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On separation axioms in (L, M)-fuzzy convex structures

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Abstract. Different from the separation axioms in the framework of (L, M)-fuzzy convex spaces defined by Liang et al.(2019). In this paper, we give some new investigations on separation axioms in (L, M)-fuzzy convex structures by L-fuzzy hull operators and r-L-fuzzy biconvex. We introduce the concepts of r-LFS_i spaces where $i = \{0, 1, 2, 3, 4\}$, and obtain various properties. In particular, we discuss the invariance of these separation properties under subspace and product.

Keywords: r-LFS_0 space, r-LFS_1 space, r-LFS_2 space, r-LFS_3 space, r-LFS_4 space

1. Introduction and preliminaries

Separation of sets constitutes one of the fundamental facets of abstract convexity theory in [27, 32] which plays an important role in various branches of mathematics where abstract convexity theory has been applied to many different mathematical research fields, such as topological spaces, lattices, metric spaces and graphs (see, for example, [10, 13, 20, 21, 26, 29, 30, 33]). In particular, convexity appears naturally in topology and has many topological properties, such as product spaces, convex variables and separation (see, for example, [1–4, 8, 9, 25, 31]).

Zadeh [36] introduced the notion of a fuzzy subset, which it have been applied to various branches of mathematics. For a generalization of a convex structure, Rosa in 1994 introduced the notion of fuzzy convex structure in [20, 21] which is called I-convex structure. Also, he studied a fuzzy topology together with a fuzzy convexity on the same underlying set X, and introduced fuzzy topology fuzzy convexity spaces and the notion of fuzzy local convexity. Recently, there has been significant research on fuzzy convex structures ([11, 14–17, 23, 34, 35]).

Separation axioms constitute one of the facets of the theory of convex structures. Jamison [8] introduced the separation axioms and gave a restricted version of the polytope screening characterization in terms of screening with half-spaces. Rosa [20] introduced the separation axioms in L-convex structures. However, separation axioms have not been defined in the setting of (L, M)-fuzzy convex. By this motivation, Liang et al. [12] introduced the separation axioms in the framework of (L, M)-fuzzy convex spaces. Sayed et al. [22] defined a new class of L-fuzzy sets called r-L-fuzzy biconvex sets in (L, M)-fuzzy convex structures. The transformation method between L-fuzzy hull operators and (L, M)-fuzzy convex structures were introduced, and

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a characterization of the product of the L-fuzzy hull operator was obtained. Different from the separation axioms in the framework of (L, M)-fuzzy convex spaces defined by Liang et al. [12], the main contributions of the present paper are to give some new investigations on separation axioms in (L, M)-fuzzy convex structures by L-fuzzy hull operators and r-L-fuzzy biconvex.

Throughout this paper, let X be a non-empty set, both L and M be a completely distributive lattices with order reversing involution $'$ where $\bot_M$ $(\bot_L)$ and $\top_M(\top_L)$ denote the least and the greatest elements in $M(L)$ respectively, and $M_{\bot_M} = M - \{\bot_M\}(L_{\bot_L} = L - \{\bot_L\})$. Recall that an order-reversing involution $'$ on L is a map $(-)' : L \longrightarrow L$ such that for any $a, b \in L$, the following conditions hold: (1) $a \leq b$ implies $b' \leq a'$. (2) $a'' = a$. The following properties hold for any subset $\{b_i : i \in I\} \in L$:

$$(1)\ (\bigvee_{i\in I} b_i)' = \bigwedge_{i\in I} b_i';\ \ (2)\ (\bigwedge_{i\in I} b_i)' = \bigvee_{i\in I} b_i'.$$

An L-fuzzy subset of X is a mapping $\mu : X \longrightarrow L$ and the family L^X denoted the set of all fuzzy subsets of a given X [5]. The least and the greatest elements in L^X are denoted by $\underline{0}$ and $\underline{1}$, respectively. For each $\alpha \in L$, let $\underline{\alpha}$ denote the constant L-fuzzy subset of X with the value α. Two L-fuzzy sets are said to be disjoint if their supports are disjoint where support of $\mu = \{x \in X : \mu(x) > 0\}$. The complementation of a fuzzy subset are defined as $\mu'(x) = (\mu(x))'$ for all $x \in X$, (e.g. $\mu'(x) = 1 - \mu(x)$ in the case of $L = [0, 1]$). Let $X = \prod_{i\in\Gamma} X_i$ and $\mu_i \in L^{X_i}$, then $\mu \in L^X$ denote the product of all $\mu_i \in L^{X_i}$ are defined as following $\mu(x) = \wedge_{i\in\Gamma}\mu_i(x_i)$ for all $x \in X$ [28].

Definition 1.1. ([7]) Let $\emptyset \neq Y \subseteq X$ and $\mu \in L^X$; the restriction of μ on Y is denoted by $\mu|Y$. An extension of $\mu \in L^Y$ on X, denoted by μ_X is defined by

$$\mu_X(x) = \begin{cases} \mu(x), & \text{if } x \in Y, \\ \bot_L, & \text{if } x \in X - Y. \end{cases}$$

Definition 1.2. ([6, 18]) A fuzzy point x_t for $t \in L_{\bot_L}$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ \bot_L, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $P_t(X)$. A fuzzy point x_t is a fuzzy singleton if $t = \top_L$ and denoted by $\chi_{\{x\}}$ for all $x \in X$. Two fuzzy points x_t and y_s are distinct if $x \neq y$.

Definition 1.3. ([24]) The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex structure ((L, M)-fcs, for short), where $\mathcal{C} : L^X \longrightarrow M$ satisfying the following axioms:

(LMC1) $\mathcal{C}(\underline{0}) = \mathcal{C}(\underline{1}) = \top_M$.

(LMC2) If $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty, then

$$\mathcal{C}(\bigwedge_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i).$$

(LMC3) If $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then

$$\mathcal{C}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i).$$

The mapping $\mathcal{C}$ is called an (L, M)-fuzzy convexity on X and $\mathcal{C}(\mu)$ can be regarded as the degree to which μ is an L-convex fuzzy set.

Definition 1.4. [19] Let $f : X \longrightarrow Y$. Then the image $f^{\rightarrow}(\mu)$ of $\mu \in L^X$ and the preimage $f^{\leftarrow}(\nu)$ of $\nu \in L^Y$ are defined by:

$$f^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) : x \in X,\ f(x) = y\}$$

and $f^{\leftarrow}(\nu) = \nu \circ f$, respectively. It can be verified that the pair $(f^{\rightarrow}, f^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$.

Definition 1.5. [24] Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex structures. A function $f : X \longrightarrow Y$ is called:

(1) An (L, M)-fuzzy convexity preserving function if $\mathcal{C}(f^{\leftarrow}(\mu)) \geq \mathcal{D}(\mu)$ for all $\mu \in L^Y$.

(2) An (L, M)-fuzzy convex-to-convex function if $\mathcal{D}(f^{\rightarrow}(\mu)) \geq \mathcal{C}(\mu)$ for all $\mu \in L^X$.

Theorem 1.6. *([24]) Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an (L, M)-fuzzy convex structure on Y where*

$$(\mathcal{C}|Y)(\mu) = \bigvee\{\mathcal{C}(\nu) : \nu \in L^X,\ \nu|Y = \mu\}$$

for each $\mu \in L^Y$. The pair $(Y, \mathcal{C}|Y)$ is called an (L, M)-fuzzy convex substructure of $(X, \mathcal{C})$.

Definition 1.7. ([24]) Let $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ be a set of (L, M)-fuzzy convex structures. Let X be the product of the sets X_i for $i \in \Gamma$, and let $\pi_i : X \longrightarrow X_i$ the projection for each $i \in \Gamma$. Define a mapping $\varphi : L^X \longrightarrow M$ by

$$\varphi(\mu) = \bigvee_{i\in\Gamma} \bigvee_{\pi_i^{\leftarrow}(\nu)=\mu} \mathcal{C}_i(\nu) \quad \text{for each } \mu \in L^X.$$

Then the product convexity $\mathcal{C}$ of X is the one generated by subbase φ. The resulting (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is called the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ and is denoted by $\prod_{i\in\Gamma}(X_i, \mathcal{C}_i)$.

Theorem 1.8. *([24]) Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then for all $i \in \Gamma$, $\pi_i : X \longrightarrow X_i$ is an (L, M)-fuzzy convexity preserving function. Moreover, $\mathcal{C}$ is the coarsest (L, M)-fuzzy convex structure such that $\{\pi_i : i \in \Gamma\}$ are (L, M)-fuzzy convexity preserving functions.*

Theorem 1.9. *([22]) Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. If $\pi_i^{\rightarrow}(\prod\limits_{i\in\Gamma} \mu_i) = \mu_i$ for any $\mu_i \in L^{X_i}$. Then for each $i \in \Gamma$, $\pi_i : X \longrightarrow X_i$ is an (L, M)-fuzzy convex-to-convex function.*

Throught this paper, we always assume that each projection π_i $(i \in \Gamma)$ is an (L, M)-fuzzy convex-to-convex function. $\forall \mu \in L^X$, $\exists \mu_i \in L^{X_i}$, such that $\mu = \prod\limits_{i\in\Gamma} \mu_i$ and $\pi_i^{\rightarrow}(\mu) = \mu_i$ for each $i \in \Gamma$.

Definition 1.10. ([22]) Let $(X, \mathcal{C})$ be (L, M)-fuzzy convex structure, $r \in M_{\perp_M}$ and $\mu \in L^X$. Then μ is called r-L-fuzzy biconvex set if $\mathcal{C}(\mu) \geq r$ and $\mathcal{C}(\mu') \geq r$.

Proposition 1.11. *([22]) Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $\emptyset \neq Y \subseteq X$ and μ is an r-L-fuzzy biconvex set in $(X, \mathcal{C})$. Then $\mu|Y$ is an r-L-fuzzy biconvex set in $(Y, \mathcal{C}|Y)$.*

Theorem 1.12. *([22]) Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure. For each $\mu \in L^X$ and $r \in M_{\perp_M}$ we define a mapping $CO_{\mathcal{C}} : L^X \times M_{\perp_M} \longrightarrow L^X$ as follows:*

$$CO_{\mathcal{C}}(\mu, r) = \bigwedge\{\nu \in L^X : \mu \leq \nu,\ \mathcal{C}(\nu) \geq r\}.$$

For $\mu, \nu \in L^X$ and $r, s \in M_{\perp_M}$ the operator $CO_{\mathcal{C}}$ satisfies the following conditions:

(1) $CO_{\mathcal{C}}(\underline{0}, r) = \underline{0}$.
(2) $\mu \leq CO_{\mathcal{C}}(\mu, r)$.
(3) If $\mu \leq \nu$, then $CO_{\mathcal{C}}(\mu, r) \leq CO_{\mathcal{C}}(\nu, r)$.
(4) If $r \leq s$, then $CO_{\mathcal{C}}(\mu, r) \leq CO_{\mathcal{C}}(\mu, s)$.
(5) $CO_{\mathcal{C}}(CO_{\mathcal{C}}(\mu, r), r) = CO_{\mathcal{C}}(\mu, r)$.
(6) For $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion,

$$CO_{\mathcal{C}}(\bigvee_{i\in\Gamma} \mu_i, r) = \bigvee_{i\in\Gamma} CO_{\mathcal{C}}(\mu_i, r).$$

A mapping $CO_{\mathcal{C}}$ is called L-fuzzy hull operator.

2. r-LFS_0 space and r-LFS_1 space

Definition 2.1. Let $x_t, y_s \in P_t(X)$ such that $x \neq y$ and $r \in M_{\perp_M}$. Then an (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is said to be:

(1) r-LFS_0 space if $CO_{\mathcal{C}}(x_t, r) \neq CO_{\mathcal{C}}(y_s, r)$.
(2) r-LFS_1 space if $CO_{\mathcal{C}}(x_t, r) \neq CO_{\mathcal{C}}(y_s, r)$ such that $x_t \notin CO_{\mathcal{C}}(y_s, r)$ and $y_s \notin CO_{\mathcal{C}}(x_t, r)$.

Proposition 2.2. *If $(X, \mathcal{C})$ is an r-LFS_0 space then for distinct fuzzy points x_t and y_s, there exists $\mu, \nu \in L^X$ such that $\mathcal{C}(\mu) \geq r$, $\mathcal{C}(\nu) \geq r$ and $\mu \neq \nu$ with $x_t \in \mu$ and $y_s \in \nu$.*

Proof. Clear by Definition 2.1. □

The next example shows that the converse of Proposition 2.2 is not true.

Example 2.3. Let $L = M = [0, 1]$ and $X = \{a, b\}$. Let μ_i be a fuzzy subsets of X where $i = \{1, 2\}$ defined as follows:

$$\mu_1(a) = 0.3, \qquad \mu_1(b) = 0.0,$$
$$\mu_2(a) = 0.7, \qquad \mu_2(b) = 1.0.$$

Define an (L, M)-fuzzy convexity $\mathcal{C} : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{C}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{6}, & \text{if } \lambda = \mu_1, \\ \frac{1}{5}, & \text{if } \lambda = \mu_2, \\ 0, & \text{otherwise.} \end{cases}$$

For $t \in [0.7, 1]$ and $s \in (0, 1]$ we obtain only two fuzzy sets which are $\underline{1}$ and μ_2 such that $a_t, b_s \in \underline{1}$ and $a_t, b_s \in \mu_2$ with $\mathcal{C}(\underline{1}) \geq \frac{1}{5}$ and $\mathcal{C}(\mu_2) \geq \frac{1}{5}$ but $(X, \mathcal{C})$ is not r-LFS_0 space because $CO_{\mathcal{C}}(a_t, \frac{1}{5}) = CO_{\mathcal{C}}(b_s, \frac{1}{5}) = \mu_2$.

Proposition 2.4. *Let $(X, \mathcal{C})$ be an r-LFS_0 space and $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is r-LFS_0 space.*

Proof. Let $(X, \mathcal{C})$ be an r-LFS_0 space, $x_t, y_s \in P_t(Y)$ such that $x \neq y$ and $\mu = CO_{\mathcal{C}}(x_t, r)$, $\nu = CO_{\mathcal{C}}(y_s, r)$ such that $\mu \neq \nu$.

First, we will prove that $\mu|Y \neq \nu|Y$. So, suppose $\mu|Y = \nu|Y$. Then, $t \leq (\mu|Y)(x) = (\nu|Y)(x)$ and $s \leq (\mu|Y)(y) = (\nu|Y)(y)$ for all $x, y \in Y$. It implies that

$$t \leq \mu(x), t \leq \nu(x), s \leq \mu(y) \text{ and } s \leq \nu(y).$$

Therefore, $t \leq (\mu \wedge \nu)(x) \leq \mu(x) = CO_{\mathcal{C}}(x_t, r)(x)$. From Definition 1.3 (2) and Theorem 1.12, we obtain $(\mu \wedge \nu)(x) = \mu(x)$. Similarly $(\mu \wedge \nu)(y) = \nu(y)$. So, $\mu = \nu$ and it is a contradiction for the assumption that $\mu \neq \nu$. Hence, $\mu|Y \neq \nu|Y$.

Second, we will prove that $CO_{\mathcal{C}|Y}(x_t, r) = \mu|Y$ and $CO_{\mathcal{C}|Y}(y_s, r) = \nu|Y$. So, suppose that there exist $\mu_1, \nu_1 \in L^Y$ such that $(\mathcal{C}|Y)(\mu_1) \geq r$ and $(\mathcal{C}|Y)(\nu_1) \geq r$, with $t \leq \mu_1(x) \leq (\mu|Y)(x)$ and $s \leq \nu_1(y) \leq (\nu|Y)(y)$ for all $x, y \in Y$. Since,

$$(\mathcal{C}|Y)(\mu_1) \geq r \text{ and } (\mathcal{C}|Y)(\nu_1) \geq r,$$

we have $\mu_1 = \lambda|Y$ and $\nu_1 = \rho|Y$ where $\lambda, \rho \in L^X$, $\mathcal{C}(\lambda) \geq r$ and $\mathcal{C}(\rho) \geq r$. It implies that

$$t \leq (\lambda|Y)(x) \leq (\mu|Y)(x)$$

and

$$s \leq (\rho|Y)(y) \leq (\nu|Y)(y). \tag{2.1}$$

Since, $x_t \in \mu = CO_{\mathcal{C}}(x_t, r)$ and $x_t \in \lambda$ we have $\mu \leq \lambda$, Similarly, $\nu \leq \rho$. Therefore,

$$t \leq (\mu|Y)(x) \leq (\lambda|Y)(x),$$

and

$$s \leq (\nu|Y)(y) \leq (\rho|Y)(y). \tag{2.2}$$

From, Equations (2.1) and (2.2) we obtain

$$t \leq (\mu|Y)(x) = (\lambda|Y)(x),$$

and

$$s \leq (\nu|Y)(y) = (\rho|Y)(y).$$

Which implies that

$$t \leq (\mu|Y)(x) = \mu_1(x), \text{ and } s \leq (\nu|Y)(y) = \nu_1(y).$$

Put $\mu_1 = CO_{\mathcal{C}|Y}(x_t, r)$ and $\nu_1 = CO_{\mathcal{C}|Y}(y_s, r)$. Then, $CO_{\mathcal{C}|Y}(x_t, r) = \mu|Y$ and $CO_{\mathcal{C}|Y}(y_s, r) = \nu|Y$. Hence, $(Y, \mathcal{C}|Y)$ is an r-LFS_0 space. □

Theorem 2.5. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C})$ is an r-LFS_0 space if $(X_i, \mathcal{C}_i)$ is an r-LFS_0 space for each $i \in \Gamma$.*

Proof. Let $(X_i, \mathcal{C}_i)$ is an r-LFS_0 space for each $i \in \Gamma$ and $x_t, y_s \in P_t(X)$ such that $x \neq y$ with $X = \prod_{i\in\Gamma} X_i$ and $\pi_i : X \longrightarrow X_i$ be the projection map for each $i \in \Gamma$. Then for some $i \in \Gamma$, $(x_i)_t$ and $(y_i)_s$ are distinct fuzzy points in X_i and

$$CO_{\mathcal{C}_i}((x_i)_t, r) \neq CO_{\mathcal{C}_i}((y_i)_s, r) \tag{2.3}$$

for each $i \in \Gamma$.

Since π_i is the projection map, $\mathcal{C}_i(CO_{\mathcal{C}_i}((x_i)_t, r)) \geq r$ and $\mathcal{C}_i(CO_{\mathcal{C}_i}((y_i)_s, r)) \geq r$, then by Theorem 1.8, we have

$$\mathcal{C}(\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))) \geq r$$

$$\text{and } \mathcal{C}(\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r))) \geq r.$$

Moreover,

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))(x) = CO_{\mathcal{C}_i}((x_i)_t, r)(\pi_i^{\rightarrow}(x)) = CO_{\mathcal{C}_i}((x_i)_t, r)(x_i) \geq t.$$

Therefore, $x_t \in \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))$. Similarly,

$$y_s \in \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)).$$

Now we will prove that

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) \neq \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)).$$

So, if possible assume that

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) = \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)).$$

Then,

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))(x) = \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r))(x)$$

for all $x \in X$.

Implies that,

$$CO_{\mathcal{C}_i}((x_i)_t, r)(\pi_i^{\rightarrow}(x)) = CO_{\mathcal{C}_i}((y_i)_s, r)(\pi_i^{\rightarrow}(x)).$$

Therefore,

$$CO_{\mathcal{C}_i}((x_i)_t, r)(x_i) = CO_{\mathcal{C}_i}((y_i)_s, r)(x_i)$$

for all $x_i \in X_i$.

So, $CO_{\mathcal{C}_i}((x_i)_t, r) = CO_{\mathcal{C}_i}((y_i)_s, r)$. It is a contradiction for Equation (2.3). Hence,

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) \neq \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)).$$

Now, to prove that

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) = CO_{\mathcal{C}}(x_t, r)$$

and

$$\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)) = CO_{\mathcal{C}}(y_s, r).$$

If possible assume that there exist $\lambda \in L^X$ such that $x_t \in \lambda \leq \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))$ with $\mathcal{C}(\lambda) \geq r$. Then,

$$\pi_i^{\rightarrow}(x_t) \in \pi_i^{\rightarrow}(\lambda) \leq CO_{\mathcal{C}_i}((x_i)_t, r)$$

i.e.,

$$(x_i)_t \in \pi_i^{\rightarrow}(\lambda) \leq CO_{\mathcal{C}_i}((x_i)_t, r).$$

Since, π_i is (L, M)-fuzzy convex-to-convex function, then $\mathcal{C}_i(\pi_i^{\rightarrow}(\lambda)) \geq r$. It is a contradiction to assumption that $CO_{\mathcal{C}_i}$ is L-fuzzy hull operator in X_i. Hence, $\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) = CO_{\mathcal{C}}(x_t, r)$. Similarly, $\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)) = CO_{\mathcal{C}}(y_s, r)$. So, we obtain $\pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) \neq \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r))$ for $x_t, y_s \in P_t(X)$. Hence, $(X, \mathcal{C})$ is an r-LFS_0 space. □

Proposition 2.6. *$(X, \mathcal{C})$ is an r-LFS_1 if and only if $\mathcal{C}(\chi_{\{x\}}) \geq r$ for all $x \in X$.*

Proof. ($\Longrightarrow$) Let $(X, \mathcal{C})$ be an r-LFS_1 and assume that there is a $x \in X$ such that $\mathcal{C}(\chi_{\{x\}}) \not\geq r$. Then, there are $y_s \in P_t(X)$ and $s \in L_{\perp_L}$ such that $y_s \in CO_{\mathcal{C}}(\chi_{\{x\}}, r)$. Therefore, the two fuzzy points $x_{\top_L}$ and y_s, $s \in L_{\perp_L}$ cannot be separated by distinct L-fuzzy hull operator which is a contradiction to the assumption that $(X, \mathcal{C})$ is an r-LFS_1. Hence, $\mathcal{C}(\chi_{\{x\}}) \geq r$.

($\Longleftarrow$) Clear by Definition. □

Proposition 2.7. *An r-LFS_1 space is always r-LFS_0 space.*

Proof. Trivial. □

The next example shows that the converse of Proposition 2.7 is not true.

Example 2.8. Let $L = M = [0, 1]$ and $X = \{a, b, c\}$. Let μ_i be fuzzy subsets of X where $i = \{1, 2, 3\}$ defined as follows:

$$\mu_1(a) = 0.4, \quad \mu_1(b) = 0.0, \quad \mu_1(c) = 0.0,$$
$$\mu_2(a) = 0.4, \quad \mu_2(b) = 1.0, \quad \mu_2(c) = 0.0,$$
$$\mu_3(a) = 0.4, \quad \mu_3(b) = 1.0, \quad \mu_3(c) = 1.0,$$

Define an (L, M)-fuzzy convexity $\mathcal{C} : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{C}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \lambda = \mu_1, \\ \frac{1}{3}, & \text{if } \lambda = \mu_2, \\ \frac{1}{2}, & \text{if } \lambda = \mu_3, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C})$ is r-LFS_0 space but it is not r-LFS_1 space because $\mu_1 = CO_{\mathcal{C}}(a_{0.4}, \frac{1}{4})$ and $a_{0.4} \in \mu_2 = CO_{\mathcal{C}}(b_{1.0}, \frac{1}{4})$.

Theorem 2.9. *Let $(X, \mathcal{C}^1)$ be an r-LFS_1 space and $\mathcal{C}^2$ be an (L, M)-fuzzy convexity on X such that $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$. Then $(X, \mathcal{C}^2)$ is also r-LFS_1 space.*

Proof. By Proposition 2.6, it can be easily proved. □

Proposition 2.10. *Let $(X, \mathcal{C})$ be an r-LFS_1 space and $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an r-LFS_1 space.*

Proof. Let $(X, \mathcal{C})$ be an r-LFS_1 space, $\emptyset \neq Y \subseteq X$ and $x_t, y_s \in P_t(Y)$. Then $CO_{\mathcal{C}}(x_t, r) \neq CO_{\mathcal{C}}(y_s, r)$ such that $x_t \notin CO_{\mathcal{C}}(y_s, r)$ and $y_s \notin CO_{\mathcal{C}}(x_t, r)$. Then we can prove as in the proof of Proposition 2.4 that $CO_{\mathcal{C}}(x_t, r)|Y \neq CO_{\mathcal{C}}(y_s, r)|Y$ and $x_t \notin CO_{\mathcal{C}}(y_s, r)|Y$, $y_s \notin CO_{\mathcal{C}}(x_t, r)|Y$. Hence $(Y, \mathcal{C}|Y)$ is an r-LFS_1 space. □

Theorem 2.11. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C})$ is an r-LFS_1 space if $(X_i, \mathcal{C}_i)$ is an r-LFS_1 space for each $i \in \Gamma$.*

Proof. Let $(X_i, \mathcal{C}_i)$ is an r-LFS_1 space for each $i \in \Gamma$ and $x_t, y_s \in P_t(X)$ such that $x \neq y$ with $X = \prod_{i\in\Gamma} X_i$ and $\pi_i : X \longrightarrow X_i$ be the projection map for all $i \in \Gamma$. Then for some $i \in \Gamma$, $(x_i)_t$ and $(y_i)_s$ are distinct fuzzy points in X_i and

$$CO_{\mathcal{C}_i}((x_i)_t, r) \neq CO_{\mathcal{C}_i}((y_i)_s, r)$$

for each $i \in \Gamma$ such that

$$(y_i)_s \notin CO_{\mathcal{C}_i}((x_i)_t, r) \text{ and } (x_i)_t \notin CO_{\mathcal{C}_i}((y_i)_s, r)$$

for each $i \in \Gamma$. Then we can prove as in the proof of Theorem 2.5 that

$$CO_{\mathcal{C}}(x_t, r) = \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r)) \neq \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r)) = CO_{\mathcal{C}}(y_s, r),$$

such that

$$y_s \notin \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((x_i)_t, r))$$

and

$$x_t \notin \pi_i^{\leftarrow}(CO_{\mathcal{C}_i}((y_i)_s, r))$$

Hence, $(X, \mathcal{C})$ is an r-LFS_1 space. □

3. r-LFS_2 space, r-LFS_3 space and r-LFS_4 space

Definition 3.1. Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $r \in M_{\perp_M}$. Then, $(X, \mathcal{C})$ is said to be an r-LFS_2 space if for distinct an L-fuzzy points $x_t, y_s \in$

$P_t(X)$, there exists r-L-fuzzy biconvex set μ such that $x_t \in \mu$ and $y_s \in \mu'$.

Theorem 3.2. *Let $(X, \mathcal{C}^1)$ be an r-LFS_2 space and $\mathcal{C}^2$ be an (L, M)-fuzzy convexity on X such that $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$. Then $(X, \mathcal{C}^2)$ is also an r-LFS_2 space.*

Proof. Let $(X, \mathcal{C}^1)$ be an r-LFS_2 space, $x_t, y_s \in P_t(X)$ such that $x \neq y$, and $\mathcal{C}^2$ be an (L, M)-fuzzy convexity on X. Then, there exists an r-L-fuzzy biconvex set μ in $(X, \mathcal{C}^1)$ such that $x_t \in \mu$ and $y_s \in \mu'$. Therefore, $\mathcal{C}^1(\mu) \geq r$ and $\mathcal{C}^1(\mu') \geq r$. By the assumption $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$ we obtain $\mathcal{C}^2(\mu) \geq r$ and $\mathcal{C}^2(\mu') \geq r$. So, μ is an r-L-fuzzy biconvex set in $(X, \mathcal{C}^2)$. Hence, $(X, \mathcal{C}^2)$ is an r-LFS_2 space. □

Proposition 3.3. *Let $(X, \mathcal{C})$ be an r-LFS_2 space and $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an r-LFS_2 space.*

Proof. Let $(X, \mathcal{C})$ be an r-LFS_2 space, $x_t, y_s \in P_t(Y)$ such that $x \neq y$. Then, there exists an r-L-fuzzy biconvex set $\mu \in L^X$ such that $x_t \in \mu$ and $y_s \in \mu'$. By Proposition 1.11, we have $\mu|Y$ is an r-L-fuzzy biconvex set in L^Y such that $x_t \in \mu$ and $y_s \in \mu'$. Hence, $(Y, \mathcal{C}|Y)$ is an r-LFS_2 space. □

Theorem 3.4. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C})$ is an r-LFS_2 space if $(X_i, \mathcal{C}_i)$ is an r-LFS_2 space for each $i \in \Gamma$.*

Proof. Let $(X_i, \mathcal{C}_i)$ is an r-LFS_2 space for each $i \in \Gamma$ and $x_t, y_s \in P_t(X)$ such that $x \neq y$ with $X = \prod_{i\in\Gamma} X_i$ and $\pi_i : X \longrightarrow X_i$ be the projection map for all $i \in \Gamma$. Then, for some $i \in \Gamma$, $(x_i)_t, (y_i)_s \in P_t(X_i)$ such that $x_i \neq y_i$. Therefore, there exists an r-L-fuzzy biconvex set μ in $(X_i, \mathcal{C}_i)$ such that $(x_i)_t \in \mu$ and $(y_i)_s \in \mu'$. Then, $\pi_i^{\leftarrow}(\mu)$ is r-L-fuzzy biconvex set in $(X, \mathcal{C})$ such that $x_t \in \pi_i^{\leftarrow}(\mu)$ and $y_s \in \pi_i^{\leftarrow}(\mu')$. Hence, $(X, \mathcal{C})$ is an r-LFS_2 space. □

Proposition 3.5. *An r-LFS_2 space is always an r-LFS_1 space.*

Proof. Clear by Definition. □

The next example shows that the converse of Proposition 3.5 is not true.

Example 3.6. Let $L = M = [0, 1]$ and $X = \{a, b, c\}$. Let μ_i be fuzzy subsets of X where $i = \{1, 2, 3, 4, 5\}$ defined as follows:

$$\mu_1(a) = 1.0, \quad \mu_1(b) = 0.0, \quad \mu_1(c) = 0.0,$$
$$\mu_2(a) = 0.0, \quad \mu_2(b) = 1.0, \quad \mu_2(c) = 0.0,$$
$$\mu_3(a) = 0.0, \quad \mu_3(b) = 0.0, \quad \mu_3(c) = 1.0,$$
$$\mu_4(a) = 0.5, \quad \mu_4(b) = 0.0, \quad \mu_4(c) = 0.0,$$
$$\mu_5(a) = 0.5, \quad \mu_5(b) = 1.0, \quad \mu_5(c) = 1.0.$$

Define an (L, M)-fuzzy convexity $\mathcal{C} : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{C}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \lambda \in \{\mu_1, \mu_2, \mu_3\}, \\ \frac{1}{3}, & \text{if } \lambda = \mu_4, \\ \frac{1}{2}, & \text{if } \lambda = \mu_5, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C})$ is r-LFS_1 space but it is not r-LFS_2 space because the only $\frac{1}{3}$-L-fuzzy biconvex set is μ_5 and $a_{0.5}, c_{1.0} \in \mu_5$.

Definition 3.7. Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $r \in M_{\perp_M}$. Then, $(X, \mathcal{C})$ is said to be an r-LFS_3 space if for an L-fuzzy point $x_t \in P_t(X)$ and $\mu \in L^X$ such that $\mathcal{C}(\mu) \geq r$ with the supports of x_t and μ are disjoint, there exists an r-L-fuzzy biconvex set λ such that $\mu \leq \lambda$ and $x_t \in \lambda'$.

Remark 3.8. If $(X, \mathcal{C}^1)$ is an r-LFS_3 space and $\mathcal{C}^2$ be an (L, M)-fuzzy convexity on X such that $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$, then $(X, \mathcal{C}^2)$ need not be an r-LFS_3 space.

Example 3.9. Let $L = M = [0, 1]$ and $X = \{a, b, c\}$. Let μ_i be fuzzy subsets of X where $i = \{1, 2, 3, 4, 5\}$ defined as follows:

$$\mu_1(a) = 1.0, \quad \mu_1(b) = 0.0, \quad \mu_1(c) = 0.0,$$
$$\mu_2(a) = 0.0, \quad \mu_2(b) = 1.0, \quad \mu_2(c) = 1.0,$$
$$\mu_3(a) = 0.0, \quad \mu_3(b) = 0.3, \quad \mu_3(c) = 0.3,$$
$$\mu_4(a) = 1.0, \quad \mu_4(b) = 0.7, \quad \mu_4(c) = 0.7,$$
$$\mu_5(a) = 0.0, \quad \mu_5(b) = 0.0, \quad \mu_5(c) = 0.8.$$

Define two mapings $\mathcal{C}^1, \mathcal{C}^2 : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{C}^1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{5}, & \text{if } \lambda \in \{\mu_1, \mu_2\}, \\ \frac{1}{4}, & \text{if } \lambda = \mu_3, \\ \frac{1}{3}, & \text{if } \lambda = \mu_4, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{C}^2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \lambda \in \{\mu_1, \mu_2, \}, \\ \frac{1}{3}, & \text{if } \lambda = \mu_3, \\ \frac{1}{2}, & \text{if } \lambda = \mu_4, \\ \frac{1}{2}, & \text{if } \lambda = \mu_5, \\ 0, & \text{otherwise.} \end{cases}$$

Then both $\mathcal{C}^1$ and $\mathcal{C}^2$ are (L, M)-fuzzy convexities, and $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$, $(X, \mathcal{C}^1)$ is an r-LFS_3 space and $(X, \mathcal{C}^2)$ is not r-LFS_3 space because μ_3 and its complement are $\frac{1}{4}$-L-fuzzy biconvex sets and $\mu_1 \leq \underline{1} - \mu_3$ and $c_{0.8} \notin \mu_3$.

Proposition 3.10. *Let $(X, \mathcal{C})$ be an r-LFS_3 space and $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an r-LFS_3 space.*

Proof. Let $(Y, \mathcal{C}|Y)$ be an (L, M)-fuzzy convex subspace of an r-LFS_3 space $(X, \mathcal{C})$, $x_t \in P_t(Y)$ and $\mu \in L^Y$ such that $(\mathcal{C}|Y)(\mu) \geq r$ with the supports of x_t and μ are disjoint. Then $\mu = \nu|Y$ where $\nu \in L^X$ such that $\mathcal{C}(\nu) \geq r$. Since the supports of x_t and μ are disjoint, we have the supports of x_t and ν are disjoint. Since $(X, \mathcal{C})$ is an r-LFS_3 space, there exists an r-L-fuzzy biconvex set $\lambda \in L^X$ such that $\nu \leq \lambda$ and $x_t \in \lambda'$. By Proposition 1.11, we have $\lambda|Y$ is an r-L-fuzzy biconvex set in L^Y such that $\nu \leq \lambda|Y$ and $x_t \in \lambda'|Y$. Hence, $(Y, \mathcal{C}|Y)$ is an r-LFS_3 space. □

Theorem 3.11. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C})$ is an r-LFS_3 space if $(X_i, \mathcal{C}_i)$ is an r-LFS_3 space for each $i \in \Gamma$.*

Proof. Let $X = \prod_{i\in\Gamma} X_i$, $\pi_i : X \longrightarrow X_i$ be the projection map for all $i \in \Gamma$ and $(X_i, \mathcal{C}_i)$ is an r-LFS_3 space for each $i \in \Gamma$. Let $x_t \in P_t(X)$ and $\mu \in L^X$ such that $\mathcal{C}(\mu) \geq r$ with the supports of x_t and μ are disjoint. Since $\mathcal{C}(\mu) \geq r$ and π_i is the projection map, we can take μ as $\mu = \bigwedge_{i\in\Gamma} \pi_i^{\leftarrow}(\nu_i)$ such that $\mathcal{C}_i(\nu_i) \geq r$. For some i, $(x_i)_t \in P_t(X_i)$ and the supports of $(x_i)_t$ and ν_i are disjoint. Since $(X_i, \mathcal{C}_i)$ is an r-LFS_3 space, there exists an r-L-fuzzy biconvex set $\lambda_i \in L^{X_i}$ such that $\nu_i \leq \lambda_i$ and $(x_i)_t \in \lambda_i'$. Then $\lambda = \pi_i^{\leftarrow}(\lambda_i)$ is an r-L-fuzzy biconvex set in X such that $\mu = \pi_i^{\leftarrow}(\nu_i) \leq \pi_i^{\leftarrow}(\lambda_i) = \lambda$ and $x_t \in \pi_i^{\leftarrow}(\lambda_i') = \lambda'$. Hence, $(X, \mathcal{C})$ is an r-LFS_3 space. □

Example 3.12. Let L, M, X and μ_i be given as Example 3.9. Define an (L, M)-fuzzy convexity $\mathcal{C} = \mathcal{C}^1 : [0, 1]^X \longrightarrow [0, 1]$ on X as Example 3.9. Then,

(1) $(X, \mathcal{C})$ is an r-LFS_3 space but it is not r-LFS_2 space because μ_2 and its complement are $\frac{1}{5}$-L-fuzzy biconvex sets where $b_{0.3} \in \mu_2$ and $c_{0.3} \notin \underline{1} - \mu_2 = \mu_1$.

(2) $(X, \mathcal{C})$ is an r-LFS_3 space but it is not r-LFS_1 space because

$$\mu_3 = CO(b_{0.3}, \frac{1}{5}) \neq CO(c_{1.0}, \frac{1}{5}) = \mu_2$$

and $b_{0.3} \in CO(c_{1.0}, \frac{1}{5})$ and $c_{1.0} \notin CO(b_{0.3}, \frac{1}{5})$.

(3) $(X, \mathcal{C})$ is an r-LFS_3 space but it is not r-LFS_0 space because

$$CO(b_{0.3}, \frac{1}{5}) = CO(c_{0.3}, \frac{1}{5}) = \mu_3.$$

Definition 3.13. An (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is said to be an r-LFS_4 space if two disjoint L-fuzzy sets $\mu, \nu \in L^X$ such that $\mathcal{C}(\mu) \geq r$ and $\mathcal{C}(\nu) \geq r$ there exist an r-L-fuzzy biconvex set λ such that $\mu \leq \lambda$ and $\nu \leq \lambda'$.

Proposition 3.14. *Let $(X, \mathcal{C})$ be an r-LFS_4 space and $\emptyset \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an r-LFS_4 space.*

Proof. Let $(X, \mathcal{C})$ be an r-LFS_4 space, $(Y, \mathcal{C}|Y)$ be an (L, M)-fuzzy convex subspace of $(X, \mathcal{C})$ and $\mu, \nu \in L^Y$ are disjoint L-fuzzy sets such that $(\mathcal{C}|Y)(\mu) \geq r$ and $(\mathcal{C}|Y)(\nu) \geq r$. Then μ, ν are disjoint L-fuzzy sets in X and there exists an r-L-fuzzy biconvex set $\lambda \in L^X$ such that $\mu \leq \lambda$ and $\nu \leq \lambda'$. By Proposition 1.11, we have $\lambda|Y$ is an r-L-fuzzy biconvex set in Y such that $\mu \leq \lambda|Y$ and $\nu \leq (\lambda|Y)'$. Hence, $(Y, \mathcal{C}|Y)$ is an r-LFS_4 space. □

Theorem 3.15. *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C})$ is an r-LFS_4 space if $(X_i, \mathcal{C}_i)$ is an r-LFS_4 space for each $i \in \Gamma$.*

Proof. Let $X = \prod_{i\in\Gamma} X_i$ and $\pi_i : X \longrightarrow X_i$ be the projection map for all $i \in \Gamma$, $(X_i, \mathcal{C}_i)$ is an r-LFS_4 space for each $i \in \Gamma$ and $\mu, \nu \in L^X$ are disjoint L-fuzzy sets such that $\mathcal{C}(\mu) \geq r$ and $\mathcal{C}(\nu) \geq r$. Then,

$$\mu = \bigwedge_{i\in\Gamma} \pi_i^{\leftarrow}(\lambda_i) \text{ and } \nu = \bigwedge_{i\in\Gamma} \pi_i^{\leftarrow}(\rho_i)$$

there exist $\lambda_i, \rho_i \in L^{X_i}$ are disjoint L-fuzzy sets such that $\mathcal{C}_i(\lambda_i) \geq r$ and $\mathcal{C}_i(\rho_i) \geq r$ for some $i \in \Gamma$. Since $(X_i, \mathcal{C}_i)$ is an r-LFS_4 space for each $i \in \Gamma$, there exists an r-L-fuzzy biconvex set $\mathcal{A}_i \in L^{X_i}$ such that $\lambda_i \leq \mathcal{A}_i$ and $\rho_i \leq \mathcal{A}_i'$. Then $\pi_i^{\leftarrow}(\mathcal{A}_i)$ is an r-L-fuzzy biconvex set in X such that $\mu \leq \pi_i^{\leftarrow}(\mathcal{A}_i)$ and $\nu \leq \pi_i^{\leftarrow}(\mathcal{A}_i')$. Hence $(X, \mathcal{C})$ is an r-LFS_4 space. □

The next example shows that

(1) An r-LFS_4 space need not be r-LFS_3 space.
(2) If $(X, \mathcal{C}^1)$ is an r-LFS_4 space and $\mathcal{C}^2$ be an (L, M)-fuzzy convexity on X such that $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$, then $(X, \mathcal{C}^2)$ need not be an r-LFS_4 space.

Example 3.16. Let $L = M = [0, 1]$ and $X = \{a, b, c\}$. Let μ_i be fuzzy subsets of X where $i = \{1, 2, 3, 4, 5, 6\}$ defined as follows:

$\mu_1(a) = 0.0,\quad \mu_1(b) = 0.0,\quad \mu_1(c) = 1.0,$
$\mu_2(a) = 1.0,\quad \mu_2(b) = 1.0,\quad \mu_2(c) = 0.0,$
$\mu_3(a) = 0.0,\quad \mu_3(b) = 0.0,\quad \mu_3(c) = 0.5,$
$\mu_4(a) = 0.0,\quad \mu_4(b) = 0.5,\quad \mu_4(c) = 0.0,$
$\mu_5(a) = 1.0,\quad \mu_5(b) = 0.0,\quad \mu_5(c) = 0.0,$
$\mu_6(a) = 1.0,\quad \mu_6(b) = 0.0,\quad \mu_6(c) = 0.5.$

Define two mappings $\mathcal{C}^1, \mathcal{C}^2 : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{C}^1(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda \in \{\mu_1, \mu_2\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_3, \\ \frac{1}{2}, & \text{if } \lambda = \mu_4, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{C}^2(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda \in \{\mu_1, \mu_2, \}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_3, \\ \frac{1}{2}, & \text{if } \lambda = \mu_4, \\ \frac{1}{2}, & \text{if } \lambda = \mu_5, \\ \frac{1}{2}, & \text{if } \lambda = \mu_6, \\ 0, & \text{otherwise.} \end{cases}$$

Then both $\mathcal{C}^1$ and $\mathcal{C}^2$ are (L, M)-fuzzy convexities, but the only $\frac{1}{3}$-L-fuzzy biconvex set is μ_2. So,

(1) $(X, \mathcal{C}^1)$ is an r-LFS_4 space but it is not r-LFS_3 space because $\mathcal{C}^1(\mu_4) \geq \frac{1}{2}$ and for $t \in (0, 1]$ we obtain $\mu_4 \leq \mu_2$ and $a_t \notin \mu_1 = \underline{1} - \mu_2$.

(2) $\mathcal{C}^1$ is coarser than $\mathcal{C}^2$, $(X, \mathcal{C}^1)$ is an r-LFS_4 space and $(X, \mathcal{C}^2)$ is not r-LFS_4 space because $\mathcal{C}^2(\mu_5) \geq \frac{1}{2}$ and $\mathcal{C}^2(\mu_6) \geq \frac{1}{2}$ where $\mu_5 \leq \mu_2$ and $\mu_6 \not\leq \mu_1 = \underline{1} - \mu_2$.

4. Conclusion

Following the notion of r-L-fuzzy biconvex sets and L-fuzzy hull operators in (L, M)-fuzzy convex structures introduced by Sayed et al.(2019), we gave some new investigations on separation axioms in (L, M)-fuzzy convex structures. Specifically, we introduced the concepts of r-LFS_i spaces where $i = \{0, 1, 2, 3, 4\}$. We discussed the relations among them, and gave a lot of examples to show the relations. In particular, we discussed the invariance of these separation properties under subspace and product.

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