \in E$ and $x \in X$. For $k \in K$, $e_x^t \in (L^X)^E$ is said to be an *L-fuzzy soft closure point* of $f \in (L^X)^E$ if for every $g \in Q_{\mathcal{T}_k}(e_x^t, r)$, we have *fqg*, where

$$Q_{\mathcal{T}_k}(e_x^t, r) = \{g \in (L^X)^E : e_x^t qg, \mathcal{T}_k(h) \geq r\}.$$

Definition 1.8. A map $\mathcal{B} : K \to L^{(L^X)^E}$ (where $\mathcal{B}_k := \mathcal{B}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an $(L, \odot, \sqcap)$-*fuzzy* (K, E)-*soft base* on *X* if it satisfies the following conditions:

(LSB1) $\mathcal{B}_k(0_X) = \mathcal{B}_X(1_X) = 1$.

(LSB2) $\mathcal{B}_k(f \odot g) \geq \mathcal{B}_k(f) \odot \mathcal{B}_k(g)$ for all $f, g \in (L^X)^E$.

(LSB3) $\mathcal{B}_k(f \sqcap g) \geq \mathcal{B}_k(f) \odot \mathcal{B}_k(g)$ for all $f, g \in (L^X)^E$.

Definition 1.9. An $(L, \odot, \sqcap)$-fuzzy (K, E)-soft neighborhood system on *X* is a set $N = \{N^x : x \in X\}$ of mappings $N^x : K \to L^{(L^X)^E}$ such that for each $k \in K$:

(1) $N_k^x(1_X) = 1$ and $N_k^x(1_X) = 0$.

(2) $N_k^x(f \odot g) \geq N_k^x(f) \odot N_k^x(g)$ for each $f, g \in (L^X)^E$.

(3) $N_k^x(f \sqcap g) \geq N_k^x(f) \odot N_k^x(g)$ for each $f, g \in (L^X)^E$.

(4) If $f \sqsubseteq g$, then $N_k^x(f) \leq N_k^x(g)$.

(5) $N_k^x(f) \leq f_e(x)$, where $f \in (L^X)^E$ and $e \in E$.

(6) $N_k^x(f) \leq \bigvee\{N_k^x(g) : g_e(y) \sqsubseteq N_k^y(g), \forall y \in X, e \in E\}$.

2. L-fuzzy (K, E)-soft Grill Space

Definition 2.1. A map $\mathcal{G} : K \to L^{(L^X)^E}$ (where $\mathcal{G}_k := \mathcal{G}(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an *L-fuzzy* (K, E)*-soft grill* on X if it satisfies the following conditions for each $k \in K$:

(LSG1) $\mathcal{G}_k(0_X) = 0$ and $\mathcal{G}_k(1_X) = 1$.

(LSG2) $\mathcal{G}_k(f \oplus g) \leq \mathcal{G}_k(f) \oplus \mathcal{G}_k(g)$ for all $f, g \in (L^X)^E$.

(LSG3) If $f \sqsubseteq g$, then $\mathcal{G}_k(f) \leq \mathcal{G}_k(g)$.

The pair $(X, \mathcal{G})$ is called an *L-fuzzy* (K, E)*-soft grill space*. If $\mathcal{G}^1$ and $\mathcal{G}^2$ are L-fuzzy (K, E)-soft grills on X, then we say that $\mathcal{G}^1$ is *finer* than $\mathcal{G}^2$ ($\mathcal{G}^2$ is coarser than $\mathcal{G}^1$) denoted by $\mathcal{G}^2 \sqsubseteq \mathcal{G}^1$ if and only if $\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)$ for each $k \in K$ and $f \in (L^X)^E$.

Remark 2.2. Let $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill on X. By Lemma 1.3(3), (LSG2) and (LSG3), we have

$$\mathcal{G}_k(f \sqcup g) \leq \mathcal{G}_k(f) \oplus \mathcal{G}_k(g) \text{ for all } k \in K.$$

Proposition 2.3. *Let* $\mathcal{G}^1, \mathcal{G}^2$ *be L-fuzzy* (K, E)*-soft grills on X. Then a mapping* $\mathcal{G} : K \to L^{(L^X)^E}$ *defined by*:

$$\mathcal{G}_k = \mathcal{G}_k^1 \bigvee \mathcal{G}_k^2 \textit{ for all } k \in K$$

is an L-fuzzy (K, E)*-soft grill on X.*

Proof. (LSG1) For all $k \in K,$ we have

$$\mathcal{G}_k(1_X) = \mathcal{G}_k^1(1_X) \bigvee \mathcal{G}_k^1(1_X) = 1 \bigvee 1 = 1.$$

Also,

$$\mathcal{G}_k(0_X) = \mathcal{G}_k^1(0_X) \bigvee \mathcal{G}_k^2(0_X) = 0 \bigvee 0 = 0.$$

(LSG2) For each $f, g \in (L^X)^E$ and $k \in K,$ we have

$$\mathcal{G}_k(f) \oplus \mathcal{G}_k(g) = \bigvee_{i \in \{1, 2\}} \mathcal{G}_k^i(f) \oplus \bigvee_{i \in \{1, 2\}} \mathcal{G}_k^i(g)$$

$$= \bigvee_{i \in \{1, 2\}} (\mathcal{G}_k^i(f) \oplus \mathcal{G}_k^i(g))$$

$$\geq \bigvee_{i \in \{1, 2\}} \mathcal{G}_k^i(f \oplus g) = \mathcal{G}_k(f \oplus g).$$

(LSG3) If $f \sqsubseteq g,$ then we have $\mathcal{G}_k^i(f) \leq \mathcal{G}_k^i(g)$ for all $k \in K$ and $i \in \{1, 2\}.$ Therefore,

$$\mathcal{G}_k(f) = \bigvee_{i \in \{1, 2\}} \mathcal{G}_k^i(f) \leq \bigvee_{i \in \{1, 2\}} \mathcal{G}_k^i(g) = \mathcal{G}_k(g).$$

The next example shows that the meet of two L-fuzzy (K, E)-soft grills on X is not an L-fuzzy (K, E)-soft grill on X, in general.

Example 2.4. Let $X = \{h_1, h_2, h_3\}$ with h_i = house for $i \in \{1, 2, 3\}$ and $E = \{e, b\}$ with e = expensive, b = beautiful. Define a binary operation $\odot$ on [0, 1] by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \to y = \min\{1 - x + y, 1\},$$

$$x \oplus y = \min\{1, x + y\}, \quad x' = 1 - x.$$

Then $([0, 1], \odot, \rightarrow, 0, 1)$ is an stsc-quantale [11, 13, 29]. Let $f_i \in ([0, 1]^X)^E$ for $i \in \{1, 2, 3, 4\}$ as follows:

$$(f_1)_e = (0.0, 1.0, 0.0), \quad (f_1)_b = (1.0, 1.0, 1.0),$$

$$(f_2)_e = (1.0, 1.0, 0.0), \quad (f_2)_b = (1.0, 1.0, 1.0),$$

$$(f_3)_e = (0.0, 1.0, 1.0), \quad (f_3)_b = (1.0, 1.0, 1.0),$$

$$(f_4)_e = (1.0, 0.0, 0.0), \quad (f_4)_b = (1.0, 1.0, 1.0).$$

For $K = \{k_1, k_2\}$, we define L-fuzzy (K, E)-soft grills $\mathcal{G}^1, \mathcal{G}^2 : K \rightarrow [0, 1]^{([0,1]^X)^E}$ as follows:

$$\mathcal{G}^1_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.3 & \text{if } g = f_1, \\ 0.6 & \text{if } g = f_2, \\ 0.5 & \text{if } g = f_3, \\ 0 & \text{otherwise,} \end{cases} \quad \mathcal{G}^2_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.9 & \text{if } g = f_4, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{G}^1_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.2 & \text{if } g = f_1, \\ 0.7 & \text{if } g = f_2, \\ 0.3 & \text{if } g = f_3, \\ 0 & \text{otherwise,} \end{cases} \quad \mathcal{G}^2_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.6 & \text{if } g = f_4, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(f_2 \oplus f_3)_e = (1.0, 1.0, 1.0), (f_2 \oplus f_3)_b = (1.0, 1.0, 1.0),$ we have

$$\begin{aligned} (\mathcal{G}^1_{k_1} \wedge \mathcal{G}^2_{k_1})(f_2 \oplus f_3) &= (\mathcal{G}^1_{k_1} \wedge \mathcal{G}^2_{k_1})(1_X) \\ &= 1 \wedge 1 = 1 \\ &\not\leq ((\mathcal{G}^1_{k_1} \wedge \mathcal{G}^2_{k_1})(f_2)) \oplus ((\mathcal{G}^1_{k_1} \wedge \mathcal{G}^2_{k_1})(f_3)) \\ &= (0.6 \wedge 0) \oplus (0.5 \wedge 0) = 0 \oplus 0 = 0, \end{aligned}$$

$$\begin{aligned}(\mathcal{G}^1_{k_2} \bigwedge \mathcal{G}^2_{k_2})(f_2 \oplus f_3) &= (\mathcal{G}^1_{k_2} \bigwedge \mathcal{G}^2_{k_2})(1_X)\\ &= 1 \bigwedge 1 = 1\\ &\not\leq ((\mathcal{G}^1_{k_2} \bigwedge \mathcal{G}^2_{k_2})(f_2)) \oplus ((\mathcal{G}^1_{k_2} \bigwedge \mathcal{G}^2_{k_2})(f_3))\\ &= (0.7 \bigwedge 0) \oplus (0.3 \bigwedge 0) = 0 \oplus 0 = 0.\end{aligned}$$

Hence, $\mathcal{G}^1_k \bigwedge \mathcal{G}^2_k$ is not an L-fuzzy (K, E)-soft grill on X.

Definition 2.5. Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space and $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill on X. Then a mapping $\phi_{\mathcal{G}} : K \times (L^X)^E \times L \to (L^X)^E$ is called an *L-fuzzy soft operator* associated with an L-fuzzy (K, E)-soft grill G and an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological $\mathcal{T}$, and is defined by

$$\phi_{\mathcal{G}}(k, f, r) = \bigvee\{e^t_x \in (L^X)^E : \mathcal{G}_k(f \odot g) \geq r, \forall g \in Q_{\mathcal{T}_k}(e^t_x, r)\}.$$

Proposition 2.6. *Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space and $\mathcal{G}^1, \mathcal{G}^2$ be two L-fuzzy (K, E)-soft grills on X. Then for all $k \in K$ and for each $f \in (L^X)^E$:*

(1) *If $\mathcal{G}^1_k(f) \leq \mathcal{G}^2_k(f)$, then $\phi_{\mathcal{G}^1}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^2}(k, f, r)$.*

(2) $\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) = \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r)$.

Proof. (1) Let $\mathcal{G}^1, \mathcal{G}^1$ be two L-fuzzy (K, E)-soft grills on X with $\mathcal{G}^1_k(f) \leq \mathcal{G}^2_k(f)$ for all $k \in K$ and $f \in (L^X)^E$ such that

$$\phi_{\mathcal{G}^1}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}^2}(k, f, r).$$

Then there exists $e^t_x \in (L^X)^E$ such that

$$\phi_{\mathcal{G}^1}(k, f, r) \sqsupseteq e^t_x \sqsupset \phi_{\mathcal{G}^2}(k, f, r).$$

It implies that $\mathcal{G}_k^1(f \odot h) \geq r$ for all $h \in Q_{\mathcal{T}_k}(e_x^t, r)$. Since $\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)$, $\mathcal{G}_k^2(f \odot h) \geq r$ for every $h \in Q_{\mathcal{T}_k}(e_x^t, r)$ and so $e_x^t \sqsubseteq \phi_{\mathcal{G}^2}(k, f, r)$. It is a contradiction.

(2) Let $f \in (L^X)^E$ and $k \in K$. Then by (1), we have

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsupseteq \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r). \tag{2.1}$$

It suffices to show that $\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r)$. So, suppose that

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r).$$

Then there exists $e_x^t \in (L^X)^E$, $k \in K$ such that

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r) \tag{2.2}$$

which implies that $e_x^t \sqsupset \phi_{\mathcal{G}^1}(k, f, r)$ and $e_x^t \sqsupset \phi_{\mathcal{G}^2}(k, f, r)$. Hence, there exist $g_1, g_2 \in Q_{\mathcal{T}}(e_x^t, r)$ such that $\mathcal{G}_k^1(f \odot g_1) = 0$ and $\mathcal{G}_k^2(f \odot g_2) = 0$ for all $k \in K$. Let $g = (g_1 \odot g_2) \in Q_{\mathcal{T}}(e_x^t, r)$ and $\mathcal{G}_k^1(f \odot g) = 0$ and $\mathcal{G}_k^2(f \odot g) = 0$. Consequently, $(\mathcal{G}^1 \vee \mathcal{G}^2)_k(f \odot g) = 0$ proving that $e_x^t \sqsupset \phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r)$. It contradicts (2.2). Hence,

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) \sqsubseteq \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r). \tag{2.3}$$

From (2.1) and (2.3), we have

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f, r) = \phi_{\mathcal{G}^1}(k, f, r) \bigvee \phi_{\mathcal{G}^2}(k, f, r).$$

Proposition 2.7. *Let* $(X, \mathcal{T})$ *be an* $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topological space and* $\mathcal{G}$ *be an L-fuzzy* (K, E)*-soft grill on X. Then for all* $k \in K$ *and for each* $f, g \in (L^X)^E$, *the following statements hold*:

(1) *If* $f \sqsubseteq g$, *then* $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$.

(2) *If* $\mathcal{G}_k(f) = 0$, *then* $\phi_{\mathcal{G}}(k, f, r) = 0_X$.

(3) $\phi_{\mathcal{G}}(k, f \oplus g, r) = \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r)$.

(4) $\phi_{\mathcal{G}}(k, f \sqcup g, r) = \phi_{\mathcal{G}}(k, f, r) \sqcup \phi_{\mathcal{G}}(k, g, r)$.

(5) $\phi_{\mathcal{G}}(k, f \odot g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r) \odot \phi_{\mathcal{G}}(k, g, r)$.

(6) $\phi_{\mathcal{G}}(k, f \sqcap g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r) \sqcap \phi_{\mathcal{G}}(k, g, r)$.

(7) *If* $\mathcal{G}_k(g) = 0$, *then we have* $\phi_{\mathcal{G}}(k, f \oplus g, r) = \phi_{\mathcal{G}}(k, f, r)$.

(8) $\phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r) = C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r)$

$\sqsubseteq C_{\mathcal{T}_k}(k, f, r)$.

(9) *If* $\mathcal{T}_k(f') \geq r$, *then* $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq f$.

Proof. (1) Suppose that $\phi_{\mathcal{G}}(k, f, r) \not\sqsubseteq \phi_{\mathcal{G}}(k, g, r)$. Then there exists $e_x^t \in (L^X)^E$ such that

$$\phi_{\mathcal{G}}(k, f, r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}}(k, g, r).$$

So, we have $\mathcal{G}_k(f \odot h) \geq r$ for all $h \in Q_{\mathcal{T}_k}(e_x^t, r)$, $k \in K$. Since $f \sqsubseteq g$, by Lemma 1.3(2) and Definition 2.1(LSG3), we have $f \odot h \sqsubseteq g \odot h$, $\mathcal{G}_k(g \odot h) \geq r$ for all $h \in Q_{\mathcal{T}_k}(e_x^t, r)$, $k \in K$. Hence, $e_x^t \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$. It is a contradiction, and hence $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$.

(2) Let $e_x^t \in (L^X)^E$ such that $e_x^t \sqsubseteq \phi_{\mathcal{G}}(k, g, r)$. Then for all $h \in Q_{\mathcal{T}_k}(e_x^t, r)$, $\mathcal{G}_k(f \odot h) \geq r$. But $f \sqsubseteq 1_X$ implies that $f \odot h \sqsubseteq f$. Hence, $\mathcal{G}_k(f) \geq r$. It is a contradiction. Thus, $\phi_{\mathcal{G}}(k, f, r) = 0_X$.

(3) Since $f \sqsubseteq f \oplus g$ and $g \sqsubseteq f \oplus g$, by (1), $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$ and $\phi_{\mathcal{G}}(k, g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$. Hence, we have

$$\phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r). \tag{2.4}$$

Conversely, suppose that

$$\phi_{\mathcal{G}}(k, f \oplus g, r) \not\sqsubseteq \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r).$$

Then there exists $e_x^t \in (L^X)^E$ such that

$$\phi_{\mathcal{G}}(k, f \oplus g, r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r).$$

It implies that there exist $h_1, h_2 \in Q_{\mathcal{T}_k}(e_x^t, r)$ such that $\mathcal{G}_k(f \odot h_1) \not\geq r$ and $\mathcal{G}_k(g \odot h_2) \not\geq r$, and hence $\mathcal{G}_k((f \odot h_1) \oplus (f \odot h_2)) \not\geq r$. Let $h = h_1 \odot h_2 \in Q_{\mathcal{T}_k}(e_x^t, r)$ and $(f \oplus g) \odot h \sqsubseteq ((f \odot h_1) \oplus (g \odot h_2))$. Then $\mathcal{G}_k((f \oplus g) \odot h) \not\geq r$. Thus, $e_x^t \not\sqsubseteq \phi_{\mathcal{G}}(k, f \oplus g, r)$. It is a contradiction. So,

$$\phi_{\mathcal{G}}(k, f \oplus g, r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r). \tag{2.5}$$

From (2.4) and (2.5), the result follows.

(4) Similar to (3).

(5) Obvious by (1).

(6) Similar to (5).

(7) Follows from (2) and (3).

(8) Suppose that $\phi_{\mathcal{G}}(k, f, r) \not\sqsubseteq C_{\mathcal{T}_k}(k, f, r)$. Then for $k \in K$, there exists $e_x^t \in (L^X)^E$ such that

$$\phi_{\mathcal{G}}(k, f, r) \sqsupseteq e_x^t \sqsupset C_{\mathcal{T}_k}(k, f, r).$$

Then there exists $g \in Q_{\mathcal{T}_k}(e_x^t, r)$ such that $f \bar{q} g$ for each $e \in E$, $x \in X$, i.e., $f \odot g = 0_X$ and hence $\mathcal{G}_k(f \odot g) = 0$. It implies that $\phi_{\mathcal{G}}(k, f, r)$

$\not\sqsupseteq e_x^t$ for $h \in K$. It is a contradiction and hence,

$$\phi_{\mathcal{G}}(k, f, r) \sqsubseteq C_{\mathcal{T}_k}(k, f, r). \tag{2.6}$$

Suppose that $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \not\sqsubseteq \phi_{\mathcal{G}}(k, f, r)$. Then for $k \in K$, there exists $e_x^t \in (L^X)^E$ such that

$$C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsupseteq e_x^t \sqsupset \phi_{\mathcal{G}}(k, f, r).$$

Then $\phi_{\mathcal{G}}(k, f, r) q g$ and $g \in Q_{\mathcal{T}_k}(e_x^t, r)$ for $e \in X$. It implies that $e_x^t \sqsubseteq \phi_{\mathcal{G}}(k, f, r)$ and $g \in Q_{\mathcal{T}_k}(e_x^t, r)$. Hence, $\mathcal{G}_k(f \odot g) \geq r$ and $e_x^t \sqsubseteq \phi_{\mathcal{G}}(k, f, r)$. It is a contradiction and hence,

$$C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r). \tag{2.7}$$

Therefore, $C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r)$. From (2.6) and (2.7), we have

$$\phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) \sqsubseteq C_{\mathcal{T}_k}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r) \sqsubseteq C_{\mathcal{T}_k}(k, f, r).$$

(9) Obvious by using (8).

The next example shows that the reverse inclusion in Proposition 2.7(5) is not true, in general.

Example 2.8. Let $X = \{h_1, h_2\}$ with $h_i =$ house for $i \in \{1, 2\}$ and $E = \{e, b\}$ with $e =$ expensive, $b =$ beautiful. Let $([0, 1], \odot, \to, 0, 1)$ be a stsc-quantal as Example 2.4. Let $f_i \in ([0, 1]^X)^E$ for $i \in \{1, 2, 3, 4, 5\}$ as follows:

$$(f_1)_e = (0.5, 0.4), \quad (f_1)_b = (0.0, 0.0),$$

$$(f_2)_e = (0.3, 0.3), \quad (f_2)_b = (0.0, 0.0),$$

$$(f_3)_e = (0.2, 0.3), \quad (f_3)_b = (0.0, 0.0),$$

$$(f_4)_e = (0.7, 0.8), \quad (f_4)_b = (0.0, 0.0),$$

$$(f_5)_e = (0.2, 0.2), \quad (f_5)_b = (0.0, 0.0).$$

Then

$$f_1 \odot 1_X = f_1, \quad f_4 \odot 1_X = f_4,$$

$$f_1 \odot f_4 = f_5, \quad f_5 \odot 1_X = f_5.$$

Let $K = \{k_1, k_2\}$ be given. Define a $([0, 1], \odot, \sqcap)$-fuzzy (K, E)-soft topology and $[0, 1]$-fuzzy (K, E)-soft grill $\mathcal{T}, \mathcal{G} : K \to [0, 1]^{([0,1]^X)^E}$ as follows:

$$\mathcal{T}_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X \text{ or } 0_X, \\ 0.5 & \text{if } g = f_1, \\ 0.6 & \text{if } g = f_2, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathcal{T}_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X \text{ or } 0_X, \\ 0.7 & \text{if } g = f_1, \\ 0.8 & \text{if } g = f_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathcal{G}_{k_1}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.3 & \text{if } f_3 \sqsubseteq f \sqsubset 1_X, \\ 0 & \text{otherwise}, \end{cases} \quad \mathcal{G}_{k_2}(g) = \begin{cases} 1 & \text{if } g = 1_X, \\ 0.4 & \text{if } f_3 \sqsubseteq f \sqsubset 1_X, \\ 0 & \text{otherwise}. \end{cases}$$

Since $1_X \in Q_{\mathcal{T}_{k_1}}(e_{h_2}^{0.4}, r)$ for all $r \in [0, 1]$, and $\mathcal{G}_{k_1}(f_1) = \mathcal{G}_{k_1}(f_4) = 0.3$, $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_1, f_1, 0.3)$ and $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_1, f_4, 0.3)$ but $e_{h_2}^{0.4} \not\sqsubseteq \phi_{\mathcal{G}}(k_1, f_1 \odot f_4, 0.3)$ because $\mathcal{G}_{k_1}(f_5) = 0$. Also, because $1_X \in Q_{\mathcal{T}_{k_2}} \cdot (e_{h_2}^{0.4}, r)$ for all $r \in [0, 1]$, and $\mathcal{G}_{k_2}(f_1) = \mathcal{G}_{k_2}(f_4) = 0.4$, then $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_2, f_1, 0.4)$ and $e_{h_2}^{0.4} \sqsubseteq \phi_{\mathcal{G}}(k_2, f_4, 0.4)$ but $e_{h_2}^{0.4} \not\sqsubseteq \phi_{\mathcal{G}}(k_2, f_1 \odot f_4, 0.4)$ because $\mathcal{G}_{k_2}(f_5) = 0$. Hence,

$$\phi_{\mathcal{G}}(k, f_1 \odot f_4, r) \not\sqsupseteq \phi_{\mathcal{G}}(k, f_1, r) \odot \phi_{\mathcal{G}}(k, f_4, r) \text{ for all } k \in K.$$

Lemma 2.9. *Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space and $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill on X. Then for $f, g \in (L^X)^E$ and for all $k \in K$,*

$$\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))' = \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$$

Proof. Let $f, g \in (L^X)^E$ and $f = (f \odot g') \oplus (f \odot g)$. Then by Proposition 2.7, we have

$$\begin{aligned}\phi_{\mathcal{G}}(k, f, r) &= \phi_{\mathcal{G}}(k, (f \odot g') \oplus (f \odot g), r)\\ &= \phi_{\mathcal{G}}(k, f \odot g', r) \oplus \phi_{\mathcal{G}}(k, f \odot g, r)\\ &\sqsubseteq \phi_{\mathcal{G}}(k, f \odot g', r) \oplus \phi_{\mathcal{G}}(k, g, r).\end{aligned}$$

Thus,

$$\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))' \sqsubseteq \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$$

Again $\phi_{\mathcal{G}}(k, f \odot g', r) \sqsubseteq \phi_{\mathcal{G}}(k, f, r)$. Hence,

$$\phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))' \sqsubseteq \phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$$

Therefore,

$$\phi_{\mathcal{G}}(k, f, r) \odot (\phi_{\mathcal{G}}(k, g, r))' = \phi_{\mathcal{G}}(k, f \odot g', r) \odot (\phi_{\mathcal{G}}(k, g, r))'.$$

Theorem 2.10. *Let $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill over an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space $(X, \mathcal{T})$. Then the following statements are equivalent for $f, g \in (L^X)^E$ and for all $k \in K$:*

(1) $f \odot \phi_{\mathcal{G}}(k, f, r) = 0_X \Rightarrow \phi_{\mathcal{G}}(k, f, r) = 0_X$.

(2) $\phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r))', r) = 0_X$.

(3) $\phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r)$.

Proof. (1) $\Rightarrow$ (2) Let $f \in (L^X)^E$ and $k \in K$. Then

$$(f \odot (\phi_{\mathcal{G}}(k, f, r))') \odot \phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r))', r) = 0_X,$$

and so, $\phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r))', r) = 0_X$.

(2) $\Rightarrow$ (3) Let $f \in (L^X)^E$ and

$$f = (f \odot (f \odot \phi_{\mathcal{G}}(k, f, r))') \oplus (f \odot \phi_{\mathcal{G}}(k, f, r)).$$

Then for all $k \in K$,

$$\begin{aligned}\phi_{\mathcal{G}}(k, f, r) &= \phi_{\mathcal{G}}(k, f \odot (f \odot \phi_{\mathcal{G}}(k, f, r))', r) \oplus \phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r)\\ &= \phi_{\mathcal{G}}(k, f \odot (\phi_{\mathcal{G}}(k, f, r))', r) \oplus \phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r)\\ &= \phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r).\end{aligned}$$

(3) $\Rightarrow$ (1) Suppose that $f \odot \phi_{\mathcal{G}}(k, f, r) = 0_X$ for all $k \in K$ and $f \in (L^X)^E$. Then we have

$$\phi_{\mathcal{G}}(k, f, r) = \phi_{\mathcal{G}}(k, f \odot \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, 0_X, r) = 0_X.$$

Definition 2.11. Let $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill on X. We define an L-fuzzy soft operator $\psi : K \times (L^X)^E \times L_0 \to (L^X)^E$ by $\psi(k, f, r) = f \oplus \phi_{\mathcal{G}}(k, f, r)$ for every $f \in (L^X)^E$ and $k \in K$.

Theorem 2.12. *For* $f, g \in (L^X)^E$ *and* $k \in K$, *the mapping* ψ *satisfies the following*:

(1) $f \sqsubseteq \psi(k, f, r)$.

(2) $\psi(k, 0_X, r) = 0_X$.

(3) $\psi(k, f \oplus g, r) = \psi(k, f, r) \oplus \psi(k, g, r)$.

(4) $\psi(k, f \sqcup g, r) = \psi(k, f, r) \sqcup \psi(k, g, r)$.

(5) $\psi(k, \psi(k, f, r), r) = \psi(k, f, r)$.

Proof. Let $f, g \in (L^X)^E$, $r \in L_0$. Hence, in view of Proposition 2.7 (2), (3), (8) for all $k \in K$, we have:

(1) Obvious.

(2) Suppose that $\mathcal{G}_k(0_X) = 0$. Then $\phi_{\mathcal{G}}(k, 0_X, r) = 0_X$. It implies that $\psi(k, 0_X, r) = 0_X$.

(3)

$$\begin{aligned}\psi(k, f \oplus g, r) &= (f \oplus g) \oplus \phi_{\mathcal{G}}(k, f \oplus g, r)\\ &= (f \oplus g) \oplus (\phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, g, r))\\ &= (f \oplus \phi_{\mathcal{G}}(k, f, r)) \oplus (g \oplus \phi_{\mathcal{G}}(k, g, r))\\ &= \psi(k, f, r) \oplus \psi(k, g, r).\end{aligned}$$

(4) Similar to that of (3).

(5)

$$\begin{aligned}&\psi(k, \psi(k, f, r), r)\\ &= \psi(k, f \oplus \phi_{\mathcal{G}}(k, f, r))\\ &= (f \oplus \phi_{\mathcal{G}}(k, f, r)) \oplus (\phi_{\mathcal{G}}(k, f \oplus \phi_{\mathcal{G}}(k, g, r), r))\\ &= (f \oplus \phi_{\mathcal{G}}(k, f, r)) \oplus (\phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r))\\ &= (f \oplus \phi_{\mathcal{G}}(k, f, r)) \oplus \phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r)\\ &= f \oplus \phi_{\mathcal{G}}(k, f, r) = \psi(k, f, r).\end{aligned}$$

Definition 2.13. For any L-fuzzy (K, E)-soft grill on an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space $(X, \mathcal{T})$, there exists a unique

$(L, \odot, \sqcap)$-fuzzy (K, E)-soft topology $\mathcal{T}_{\mathcal{G}}$ on X given by $(\mathcal{T}_{\mathcal{G}})_k(k) = \bigvee\{r : \psi(k, f', r) = f'\}$ such that $\psi(k, f, r) = C_{\mathcal{T}_{\mathcal{G}}}(k, f, r)$ for every $f \in (L^X)^E$ and $k \in K$.

Theorem 2.14. *Let* $(X, \mathcal{T})$ *be an* $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topological space. Then* $\mathcal{T}_k(k) \leq (\mathcal{T}_{\mathcal{G}})_k(f)$ *for any L-fuzzy* (K, E)*-soft grill* $\mathcal{G}$ *on X,* $f \in (L^X)^E$ *and* $k \in K$.

Proof. Suppose that $\mathcal{T}_k(f) \not\leq (\mathcal{T}_{\mathcal{G}})_k(f)$ for all $f \in (L^X)^E$ and $k \in K$. Then there exists $r \in L$ such that

$$\mathcal{T}_k(f) \geq r > (\mathcal{T}_{\mathcal{G}})_k(f) \text{ for all } k \in K. \tag{2.8}$$

Since $\mathcal{T}_k(f) \geq r$ for all $k \in K$, $C_{\mathcal{T}}(k, f', r) = f'$ for all $k \in K$. By Proposition 2.7, it follows that $\phi_G(k, f', r) \sqsubseteq f'$. Hence, $C_{\mathcal{T}_{\mathcal{G}}}(k, f', r) = \psi(k, f', r) = f'$ and so $(\mathcal{T}_{\mathcal{G}})_k(f) \geq r$. It contradicts (2.8). Hence, $\mathcal{T}_k(f) \leq (\mathcal{T}_{\mathcal{G}})_k(f)$.

Theorem 2.15. *Let* $(X, \mathcal{T})$ *be an* $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topological space and* $\mathcal{G}$ *be an L-fuzzy* (K, E)*-soft grill on X. Then for each* $f \in (L^X)^E$ *and for all* $k \in K$*, the following statements hold*:

(1) $\mathcal{T}_{\mathcal{G}}(f') \geq r$ *if and only if* $\phi_{\mathcal{G}}(k, f, r) \sqsubseteq f$.

(2) *If* $\mathcal{G}(f) = 0$*, then* $\mathcal{T}_{\mathcal{G}}(f') \geq r$.

(3) $\mathcal{T}_{\mathcal{G}}((\phi_{\mathcal{G}}(k, f, r))') \geq r$.

Proof. (1) By Definition 2.13, it is obvious.

(2) Let $\mathcal{G}(f') = 0$. Then by Proposition 2.7(2), we have $\phi_{\mathcal{G}}(k, f, r) = 0_X$ and so

$$C_{\mathcal{T}_{\mathcal{G}}}(k, f, r) = \psi(k, f, r) = f \oplus \phi_{\mathcal{G}}(k, f, r) = f \text{ for all } k \in K.$$

Hence, $\mathcal{T}_{\mathcal{G}}(f') \geq r$.

(3) Let $f \in (L^X)^E$ and $k \in K$. Then from the definition of $\psi_{\mathcal{G}}$ and by Proposition 2.7(8), we have

$$\psi(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r) \oplus \phi_{\mathcal{G}}(k, \phi_{\mathcal{G}}(k, f, r), r) = \phi_{\mathcal{G}}(k, f, r).$$

Consequently, $\mathcal{T}_{\mathcal{G}}((\phi_{\mathcal{G}}(k, f, r))') \geq r$.

Theorem 2.16. *Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space and $\mathcal{G}^1, \mathcal{G}^2$ be two L-fuzzy (K, E)-soft grills on X such that $\mathcal{G}^1_k(f) \leq \mathcal{G}^2_k(f)$ for all $k \in K$ and for each $f \in (L^X)^E$. Then the following statements hold*:

(1) $(\mathcal{T}_{\mathcal{G}^2})_k(f) \leq (\mathcal{T}_{\mathcal{G}^1})_k(f)$.

(2) *If $\mathcal{T}_k = (\mathcal{T}_{\mathcal{G}^1})_k$, then $(\mathcal{T}_{\mathcal{G}^1})_k = (\mathcal{T}_{\mathcal{G}^2})_k$.*

Proof. (1) Suppose that $(\mathcal{T}_{\mathcal{G}^2})_k \not\leq (\mathcal{T}_{\mathcal{G}^1})_k$ for all $f \in (L^X)^E$. Then there exists $r \in L$ such that

$$(\mathcal{T}_{\mathcal{G}^2})_k(f) \geq r > (\mathcal{T}_{\mathcal{G}^1})_k(f) \text{ for all } k \in K. \tag{2.9}$$

Since $(\mathcal{T}_{\mathcal{G}^2})_k(f) \geq r$ for all $k \in K$,

$$C_{\mathcal{T}_{\mathcal{G}^2}}(k, f', r) = \psi_{\mathcal{G}^2}(k, f', r)$$

$$= f' \oplus \phi_{\mathcal{G}^2}(k, f', r) = f'$$

$$\Rightarrow \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.$$

Since $\mathcal{G}_k^1(f) \leq \mathcal{G}_k^2(f)$, by Proposition 2.6, we have

$$\phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.$$

Thus, $C_{\mathcal{T}_{\mathcal{G}^1}}(k, f', r) = f'$. It implies that $(\mathcal{T}_{\mathcal{G}^1})_k(f) \geq r$. It contradicts (2.9). Hence, $(\mathcal{T}_{\mathcal{G}^2})_k(f) \leq (\mathcal{T}_{\mathcal{G}^1})_k(f)$.

(2) Follows from Theorem 2.14 and (1) of this theorem.

Corollary 2.17. *Let* $(X, \mathcal{T})$ *be an* $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topological space and* $\mathcal{G}^1, \mathcal{G}^2$ *be two L-fuzzy* (K, E)*-soft grills on X. Then for all* $k \in K$ *and for each* $f \in (L^X)^E$, $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) = (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f)$.

Proof. By Theorem 2.16, we have $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \leq (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f)$. So, we only prove that $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \geq (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f)$. Suppose that

$$(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \ngeq (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f).$$

Then there exists $r \in L$ such that

$$(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) < r \leq (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f).$$

Note that $r \leq (\mathcal{T}_{\mathcal{G}^1})_k(f) \bigvee (\mathcal{T}_{\mathcal{G}^2})_k(f)$ implies that $(\mathcal{T}_{\mathcal{G}^1})_k(f) \geq r$ or $(\mathcal{T}_{\mathcal{G}^2})_k(f) \geq r$. By Theorem 2.15, we have

$$\phi_{\mathcal{G}^1}(k, f', r) \sqsubseteq f', \quad \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.$$

Also, by Proposition 2.6, we have

$$\phi_{\mathcal{G}^1 \vee \mathcal{G}^2}(k, f', r) = \phi_{\mathcal{G}^1}(k, f', r) \bigvee \phi_{\mathcal{G}^2}(k, f', r) \sqsubseteq f'.$$

Consequently, $(\mathcal{T}_{\mathcal{G}^1 \vee \mathcal{G}^2})_k(f) \geq r$. It is a contradiction.

Definition 2.18. Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space. Define a mapping $\mathcal{P} : K \to L^{(L^X)^E}$ by

$$\mathcal{P}_k^f(f) = \begin{cases} 1, & \text{if } fgq, \forall f, g \in (L^X)^E \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}$ is an L-fuzzy (K, E)-soft grill on X. We call this L-fuzzy (K, E)-soft grill the *L-fuzzy (K, E)-soft principle grill* generated by an L-fuzzy soft set f.

Lemma 2.19. *Let $(X, \mathcal{T})$ be an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space and $f, g \in (L^X)^E$. Then*

(1) *If $f \sqsubseteq g$, then $\mathcal{P}_k^f(h) \leq \mathcal{P}_k^g(h)$ for all $k \in K$, $h \in (L^X)^E$.*

(2) *If $\mathcal{G} = \mathcal{P}^f$, then $\phi_{\mathcal{P}^f}(k, f, r) = C_{\mathcal{T}}(k, f, r)$.*

(3) *If $f \sqsubseteq g$ and $\mathcal{T}_k = (\mathcal{T}_{\mathcal{P}^f})_k$, then $(\mathcal{T}_{\mathcal{P}^f})_k = (\mathcal{T}_{\mathcal{P}^g})_k$.*

Proof. Obvious.

Theorem 2.20. *Let $\mathcal{G}$ be an L-fuzzy (K, E)-soft grill on an $(L, \odot, \sqcap)$-fuzzy (K, E)-soft topological space $(X, \mathcal{T})$. Define a mapping $\mathcal{B}(\mathcal{G}, \mathcal{T})$: $K \to L^{(L^X)^E}$ as follows*:

$$(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) = \bigvee\{r : f = g \odot h', \mathcal{T}_k(g) \geq r, \mathcal{G}_k(h) = 0\}.$$

Then $\mathcal{B}(\mathcal{G}, \mathcal{T})$ is an $(L, \odot, \sqcap)$-fuzzy-(K, E)-soft base on X.

Proof. (LSB1) Since $1_X = 1_X \odot 0'_X$ and $0_X = 0_X \odot 0'_X$ with $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$ and $\mathcal{G}_k(0_X) = 0$ for $k \in K$, we have $(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(0_X) = (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(1_X) = 1$ for all $r \in L$.

(LSB2) Suppose that

$$(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f \odot g) \ngeq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(g)$$

for all $f, g \in (L^X)^E$ and $k \in K$. Then there exists $r \in L$ such that

$$(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f \odot g) < r \leq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(g).$$

Since $r \leq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(g)$, there exist $f_1, g_1, h_1, h_2 \in (L^X)^E$ such that $\mathcal{T}_k(f_1) \geq r$, $\mathcal{T}_k(g_1) \geq r$ and $\mathcal{G}_k(h_1) = \mathcal{G}_k(h_2) = 0$, where $f = f_1 \odot h_1'$ and $g = g_1 \odot h_2'$. It implies that $\mathcal{T}_k(f_1 \odot g_1) \geq r$ and $\mathcal{G}_k(h_1 \oplus h_2) = 0$. Since

$$\begin{aligned} f \odot g &= (f_1 \odot h_1') \odot (g_1 \odot h_2') \\ &= (f_1 \odot g_1) \odot (h_1 \oplus h_2)'. \end{aligned}$$

We have $(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f \odot g) \geq r$. It is a contradiction. Hence,

$$(\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f \odot g) \geq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) \odot (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(g).$$

(LSB3) Similar to (LSB2).

Corollary 2.21. *For any L-fuzzy* (K, E)*-soft grill* $\mathcal{G}$ *on an* $(L, \odot, \sqcap)$*-fuzzy* (K, E)*-soft topological space* $(X, \mathcal{T})$, *we have* $\mathcal{T}_k(f) \leq (\mathcal{B}(\mathcal{G}, \mathcal{T}))_k(f) \leq (\mathcal{T}_{\mathcal{G}})_k(f)$ *for each* $f \in (L^X)^E$, $k \in K$.

Proof. Obvious.

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