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Suitable interval-valued intuitionistic fuzzy topological spaces

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Abstract. In the present paper, a characterization of the intuitionistic fuzzy sets, the interval-valued intuitionistic fuzzy sets and their set-operations are given. By making use of these characterizations, the relationships between the interval-valued intuitionistic fuzzy topology and four fuzzy topologies associated to it are studied. For this reason, some subclasses of the family of interval-valued intuitionistic fuzzy topologies on a set which we call pre-suitable and suitable are introduced. Furthermore, the concepts of homeomorphism functions and compactness in the framework of interval-valued intuitionistic fuzzy topological spaces are introduced and studied.

Keywords: Interval-valued intuitionistic fuzzy set, interval-valued intuitionistic fuzzy topology, fuzzy topology, homeomorphism, compactness

1. Introduction and preliminaries

After the introduction of the concept of fuzzy set (see [38]) several researches were conducted on the generalizations of the notion of fuzzy sets. The notion of a interval-valued fuzzy set has been introduced by different authors (see [18], [24] and [39]). In 1985, the interval representation of language value was discussed by Schwarz in [32]. In 1986, interval-valued fuzzy sets which based on the normal forms were studied by Turksen in [34]. In 1987, a method about interval-valued fuzzy inference was given by Gorzalczany in [20]. In the papers [15, 21, 36–38], the basic research of interval-valued fuzzy sets was studied. In 1983, Atanassov proposed a generalization of the notion of fuzzy set: the concept of intuitionistic fuzzy set [1]. Some basic results on intuitionistic fuzzy sets were published in [2, 3], and the book [5] provides a comprehensive coverage of virtually all results in the area of the theory and applications of intuitionistic fuzzy sets. Actually, intuitionistic fuzzy sets are an object of research by many scientists (see for example, [14, 23]). In particular, intuitionistic fuzzy logic and, in the area of applications, intuitionistic fuzzy generalized nets and intuitionistic fuzzy programs, have been studied by Atanassov and co-workers (see [5]). Çoker and Demirci [12] defined and studied the basic concept of intuitionistic fuzzy point. Later Çoker [10, 11] constructed the fundamental theory on intuitionistic fuzzy topological spaces, and Çoker and others [12, 13, 16, 17, 22, 30, 31, 33] studied compactness, connectedness and continuity in intuitionistic fuzzy topological spaces and intuitionistic gradation of neighborhoodness and other topics. The author in [27] defined the notion of Hausdorffness and obtained some results of nets and filters in intuitionistic fuzzy topological spaces [28]. Lee and Lee [25] showed that the category of fuzzy topological spaces in the sense of Chang [9] (which redefined by Lowen [26] and now known as a stratified fuzzy topology) is a

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bireflective full subcategory of that of intuitionistic fuzzy topological spaces, and Wang and He [35] showed that every intuitionistic fuzzy set may be regarded as an L-fuzzy set [19] for some appropriate lattice L. In 1989 [6], Atanassov and Gargov presented the basic preliminaries of interval-valued intuitionistic fuzzy set theory. Also, in [4, 7, 8] some types of operators were defined over interval-valued intuitionistic fuzzy set. In [29], topology of interval-valued intuitionistic fuzzy sets was defined and some of its properties were studied. In the present paper, we will give a characterization of the concept of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets and their set-operations [6]. We prove that for an interval-valued intuitionistic fuzzy topology there exist four fuzzy topologies in the sense of Chang [9]. Also, to study the interactions between these types of topologies, we investigate the concepts of pre-suitable and suitable interval-valued intuitionistic fuzzy topologies and study some basic concepts of these concepts. Furthermore, we establish the concept of interval-valued intuitionistic fuzzy homeomorphism and the concept of interval-valued intuitionistic fuzzy compactness.

For the references of definitions and results used in this paper concerning fuzzy sets (resp. fuzzy topology (F topology, for short), intuitionistic fuzzy set (IF set, for short), intuitionistic fuzzy topology (IF topology, for short)) we refer to [9, 38] (resp. [2, 10, 13]).

Definition 1.1. [31]. Let X be a nonempty set, $B \subseteq X$ and $\alpha \in [0, 1]$. We define the fuzzy set $\alpha 1_B$ by $\alpha 1_B = \alpha$ when $x \in B$ and $\alpha 1_B = 0$ when $x \notin B$. We use 1_X and 1_ϕ instead of 11_X and 01_X, respectively. We write $\alpha 1_x$ instead of $\alpha 1_{\{x\}}$. For each fuzzy topological space (X, τ) and fuzzy set B in X, the trivial case $01_x = 1_\phi$ is an interior point of B will be considered. $\alpha 1_x$ is quasi-coincident with a fuzzy set A (written $\alpha 1_x \, q \, A$) if and only if $\alpha 1_x \notin A^c$ if and only if $A(x) + \alpha > 1$, where A^c is the complement of the fuzzy set A. If (X, τ) is fuzzy topological space and $A \subseteq X$, then A is a τ-Q-neighborhood of a fuzzy point $\alpha 1_x$ in X or $\alpha 1_x$ is a τ-Q-interior point of A if and only if there exists $B \in \tau$ such that $\alpha 1_x \, q \, B \subseteq A$.

Definition 1.2. [2]. An intuitionistic fuzzy set A over the universe of discourse X is an expression given by $A = \{\langle x, \ \mu_A(x), \upsilon_A(x) \ \rangle : x \in X\}$, where $\mu_A : X \to [0, \ 1]$, $\upsilon_A : X \to [0, \ 1]$ with the condition $0 \le \mu_A(x) \ + \upsilon_A(x) \ \le 1$. The values $\mu_A(x)$ and $\upsilon_A(x)$ denote, respectively, the degree of membership and the degree of nonmembership of the element x to the set A. We will denote by IF(X) the set of all intuitionistic fuzzy sets of X (IF sets of X, for short). For every intuitionistic fuzzy set A we have

$\Box A = \{\langle x, \mu_A(x)\rangle : x \in X\} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$,

$\Diamond A = \{\langle x, 1 - \upsilon_A(x)\rangle : x \in X\} = \{\langle x, 1 - \upsilon_A(x), \upsilon_A(x)\rangle : x \in X\}$.

Definition 1.3. Let $D[0, \ 1]$ be the set of all closed subintervals of the interval $[0, \ 1]$ and $X(\neq \phi)$ be a given set. Following Atanassov and Gargov [6], an interval-valued intuitionistic fuzzy set (IIF set for short) in X is an expression given by $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$, where $M_A : X \to D[0, \ 1]$, $N_A : X \to D[0, \ 1]$ with the condition $0 < \sup M_A(x) + \sup N_A(x) \le 1$. The intervals $M_A(x)$ and $N_A(x)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x to the set A. Thus for each $x \in X$, $M_A(x)$ and $N_A(x)$ are closed intervals whose lower and upper end points are, respectively, denoted by $M_A^L(x)$, $M_A^U(x)$ and $N_A^L(x)$, $N_A^U(x)$. We will denote by II(X) the set of all IIF sets of X. For every interval-valued intuitionistic fuzzy set A we have

$\Box A = \{\langle x, M_A(x), [\inf N_A(x), 1 - \sup M_A(x)]\rangle : x \in X\}$,

$\Diamond A = \{\langle x, [\inf M_A(x), 1 - \sup N_A(x)], N_A(x)\rangle : x \in X\}$.

Definition 1.4. Following Mondal and Samanta [29], a topological space of IIF sets is a pair (X, τ), where X is a nonempty set and τ is a subfamily of II(X) satisfies the basic conditions of classical topology. τ is called a topology of IIF sets on X (IIF topology, for short). Every member of τ is called open (IIF open set). $B \in$ II(X) is said to be closed (IIF closed set) in (X, τ) if and only if $B^c \in \tau$. The family of all interval-valued intuitionistic fuzzy topologies on X will be denoted by IT(X).

2. A characterization of IF set and IIF set

First, we give a characterization of intuitionistic fuzzy sets and their set-operations.

Definition 2.1. An ordered pair (μ, ν) of fuzzy sets of a nonempty set X such that $\mu \le \nu$ is called a suitable intuitionistic fuzzy set of X (SIF set, for short). The family of all suitable intuitionistic fuzzy sets of X will be denoted by SIF (X).

Theorem 2.2. *There exists a bijection between IF (X) and SIF(X).*

Proof. The functions $H : IF(X) \to SIF(X)$ and $M : SIF(X) \to IF(X)$ defined by:
$H(A) = H(\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}) = (\mu_A, 1 - \nu_A)$ and $M((\mu, \nu)) = \{\langle x, \mu(x), 1 - \nu(x)\rangle : x \in X\}$ are well defined, $(H \circ M)((\mu, \nu)) = (\mu, \nu)$ and $(M \circ H)(A) = A$, i.e., $M = H^{-1}$. □

By making use of our characterization of the IF set one can have the following three theorems.

Theorem 2.3. *Let* $A = (A_1, A_2), B = (B_1, B_2) \in \mathrm{IF}(X)$, *and* $\{A^j : j \in \Lambda\} \subseteq \mathrm{IF}(X)$. *Then:*

(1) $A \subseteq B$ *if and only if* $A_i \leq B_i$ *for each* $i \in \{1, 2\}$;

(2) $A = B$ *if and only if* $A_i = B_i$ *for each* $i \in \{1, 2\}$;

(3) $\bigcup_{j\in\Lambda} A^j = (\bigcup_{j\in\Lambda} A_1^j, \bigcup_{j\in\Lambda} A_2^j)$;

(4) $\bigcap_{j\in\Lambda} A^j = (\bigcap_{j\in\Lambda} A_1^j, \bigcap_{j\in\Lambda} A_2^j)$;

(5) $A^c = (A_2^c, A_1^c)$;

(6) $\tilde{1} = (1_X, 1_X)$;

(7) $\tilde{0} = (1_\phi, 1_\phi)$;

(8) $\Box A = (A_1, A_1)$;

(9) $\Diamond A = (A_2, A_2)$.

Theorem 2.4. *Let* $\{A^j : j \in \Lambda\} \subseteq \mathrm{IF}(X)$. *Then:*

(1) $\bigcup_{j\in\Lambda} (\Box A^j) = \Box(\bigcup_{j\in\Lambda} A^j)$;

(2) $\bigcap_{j\in\Lambda} (\Box A^j) = \Box(\bigcap_{j\in\Lambda} A^j)$;

(3) $\bigcup_{j\in\Lambda} (\Diamond A^j) = \Diamond(\bigcup_{j\in\Lambda} A^j)$;

(4) $\bigcap_{j\in\Lambda} (\Diamond A^j) = \Diamond(\bigcap_{j\in\Lambda} A^j)$.

Theorem 2.5. *Let* $f : X \to Y$, $A = (A_1, A_2), \in \mathrm{IF}(X)$, *and* $C = (C_1, C_2) \in \mathrm{IF}(Y)$. *Then:*

(1) $f(A) = (f(A_1), f(A_2))$;

(2) $f^{-1}(C) = (f^{-1}(C_1), f^{-1}(C_2))$;

(3) $f(\Box A) = \Box f(A)$;

(4) $f(\Diamond A) = \Diamond f(A)$;

(5) $f^{-1}(\Box C) = \Box f^{-1}(C)$;

(6) $f^{-1}(\Diamond C) = \Diamond f^{-1}(C)$.

Second, we give a characterization of interval-valued intuitionistic fuzzy sets and their set-operations.

Definition 2.6. An ordered quadrable $(\kappa, \lambda, \mu, \nu)$ of fuzzy sets of a nonempty set X such that $\kappa \leq \lambda \leq \nu \leq \mu$ is called a suitable interval-valued intuitionistic fuzzy set of X (SIIF set for short). The family of all suitable interval-valued intuitionistic fuzzy sets of X will be denoted by $\mathrm{SI}(X)$.

In the following theorem we point out a bijection between $\mathrm{II}(X)$ and $\mathrm{SI}(X)$. In other words, the concept of interval-valued intuitionistic fuzzy set of X can be determined in a uniquely manner as an ordered quadrable $(\kappa, \lambda, \mu, \nu)$ of fuzzy sets of X such that $\kappa \leq \lambda \leq \nu \leq \mu$.

Theorem 2.7. *There exist two functions* $\eta_1 : \mathrm{II}(X) \to \mathrm{SI}(X)$ *and* $\eta_2 : \mathrm{SI}(X) \to \mathrm{II}(X)$ *such that* $\eta_1 \circ \eta_2 = \mathrm{id}_{\mathrm{SI}(X)}$ *and* $\eta_2 \circ \eta_1 = \mathrm{id}_{\mathrm{II}(X)}$, *i.e.,* $\eta_1 = {\eta_2}^{-1}$.

Proof. Define η_1 and η_2 as follows:
$\eta_1(A) = \eta_1(\{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\upsilon_A^L(x), \upsilon_A^U(x)]\rangle : x \in X\}) = (\mu_A^L, \mu_A^U, 1 - \upsilon_A^L, 1 - \upsilon_A^U)$ for each $A \in \mathrm{II}(X)$, and $\eta_2((\kappa, \lambda, \mu, \nu)) = \{\langle x, [\kappa(x), \lambda(x)], [1 - \mu(x), 1 - \nu(x)]\rangle : x \in X\}$ for each $(\kappa, \lambda, \mu, \nu) \in \mathrm{SI}(X)$. It is clear that η_1 and η_2 are well defined and $\eta_1 \circ \eta_2 = \mathrm{id}_{\mathrm{SI}(X)}$ and $\eta_2 \circ \eta_1 = \mathrm{id}_{\mathrm{II}(X)}$, i.e., $\eta_1 = {\eta_2}^{-1}$. □

By making use of our characterization of the interval-valued intuitionistic fuzzy set we have the following theorem.

Theorem 2.8. *Let* $A = (A_1, A_2, A_3, A_4), B = (B_1, B_2, B_3, B_4) \in \mathrm{II}(X)$ *and* $\{A^j : j \in \Lambda\} \subseteq \mathrm{II}(X)$. *Then:*

(1) $A \subseteq B$ *if and only if* $A_i \leq B_i$ *for each* $i \in \{1, 2, 3, 4\}$;

(2) $A = B$ *if and only if* $A_i = B_i$ *for each* $i \in \{1, 2, 3, 4\}$;

(3) $\bigcup_{j\in\Lambda} A^j = (\bigcup_{j\in\Lambda} A_1^j, \bigcup_{j\in\Lambda} A_2^j, \bigcup_{j\in\Lambda} A_3^j, \bigcup_{j\in\Lambda} A_4^j)$;

(4) $\bigcap_{j\in\Lambda} A^j = (\bigcap_{j\in\Lambda} A_1^j, \bigcap_{j\in\Lambda} A_2^j, \bigcap_{j\in\Lambda} A_3^j, \bigcap_{j\in\Lambda} A_4^j)$;

(5) $A^c = (A_3^c, A_4^c, A_1^c, A_2^c)$;

(6) $\tilde{1} = (1_X, 1_X, 1_X, 1_X)$;

(7) $\tilde{0} = (1_\phi, 1_\phi, 1_\phi, 1_\phi)$;

(8) $\Box A = (A_1, A_2, A_3, A_2)$;

(9) $\Diamond A = (A_1, A_4, A_3, A_4)$.

Proof. Applying Theorem 2.7. we obtain

(1) $A \subseteq B$

$$\Leftrightarrow \left\{\langle x, [A_1(x), A_2(x)], [1 - A_3(x), 1 - A_4(x)]\rangle : x \in X\right\}$$
$$\subseteq \left\{\langle x, [B_1(x), B_2(x)], [1 - B_3(x), 1 - B_4(x)]\rangle : x \in X\right\}$$
$$\Leftrightarrow \forall x \in X, A_1(x) \leq B_1(x), A_2(x) \leq B_2(x), 1 - A_3(x) \geq 1 - B_3(x), 1 - A_4(x) \geq 1 - B_4(x)$$
$$\Leftrightarrow A_i \leq B_i, \forall i \in \{1, 2, 3, 4\}.$$

(2) Obvious.

(3) $\bigcup_{j\in J} A^j$

$$= \bigcup_{j\in J}\left\{\left\langle x, \left[A_1^j(x), A_2^j(x)\right], \left[1 - A_3^j(x), 1 - A_4^j(x)\right]\right\rangle : x \in X\right\}$$
$$= \left\{\left\langle x, \left[\bigcup_{j\in J} A_1^j(x), \bigcup_{j\in J} A_2^j(x)\right], \left[\bigcap_{j\in J}\left(1 - A_3^j(x)\right), \bigcap_{j\in J}\left(1 - A_4^j(x)\right)\right]\right\rangle : x \in X\right\}$$
$$= \left(\bigcup_{j\in J} A_1^j, \bigcup_{j\in J} A_2^j, \bigcup_{j\in J} A_3^j, \bigcup_{j\in J} A_4^j\right).$$

(4) Similar to (3).

(5) A^c

$$= \left\{\langle x, [1 - A_3(x), 1 - A_4(x)], [A_1(x), A_2(x)]\rangle : x \in X\right\}$$
$$= (1 - A_3, 1 - A_4, 1 - A_1, 1 - A_2)$$
$$= (A_3^c, A_4^c, A_1^c, A_2^c).$$

The proofs of (6)-(9) are straightforward. □

Theorem 2.9. *Let* $\{A^j : j \in \Lambda\} \subseteq \text{II}(X)$. *Then:*

(1) $\bigcup_{j\in\Lambda} (\Box A^j) = \Box(\bigcup_{j\in\Lambda} A^j)$;

(2) $\bigcap_{j\in\Lambda} (\Box A^j) = \Box(\bigcap_{j\in\Lambda} A^j)$;

(3) $\bigcup_{j\in\Lambda} (\Diamond A^j) = \Diamond(\bigcup_{j\in\Lambda} A^j)$;

(4) $\bigcap_{j\in\Lambda} (\Diamond A^j) = \Diamond(\bigcap_{j\in\Lambda} A^j)$.

Proof. Applying Theorem 2.8.(8) and (9), we prove (1) as (2)-(4) are similar.

(1) $\bigcup_{j\in J} (\Box A^j) = \bigcup_{j\in J} (A_1^j, A_2^j, A_3^j, A_2^j)$

$$= \left(\bigcup_{j\in J} A_1^j, \bigcup_{j\in J} A_2^j, \bigcup_{j\in J} A_3^j, \bigcup_{j\in J} A_2^j\right)$$
$$= \Box\left(\bigcup_{j\in J} A^j\right).$$

□

Theorem 2.10. *Let* $f : X \to Y$ *and* $A = (A_1, A_2, A_3, A_4) \in \text{II}(X)$, $C = (C_1, C_2, C_3, C_4) \in \text{II}(Y)$. *Then:*

(1) $f(A) = (f(A_1), f(A_2), f(A_3), f(A_4))$;

(2) $f^{-1}(C) = (f^{-1}(C_1), f^{-1}(C_2), f^{-1}(C_3), f^{-1}(C_4))$;

(3) $f(\Box A) = \Box f(A)$;

(4) $f(\Diamond A) = \Diamond f(A)$;

(5) $f^{-1}(\Box C) = \Box f^{-1}(C)$;

(6) $f^{-1}(\Diamond C) = \Diamond f^{-1}(C)$.

Proof.

(1) $f(A)$

$$= \left\{\langle y, [f(A_1)(y), f(A_2)(y)], [1 - f(A_3)(y), 1 - f(A_4)(y)]\rangle : y \in Y\right\}$$
$$= (f(A_1), f(A_2), f(A_3), f(A_4)).$$

(2) $f^{-1}(B) = \left\{\langle x, [f^{-1}(B_1)(x), f^{-1}(B_2)(x)], [f^{-1}(1 - B_3)(x), f^{-1}(1 - B_4)(x)]\rangle : x \in X\right\}$

$$= \left\{\langle x, [f^{-1}(B_1)(x), f^{-1}(B_2)(x)], [1 - f^{-1}(B_3)(x), 1 - f^{-1}(B_4)(x)]\rangle : x \in X\right\}$$
$$= (f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3), f^{-1}(B_4)).$$

The proofs of (3)-(6) are straightforward. □

Remark 2.11.

(1) An IIF set (A_1, A_2, A_3, A_4) is identified with an IF set if and only if $A_1 = A_2$ and $A_3 = A_4$. For this reason Theorems 2.2., 2.3., 2.4. and 2.5. can be obtained as corollaries from corresponding Theorems 2.7., 2.8., 2.9. and 2.10., respectively;

(2) An IIF set (A_1, A_2, A_3, A_4) is identified with a fuzzy set if and only if $A_1 = A_2 = A_3 = A_4$;

(3) An IIF set (A_1, A_2, A_3, A_4) is identified with an ordinary set if and only if $A_1 = A_2 = A_3 = A_4$ and $A(X) \subseteq \{0, 1\}$.

Theorem 2.12. *Let τ be an IIF topology on a nonempty set X and let τ_{IF}(resp. τ_F, τ_O) $= \{A : A \in \tau\}$ and A is identified with an intuitionistic fuzzy set (resp. fuzzy set; ordinary set)$\}$. Then τ_{IF} is an IF topology, τ_F is a fuzzy topology and τ_O is a topology on X.*

Theorem 2.13. *Let τ be an IIF topology on a nonempty set X and $\tau_\Box$ (resp. $\tau_\Diamond = \{\Box A$(resp.$\Diamond A) : A \in \tau\}$. Then both $\tau_\Box$ and $\tau_\Diamond$ is an IIF topology on X.*

3. SIIF and PIIF topologies

Definition 3.1. Let X be a nonempty set. Then:

(1) An ordered quadrable $(\tau_1, \tau_2, \tau_3, \tau_4)$ of fuzzy topologies on X is called a pre-suitable interval-valued intuitionistic fuzzy topology (PIIF topology for short) on X. The family of all pre-suitable interval-valued intuitionistic fuzzy topologies on X will be denoted by PT(X);
(2) A pre-suitable interval-valued intuitionistic fuzzy topology $(\tau_1, \tau_2, \tau_3, \tau_4)$ on X is called suitable interval-valued intuitionistic fuzzy topology (SIIF topology for short) on X if $\tau_1 \subseteq \tau_2 \subseteq \tau_4 \subseteq \tau_3$. The family of all suitable interval-valued intuitionistic fuzzy topologies on X will be denoted by ST(X).

We present now the following problem: Is there a bijection between IT(X) and PT(X)?

In the following we point out a bijection between a subfamily of IT(X) and PT (X)).

Theorem 3.2.

(1) *For each $\tau \in$ IT(X), there exist four fuzzy topologies on X defined as $\tau_i = \{A_i : A \in \tau\}$ for each $i \in \{1, 2, 3, 4\}$ and $A = (A_1, A_2, A_3, A_4) \in$ II(X), i.e., there exists a function ξ from IT(X) into PT(X) such that $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$.*
(2) *There exists a function $\eta :$ PT(X) $\to$ IT(X) defined as follows:*
$\eta((\theta_1, \theta_2, \theta_3, \theta_4)) = \{A : A \in$ II(X), $A_i \in \theta_i$, $i \in \{1, 2, 3, 4\}\}$.
(3) *The function η defined in (2) above satisfies the following statements:*
(a) η is injection;
(b) If $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$, then $\eta(\xi(\tau)) \supseteq \tau$;
(c) If $(\theta_1, \theta_2, \theta_3, \theta_4) \in$ PT(X), then $\xi(\eta((\theta_1, \theta_2, \theta_3, \theta_4))) = (\theta_1, \theta_2, \theta_3, \theta_4)$.

Proof. (1) We now prove that $\tau_i = \{A_i : A \in \tau\}$ is a fuzzy topology for each $i \in \{1, 2, 3, 4\}$.

(a) Since $\tilde{1}, \tilde{0} \in \tau$, then $1_X, 1_\phi \in \tau_i$;

(b) Suppose that $H, K \in \tau_i$. Then there exist $A, B \in \tau$ such that $A_i = H$ and $B_i = K$. Since $A \cap B \in \tau$, then we have $H \cap K = A_i \cap B_i = (A \cap B)_i \in \tau_i$;

(c) Suppose that $\{H^j : j \in \Lambda\} \subseteq \tau_i$. Then there exist $\{A^j : j \in \Lambda\} \subseteq \tau$ such that $H^j = A_i^j$ for each $j \in \Lambda$. Since $\bigcup_{j\in\Lambda} A^j \in \tau$, then $\bigcup_{j\in\Lambda} H^j = \bigcup_{j\in\Lambda} A_i^j = \left(\bigcup_{j\in\Lambda} A^j\right)_i \in \tau_i$.

(2) It is clear that $\eta((\theta_1, \theta_2, \theta_3, \theta_4))$ is uniquely determined. Now, we prove that $\eta((\theta_1, \theta_2, \theta_3, \theta_4)) \in$ IT(X):

(a) Since $1_X, 1_\phi \in \theta_i$ for each $i \in \{1, 2, 3, 4\}$, then we have $\tilde{1}, \tilde{0} \in \eta((\theta_1, \theta_2, \theta_3, \theta_4))$;

(b) Suppose $A, B \in \eta((\theta_1, \theta_2, \theta_3, \theta_4))$. Then $A_i, B_i \in \theta_i$ for each $i \in \{1, 2, 3, 4\}$. Thus $A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3, A_4 \cap B_4) \in$ II(X). Therefore $A \cap B \in \eta((\theta_1, \theta_2, \theta_3, \theta_4))$;

(c) Suppose that $\{A^j : j \in \Lambda\} \subseteq \eta((\theta_1, \theta_2, \theta_3, \theta_4))$. Then $A_i^j \in \theta_i$ for each $i \in \{1, 2, 3, 4\}$. Hence $\bigcup_{j\in\Lambda} A^j = (\bigcup_{j\in\Lambda} A_1^j, \bigcup_{j\in\Lambda} A_2^j, \bigcup_{j\in\Lambda} A_3^j, \bigcup_{j\in\Lambda} A_4^j) \in$ II(X). Therefore $\bigcup_{j\in\Lambda} A^j \in$ IT(X).

(3) (a) Suppose that $(\theta_1, \theta_2, \theta_3, \theta_4) \neq (\delta_1, \delta_2, \delta_3, \delta_4)$. Then $\theta_i \neq \delta_i$ for some $i \in \{1, 2, 3, 4\}$. If $\theta_1 \neq \delta_1$ (resp. $\theta_2 \neq \delta_2$, $\theta_3 \neq \delta_3$, $\theta_4 \neq \delta_4$), then there exists $A \in \theta_1$ (resp. θ_2, θ_3, θ_4), say, such that $A \notin \delta_1$ (resp. δ_2, δ_3, δ_4). Then $(A, 1_X, 1_X, 1_X)$ (resp. $(1_\phi, A, 1_X, 1_X)$, $(1_\phi, 1_\phi, A, 1_\phi)$, $(1_\phi, 1_\phi, 1_X, A)) \in \eta((\theta_1, \theta_2, \theta_3, \theta_4))$ but $\notin \eta((\delta_1, \delta_2, \delta_3, \delta_4))$. Therefore η is injection.

(b) Immediate.

(c) One direction is obvious. For the other direction, suppose that $H \in \theta_1$ (resp. θ_2, θ_3, θ_4). Then $(H, 1_X, 1_X, 1_X)$ (resp. $(1_\phi, H, 1_X, 1_X)$, $(1_\phi, 1_\phi, H, 1_\phi), (1_\phi, 1_\phi, 1_X, H)) \in \eta((\theta_1, \theta_2, \theta_3, \theta_4))$. Therefore
$H \in (\eta((\theta_1, \theta_2, \theta_3, \theta_4)))_1$(resp. $(\eta((\theta_1, \theta_2, \theta_3, \theta_4)))_2$, $(\eta((\theta_1, \theta_2, \theta_3, \theta_4)))_3$, $(\eta((\theta_1, \theta_2, \theta_3, \theta_4)))_4$). □

Remark 3.3. There exists a bijection between η(PT(X)) and PT(X) (See Theorem 3.2.(2) and (3) (a)).

The following example illustrates that the converse of Theorem 3.2.(3) (b) may not be true.

Example 3.4. Let $X = [0,\ 1]$ and $\tau = \{\tilde{1}, \tilde{0}, A, B\}$, where $A_1(x) = \frac{1}{4}$, $A_2(x) = B_1(x) = B_2(x) = \frac{1}{2}$, $A_4(x) = \frac{2}{3}$, and $A_3(x) = B_3(x) = B_4(x) = 1$ for each $x \in [0,\ 1]$. Then $\tau \in \mathrm{IT}([0,\ 1])$. Since $(1_\phi, 1_\phi, 1_X, \frac{2}{3}1_X) \in \eta(\xi(\tau))$ but $(1_\phi, 1_\phi, 1_X, \frac{2}{3}1_X) \notin \tau$, then $\eta(\xi(\tau)) \nsubseteq \tau$.

Theorem 3.5. *If* $\tau = \eta((\tau_1, \tau_2, \tau_3, \tau_4))$*, then* $\eta(\xi(\tau)) = \tau$.

Proof. From Theorem 3.2.(3) (c)
$\eta(\xi(\tau)) = \eta(\xi(\eta((\tau_1, \tau_2, \tau_3, \tau_4))))$
$= \eta((\tau_1, \tau_2, \tau_3, \tau_4)) = \tau$. □

Theorem 3.6. *The pre-suitable equality relation* R_{PSE} *on* $\mathrm{IT}(X)$ *is an equivalence relation on* $\mathrm{IT}(X)$.

4. Basic concepts of SIIF topology

Theorem 4.1. *Let* $\tau = \eta((\tau_1, \tau_2, \tau_3, \tau_4))$, $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$ *and* $A \in \mathrm{II}(X)$. *Then* $Int_\tau(A)$
$= (Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4))$.

Proof. (1) $Int_\tau(A) = \bigcup\{B : B \in \tau, B \subseteq A\}$
$= \Big(\bigcup_{B\in\tau, B\subseteq A} B_1, \bigcup_{B\in\tau, B\subseteq A} B_2, \bigcup_{B\in\tau, B\subseteq A} B_3, \bigcup_{B\in\tau, B\subseteq A} B_4\Big)$
From Theorem 3.2.(3) (c), $\bigcup_{B\in\tau, B\subseteq A} B_i \in \tau_i$ for each $i \in \{1, 2, 3, 4\}$. Since $\bigcup_{B\in\tau, B\subseteq A} B_i \subseteq A_i$ for each $i \in \{1, 2, 3, 4\}$, then $\bigcup_{B\in\tau, B\subseteq A} B_i \subseteq Int_{\tau_i}(A_i)$ for each $i \in \{1, 2, 3, 4\}$. Therefore, the suitability of $(\tau_1, \tau_2, \tau_3, \tau_4)$ implies that $(Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4)) \in \mathrm{II(X)}$. Hence $Int_\tau(A) \subseteq (Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4))$.
(2) Since $(Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4)) \in \mathrm{II}(X)$, then it belongs to τ. Now, $(Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4))$
$\subseteq (A_1, A_2, A_3, A_4) = A$. Therefore
$(Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4))$
$\subseteq Int_\tau(A)$. □

Theorem 4.2. *Let* $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$ *and* $A \in \mathrm{II}(X)$. *Then*
$Cl_\tau(A) = (Cl_{\tau_3}(A_1), Cl_{\tau_4}(A_2), Cl_{\tau_1}(A_3), Cl_{\tau_2}(A_4))$, *where* $\tau = \eta((\tau_1, \tau_2, \tau_3, \tau_4))$.

Proof. From Theorem 2.20. [32], we have
$Cl_\tau(A) = (Int_\tau(A^c))^c$
$= (Int_\tau((A_3^c, A_4^c, A_1^c, A_2^c)))^c$
$= (Int_{\tau_1}(A_3^c), Int_{\tau_2}(A_4^c), Int_{\tau_3}(A_1^c), Int_{\tau_4}(A_2^c))^c$
$= \big((Int_{\tau_3}(A_1^c))^c, (Int_{\tau_4}(A_2^c))^c, (Int_{\tau_1}(A_3^c))^c, (Int_{\tau_2}(A_4^c))^c\big)$
$= (Cl_{\tau_3}(A_1), Cl_{\tau_4}(A_2), Cl_{\tau_1}(A_3), Cl_{\tau_2}(A_4))$. □

Remark 4.3. One can deduce that an interval-valued intuitionistic fuzzy point p in X ($p \in \mathrm{IP}(X)$, for short) can be uniquely determined as $(t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$, where $t_1 \leq t_2 \leq t_4 \leq t_3$ and $t_2 > 0$. Note that t_1 may be equal 0.

Definition 4.4. Let $\tau \in \mathrm{IT}(X)$ and $A \in \mathrm{II}(X)$. Then A is a τ-Q-neighborhood of $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x) \in \mathrm{IP}(X)$ (or equivalently, p is a τ-Q-interior point of A) if and only if

(1) $t_1 = 0$ and A_i is a τ_i-Q-neighborhood of $t_i 1_x$ for each $i \in \{2, 3, 4\}$; or
(2) $t_1 > 0$ and A_i is a τ_i-Q-neighborhood of $t_i 1_x$ for each $i \in \{1, 2, 3, 4\}$, where $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$.

The family of all τ-Q-neighborhoods of p will be denoted by $N_\tau^Q(p)$.

Theorem 4.5. *Let* $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$*,* $A \in \mathrm{II}(X)$ *and* $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x) \in \mathrm{IP}(X)$. *Then*

(1) *A is an $\eta((\tau_1, \tau_2, \tau_3, \tau_4))$-Q-neighborhood of p if and only if $t_1 = 0$ and A_2 is a τ_2-Q-neighborhood of $t_2 1_x$.*
(2) *A is an $\eta((\tau_1, \tau_2, \tau_3, \tau_4))$-Q-neighborhood of p if and only if $t_1 > 0$ and A_1 is a τ_1-Q-neighborhood of $t_1 1_x$.*

Proof. One direction in the two statements is obvious and the other is obtained since $\tau_1 \subseteq \tau_2 \subseteq \tau_4 \subseteq \tau_3$. □

Remark 4.6.
We write $p\ q\ A$ to mean that:

(1) $t_1 = 0$ and $t_i 1_x\ q\ A_i$ for each $i \in \{2, 3, 4\}$; or
(2) $t_1 > 0$ and $t_i 1_x\ q\ A_i$ for each $i \in \{1, 2, 3, 4\}$.

Theorem 4.7. *Let* $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$*,* $A, B \in \mathrm{II}(X)$ *and* $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x) \in \mathrm{IP}(X)$. *Denote* $\eta((\tau_1, \tau_2, \tau_3, \tau_4))$ *by* τ. *Then we have:*

(1) *(i)* $\tilde{1} \in N_\tau^Q(p)$ *for every* $p \in \mathrm{IP}(X)$*;*
(ii) If $A \in N_\tau^Q(p)$*, then* $p\ q\ A$*;*
(iii) If $A, B \in N_\tau^Q(p)$*, then* $A \cap B \in N_\tau^Q(p)$*;*
(iv) If $A \in N_\tau^Q(p)$ *and* $A \subseteq B$*, then* $B \in N_\tau^Q(p)$*;*
(v) If $A \in N_\tau^Q(p)$*, then there exists* $B \in N_\tau^Q(p)$ *such that* $B \subseteq A$ *and* $B \in N_\tau^Q(s)$ *for every* $s \in \mathrm{IP}(X)$ *and* $s\ q\ B$*.*
$t_1 = 0$ *and* $t_i 1_x\ q\ A_i$ *for each* $i \in \{2, 3, 4\}$*; or*

(2) *For every* $p \in \mathrm{IP}(X)$*, suppose there exists* $U_p \subseteq \mathrm{II}(X)$ *satisfying (i)-(iv), then* $\theta = \{A \in U_p : p\ q\ A\} \in \mathrm{IT}(X)$*. If in addition* U_p *satisfies (v), then* $U_p = N_\theta^Q(p)$*.*

Proof. (1) (i)-(iv) are immediate.

(v) Suppose that $A \in N_\tau^Q(p)$. If $t_1 = 0$ we have that A_2 is a τ_2-Q-neighborhood of $t_2 1_x$. Thus, there exists $O_2 \in \tau_2$ such that $t_2 1_x\ q\ O_2 \subseteq A_2$. Now, we obtain $B = (Int_{\tau_1}(A_1), Int_{\tau_2}(A_2), Int_{\tau_3}(A_3), Int_{\tau_4}(A_4)) \in N_\tau^Q(p)$ and $B \subseteq A$. If $s\ q\ B$, $s \in \mathrm{IP}(X)$, one can easily have that $B \in N_\tau^Q(s)$.

(2) First, we prove that $\theta \in \mathrm{IT}(X)$ as follows:

(a) It is obvious that $\tilde{0},\ \tilde{1} \in \theta$;

(b) Suppose $A, B \in \theta$. Then if $t_1 = 0$ and $t_i 1_x\ q\ (A \cap B)$ for each $i \in \{2, 3, 4\}$, then from (iii), $A \cap B \in U_p$. Therefore $A \cap B \in \theta$. One can deduce in a similar way that $A \cap B \in \theta$ when $t_1 > 0$.

(c) Suppose $\{A^j : j \in \Lambda\} \subseteq \theta$. If $t_1 = 0$ and $t_i 1_x\ q\ \bigcup_{j\in\Lambda} A^j$ for each $i \in \{2, 3, 4\}$, then $\bigvee_{j\in\Lambda} A^j(x) + t_i > 1$ for each $i \in \{2, 3, 4\}$. Thus, there exists $j_\circ \in \Lambda$ such that $A^{j_\circ}(x) + t_i > 1$ for each $i \in \{2, 3, 4\}$. So, $A^{j_\circ} \in U_p$. From (I) (iv) we have $\bigcup_{j\in\Lambda} A^j \in U_p$. Therefore $\bigcup_{j\in\Lambda} A^j \in \theta$. On the other hand, if $t_1 > 0$, one can prove in a similar way that $\bigcup_{j\in\Lambda} A^j \in \theta$.

Now, suppose that U_p satisfies (v) and $C \in U_p$. Then from (v), there exists $B \in U_p$ such that $B \subseteq C$ and $B \in U_s$ for every $s\ q\ B$, $s \in \mathrm{IP}(X)$. Thus $B \in \theta$. So one can deduce that $C \in N_\theta^Q(p)$. Again, suppose that $C \in N_\theta^Q(p)$. Then there exists $D \in \theta$ such that $p\ q\ D \subseteq C$. Then $D \in U_p$. Therefore from (iv), $C \in U_p$. □

Theorem 4.8. *Let* $\tau \in$ IT(X)*. If* $\mathcal{B}$ *(resp.* $\mathcal{S}$*)* $\subseteq \tau$ *is a base (resp. subbase) for* τ*, then* $\mathcal{B}^i = \{A_i : A \in \mathcal{B}\}$ *(resp.* $\mathcal{S}^i = \{A_i : A \in \mathcal{S}\}$*) is a base (resp. subbase) for* τ_i *for each* $i \in \{1, 2, 3, 4\}$*, where* $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$*.*

Proof. Suppose $i \in \{1, 2, 3, 4\}$.

(1) For the base case, suppose $M \in \tau_i$. Then there exists $A \in \tau$ such that $A_i = M$. Thus, there exists a subfamily $\{\mathcal{B}^k : k \in \Lambda\}$ of $\mathcal{B}$ such that $\bigcup_{k\in\Lambda} \mathcal{B}^k = A$ and so there exists a subfamily $\{\mathcal{B}_i^k : k \in \Lambda\}$ of $\mathcal{B}^i$ such that $\bigcup_{k\in\Lambda} \mathcal{B}_i^k = A_i = M$. Hence $\mathcal{B}^i$ is a base for τ_i .

(2) For the subbase case, suppose $M \in \tau_i$. Then there exists $A \in \tau$ such that $A_i = M$. Thus, there exist finite sets Λ_l and arbitrary set ℓ such that $\bigcup_{l\in\ell} \bigcap_{k\in\Lambda_l} \mathcal{B}^k = A$ and $\{\mathcal{B}^k : k \in \Lambda_l\} \subseteq \mathcal{S}$ for each $l \in \ell$. Hence there exist finite sets Λ_l and arbitrary set ℓ such that $\bigcup_{l\in\ell} \bigcap_{k\in\Lambda_l} \mathcal{B}_i^k = A_i = M$ and $\{\mathcal{B}_i^k : k \in \Lambda_l\} \subseteq \mathcal{S}^i$ for each $l \in \ell$. Hence $\mathcal{S}^i$ is a subbase for τ_i. □

Remark 4.9. In the following example we illustrate that if $\mathcal{B}^i$ is a base for τ_i for each $i \in \{1, 2, 3, 4\}$, then $\mathcal{B} = \{A \in \mathrm{II}(X) : A_i \in \mathcal{B}^i,\ i \in \{1, 2, 3, 4\}\}$ need not be a base for τ even $\tau = \eta((\tau_1, \tau_2, \tau_3, \tau_4))$ and $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$.

Example 4.10. Let $X = \{a, b, c\}$ and $\tau = \{\mu : \mu \in \mathrm{II}(X),\ \mu_i \in \tau_i,\ i \in \{1, 2, 3, 4\}\}$, where $\tau_1 = \{1_X, 1_\phi\} \cup \{\alpha, \beta, \gamma, \delta\}$ and $\tau_2 = \tau_3 = \tau_4 = \{1_X, 1_\phi\} \cup \{\alpha, \beta, \gamma, \delta, \zeta, \xi, \lambda, \nu, \varphi\}$, where $\alpha(a) = \frac{1}{4}, \alpha(b) = \frac{1}{2}, \alpha(c) = 1$, $\beta(a) = \frac{1}{3}, \beta(b) = 0, \beta(c) = \frac{1}{2}$, $\gamma(a) = \frac{1}{4}, \gamma(b) = 0, \gamma(c) = \frac{1}{2}$, $\delta(a) = \frac{1}{3}, \delta(b) = \frac{1}{2}, \delta(c) = 1, \zeta(a) = 0, \zeta(b) = 0, \zeta(c) = 1, \xi(a) = \frac{1}{4}, \xi(b) = 0, \xi(c) = 1$, $\lambda(a) = 0, \lambda(b) = 0, \lambda(c) = \frac{1}{2}$, $\nu(a) = 0, \nu(b) = \frac{1}{2}, \nu(c) = 1$, $\varphi(a) = \frac{1}{3}$, $\varphi(b) = 0$, $\varphi(c) = 1$. Hence, τ_1, τ_2, τ_3 and τ_4 are fuzzy topologies on X and $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$.

Now, $\mathcal{B}^1 = \{1_X, 1_\phi, \alpha, \beta, \gamma\}$ is a base for τ_1 and $\mathcal{B}^i = \{1_X, 1_\phi, \beta, \gamma, \zeta, \xi, \lambda, \nu\}$ is a base for τ_i where $i \in \{2, 3, 4\}$. It is clear that $(\alpha, \alpha, \alpha, \alpha) \in \eta((\tau_1, \tau_2, \tau_3, \tau_4))$ but there exists no IIF sets can be constructed from the bases $\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3$ and $\mathcal{B}^4$ such that $(\alpha, \alpha, \alpha, \alpha)$ can be written as a union of them.

5. IIF homeomorphisms and IIF compactness

We suppose that the definitions of continuous (resp. open, closed, homeomorphism) functions in the frameworks of F topology and IF topology are well known.

Definition 5.1. Let $\tau \in \mathrm{IT}(X)$, $\sigma \in \mathrm{IT}(Y)$ and $f : X \to Y$ be a function. Then

(1) f is said to be interval-valued intuitionistic fuzzy continuous (IIF continuous, for short) if and only if $(f(B))^{-1} \in \tau)$ for each $B \in \sigma$.
(2) f is said to be interval-valued intuitionistic fuzzy open (IIF open, for short) (resp. closed) (IIF closed, for short) if and only if for each $A \in \tau$ (resp. $B^c \in \tau$), $f(A) \in \sigma$ (resp. $(f(B))^c \in \sigma$).
(3) f is said to be an interval-valued intuitionistic fuzzy homeomorphism (IIF homeomorphism, for short) if and only if
  (a) f is a bijection, and
  (b) f and f^{-1} are IIF continuous.

Theorem 5.2. *Let $\tau \in \mathrm{IT}(X)$, $\sigma \in \mathrm{IT}(Y)$ and $f : X \to Y$ be a function. Then*

(1) *f is IIF open if and only if $f(Int_\tau(A)) \subseteq Int_\sigma(f(A))$ for every $A \in \mathrm{II}(X)$;*
(2) *f is IIF closed if and only if $f(Cl_\tau(A)) \supseteq Cl_\sigma(f(A))$ for every $A \in \mathrm{II}(X)$;*
(3) *If f is a bijection, then f is IIF open if and only if f^{-1} is IIF continuous if and only if f is IIF closed.*

Proof.

(1) $\Rightarrow$ Since $Int_\tau(A) \in \tau$, then $f(Int_\tau(A)) \subseteq Int_\sigma(f(Int_\tau(A))) \subseteq Int_\sigma(f(A))$.
$\Leftarrow$ Let $A \in \tau$. Then $f(A) = f(Int_\tau(A)) \subseteq Int_\sigma(f(A))$. Hence $f(A) \in \sigma$.
(2) $\Rightarrow$ Since $(Cl_\tau(A))^c \in \tau$, then $(f(Cl_\tau(A)))^c \in \sigma$ and so, $f(Cl_\tau(A)) \supseteq Cl_\sigma(f(Cl_\tau(A))) \supseteq Cl_\sigma(f(A))$.
$\Leftarrow$ Let $A^c \in \tau$. Then $f(A) = f(Cl_\tau(A)) \supseteq Cl_\sigma(f(A))$.
Hence $(f(A))^c \in \sigma$.
(3) The proof of the statements is obtained from the facts: $(f^{-1})^{-1} = f$ and $(f(A))^c = f(A^c)$.

□

Theorem 5.3. *Let $\tau \in \mathrm{IT}(X)$, $\sigma \in \mathrm{IT}(Y)$ and $f : X \to Y$ be a function. Then the following statements are equivalent:*

(1) *f is an IIF homeomorphism;*
(2) *f is a bijection, IIF open and IIF continuous;*
(3) *f is a bijection and $f(Cl_\tau(A)) = Cl_\sigma(f(A))$ for every $A \in \mathrm{II}(X)$;*
(4) *f is a bijection, IIF closed and IIF continuous.*

Theorem 5.4.

(1) *Let $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$ and $\xi(\sigma) = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. If $f : (X, \tau) \to (Y, \sigma)$ is IIF continuous (resp. open, closed, homeomorphism), then $f_i : (X, \tau_i) \to (Y, \sigma_i)$ is fuzzy continuous (resp. open, closed, homeomorphism) for each $i \in \{1, 2, 3, 4\}$.*
(2) *If $f_i : (X, \tau_i) \to (Y, \sigma_i)$ is fuzzy continuous (resp. open, closed, homeomorphism) for each $i \in \{1, 2, 3, 4\}$, then $f : (X, \tau) \to (Y, \sigma)$ is IIF continuous (resp. open, closed, homeomorphism), where $\tau = \eta((\tau_1, \tau_2, \tau_3, \tau_4))$ and $\sigma = \eta((\sigma_1, \sigma_2, \sigma_3, \sigma_4))$.*

Theorem 5.5. Let $f : (X, \tau) \to (Y, \sigma)$ be a function, where (X, τ) and (Y, σ) are IIF topological spaces. Then:

(1) *f is IIF continuous (resp. open , closed , homeomorphism) implies $f : (X, \tau_{IF}) \to (Y, \sigma_{IF})$ is IF continuous (resp. open, closed, homeomorphism) implies $f : (X, \tau_F) \to (Y, \sigma_F)$ is fuzzy continuous (resp, open, closed, homeomorphism) implies $f : (X, \tau_O) \to (Y, \sigma_O)$ is continuous (resp. open, closed, homeomorphism);*
(2) *If f is IIF continuous (resp. open , closed , homeomorphism), then $f : (X, \tau_\Box) \to (Y, \sigma_\Box)$ and $f : (X, \tau_\Diamond) \to (Y, \sigma_\Diamond)$ so are.*

Definition 5.6. Let (X, τ) be an IIF topological space. Then:

(1) A subfamily $\mathcal{U}$ of τ is called an interval-valued intuitionistic fuzzy open cover (IIF open cover for short) of X if and only if $\bigcup\{A : A \in \mathcal{U}\} = \tilde{1}$;
(2) A finite subfamily $\mathcal{U}_\circ$ of an IIF open cover $\mathcal{U}$ of X which is also an IIF open cover of X is called a finite subcover of $\mathcal{U}$;
(3) A subfamily $\mathcal{M}$ of IIF closed sets of X has the finite intersection property (FIP for short) if and only if every finite subfamily $\mathcal{M}_\circ$ of $\mathcal{M}$ satisfies the condition $\bigcap_{A \in \mathcal{M}_\circ} A \neq \tilde{0}$;
(4) (X, τ) is called an interval-valued intuitionistic fuzzy compact topological space (IIF compact space for short) if and only if every IIF open cover of X has a finite subcover.

Theorem 5.7. (X, τ) *is an IIF compact space if and only if every subfamily of IIF closed sets of X has the FIP has a nonempty intersection.*

Theorem 5.8. *(X, τ) is an IIF compact space if and only if $(X, \tau_\Box)$ so is if and only if $(X, \tau_\Diamond)$ so is.*

Proof. Necessity: Follows from the fact that $\Box A \subseteq A \subseteq \Diamond A$.
Sufficiency: Suppose that $(X, \tau_\Box)$ is an IIF compact space and $\mathcal{A} = \{\Diamond G^j : j \in \Lambda\} \subseteq \tau_\Diamond$ is a cover of $\tilde{1}$. Then $(\bigcup_{j\in\Lambda} G_1^j, \bigcup_{j\in\Lambda} G_2^j, \bigcup_{j\in\Lambda} G_3^j, \bigcup_{j\in\Lambda} G_4^j) = (1_X, 1_X, 1_X, 1_X)$. Since $G_1^j \leq G_2^j \leq G_4^j$ for each $j \in \Lambda$ and $\bigcup_{j\in\Lambda} G_1^j = 1_X$, then
$\big(\bigcup_{j\in\Lambda} G_1^j, \bigcup_{j\in\Lambda} G_2^j, \bigcup_{j\in\Lambda} G_3^j, \bigcup_{j\in\Lambda} G_2^j\big)$ $= \bigcup\{\Box G^j : j \in \Lambda\} \subseteq \tau_\Box$ is a cover of $\tilde{1}$. Thus, there exists $\Box G^1, \Box G^2, \ldots, \Box G^n \in \{\Box G^j : j \in \Lambda\}$ such that $\big(\bigcup_{i=1}^n G_1^i, \bigcup_{i=1}^n G_2^i, \bigcup_{i=1}^n G_3^i, \bigcup_{i=1}^n G_2^i\big) = (1_X, 1_X, 1_X, 1_X)$. Hence there exists $\Diamond G^1, \Diamond G^2, \ldots, \Diamond G^n \in \{\Diamond G^j : j \in \Lambda\}$ such that $\big(\bigcup_{i=1}^n G_1^i, \bigcup_{i=1}^n G_4^i, \bigcup_{i=1}^n G_3^i, \bigcup_{i=1}^n G_4^i\big)$ $= \bigcup_{i=1}^n \Diamond G^i = \tilde{1}$. Therefore $(X, \tau_\Diamond)$ is an IIF compact space. □

Theorem 5.9. *Let (X, τ) be an IIF compact space, (Y, σ) be IIF space and $f : (X, \tau) \to (Y, \sigma)$ be a continuous surjective function. Then (Y, σ) is IIF compact.*

Theorem 5.10. *The following statements are true:*

(1) *If (X, τ) is IIF compact space, then (X, τ_{IF}) is an IF compact;*
(2) *If (X, τ_{IF}) is IF compact space, then (X, τ_F) is an F compact;*
(3) *If (X, τ_F) is an F compact space, then (X, τ_O) is compact.*

The converse of each of the above statements is not true in general as shown by the following example.

Example 5.11.

(1) Suppose $A_n(x) = (1 - \frac{1}{n}, 1 - \frac{1}{2n}, 1 - \frac{1}{4n}, 1 - \frac{1}{3n})$ for each $x \in [0, 1]$ and $n \in \mathrm{N}$ and $\tau = \{\tilde{1}, \tilde{0}\} \cup \{A_n : n \in \mathrm{N}\}$. Thus $([0, 1], \tau_{IF})$ is IF compact space but $([0, 1], \tau)$ is not IIF compact space;
(2) Suppose $A_n(x) = (1 - \frac{1}{n}, 1 - \frac{1}{n}, 1 - \frac{1}{2n}, 1 - \frac{1}{2n})$ for each $x \in [0, 1]$ and $n \in \mathrm{N}$ and $\tau = \{\tilde{1}, \tilde{0}\} \cup \{A_n : n \in \mathrm{N}\}$. Thus $([0, 1], \tau_F)$ is F compact space but $([0, 1], \tau_{IF})$ is not IF compact space;
(3) Suppose $A_n(x) = (1 - \frac{1}{n}, 1 - \frac{1}{n}, 1 - \frac{1}{n}, 1 - \frac{1}{n})$ for each $x \in [0, 1]$ and $n \in \mathrm{N}$ and $\tau = \{\tilde{1}, \tilde{0}\} \cup \{A_n : n \in \mathrm{N}\}$. Thus $([0, 1], \tau_O)$ is compact space but $([0, 1], \tau_F)$ is not F compact space.

Theorem 5.12. *Let $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{PT}(X)$.*

(1) *If (X, τ_i) is F-compact spaces for each $i \in \{1, 2, 3, 4\}$, then $(X, \eta((\tau_1, \tau_2, \tau_3, \tau_4)))$ is IIF compact;*
(2) *If $(X, \eta((\tau_1, \tau_2, \tau_3, \tau_4)))$ is IIF compact, then (X, τ_1) is F compact.*

Proof. (1) Suppose $H \subseteq \eta((\tau_1, \tau_2, \tau_3, \tau_4))$ and $\bigcup_{h\in H} h = \tilde{1}$. Then $H^i = \{h_i : h \in H\} \subseteq \tau_i$ and $\bigcup_{h_i\in H} h_i = 1_X$ for each $i \in \{1, 2, 3, 4\}$. Thus, there exists a finite subset $\{h_i^1, h_i^2, \ldots, h_i^{n_i}\}$ of H^i such that $\bigcup_{j=1}^{n_i} h_i^j = 1_X$ for each $i \in \{1, 2, 3, 4\}$. Choose $S = \bigcup\{j : j \in n_i, i \in \{1, 2, 3, 4\}\}$. Therefore, S is finite, and we have $\bigcup_{s\in S} h^s = \tilde{1}$.
(2) Suppose $M \subseteq \tau_1$ such that $\bigcup_{B\in M} B = 1_X$. Then $M^* = \{(B, 1_X, 1_X, 1_X) : B \in M\} \subseteq \eta((\tau_1, \tau_2, \tau_3, \tau_4))$. Then there exists a finite subset $\{(B^1, 1_X, 1_X, 1_X), \ldots, (B^n, 1_X, 1_X, 1_X)\}$ of M^* such that $\bigcup_{k=1}^n (B^k, 1_X, 1_X, 1_X) = \tilde{1}$. Hence there exists a finite subset $\{B^1, \ldots, B^k\}$ of M such that $\bigcup_{k=1}^n B^k = 1_X$. □

Corollary 5.13. *If $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$ and (X, τ_3) is F compact, then $(X, \eta((\tau_1, \tau_2, \tau_3, \tau_4)))$ is an IIF compact.*

The following example illustrate that there exists $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$ and $(X, \eta((\tau_1, \tau_2, \tau_3, \tau_4)))$ is IIF compact space but (X, τ_i) is not F compact for $i \in \{2, 3, 4\}$.

Example 5.14. Let X be any nonempty set. Define $(\tau_1, \tau_2, \tau_3, \tau_4) \in \mathrm{ST}(X)$, where $\tau_1 = \{1_X, 1_\phi\}$ and $\tau_2 = \tau_3 = \tau_4 = \{1_X, 1_\phi\} \cup \{f_n : n \in \mathrm{N}\}$, where $f_n(x) = 1 - \frac{1}{n}$ for every $n \in \mathrm{N}$ and $x \in X$. Then $(X, \eta((\tau_1, \tau_2, \tau_3, \tau_4)))$ is IIF compact space but (X, τ_i) is not F compact for $i \in \{2, 3, 4\}$.

6. Conclusion

In this paper, some new results about intuitionistic fuzzy topological spaces are obtained. This subject has been studied by many mathematicians and have applications to medicine, photography, and other. By making use of a characterization of the concept of interval-valued intuitionistic fuzzy sets (Theorem 2.5.), the concepts of homeomorphism functions (Theorem 5.3.), and compactness (Theorems 5.7.-5.10.) are introduced in interval-valued intuitionistic fuzzy topological spaces. Furthermore, the concepts of the base and subbase in interval-valued intuitionistic fuzzy topological spaces (Theorem 4.8.) are given.

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References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, in: VII ITKR's Session, Sofia, June 1983.

[2] K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), 87–96.

[3] K.T. Atanassov, Review and new results on intuitionistic fuzzy sets, preprint IM-MFAIS-1–88, Sofia, 1988.

[4] K.T. Atanassov, Operators over interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **64** (1994), 159–174.

[5] K.T. Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Springer-Verlag, Heidelberg, New York, 1999.

[6] K. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **31**(3) (1989), 343–349.

[7] P. Burillo and H. Bustince, Two operators on interval-valued intuitionistic fuzzy sets: Part I, *Comptes rendus de l'Académie bulgare des sciences* **47**(12) (1994), 9–12.

[8] P. Burillo and H. Bustince, Two operators on interval-valued intuitionistic fuzzy sets: Part II, *Comptes rendus de l'Académie bulgare des sciences* **48**(1) (1995), 17–20.

[9] C.L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications* **24** (1968), 182–190.

[10] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems* **88** (1997), 81–89.

[11] D. Çoker, An introduction to fuzzy subspaces in intuitionistic fuzzy topological spaces, *Journal Fuzzy Mathematics* **4** (1996), 749–764.

[12] D. Çoker and M. Demirci, On intuitionistic fuzzy points, *Notes IFS* **1**(2) (1995), 79–84.

[13] D. Çoker and A.H. Eş, On fuzzy compactness in intuitionistic fuzzy topological spaces, *Journal Fuzzy Mathematics* **3** (1995), 899–909.

[14] E. Coşkun, Systems on intuitionistic fuzzy special sets and intuitionistic fuzzy special measures, *Information Sciences* **128** (2000), 105–118.

[15] G. Deschrijver, Generalized arithmetic operators and their relationship to t-norms in interval-valued fuzzy set theory, *Fuzzy Sets and Systems* **160** (2009), 3080–3102.

[16] A.H. Eş and D. Çoker, More on fuzzy compactness in intuitionistic fuzzy topological spaces, *Notes IFS* **2**(1) (1996), 4–10.

[17] A.H. Eş and D. Çoker, On several types of degree of fuzzy compactness, *Fuzzy Sets and Systems* **87** (1997), 349–359.

[18] I. Grattan-Guiness, Fuzzy membership mapped onto interval and many valued quantities, *Z Math Logik Grundlag Math* **22** (1975), 149–160.

[19] J.A. Goguen, L-fuzzy sets, *Journal of Mathematical Analysis and Applications* **18** (1967), 145–174.

[20] M.B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, *Fuzzy Sets and Systems* **21** (1987), 1–17.

[21] M. Guangwu, The basic theory of interval-valued fuzzy sets, *Applied Mathematics* (in Chinese) **6**(2) (1993), 212–217.

[22] H. Gürçay, D. Çoker and A.H. Eş, On fuzzy continuity in intuitionistic fuzzy topological spaces, *Journal Fuzzy Mathematics* **5** (1997), 365–378.

[23] W.-L. Hung and J.-W. Wu, Correlation of intuitionistic fuzzy sets by centroid method, *Information Sciences* **114** (2002), 219–225.

[24] K.U. Jahn, Interval-wertige mengen, *Math Nachr* **68** (1975), 115–132.

[25] S.J. Lee and E.P. Lee, The category of intuitionistic fuzzy topological spaces, *Bulletin of the Korean Mathematical Society* **37** (2000), 63–76.

[26] R. Lowen, Fuzzy topological spaces and fuzzy compactness, *Journal of Mathematical Analysis and Applications* **56** (1976), 621–633.

[27] F.G. Lupiaññez, Hausdorffness in intuitionistic fuzzy topological spaces, *Mathware & Soft Computing* **10** (2003), 17–22.

[28] F.G. Lupiaññez, Nets and filters in intuitionistic fuzzy topological spaces, *Information Sciences* **176** (2006), 2396–2404.

[29] T.K. Mondal and S.K. Samanta, Topology of interval-valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **119** (2001), 483–494.

[30] K.K. Mondal and S.K. Samanta, A study on intuitionistic fuzzy topological spaces, *Notes IFS* **9** (2003), 1–32.

[31] P.-M. Pu and Y.-M. Liu, Fuzzy Topology I, Neighborhood structure of a fuzzy point and Moore Smith convergence, *Journal of Mathematical Analysis Applications* **78** (1980), 571–599.

[32] D.G. Schwarz, The case for an interval based representation of ligiistic truth, *Fuzzy Sets and Systems* **20** (1985), 153–165.

[33] N. Turanh and D. Çoker, On some types of fuzzy connectedness in fuzzy topological spaces, *Fuzzy Sets and Systems* **60** (1993), 97–102.

[34] I.B. Turksen, Interval valued fuzzy sets based on normal forms, *Fuzzy Sets and Systems* **20** (1986), 191–210.

[35] G.-J. Wang, Y.Y. He, Intuitionistic fuzzy sets and L-fuzzy sets, *Fuzzy Sets and Systems* **110** (2000), 271–274.

[36] Z. Wenyi, L. Hongxing and S. Yu, Decomposition theorem of interval-valued fuzzy sets, *Journal of Beijing Normal University* (in Chinese), **39**(3) (2003), 171–177.

[37] Z. Wenyi, L. Hongxing and S. Yu, Representation theorem of interval-valued fuzzy sets, *Journal of Beijing Normal University* **39**(4) (2003), 444–447.
[38] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.
[39] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, interval-valued fuzzy sets, *IEEE Trans Systems Man Cybernet* **3**(1) (1973), 28–44.