Universal Journal of Mathematics and Mathematical Sciences

http://www.pphmj.com
http://dx.doi.org/10.17654/UM014010013
Volume 14, Number 1, 2021, Pages 13-28 **ISSN: 2277-1417**

DELTA GENERALIZED CLOSED AND GENERALIZED DELTA CLOSED SETS IN BITOPOLOGICAL SPACES

F. H. Khedr and O. R. Sayed

Department of Mathematics
Faculty of Sciences
University of Assiut
Assiut 715161, Egypt
e-mail: khedrfathi@gmail.com
o_r_sayed@yahoo.com
o_sayed@aun.edu.eg

Abstract

In this paper, we introduce the concepts of delta generalized closed, generalized delta closed and delta star generalized closed sets in bitopological spaces and study some of its properties and its relations with other types of closed sets. Also we study some properties of weak forms of separation axioms.

1. Introduction

Throughout this paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) mean bitopological spaces (or simply spaces) on which no separation axiom is assumed unless explicitly stated. Also, $i, j = 1, 2,$ and $i \neq j$. Let A be a

Received: January 24, 2021; Accepted: February 28, 2021

2020 Mathematics Subject Classification: 54A10, 54C05, 54E55.

Keywords and phrases: bitopological space, $ij-\delta g$-closed set, $ij-g\delta$-closed set, $ij-\delta g^*$-closed set, $ij-T_{\frac{3}{4}}$-space, pairwise $T_{\frac{1}{2}}$ space, pairwise almost weakly Hausdorff space, pairwise T_δ space.

subset of a space (X, τ_1, τ_2). Then the *closure* of A and the *interior* of A in the topological space (X, τ_i) are denoted by $i - Cl(A)$ and $i - Int(A)$, respectively. We write *i-open* (resp. *i-closed*) set to mean that the set is open (resp. closed) in the topological space (X, τ_i). A subset A of a space (X, τ_1, τ_2) is said to be *ij-regular open* (resp. *ij-regular closed*) [1] if $A = i - Int(j - Cl(A))$ (resp. $A = i - Cl(j - Int(A))$). The collection of all *ij*-regular open sets form a base for a topology τ_i^* coarser than τ_i [5]. The bitopological space (X, τ_1^*, τ_2^*) is called the *semi-regularization* of (X, τ_1, τ_2). If $\tau_i^* = \tau_i$, then (X, τ_1, τ_2) is said to be *pairwise semi-regular*. The $ij - \delta$*-interior* [5] of a subset A of a space X is the union of all *ij*-regular open sets of X contained in A and is denoted by $ij - \delta - Int(A)$. The subset A of X is called $ij - \delta$*-open* if $A = ij - \delta - Int(A)$, i.e., a set is $ij - \delta$-open if it is the union of *ij*-regular open sets. The complement of an $ij - \delta$-open set is called $ij - \delta$*-closed*. A point $p \in X$ is in the $ij - \delta$-closure of A [1] if $i - Int(j - Cl(U)) \cap A \neq \emptyset$ for every $U \in \tau_i$ and $p \in U$. The set of all $ij - \delta$-closure points of A is denoted by $ij - \delta - Cl(A)$. Obviously, A is $ij - \delta$-closed if and only if $A = ij - \delta - Cl(A)$. The family of all $ij - \delta$-open sets forms a topology on X denoted by τ_i^δ [5]. It is well known that $\tau_i^* = \tau_i^\delta$ [5]. A subset A of X is called *ij-semiopen* [2] (resp. $ij - \alpha$*-open* [4], *ij-preopen* [4]) if $A \subset j - Cl(i - Int(A))$ (resp. $A \subset i - Int(j - Cl(i - Int(A)))$, $A \subset i - Int(j - Cl(A))$. The complement of an *ij*-semiopen (resp. $ij - \alpha$-open, *ij*-preopen) set is called *ij-semiclosed* (resp. $ij - \alpha$*-closed*, *ij- preclosed*).

2. $ij - \delta g$ -closed Sets

In this section, we introduce the notion of $ij - \delta g$-closed sets and study some of its properties and its relations with other sets.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is called:

1. $ij-g$*-closed* [3], if $j-Cl(A) \subset U$ whenever $A \subset U$ and U is i-open in X.

2. $ij-sg$*-closed* [6], if $ji-sCl(A) \subset U$ whenever $A \subset U$ and U is ij-semiopen in X.

3. $ij-g\alpha$*-closed* [6] if $ji-\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is $ij-\alpha$-open in X.

4. $ij-rg$*-closed* if $j-Cl(A) \subset U$ whenever $A \subset U$ and U is ij-regular open in X.

Definition 2.2. A subset A of a bitopological space (X, τ_1, τ_2) is called $ij-\delta$*-generalized closed* (briefly $ij-\delta g$*-closed*) if $ji-\delta-Cl(A) \subset U$ whenever $A \subset U$ and U is i-open in X.

Theorem 2.3. *In a bitopological space* (X, τ_1, τ_2), *the following hold*:

(1) *Every* $ji-\delta$*-closed set is* $ij-\delta g$*-closed*.

(2) *Every* $ij-\delta g$*-closed set is* $ij-g$*-closed in* (X, τ_1^*, τ_2^*).

(3) *Every* $ij-\delta g$*-closed set is* $ij-g$*-closed in* (X, τ_1, τ_2) *and hence* $ij-\alpha g$*-closed*, $ij-gs$*-closed and* $ij-rg$*-closed*.

Remark 2.4. The converses of implications in Theorem 2.3 are not true in general.

Example 2.5. Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{e\}, \{a, b\}, \{c, d\}, \{a, e\}, \{a, c, d\}, \{a, b, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}\}$ and $\tau_2 = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{b, d\}, \{a, c\}, \{a, b, c\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}\}$. Then $\{b, c, e\}$ is $12-\delta g$-closed but not $21-\delta$-closed. Also, $\{d, e\}$ and $\{a, d\}$ are $12-g$-closed but not $12-\delta g$-closed.

Recall that (X, τ_1, τ_2) is pairwise $T_{\frac{1}{2}}$ if each singleton is either *j*-open or *i*-closed [3].

Definition 2.6. A bitopological space (X, τ_1, τ_2) is called *pairwise almost weakly Hausdorff* if (X, τ_1^*, τ_2^*) is pairwise $T_{\frac{1}{2}}$.

Theorem 2.7. *Let A be a subset of a pairwise semi-regular bitopological space* (X, τ_1, τ_2). *Then*

(1) *A is* $ij-\delta g$*-closed if and only if A is* $ij-g$*-closed*.

(2) *If, in addition, X is pairwise* $T_{\frac{1}{2}}$, *then A is* $ij-\delta g$*-closed if and only if A is j-closed*.

Theorem 2.8. *In an ij-almost weakly Hausdorff space* (X, τ_1, τ_2), *the* $ij-g$*-closed sets of* (X, τ_1^*, τ_2^*) *are* $ji-\delta$*-closed in* (X, τ_1, τ_2), *and thus are* $ij-\delta g$*-closed in* (X, τ_1, τ_2).

Proof. Let $A \subset X$ be an $ij-g$-closed subset of (X, τ_1^*, τ_2^*). Let $x \in ji-\delta-Cl(A)$. If $\{x\}$ is $ji-\delta$-open, then $x \in A$. If not, then $X/\{x\}$ is $ij-\delta$-open, since X is *ij*-almost weakly Hausdorff. Assume that $x \notin A$. Since A is $ij-g$-closed in (X, τ_1^*, τ_2^*), $ji-\delta-Cl(A) \subset X \setminus \{x\}$, i.e., $x \notin ji-\delta-Cl(A)$, a contradiction. Then $x \in A$. Thus $ji-\delta-Cl(A) = A$ or equivalently *A* is $ji-\delta$-closed and hence $ij-\delta g$-closed in (X, τ_1, τ_2).

Definition 2.9 [10]**.** A bitopological space (X, τ_1, τ_2) is called a *pairwise* R_1 *space* if for every two distinct points, $x, y \in X$ such that $i-C(\{x\}) \neq j-Cl(\{y\})$, there exist an *i*-open set *U* and a *j*-open set *V* such that $x \in V$, $y \in U$ and $U \cap V = \emptyset$.

Theorem 2.10. *For an i-compact subset of a pairwise* R_1 *space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *A is an* $ij - \delta g$ *-closed set.*

(2) *A is an* $ij - g$ *-closed set.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Note that in a pairwise R_1 space, the concepts of *j*-closure and $ji - \delta$-closure coincide for *i*-compact spaces: see Theorem 3.9 from [8]. The rest of the proof is obvious.

Definition 2.11. A bitopological space (X, τ_1, τ_2) is called *pairwise partition* if every *i*-open set is *j*-closed.

Theorem 2.12. *A bitopological space* (X, τ_1, τ_2) *is a pairwise partition space if and only if every subset of X is* $ij - \delta g$ *-closed.*

Proof. Let *X* be a pairwise partition space and $A \subset X$ be any subset. Let $A \subset U$, where *U* is *i*-open in *X*. Then *U* is *ji*-clopen. Thus $ji - \delta - Cl(A) \subset ji - \delta - Cl(U) = U$. Conversely, let $U \subset X$ be an *i*-open set. Then $ji - \delta - Cl(U) \subset U$, since every subset of *X* is $ij - \delta g$ -closed. Therefore, *U* is $ji - \delta$-closed and hence *j*-closed.

Theorem 2.13. *Let* (X, τ_1, τ_2) *be a bitopological space. Then*

(1) *Finite union of* $ij - \delta g$ *-closed sets is always* $ij - \delta g$ *-closed.*

(2) *Countable union of* $ij - \delta g$ *-closed sets need not be* $ij - \delta g$ *-closed.*

(3) *Finite intersection of* $ij - \delta g$ *-closed sets may fail to be* $ij - \delta g$ *-closed.*

Proof. (1) Let $A, B \subset X$ be $ij - \delta g$ -closed and $A \cup B \subset U$, where *U* is an *i*-open set. Then $ji - \delta - Cl(A \cup B) \subset ji - \delta - Cl(A) \cup ji - \delta - Cl(B) \subset U$. This shows that $A \cup B$ is $ij - \delta g$ -closed.

(2) Let $X = \mathbb{R}$, $\tau_1 = \tau_2 =$ the usual topology on $\mathbb{R}$. Then *X* is pairwise semi-regular, and by Theorem 2.7 every singleton in *X* is $ij - \delta g$ -closed. Let

$\mathbb{N}$ be the set of all positive integers, set $A = \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \right\}$. Clearly, A is a countable union of $ij - \delta g$-closed sets but A is not $ij - \delta g$-closed, since $A \subset (0, \infty)$ but $0 \in ji - \delta - Cl(A)$.

(3) See the following example.

Example 2.14. Consider (X, τ_1, τ_2) as in Example 2.5. Let $A = \{a, b, c, e\}$ and $B = \{a, b, d, e\}$. Then A and B are $12 - \delta g$-closed but $A \bigcap B = \{a, b, e\}$ is not $12 - \delta g$-closed.

Theorem 2.15. *For a subset A of a bitopological space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *A is ij-clopen*.

(2) *A is* $ij - \delta g$ *-closed*, *ij-preopen and ji-semiclosed*.

(3) *A is* $ij - \alpha g$ *-closed and ij-regular open*.

(4) *A is* $ij - \alpha g$ *-closed and ij-regular open*.

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Since A is $ij - \alpha g$-closed, $ji - \alpha Cl(A) \subset A$ and thus A is $ji - \alpha$-closed. Therefore, $j - Cl(i - Int(j - Cl(A))) \subset A.$ Since A is *ij*-regular open, $j - Cl(A) \subset A,$ and so A is *j*-closed. Thus A is *ji*-clopen.

3. $ij - T_{\frac{3}{4}}$ -space

Definition 3.1. A bitopological space (X, τ_1, τ_2) is called an $ij - T_{\frac{3}{4}}$ *space* if every $ij - \delta g$-closed subset of X is $ji - \delta$-closed.

Lemma 3.2. *Let A be an* $ij - \delta g$ *-closed set of a bitopological space* (X, τ_1, τ_2). *Then* $ji - \delta - Cl(A) \backslash A$ *does not contain any nonempty i-closed set*.

Proof. Assume that F is an i-closed set of $ji - \delta - Cl(A)\backslash A$. Then clearly $A \subset X\backslash F$, where A is $ij - \delta g$-closed and $X\backslash F$ is i-open. Thus $ji - \delta - Cl(A) \subset X\backslash F$ or equivalently $F \subset X\backslash ji - \delta - Cl(A)$. Since by assumption $F \subset ji - \delta - Cl(A)$, F must be the empty set.

Lemma 3.3. *In any bitopological space* (X, τ_1, τ_2) *a singleton* $\{x\}$ *is* $ij - \delta$*-open if and only if it is ij-regular open.*

Theorem 3.4. *For a bitopological space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *X is pairwise* $ij - T_{\frac{3}{4}}$*-space.*

(2) *Every singleton set* $\{x\}$ *is* $ji - \delta$*-open or i-closed.*

(3) *Every singleton set* $\{x\}$ *is ji-regular open or i-closed.*

Proof. (1) $\Rightarrow$ (2) If $\{x\}$ is not i-closed, then $X\backslash\{x\}$ is not i-open and thus $ij - \delta g$-closed. By (1), $X\backslash\{x\}$ is $ji - \delta$-closed, i.e., $\{x\}$ is $ji - \delta$-open.

(2) $\Rightarrow$ (1) Let $A \subset X$ be $ij - \delta g$-closed. Let $x \in ji - \delta - Cl(A)$. We consider two cases:

Case 1. Let $\{x\}$ be $ji - \delta$-open. Since x belongs to the $ji - \delta$-closure of A, $\{x\} \cap A \neq \emptyset$. This shows that $x \in A$.

Case 2. Let $\{x\}$ be i-closed. If we assume that $x \notin A$, then we would have $x \in ji - \delta - Cl(A)\backslash A$ which cannot happen according to Lemma 3.2. Hence $x \in A$.

So in both the cases, we have $ji - \delta - Cl(A) \subset A$ and so A is $ij - \delta$-closed.

(2) $\Rightarrow$ (3) Follows from Lemma 3.3.

Corollary 3.5. *Every pairwise* T_1 *space is pairwise* $T_{\frac{3}{4}}$.

Corollary 3.6. *Every pairwise* $T_{\frac{3}{4}}$ *space is pairwise* $T_{\frac{1}{2}}$.

Definition 3.7 [6]**.** A bitopological space (X, τ_1, τ_2) is called *pairwise semi* T_1 if every singleton set is pairwise semiclosed.

Theorem 3.8. *Every pairwise* $T_{\frac{3}{4}}$ *bitopological space is pairwise semi* T_1.

Proof. Every *i*-closed set and every *ji*-regular open set are *ji*-semiclosed.

Lemma 3.9. *In a bitopological space* (X, τ_1, τ_2), *every singleton set* $\{x\}$ *is either ij-nowhere dense or ij-preopen*.

Lemma 3.10. *For a bitopological space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *Each ij-preopen singleton is ij-regular open*.

(2) *X is a pairwise semi* T_1 *space*.

Proof. (1) $\Rightarrow$ (2) Every singleton is either *ij*-nowhere dense or *ij*-preopen. In the first case (2) is clear, since *ij*-nowhere dense subsets are *ji*-semi closed. In the second case, the singleton, by (1), is *ij*-regular open and hence *ji*-semi closed. Thus *X* is a pairwise semi T_1 space.

(2) $\Rightarrow$ (1) Let $x \in X$ be such that $\{x\}$ is *ij*-preopen. Since X is pairwise semi T_1, $\{x\}$ is *ji*-semi closed and so $i = Int(j - Cl(\{x\}) \subset \{x\}$ $\subset i - Int(j - Cl(\{x\})))$ Hence, $\{x\}$ is *ij*-regular open.

Theorem 3.11. *For a bitopological space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *X is pairwise* $T_{\frac{3}{4}}$.

(2) *X is pairwise* $T_{\frac{1}{2}}$ *and pairwise semi* T_1.

Proof. (1) $\Rightarrow$ (2) follows from Corollary 3.6 and Theorem 3.8.

(2) $\Rightarrow$ (1) Since X is pairwise $T_{\frac{1}{2}}$, every singleton is *j*-open or *i*-closed. Since *j*-open *ij*-semi closed sets are *ji*-regular open sets, all *j*-open singletons are *ji*-regular open. Thus X is pairwise $T_{\frac{3}{4}}$.

4. $ij - g\delta$-closed Sets

Definition 4.1. A subset A of a bitopological space (X, τ_1, τ_2) is called *ij-generalized* δ*-closed* (briefly $ij - g\delta$*-closed*) if $j - Cl(A) \subset U$ whenever $A \subset U$ and U is $ij - \delta$-open in X.

A family $\{A_k\}_{k \in I}$ of subsets of a bitopological space (X, τ_1, τ_2) is called *i-locally finite* if for each point $x \in X$, there exists an *i*-open set U containing x such that $\{k \in I : U \cap A_k \neq \emptyset\}$ is a finite set. The family is called *pairwise locally finite* if it is 1-locally finite and 2-locally finite.

Lemma 4.2. *For an i-locally finite family* $\{A_k\}_{k \in I}$ *of subsets of a bitopological space* (X, τ_1, τ_2),

$$i - Cl\left(\bigcup_{k \in I} A_k\right) = \bigcup_{k \in I} i - Cl(A_k).$$

Theorem 4.3. *Let* $\{A_k\}_{k \in I}$ *be an i-locally finite family of* $ij - g\delta$*-closed sets. Then* $A = \bigcup_{k \in I} A_k$ *is also* $ij - g\delta$*-closed.*

Theorem 4.4. *Every* $ij - g$*-closed set is* $ij - g\delta$*-closed. But not conversely.*

Proof. Follows directly since every $ij - \delta$-open is *i*-open. For the converse consider the following example.

Example 4.5. Let (X, τ_1, τ_2) as in Example 2.5. Then $\{a, b, c, d\}$ is $12 - g\delta$-closed but not $12 - g$-closed.

Theorem 4.6. *A subset A of a pairwise semi-regular bitopological space is* $ij-g$*-closed if and only if it is* $ij-g\delta$*-closed.*

Proof. Follows from the fact that in a pairwise semi-regular space, we have $\tau_i = \tau_i^{\delta}$.

From the above theorem and the definition of a pairwise $T_{\frac{1}{2}}$ space, we can see that in a pairwise semi-regular pairwise $T_{\frac{1}{2}}$ space, the concepts of $ij-g\delta$-closed, $ij-g$-closed and *j*-closed sets coincide.

Lemma 4.7. *Let* (X, τ_1, τ_2) *be a bitopological space. If* $A \subset X$ *is* $ij-g\delta$*-closed and* $A \subset B \subset j-Cl(A)$, *then B is also* $ij-g\delta$*-closed.*

Theorem 4.8. *Let* (X, τ_1, τ_2) *be a bitopological space.*

(a) *If* $A \subset X$ *is* $ij-g\delta$*-closed, then* $j-Cl(A)\backslash A$ *does not contain any nonempty* $ij-\delta$*-closed set.*

(b) *If* $A \subset X$ *is* $ij-g\delta$*-closed and* $A \subset B \subset j-Cl(A)$, *then* $j-Cl(B)\backslash B$ *does not contain any nonempty* $ij-\delta$*-closed set.*

Theorem 4.9. *Suppose that* $B \subset A \subset X$ *and B is* $ij-g\delta$*-closed in X. Then A is* $ij-g\delta$*-closed relative to Y provided Y is i-open or j-dense in X.*

Proof. Let $A \subset Y \cap G$, where *G* is $ij-\delta$-open in *X*. Then $A \subset G$. Since *A* is $ij-g\delta$-closed, $j-Cl(A) \subset G$. This implies that $j-Cl(A) \cap Y = j-Cl_Y(A) \subset G \cap Y$. Thus *A* is $ij-g\delta$-closed in *Y*.

Theorem 4.10. *For a subset A of a bitopological space* (X, τ_1, τ_2), *the following two conditions are equivalent:*

(1) *A is ji-clopen* (*i-open and j-closed*).

(2) *A is* $ij-\delta$*-open and* $ij-g\delta$*-closed.*

Definition 4.11. A bitopological space (X, τ_1, τ_2) is called *pairwise irreducible* or *pairwise hyperconnected* if every *i*-open subset of *X* is *j*-dense, i.e., if *U* is an *i*-open subset of *X*, then $j - Cl(U) = X$.

Definition 4.12 [9]**.** A bitopological space (X, τ_1, τ_2) is called *pairwise connected* if *X* cannot be expressed as the union of two nonempty subsets *A* and *B* such that $A \bigcap i - Cl(B) = (j - Cl(A)) \bigcap B = \varnothing$. Equivalently, if *X* cannot be written as a union of two nonempty proper subsets *A* and *B*, where *A* is *i*-open and *B* is *j*-closed.

Corollary 4.13. *For a bitopological space* (X, τ_1, τ_2), *the following conditions are equivalent*:

(1) *X is pairwise hyperconnected*.

(2) *Every subset of X is* $ij - g\delta$ *-closed and X is pairwise connected*.

Proof. (1) $\Rightarrow$ (2) Since *X* is pairwise hyperconnected, the only *ij*-regular open subsets of *X* are the trivial ones. Hence, every subset of *X* is trivially $ij - g\delta$-closed. On the other hand, every pairwise hyperconnected space is pairwise connected.

(2) $\Rightarrow$ (1) Let *A* be a nonempty proper $ij - \delta$-open subset of *X*. By (2), *A* is $ij - g\delta$-closed. From Theorem 4.10, it follows that *A* is *ji*-clopen. According to (2), *X* must be pairwise disconnected, a contradiction. Then *X* is pairwise hyperconnected.

5. $ij - \delta g^*$-closed Sets

In this section, we introduce the concept of $ij - \delta g^*$-closed sets and study some properties of weak forms of separation axioms.

Definition 5.1. A subset *A* of a bitopological space (X, τ_1, τ_2) is called $ij - \delta g^*$*-closed* if $j - \delta - Cl(A) \subset U$ whenever $A \subset U$ and *U* is $ij - \delta$-open set in *X*.

Remark 5.2. Every $ij - \delta g$-closed set is $ij - \delta g^*$-closed and every $ij - \delta g^*$-closed set is $ij - g\delta$-closed, but not conversely.

Example 5.3. Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$, $\tau_2 = \{X, \varnothing, \{b\}, \{d\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. Then in (X, τ_1, τ_2), $\{b, c, d\}$ is $12 - \delta g^*$-closed but not $12 - \delta g$-closed.

Theorem 5.4. *Let* (X, τ_1, τ_2) *be a bitopological space and* $A, B \subset X$. *Then*

(a) *If* A *is* $ij - \delta g^*$*-closed, then* $ji - \delta - Cl(A)\backslash A$ *does not contain a nonempty* $ij - \delta$*-closed set.*

(b) *If* A *is* $ij - \delta g^*$*-closed and* $A \subset B \subset ji - \delta - Cl(A)$, *then* B *is* $ij - \delta g^*$*-closed.*

Theorem 5.5. *For a bitopological space* (X, τ_1, τ_2), *the following statements are equivalent*:

(1) *X is pairwise almost weakly Hausdorff.*

(2) *Every singleton* $\{x\}$ *is* $ij - \delta$*-closed or* $ji - \delta$*-open.*

(3) *Every* $ij - \delta g^*$*-closed set is* $ji - \delta$*-closed.*

(4) *Every singleton* $\{x\}$ *is* $ji - \delta$*-closed or ji-regular open.*

(5) *Every* $ij - g\delta$*-closed set is j-closed.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $A \subset X$ be an $ij - \delta g^*$-closed set. Let $x \in ji - \delta - Cl(A)$. We consider the following two cases:

Case (1). Let $\{x\}$ be $ji - \delta$-open. Since $x \in ji - \delta - Cl(A)$, $\{x\} \cap A \neq \varnothing$. This shows that $x \in A$.

Case (2). Let $\{x\}$ be $ij-\delta$-closed. If we assume that $x \notin A,$ then we would have $x \in ji-\delta-Cl(A)\backslash A$ which is impossible according to Theorem 5.4(a). Hence $x \in A.$ In both cases we have $ji-\delta-Cl(A) \subset A$ and so A is $ji-\delta$-closed.

(3) $\Rightarrow$ (2) If $\{x\}$ is not $ij-\delta$-closed, then $X\backslash\{x\}$ is not $ij-\delta$-open and thus $X\backslash\{x\}$ is trivially $ij-\delta g^{*\sim}$-closed. By (3) $X\backslash\{x\}$ is $ji-\delta$-closed or equivalently $\{x\}$ is $ji-\delta$-open.

(2) $\Rightarrow$ (4) It is straightforward.

(4) $\Rightarrow$ (5) Similar to (2) $\Rightarrow$ (3).

(5) $\Rightarrow$ (4) Let $x \in X$. If $\{x\}$ is not $ij-\delta$-closed, then $X\backslash\{x\}$ is not $ij-\delta$-open and thus $X\backslash\{x\}$ is trivially $ij-g\delta$-closed. By (5) $X\backslash\{x\}$ is j-closed or equivalently $\{x\}$ is j-open.

Definition 5.6 [6]**.** A bitopological space (X, τ_1, τ_2) is called *pairwise semi Hausdorff* if for every two distinct points $x, y \in X$, there exist an *ij*-semiopen set U and a *ji*-semiopen set V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Clearly, every pairwise Hausdorff space is pairwise semi Hausdorff but not conversely.

Theorem 5.7. *Every pairwise almost weakly Hausdorff space is pairwise semi Hausdorff.*

Proof. Let $x, y \in X, x \neq y$ and assume that X is pairwise almost weakly Hausdorff. By Theorem 5.5, we have that each point is either j-open or $ij-\delta$-closed. If $\{x\}$ is j-open and $\{y\}$ is i-open, then we are done. If $\{x\}$ is $ij-\delta$-closed, then $\{x\}$ is the intersection of *ij*-regular closed sets and hence for some *ij*-regular closed set R containing x, we have $y \notin R.$ Clearly, R is *ji*-semiopen set containing x but not y and $X\backslash R$ is *ij*-semiopen set containing y but not x. Also, $R \cap (X\backslash R) = \varnothing.$

Definition 5.8. A bitopological space (X, τ_1, τ_2) is called a *pairwise* T_δ*-space* if every $ij - g\delta$-closed set is $ij - \delta$-closed.

Theorem 5.9. *A bitopological space* (X, τ_1, τ_2) *is pairwise* T_δ*-space if and only if it is pairwise almost weakly Hausdorff and pairwise semi-regular.*

Proof. Let X be a pairwise T_δ-space and $A \subset X$ be $ij - \delta g^*$-closed. By Theorem 5.4(b), A is $ij - g\delta$-closed. So, by assumption, A is $ji - \delta$-closed. From Theorem 5.5, X is pairwise almost weakly Hausdorff. Next, we show that X is pairwise semi-regular. Let $A \subset X$ be j-closed. Then A is $ij - g$-closed and thus by Theorem 4.4, A is $ij - g\delta$-closed. By assumption, A is $ji - \delta$-closed. This shows that τ_i-coincides with its semi-regularization τ_i^*.

Conversely, let $A \subset X$ be $ij - g\delta$-closed. We first show that A is j-closed. Let $x \in j - Cl(A)$. By Theorem 5.5, we have the following two cases:

Case (1). Let $\{x\}$ be $ji - \delta$-open. Since $x \in j - Cl(A)$ and $\{x\}$ is clearly j-open, $\{x\} \cap A \neq \emptyset$. This shows that $x \in A$.

Case (2). Let $\{x\}$ be $ji - \delta$-closed. If we assume that $x \notin A$, then we have $x \in j - Cl(A) \setminus A$, which contradicts Theorem 5.4(a). Hence $x \in A$.

So in both the cases we have $j - Cl(A) \subset A$. Thus A is j-closed. Since X is pairwise semi-regular, A is $ji - \delta$-closed. Thus X is pairwise T_δ.

Definition 5.10. A bitopological space (X, τ_1, τ_2) is called a *pairwise pointwise semi-regular space* if every j-closed singleton of X is $ji - \delta$-closed.

Theorem 5.11. *Every pairwise semi-regular space and every pairwise almost weakly Hausdorff space are pairwise pointwise semi-regular.*

Theorem 5.12. *For a bitopological space* (X, τ_1, τ_2), *the following hold*:

(a) *If X is pairwise semi-regular*, *then every* $ij - g\delta$*-closed set is* $ij - g$*-closed*.

(b) *If every* $ij - g\delta$*-closed set is* $ij - g$*-closed*, *then X is pairwise pointwise semi-regular*.

Proof. (a) Let $A \subset X$ be $ij - g\delta$-closed and $A \subset U$, where U is *i*-open. Since X is pairwise semi-regular, U is $ij - \delta$-open. Since A is $ij - g\delta$-closed, we have $j - Cl(A) \subset U$ and therefore A is $ij - g$-closed.

(b) Let $x \in X$ and assume that $\{x\}$ is *j*-closed. If $\{x\}$ is not $ji - g\delta$-closed, then $X \setminus \{x\}$ is not $ji - \delta$-open and hence trivially it is $ji - g\delta$-closed. By assumption it is $ji - g$-closed and thus $\{x\}$ is $ji - g$-open. But every $ji - g$-open *j*-closed set is *ji*-clopen and hence $ji - \delta$-closed. This shows that *X* is pairwise pointwise semi-regular.

References

[1] G. K. Banerjee, On pairwise almost strongly θ-continuous mappings, Bull. Calcutta Math. Soc. 79 (1987), 314-320.

[2] S. Bose, Semi open sets, semicontinuity and semi open mappings in bitopological spaces, Bull. Calcutta Math. Soc. 73 (1981), 237-246.

[3] T. Fukutake, On generalized closed sets in bitopological spaces, Bull. Fukuoko Univ. Ed. III 35 (1986), 19-28.

[4] F. H. Khedr, $c\alpha$-continuity in bitopological spaces, Arabian J. Science Engineering 179(1) (1992), 85-89.

[5] F. H. Khedr and S. M. Al-Areefi, θ-connectedness and δ-connectedness in bitopological spaces, The Arabian J. Sci. Engineering 18(1) (1993), 51-56.

[6] F. H. Khedr and H. S. Al-Saadi, On pairwise semi-generalized closed sets, JKAU: Sci. 21(2) (2009), 269-295.

[7] F. H. Khedr and H. S. Al-Saadi, On pairwise θ-generalized closed sets, Journal of International Mathematical Virtual Institute 1 (2011), 37-51.

[8] F. H. Khedr, A. M. Al-Shibani and T. Noiri, On δ-continuity in bitopological spaces, J. Egyptian Math. Soc. 5(1) (1997), 57-63.

[9] W. J. Pervin, Connectedness in bitopological spaces, Proc. Kor. Ned. Akad. Van. Wetensch. A70 (Indag. Math. 29) (1967), 369-372.

[10] I. L. Reilly, On essentially pairwise Hausdorff spaces, Rend. Circ. Mat. Palermo (2) 25 (1976), 47-52.

[11] A. R. Singal and S. P. Arya, On pairwise almost regular spaces, Glasnik Mat. 6(26) (1971), 335-343.