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pre-Continuity and $D(c, p)$-continuity in fuzzifying topology

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Abstract

The concepts of fuzzy *pre*-continuity and fuzzy *cpre*-continuity are introduced and studied in fuzzifying topology essentially in order to give decompositions of fuzzy continuity.

Keywords: Fuzzy logic; Fuzzifying topology; Fuzzy continuity; *pre*-continuity; *cpre*-continuity

1. Introduction

Ying pointed out a new approach for fuzzy topology with fuzzy logic [7]. The concept of fuzzifying topology with the semantic method of continuous-valued logic was discussed by him [7] as a preliminary of the research on bifuzzy topology. All the conventions in [7–9] are good in this paper. The concept of *pre*-continuity was introduced by Mashhour et al. [5] and the concept of $D(c, p)$-continuity was introduced by Przemski [6]. The concept of $D(c, p)$-continuity will be renamed in the present paper as *cpre*-continuity. It is worth mentioning that the concept of *pre*-continuity was introduced in fuzzy topology [2] by Bin Shahna [1]. In the present paper we extend and study the concepts of *pre*-continuity and *cpre*-continuity in fuzzifying topology. The main result gives some decompositions of fuzzy continuity in fuzzifying topology. Furthermore, the concept of *cpre*-neighborhood system is presented and a fuzzifying topology induced by it is introduced. Also comparisons of some types of fuzzy continuity and fuzzy *cpre*-continuity are studied.

2. Preliminaries

For the fuzzy logical and corresponding set theoretical notations we refer to [7,8]. We note that the set of truth values is the unit interval and we often do not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. For the definitions and results in fuzzifying topology which are used in the sequel we refer to [7–9].

We now give some definitions and results as introduced in [4] which are useful in the rest of the present paper.

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Definition 2.1. For any $\tilde{A} \in \mathscr{F}(X)$,

$$\models (\tilde{A})^{\circ} \equiv X \sim \overline{(X \sim \tilde{A})}.$$

Lemma 2.1. *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then*
(1) $\models \bar{\tilde{A}} \subseteq \bar{\tilde{B}}$;
(2) $\models (\tilde{A})^{\circ} \subseteq (\tilde{B})^{\circ}$.

Lemma 2.2. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A}, \tilde{B}$,
(1) $\models X^{\circ} \equiv X$;
(2) $\models (\tilde{A})^{\circ} \subseteq \tilde{A}$;
(3) $\models (\tilde{A} \cap \tilde{B})^{\circ} \equiv (\tilde{A})^{\circ} \cap (\tilde{B})^{\circ}$;
(4) $\models (\tilde{A})^{\circ\circ} \supseteq (\tilde{A})^{\circ}$.

One can add the following lemma.

Lemma 2.3. *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A} \in \mathscr{F}(X)$,
(1) $\models X \sim (\tilde{A})^{-\circ} \equiv (X \sim \tilde{A})^{\circ -}$;
(2) *if* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then* $\models (\tilde{A})^{-\circ} \subseteq (\tilde{B})^{-\circ}$.

3. Fuzzifying *pre*-open sets and fuzzifying *cpre*-open sets

Definition 3.1. Let (X, τ) be a fuzzifying topological space.
(1) The family of fuzzifying *pre*- (resp. *cpre*-) open sets is denoted by $p\tau$ (resp. $cp\tau) \in \mathscr{F}(P(X))$ and defined as follows:

$$A \in p\tau := \forall x (x \in A \to x \in A^{-\circ}) \text{ (resp. } A \in cp\tau := \forall x (x \in A \cap A^{-\circ} \to x \in A^{\circ})).$$

(2) The family of fuzzifying *pre*- (resp. *cpre*-) closed sets is denoted by pF (resp. $cpF) \in \mathscr{F}(P(X))$ and defined as follows:

$$A \in pF \text{ (resp. } cpF) := X \sim A \in p\tau \text{ (resp. } cp\tau).$$

Lemma 3.1. *For any* $\alpha, \beta, \gamma, \delta \in I$,

$$(1 - \alpha + \beta) \wedge (1 - \gamma + \delta) \leqslant 1 - (\alpha \wedge \gamma) + (\beta \wedge \delta).$$

Lemma 3.2. *For any* $A \in P(X)$,

$$\models A^{\circ} \subseteq A^{-\circ}.$$

Proof. From Theorem 5.3 [7] we have $[A \subseteq A^{-}] = 1$ and from Lemma 2.1(2), $[A^{\circ} \subseteq A^{-\circ}] = 1$. $\square$

Theorem 3.1. *Let* (X, τ) *be a fuzzifying topological space. Then*
(1) (a) $p\tau(X) = 1, p\tau(\phi) = 1$;
(b) *for any* $\{A_{\lambda}: \lambda \in \Lambda\}$, $p\tau(\bigcup_{\lambda \in \Lambda} A_{\lambda}) \geqslant \bigwedge_{\lambda \in \Lambda} p\tau(A_{\lambda})$;
(2) (a) $cp\tau(X) = 1, cp\tau(\emptyset) = 1$;
(b) $cp\tau(A \cap B) \geqslant cp\tau(A) \wedge cp\tau(B)$.

Proof. The proof of (a) in (1) and (a) in (2) are straightforward.

(1) (b) From Lemma 2.3, $\models A_\lambda^{-\circ} \subseteq (\bigcup_{\lambda\in\Lambda} A_\lambda)^{-\circ}$. So,

$$p\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \inf_{x\in\bigcup_{\lambda\in\Lambda} A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ}(x) = \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda} \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ}(x)$$
$$\geqslant \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda} A_\lambda^{-\circ}(x) = \bigwedge_{\lambda\in\Lambda} p\tau(A_\lambda).$$

(2) (b) Applying Lemmas 2.2(3), 2.3(2) and 3.1 we have

$$\begin{aligned} cp\tau(A)\wedge cp\tau(B) &= \inf_{x\in A}(1-A^{-\circ}(x)+A^{\circ}(x)) \wedge \inf_{x\in B}(1-B^{-\circ}(x)+B^{\circ}(x)) \\ &\leqslant \inf_{x\in A\cap B}((1-A^{-\circ}(x)+A^{\circ}(x))\wedge(1-B^{-\circ}(x)+B^{\circ}(x))) \\ &\leqslant \inf_{x\in A\cap B}(1-(A^{-\circ}\cap B^{-\circ})(x)+(A^{\circ}\cap B^{\circ})(x)) \\ &\leqslant \inf_{x\in A\cap B}(1-(A\cap B)^{-\circ}(x)+(A\cap B)^{\circ}(x)) \\ &= cp\tau(A\cap B). \quad \square \end{aligned}$$

From Theorem 3.1, we can have the following theorem.

Theorem 3.2. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) (a) $pF(X)=1, pF(\emptyset)=1$;
  (b) $pF(\bigcap_{\lambda\in\Lambda} A_\lambda) \geqslant \bigwedge_{\lambda\in\Lambda} pF(A_\lambda)$;

(2) (a) $cpF(X)=1,\ cpF(\emptyset)=1$;
  (b) $cpF(A\cup B)\geqslant cpF(A)\wedge cpF(B)$.

Theorem 3.3. *Let* (X,τ) *be a fuzzifying topological space. Then*

(1) (a) $\models \tau\subseteq p\tau$;
  (b) $\models \tau\subseteq cp\tau$;

(2) (a) $\models F\subseteq pF$;
  (b) $\models F\subseteq cpF$.

Proof. From Theorems 2.2(3) [8] and 3.2, we have

(1) (a) $[A\in\tau]=[A\subseteq A^{\circ}]\leqslant[A\subseteq A^{-\circ}]=[A\in p\tau]$.
  (b) $[A\in\tau]=[A\subseteq A^{\circ}]\leqslant[A\cap A^{-\circ}\subseteq A^{\circ}]=[A\in cp\tau]$.

(2) The proof is obtained from (1). $\square$

Remark 3.1. In crisp setting, i.e., if the underlying fuzzifying topology is the ordinary topology, one can have

$$\models (A\in p\tau\wedge A\in cp\tau)\rightarrow A\in\tau.$$

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 3.1. Let $X=\{a,b,c\}$ and let τ be a fuzzifying topology on X defined as follows: $\tau(X)=\tau(\emptyset)=\tau(\{a\})=\tau(\{a,c\})=1;\ \tau(\{b\})=\tau(\{a,b\})=0$ and $\tau(\{c\})=\tau(\{b,c\})=\frac{1}{6}$. One can have that $p\tau(\{a,b\})=\frac{5}{6}, cp\tau(\{a,b\})=\frac{1}{6}$ and hence, $p\tau(\{a,b\})\wedge cp\tau(\{a,b\})=\frac{5}{6}\wedge\frac{1}{6}=\frac{1}{6}\not\leqslant 0=\tau(\{a,b\})$.

Theorem 3.4. *Let* (X,τ) *be a fuzzifying topological space.*
(1) $\models A\in\tau\to(A\in p\tau\wedge A\in cp\tau)$.
(2) *If* $[A\in p\tau]=1$ *or* $[A\in cp\tau]=1$, *then* $\models A\in\tau\leftrightarrow(A\in p\tau\wedge A\in cp\tau)$.

Proof. (1) Obtained from Theorem 3.3(1).

(2) If $[A\in p\tau]=1$, then for each $x\in A$, $A^{-\circ}(x)=1$ and so for each $x\in A, 1-A^{-\circ}(x)+A^{\circ}(x)=A^{\circ}(x)$. Thus from Lemma 3.2 $\models A^{\circ}\subseteq A^{-\circ}$ and so we have, $[A\in p\tau]\wedge[A\in cp\tau]=[A\in cp\tau]=[A\in\tau]$. If $[A\in cp\tau]=1$ then for each $x\in A$, $1-A^{-\circ}(x)+A^{\circ}(x)=1$ and so for each $x\in A$, we have $A^{-\circ}(x)=A^{\circ}(x)$. Thus $[A\in p\tau]\wedge[A\in cp\tau]=[A\in p\tau]=[A\in\tau]$. □

Theorem 3.5. *Let* (X,τ) *be a fuzzifying topological space. Then*

$$\models(A\in p\tau\ \dot{\wedge}\ A\in cp\tau)\to A\in\tau.$$

Proof.

$$\begin{aligned}p\tau(A)\ \dot{\wedge}\ cp\tau(A)&=\inf_{x\in A}A^{-\circ}(x)\ \dot{\wedge}\ \inf_{x\in A}(1-A^{-\circ}(x)+A^{\circ}(x))\\&=\max\left(0,\inf_{x\in A}A^{-\circ}(x)+\inf_{x\in A}(1-A^{-\circ}(x)+A^{\circ}(x))-1\right)\\&\leqslant\inf_{x\in A}A^{\circ}(x)=[A\in\tau].\qquad\square\end{aligned}$$

4. Fuzzifying *pre*- (resp. *cpre*-) neighborhood structure of a point

Definition 4.1. Let $x\in X$. The *pre*- (resp. *cpre*-) neighborhood of x is denoted by pN_x (resp. $cpN_x)\in\mathscr{F}(P(X))$ and defined as

$$pN_x(A)=\sup_{x\in B\subseteq A}p\tau(B)\quad\left(\text{resp. } cpN_x(A)=\sup_{x\in B\subseteq A}cp\tau(B)\right).$$

Theorem 4.1. (1) $\models A\in p\tau\leftrightarrow\forall x(x\in A\to\exists B(B\in p\tau\wedge x\in B\subseteq A))$;
(2) $\models A\in p\tau\leftrightarrow\forall x(x\in A\to\exists B(B\in pN_x\wedge B\subseteq A))$.

Proof. (1) Now, $[\forall x(x\in A\to\exists B(B\in p\tau\wedge x\in B\subseteq A))]=\inf_{x\in A}\sup_{x\in B\subseteq A}p\tau(B)$. It is clear that $\inf_{x\in A}\sup_{x\in B\subseteq A}p\tau(B)\geqslant p\tau(A)$. In the other hand, let $\beta_x=\{B\colon x\in B\subseteq A\}$. Then, for any $f\in\prod_{x\in A}B_x$ we have $\bigcup_{x\in A}f(x)=A$ and so $p\tau(A)=p\tau(\bigcup_{x\in A}f(x))\geqslant\inf_{x\in A}p\tau(f(x))$. Thus,

$$p\tau(A)\geqslant\sup_{f\in\prod_{x\in A}B_x}\inf_{x\in A}p\tau(f(x))=\inf_{x\in A}\sup_{x\in B\subseteq A}p\tau(B).$$

(2) From (1) we have,

$$\begin{aligned}[\forall x(x\in A\to\exists B(B\in pN_x\wedge B\subseteq A))]&=\inf_{x\in A}\sup_{B\subseteq A}pN_x(B)\\&=\inf_{x\in A}\sup_{B\subseteq A}\sup_{x\in C\subseteq B}p\tau(C)\\&=\inf_{x\in A}\sup_{x\in C\subseteq A}p\tau(C)=[A\in p\tau].\qquad\square\end{aligned}$$

Corollary 4.1. $\inf_{x\in A} pN_x(A) = p\tau(A)$.

Theorem 4.2. *The mapping* $pN : X \to \mathscr{F}^N(P(X))$, $x \mapsto pN_x$ *where* $\mathscr{F}^N(P(X))$ *is the set of all normal fuzzy subset of* $P(X)$ *has the following properties*:
(1) *for any* x, A, $\models A \in pN_x \to x \in A$;
(2) *for any* x, A, B, $\models A \subseteq B \to (A \in pN_x \to B \in pN_x)$;
(3) *for any* x, A, $\models A \in pN_x \to \exists H(H \in pN_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in pN_y))$.

Proof. One can easily have that for each $x \in X$, $pN_x(X) = 1$, i.e. each pN_x is normal.
(1) If $pN_x(A) = 0$, the result holds. Suppose $pN_x(A) > 0$, then $\sup_{x\in H\subseteq A} p\tau(H) > 0$ and so there exists H_0 such that $x \in H_0 \subseteq A$. Thus $[x \in A] = 1 \geqslant pN_x(A)$.
(2) Immediate.
(3)

$$\begin{aligned}
&[\exists H(H \in pN_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in pN_y))]\\
&\quad = \sup_{H\subseteq A}\left(pN_x(H) \wedge \inf_{y\in H} pN_y(H)\right)\\
&\quad = \sup_{H\subseteq A}(pN_x(H) \wedge p\tau(H))\\
&\quad = \sup_{H\subseteq A} p\tau(H) \geqslant \sup_{x\in H\subseteq A} p\tau(H) = [A \in pN_x]. \qquad \square
\end{aligned}$$

Theorem 4.3. *The mapping* $cpN : X \to \mathscr{F}^N(P(X))$, $x \mapsto cpN_x$, *where* $\mathscr{F}^N(P(X))$, *is the set of all normal fuzzy subsets of* $P(X)$ *has the following properties*:
(1) *for any* x, A, $\models A \in cpN_x \to x \in A$;
(2) *for any* x, A, B, $\models A \subseteq B \to (A \in cpN_x \to B \in cpN_x)$;
(3) *for any* x, A, B, $\models A \in cpN_x \wedge B \in cpN_x \to A \cap B \in cpN_x$.
Conversely, if a mapping cpN satisfies (2) *and* (3), *then cpN assigns a fuzzifying topology on X, denoted by* $\tau_{cpN} \in \mathscr{F}(P(X))$ *and defined as*

$$A \in \tau_{cpN} := \forall x(x \in A \to A \in cpN_x).$$

Proof. It is clear that each cpN_x is normal.
The proof of (1) and (2) are similar to the corresponding results in Theorem 4.2.
(3) From Theorem 3.1(2)(b) we have

$$\begin{aligned}
[A \cap B \in cpN_x] &= \sup_{x\in H\subseteq A\cap B} cp\tau(H)\\
&= \sup_{x\in H_1\subseteq A, x\in H_2\subseteq B} cp\tau(H_1 \cap H_2)\\
&\geqslant \sup_{x\in H_1\subseteq A, x\in H_2\subseteq B} (cp\tau(H_1) \wedge cp\tau(H_2))\\
&= \sup_{x\in H_1\subseteq A} cp\tau(H_1) \wedge \sup_{x\in H_2\subseteq B} cp\tau(H_2)\\
&= cpN_x(A) \wedge cpN_x(B).
\end{aligned}$$

Conversely, we need to prove that $\tau_{cpN} = \inf_{x\in A} cpN_x(A)$ is a fuzzifying topology. From Theorem 3.2 [7] and since τ_{cpN} satisfies properties (2) and (3), then τ_{cpN} is a fuzzifying topology. $\square$

Theorem 4.4. *Let (X,τ) be a fuzzifying topological space. Then*

$$\models cp\tau \subseteq \tau_{cpN}.$$

Proof. Let $B \in P(X)$, $\tau_{cpN}(B) = \inf_{x\in B} cpN_x(B) = \inf_{x\in B} \sup_{x\in A\subseteq B} cp\tau(A) \geqslant cp\tau(B)$.

5. *pre*- (resp. *cpre*-) Closure and *pre*- (resp. *cpre*-) interior

Definition 5.1. (1) The *pre*- (resp. *cpre*-) closure of A is denoted by $p\text{-}cl$ (resp. $cp\text{-}cl) \in \mathscr{F}(P(X))$ and defined as follows:

$$p\text{-}cl(A)(x) = \inf_{x\notin B\supseteq A}(1 - pF(B)) \left(\text{resp. } cp\text{-}cl(A)(x) = \inf_{x\notin B\supseteq A}(1 - cpF(B))\right).$$

(2) The *pre*- (resp. *cpre*-) interior of A is denoted by $p\text{-}int(A)$ (resp. $cp\text{-}int(A)) \in \mathscr{F}(P(X))$ and defined as follows:

$$p\text{-}int(A)(x) = pN_x(A) \text{ (resp. } cp\text{-}int(A)(x) = cpN_x(A)).$$

Theorem 5.1.
(1) (a) $p\text{-}cl(A)(x) = 1 - pN_x(X \sim A)$;
(b) $\models p\text{-}cl(\emptyset) \equiv \emptyset$;
(c) $\models A \subseteq p\text{-}cl(A)$;
(d) $\models x \in p\text{-}cl(A) \leftrightarrow \forall B(B \in pN_x \to A \cap B \neq \emptyset)$;
(e) $\models A \equiv p\text{-}cl(A) \leftrightarrow A \in pF$;
(f) $\models B \doteq p\text{-}cl(A) \to B \in pF$.
(2) (a) $cp\text{-}cl(A)(x) = 1 - cpN_x(X \sim A)$;
(b) $\models cp\text{-}cl(\emptyset) \equiv \emptyset$;
(c) $\models A \subseteq cp\text{-}cl(A)$;
(d) $\models x \in cp\text{-}cl(A) \leftrightarrow \forall B(B \in cpN_x \to A \cap B \neq \emptyset)$;
(e) $\models A \equiv cp\text{-}cl(A) \leftrightarrow A \in F\tau_{cpN}$;
(f) $\models B \doteq cp\text{-}cl(A) \to B \in F\tau_{cpN}$.

Proof. (1) (a) $p\text{-}cl(A)(x) = \inf_{x\notin B\supseteq A}(1 - pF(B)) = \inf_{x\in X\sim B\subseteq X\sim A}(1 - p\tau(X \sim B)) = 1 - \sup_{x\in X\sim B\subseteq X\sim A} p\tau(X \sim B) = 1 - pN_x(X \sim A)$.

(b) $p\text{-}cl(\emptyset)(x) = 1 - pN_x(X \sim \emptyset) = 0$.

(c) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $pN_x(A) = 0$. If $x \in A$, then $p\text{-}cl(A)(x) = 1 - pN_x(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq p\text{-}cl(A)] = 1$.

(d) $[\forall B(B \in pN_x \to A \cap B \neq \emptyset)] = \inf_{B\subseteq X\sim A}(1 - pN_x(B)) = 1 - pN_x(X \sim A) = [x \in p\text{-}cl(A)]$.

(e) From Corollary 4.1 and from (a) and (c) above we have

$$\begin{aligned}[A \equiv p\text{-}cl(A)] &= \inf_{x\in X\sim A}(1 - (p\text{-}cl(A))(x)) \\ &= \inf_{x\in X\sim A} pN_x(X \sim A) = p\tau(X \sim A) = [A \in pF].\end{aligned}$$

(f) If $[A \subseteq B] = 0$, then $[B \doteq p\text{-}cl(A)] = 0$. Now, we suppose $[A \subseteq B] = 1$, and have $[B \subseteq p\text{-}cl(A)] = 1 - \sup_{x\in B\sim A} pN_x(X \sim A)$, $[p\text{-}cl(A) \subseteq B] = \inf_{x\in X\sim B} pN_x(X \sim A)$. So, $[B \doteq p\text{-}cl(A)] = \max(0, \inf_{x\in X\sim B} pN_x(X \sim A) - \sup_{x\in B\sim A} pN_x(X \sim A))$. If $[B \doteq p\text{-}cl(A)] > t$, then $\inf_{x\in X\sim B} pN_x(X \sim A) > t + \sup_{x\in B\sim A} pN_x(X \sim A)$. For any $x \in X \sim B$, $\sup_{x\in C\subseteq X\sim A} p\tau(C) > t + \sup_{x\in B\sim A} pN_x(X \sim A)$, i.e., there exists C_x such that $x \in C_x \subseteq X \sim A$

and $p\tau(C_x) > t + \sup_{x\in B\sim A} pN_x(X\sim A)$. Now we want to prove that $C_x \subseteq X\sim B$. If not, then there exists $x' \in B\sim A$ with $x' \in C_x$. Hence, we obtain $\sup_{x\in B\sim A} pN_x(X\sim A) \geqslant pN_{x'}(X\sim A) \geqslant p\tau(C_x) > t + \sup_{x\in B\sim A} pN_x(X\sim A)$, a contradiction. Therefore, $pF(B) = p\tau(X\sim B) = \inf_{x\in X\sim B} pN_x(X\sim B) \geqslant \inf_{x\in X\sim B} p\tau(C_x) \geqslant t + \sup_{x\in B\sim A} pN_x(X\sim A) \geqslant t$. Since t is arbitrary, it holds that $[B \doteq p\text{-}cl(A)] \leqslant [B \in pF]$.

(2) The proof is similar to (1). □

Theorem 5.2. *For any* x, A, B,

(1) (a) $\models p\text{-}int(A) \equiv X \sim p\text{-}cl(X\sim A)$;
 (b) $\models p\text{-}int(X) \equiv X$;
 (c) $\models p\text{-}int(A) \subseteq A$;
 (d) $\models B \doteq p\text{-}int(A) \to B \in p\tau$;
 (e) $\models B \in p\tau \wedge B \subseteq A \to B \subseteq p\text{-}int(A)$;
 (f) $\models A \equiv p\text{-}int(A) \leftrightarrow A \in p\tau$.

(2) (a) $\models cp\text{-}int(A) \equiv X \sim cp\text{-}cl(X\sim A)$;
 (b) $\models cp\text{-}int(X) \equiv X$;
 (c) $\models cp\text{-}int(A) \subseteq A$;
 (d) $\models B \doteq cp\text{-}int(A) \to B \in \tau_{cpN}$;
 (e) $\models B \in \tau_{cpN} \wedge B \subseteq A \to B \subseteq cp\text{-}int(A)$;
 (f) $\models A \equiv cp\text{-}int(A) \leftrightarrow A \in \tau_{cpN}$.

Proof. (1) (a) From Theorem 5.1(a) $p\text{-}cl(X\sim A)(x) = 1 - pN_x(A) = 1 - (p\text{-}int(A))(x)$. Then, $[p\text{-}int(A) \equiv X \sim p\text{-}cl(X\sim A)] = 1$.

(b) and (c) are obtained from (a) above and from Theorem 5.1(1)(b) and (1)(c).

(d) From (a) above and from Theorem 5.1(1)(f) we have $[B \doteq p\text{-}int(A)] = [X\sim B \doteq p\text{-}cl(X\sim A)] \leqslant [X\sim B \in pF] = [B \in p\tau]$.

(e) If $[B\subseteq A] = 0$, then the result holds. If $[B\subseteq A] = 1$, then we have that $[B \subseteq p\text{-}int(A)] = \inf_{x\in B}(p\text{-}int(A))(x) = \inf_{x\in B} pN_x(A) \geqslant \inf_{x\in B} pN_x(B) = p\tau(B) = [B\in p\tau \wedge B\subseteq A]$.

(f) From Corollary 4.1, we have

$$[A \equiv p\text{-}int(A)] = \min\left(\inf_{x\in A}(p\text{-}int(A))(x), \inf_{x\in X\sim A}(1 - (p\text{-}int(A))(x))\right)$$
$$= \inf_{x\in A}(p\text{-}int(A))(x) = \inf_{x\in A} pN_x(A) = p\tau(A) = [A \in p\tau].$$

(2) The proof is similar to (1). □

6. pre-Continuous functions and cpre-continuous functions

Definition 6.1. Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces.

(1) A unary fuzzy predicate $pC \in \mathscr{F}(Y^X)$ called fuzzy *pre*-continuity, is given as

$$pC(f) := \forall u(u\in U \to f^{-1}(u) \in p\tau).$$

(2) A unary fuzzy predicate $cpC \in \mathscr{F}(Y^X)$ called fuzzy *cpre*-continuity, is given as

$$cpC(f) := \forall u(u\in U \to f^{-1}(u) \in cp\tau).$$

Definition 6.2. Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. For any $f\in Y^X$, we define the unary fuzzy predicates $pH_j, cpH_j\in\mathscr{F}(Y^X)$ where $j=1,2,\ldots,5$ as follows:

(1) (a) $pH_1(f):=\forall B(B\in F_Y\to f^{-1}(B)\in pFx)$;

(b) $cpH_1(f):=\forall B(B\in F_Y\to f^{-1}(B)\in cpFx)$;

where F_Y is the family of closed subsets of Y; and pF_X and cpF_X are the families of *pre*-closed and *cpre*-closed subsets of X, respectively.

(2) (a) $pH_2(f):=\forall x\forall u(u\in N^Y_{f(x)}\to f^{-1}(u)\in pN^X_x)$;

(b) $cpH_2(f):=\forall x\forall u(u\in N^Y_{f(x)}\to f^{-1}(u)\in cpN^X_x)$;

where N^Y is the neighborhood system of Y; and pN^X and cpN^X are the *pre*-neighborhood and *cpre*-neighborhood systems of X respectively.

(3) (a) $pH_3(f):=\forall x\forall u(u\in N^Y_{f(x)}\to\exists v(f(v)\subseteq u\to v\in pN^X_x))$;

(b) $cpH_3(f):=\forall x\forall u(u\in N^Y_{f(x)}\to\exists v(f(v)\subseteq u\to v\in cpN^X_x))$.

(4) (a) $pH_4(f):=\forall A(f(p\text{-}cl_X(A))\subseteq cl_Y(f(A)))$;

(b) $cpH_4(f):=\forall A(f(cp\text{-}cl_X(A))\subseteq cl_Y(f(A)))$.

(5) (a) $pH_5(f):=\forall B(p\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B)))$;

(b) $cpH_5(f):=\forall B(cp\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B)))$.

Theorem 6.1. (1) $\models f\in pC\leftrightarrow f\in pH_j,\ j=1,2,3,4,5$;

(2) $\models f\in cpC\leftrightarrow f\in cpH_1$.

Proof. We will prove (1) only since the proof of (2) is similar to the corresponding result in (1).

(a) We prove that $\models f\in pC\leftrightarrow f\in pH_1$.

$$\begin{aligned}[f\in pH_1] &= \inf_{B\in P(Y)}\min(1,1-F_Y(B)+pF_X(f^{-1}(B)))\\ &= \inf_{B\in P(Y)}\min(1,1-U(Y\sim B)+p\tau(X\sim f^{-1}(B)))\\ &= \inf_{B\in P(U)}\min(1,1-U(Y\sim B)+p\tau(f^{-1}(Y\sim B)))\\ &= \inf_{u\in P(Y)}\min(1,1-U(u)+p\tau(f^{-1}(u)))\\ &= [f\in pC].\end{aligned}$$

(b) We want to prove that $\models f\in pC\leftrightarrow f\in pH_2$.

First, we prove that $pH_2(f)\geqslant pC(f)$. If $N^Y_{f(x)}(u)\leqslant pN^X_x(f^{-1}(u))$ the result holds. Suppose $N^Y_{f(x)}(u)>pN^X_x(f^{-1}(u))$. It is clear that, if $f(x)\in A\subseteq u$, then $x\in f^{-1}(A)\subseteq f^{-1}(u)$. Then,

$$\begin{aligned}N^Y_{f(x)}(u)-pN^X_x(f^{-1}(u)) &= \sup_{f(x)\in A\subseteq u}U(A)-\sup_{x\in B\subseteq f^{-1}(u)}p\tau(B)\\ &\leqslant \sup_{f(x)\in A\subseteq u}U(A)-\sup_{f(x)\in A\subseteq u}p\tau(f^{-1}(A))\\ &\leqslant \sup_{f(x)\in A\subseteq u}(U(A)-p\tau(f^{-1}(A))).\end{aligned}$$

So,

$$1-N^Y_{f(x)}(u)+pN^X_x(f^{-1}(u))\geqslant\inf_{f(x)\in A\subseteq u}(1-U(A)+p\tau(f^{-1}(A)))$$

and thus,

$$\min(1, 1 - N^Y_{f(x)}(u) + pN^X_x(f^{-1}(u))) \geqslant \inf_{f(x)\in A\subseteq u} \min(1, 1 - U(A) + p\tau(f^{-1}(A)))$$
$$\geqslant \inf_{v\in P(Y)} \min(1, 1 - U(v) + p\tau(f^{-1}(v))) = pC(f).$$

Hence,

$$\inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + pN^X_x(f^{-1}(u))) \geqslant [f \in pC].$$

Secondly, we prove that $pC(f) \geqslant pH_2(f)$. From Corollary 4.1, we have

$$\begin{aligned} pC(f) &= \inf_{u\in P(Y)} \min(1, 1 - U(u) + p\tau(f^{-1}(u))) \\ &\geqslant \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{f(x)\in u} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} pN^X_x(f^{-1}(u))\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1 - \inf_{x\in f^{-1}(u)} N^Y_{f(x)}(u) + \inf_{x\in f^{-1}(u)} pN^X_x(f^{-1}(u))\right) \\ &\geqslant \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + pN^X_x(f^{-1}(u))) = pH_2(f). \end{aligned}$$

(c) We prove that $\models f \in pH_2 \leftrightarrow f \in pH_3$. From Theorem 4.2(2) we have

$$\begin{aligned} pH_3(f) &= \inf_{x\in X} \inf_{u\in P(Y)} \min\left(1, 1 - N^Y_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} pN^X_x(v)\right) \\ &= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N^Y_{f(x)}(u) + pN^X_x(f^{-1}(u))) = pH_2(f). \end{aligned}$$

(d) We prove that $\models f \in pH_4 \leftrightarrow f \in pH_5$.
First, for any $B \in P(Y)$ one can deduce that

$$[f^{-1}(f(p\text{-}cl_X(f^{-1}(B)))) \supseteq p\text{-}cl_X(f^{-1}(B))] = 1, \quad [cl_Y(f(f^{-1}(B))) \subseteq cl_Y(B)] = 1$$

and

$$[f^{-1}(cl_Y(f(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] = 1.$$

Then from Lemma 1.2(2) [9] we have

$$\begin{aligned} [p\text{-}cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] &\geqslant [f^{-1}(f(p\text{-}cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] \\ &\geqslant [f^{-1}(f(p\text{-}cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \\ &\geqslant [f(p\text{-}cl_X(f^{-1}(B))) \subseteq cl_Y(f(f^{-1}(B)))]. \end{aligned}$$

Therefore,

$$\begin{aligned} pH_5(f) &= \inf_{B\in P(Y)} [p\text{-}cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \\ &\geqslant \inf_{B\in P(Y)} [f^{-1}(f(p\text{-}cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(B))] \\ &\geqslant \inf_{B\in P(Y)} [f^{-1}(f(p\text{-}cl_X(f^{-1}(B)))) \subseteq f^{-1}(cl_Y(f(f^{-1}(B))))] \end{aligned}$$

$$\geqslant \inf_{B\in P(Y)}[f(p\text{-}cl_X(f^{-1}(B)))\subseteq cl_Y(f(f^{-1}(B)))]$$
$$\geqslant \inf_{A\in P(X)}[f(p\text{-}cl_X(A))\subseteq cl_Y(f(A))]=pH_4(f).$$

Secondly, for each $A\in P(X)$, there exists $B\in P(Y)$ such that $f(A)=B$ and $f^{-1}(B)\supseteq A$. Hence,

$$\begin{aligned}[p\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))] &\leqslant [p\text{-}cl_X(A)\subseteq f^{-1}(cl_Y(f(A)))]\\ &\leqslant [f(p\text{-}cl_X(A))\subseteq f(f^{-1}(cl_Y(f(A))))]\\ &\leqslant [f(p\text{-}cl_X(A))\subseteq cl_Y(f(A))].\end{aligned}$$

Thus,

$$\begin{aligned}pH_4(f) &= \inf_{A\in P(X)}[p\text{-}cl_X(A)\subseteq f^{-1}(cl_Y(f(A))]\geqslant \inf_{B\in P(Y),B=f(A)}[p\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))]\\ &\geqslant \inf_{B\in P(Y)}[p\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B))]=pH_5(f).\end{aligned}$$

(e) We want to prove that $\models f\in pH_5\leftrightarrow f\in pH_2$,

$$\begin{aligned}pH_5(f) &= [\forall B(p\text{-}cl_X(f^{-1}(B))\subseteq f^{-1}(cl_Y(B)))]\\ &= \inf_{B\in P(Y)}\inf_{x\in X}\min(1,1-(1-pN_x(X\sim f^{-1}(B)))+1-N_{f(x)}(Y\sim B))\\ &= \inf_{B\in P(Y)}\inf_{x\in X}\min(1,1-N_{f(x)}(Y\sim B))+pN_x(f^{-1}(Y\sim B))\\ &= \inf_{u\in P(Y)}\inf_{x\in X}\min(1,1-N_{f(x)}(u))+pN_x(f^{-1}(u))=pH_2(f).\qquad\square\end{aligned}$$

Theorem 6.2. (1) $\models f\in cpH_2\leftrightarrow f\in cpH_j,\ j=3,4,5$;
(2) $\models f\in cpC\rightarrow f\in cpH_2$.

Proof. (1) It is similar to the proof of (c)–(e) in the proof of Theorem 6.1. $\square$
(2) It is similar to the proof of the first part in (b) in Theorem 6.1.

Remark 6.1. In the following theorem, we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous or cpC-continuous. Thus, the symbols (τ,U)-$C(f)$, (τ_{cpN},U)-$C(f)$, (τ,U_{cpN})-$cpC(f)$, etc. will be understood.

Applying Theorems 3.4(1) and 4.4 one can deduce the following theorem.

Theorem 6.3. (1) $\models f\in(\tau,U_{cpN})\text{-}C\rightarrow f\in(\tau,U)\text{-}C$;
(2) $\models f\in(\tau,U)\text{-}cpC\rightarrow f\in(\tau_{cpN},U)\text{-}C$;
(3) $\models f\in(\tau,U)\text{-}C\rightarrow f\in(\tau,U)\text{-}cpC$.

7. Decompositions of fuzzy continuity in fuzzifying topology

Theorem 7.1. *Let $(X,\tau),(Y,U)$ be two fuzzifying topological spaces. Then for each $f\in Y^X$.*

$$\models C(f)\rightarrow(pC(f)\wedge cpC(f)).$$

Proof. The proof is obtained from Theorem 3.3(1). □

Remark 7.1. In crisp setting, i.e., if the underlying fuzzifying topology is the ordinary topology, one can have

$$\models pC(f) \wedge cpC(f) \to C(f).$$

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 7.1. Let (X,τ) be the fuzzifying topological space defined in Counterexample 3.1. Consider the identity function f from (X,τ) onto (X,σ) where σ is a fuzzifying topology on X defined as follows:

$$\sigma(A) = \begin{cases} 1, & A \in \{X, \emptyset, \{a,b\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\frac{5}{6} \wedge \frac{1}{6} = pC(f) \wedge cpC(f) \not\leqslant C(f) = 0$.

Theorem 7.2. *Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces and let $f \in Y^X$. Then*

$$\models C(f) \to (pC(f) \leftrightarrow cpC(f)).$$

Proof. $[pC(f) \to cpC(f)] = \min(1, 1 - pC(f) + cpC(f)) \geqslant pC(f) \wedge cpC(f)$.

Also, $[cpC(f) \to pC(f)] = \min(1, 1 - cpC(f) + pC(f)) \geqslant cpC(f) \wedge pC(f)$. Then from Theorem 7.1 we have $cpC(f) \wedge pC(f) \geqslant C(f)$ and so the result holds. □

Theorem 7.3. *Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces and let $f \in Y^X$. If $[p\tau(f^{-1}(u))] = 1$ or $[cp\tau(f^{-1}(u))] = 1$ for each $u \in P(Y)$, Then*

$$\models C(f) \leftrightarrow (pC(f) \wedge cpC(f)).$$

Proof. Now, we need to prove that $C(f) = pC(f) \wedge cpC(f)$. Applying Theorem 3.4(2) we have

$$\begin{aligned} pC(f) \wedge cpC(f) &= \inf_{u \in P(Y)} \min(1, 1 - U(u) + p\tau(f^{-1}(u))) \wedge \inf_{u \in P(Y)} \min(1, 1 - U(u) + cp\tau(f^{-1}(u))) \\ &= \inf_{u \in P(Y)} \min(1, (1 - U(u) + p\tau(f^{-1}(u))) \wedge (1 - U(u) + cp\tau(f^{-1}(u)))) \\ &= \inf_{u \in P(Y)} \min(1, 1 - U(u) + (p\tau(f^{-1}(u))) \wedge cp\tau(f^{-1}(u))) \\ &= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \tau(f^{-1}(u)) = C(f). \qquad \square \end{aligned}$$

Theorem 7.4. *Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces and let $f \in Y^X$. Then,*
(1) *if $[p\tau(f^{-1}(u))] = 1$ for each $u \in P(Y)$, then*

$$\models pC(f) \to (cpC(f) \leftrightarrow C(f)).$$

(2) *if $[cp\tau(f^{-1}(u))] = 1$ for each $u \in P(Y)$, then*

$$\models cpC(f) \to (pC(f) \leftrightarrow C(f)).$$

Proof. (1) Since $[p\tau(f^{-1}(u))]=1$ and so $[f^{-1}(u)\subseteq(f^{-1}(u))^{-\circ}]=1$, then $[f^{-1}(u)\cap(f^{-1}(u))^{-\circ}\subseteq(f^{-1}(u))^{\circ}]=[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]$. Thus,

$$\begin{aligned} cpC(f) &= \inf_{u\in P(Y)} \min(1, 1-U(u)+cp\tau(f^{-1}(u))) \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\cap(f^{-1}(u))^{-\circ}\subseteq(f^{-1}(u))^{\circ}]) \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]) \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+\tau(f^{-1}(u)))=C(f). \end{aligned}$$

(2) Since $[cp\tau(f^{-1}(u))]=1$ one can deduce that $(f^{-1}(u))^{-\circ}=(f^{-1}(u))^{\circ}$. So,

$$\begin{aligned} pC(f) &= \inf_{u\in P(Y)} \min(1, 1-U(u)+p\tau(f^{-1}(u))). \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\subseteq(f^{-1}(u))^{-\circ}]) \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]) \\ &= \inf_{u\in P(Y)} \min(1, 1-U(u)+\tau(f^{-1}(u)))=C(f). \qquad \square \end{aligned}$$

Theorem 7.5. *Let $(X,\tau),(Y,U),(Z,V)$ be three fuzzifying topological spaces. For any $f\in Y^X$, $g\in Z^Y$.*
(1) $\models pC(f)\to(C(g)\to pC(g\circ f))$;
(2) $\models C(g)\to(pC(f)\to pC(g\circ f))$;
(3) $\models cpC(f)\to(C(g)\to cpC(g\circ f))$;
(4) $\models C(g)\to(cpC(f)\to cpC(g\circ f))$.

Proof. (1) We need to prove that $[pC(f)]\leqslant[C(g)\to pC(g\circ f)]$. If $[C(g)]\leqslant[pC(g\circ f)]$, the result holds; if $[C(g)]>[pC(g\circ f)]$, then

$$\begin{aligned} [C(g)]-[pC(g\circ f)] &= \inf_{v\in P(Z)} \min(1, 1-V(v)+U(g^{-1}(v))) \\ &\quad - \inf_{v\in P(Z)} \min(1, 1-V(v)+p\tau((g\circ f)^{-1}(v))) \\ &\leqslant \sup_{v\in P(Z)} (U(g^{-1}(v))-p\tau(g\circ f)^{-1}(v)) \\ &\leqslant \sup_{u\in P(Y)} (U(u)-p\tau(f^{-1}(u))). \end{aligned}$$

Therefore,

$$\begin{aligned} [C(g)\to pC(g\circ f)] &= \min(1, 1-[C(g)]+[pC(g\circ f)]) \\ &\geqslant \inf_{u\in P(Y)} \min(1, 1-U(u)+p\tau(f^{-1}(u)))=pC(f). \end{aligned}$$

(2)

$$\begin{aligned} &[C(g)\to(pC(f)\to pC(g\circ f))] \\ &\quad=[\neg(C(g)\mathbin{\&}\neg(pC(f)\to pC(g\circ f)))] \end{aligned}$$

$$= [\neg(C(g) \wedge \neg\neg(pC(f) \wedge \neg(pC(g \circ f))))] = [\neg(C(g) \wedge pC(f) \wedge \neg(pC(g \circ f)))]$$
$$= [\neg(pC(f) \wedge C(g) \wedge \neg pC(g \circ f))] = [\neg(pC(f) \wedge \neg\neg(C(g) \wedge \neg(pC(g \circ f))))]$$
$$= [\neg(pC(f) \wedge \neg(C(g) \to pC(g \circ f)))] = [pC(f) \to (C(g) \to pC(g \circ f))] = 1.$$

The proofs of (3) and (4) are similar to (1) and (2), respectively. □

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