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Concave (L, M)-fuzzy interior operators and (L, M)-fuzzy hull operators

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Abstract
In this paper, the notions of (concave) (L, M)-fuzzy interior operators are introduced. It is proved that the category of (L, M)-fuzzy concave spaces and the category of concave (L, M)-fuzzy interior spaces is isomorphic, and there is a Galois correspondence between the category of (L, M)-fuzzy concave spaces and the category of (L, M)-fuzzy interior spaces. In addition, (L, M)-fuzzy hull operators proposed by Sayed et al. (Filomat 33(13):4151–4163, 2019) are further studied. Particularly, some results in Sayed et al. (2019) are corrected.

Keywords Concave (L, M)-fuzzy interior operator · (L, M)-fuzzy hull operator · (L, M)-fuzzy convex structure · (L, M)-fuzzy convexity preserving function · Galois correspondence

Mathematics Subject Classification 52A01

1 Introduction and preliminaries

Abstract convexity theory (Soltan 1984; Van de Vel 1993) is one of the important branches of mathematics, it deals with set-theoretic structures which satisfies axioms similar to that usual convex sets fulfill. It plays an important role in various branches of mathematics. There are many different mathematical research fields that can be applied to axiomatic convexity,

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such as lattices, topological spaces, metric spaces and graphs (see, for example, Lassak 1977; Maruyama 2009; Šostak 1985; Soltan 1983; Van Mill 1977; Varlet 1975; Wang and Shi 2018; Xiu and Shi 2017).

Rosa (1994a, b) first generalized convex structure to I-convex structure. Also, introduced fuzzy topology fuzzy convexity spaces and the notion of fuzzy local convexity. Subsequently, many scholars generalize convex structures to other fuzzy context from different viewpoints. Generally speaking, there are three approaches to extensions of convex structures to the fuzzy context, they are called L-convex structures (see, for example, Jin and Li 2016; Li et al. 2019; Pang and Shi 2017, 2018, 2019), M-fuzzifying convex structures (see, for example, Li and Shi 2018; Shi and Li 2015; Pang 2020; Wang and Pang 2019; Xiu and Pang 2017, 2018; Xiu and Shi 2017) and (L, M)-fuzzy convex structures (see, for example, Sayed et al. 2019; Shi and Xiu 2017), respectively. Recently, there has been significant research on fuzzy convex structures (Li 2017; Pang and Xiu 2019; Pang and Zhao 2016; Shen and Shi 2020; Wu and Li 2019; Xiu and Li 2019a; Zhao et al. 2021a, b; Zhong et al. 2019).

Sayed et al. (2019) defined r-L-fuzzy biconvex sets in (L, M)-fuzzy convex structures. The transformation method between (L, M)-fuzzy hull spaces and (L, M)-fuzzy convex spaces were introduced, and a characterization of the product of the (L, M)-fuzzy hull operator was obtained. However, there are some problems in Sayed et al. (2019) as follows:

(1) Notice that L (M) is a completely distributive lattice, not a unit interval [0, 1]. So, some results on the proof on (L, M)-fuzzy hull operators need not be true;
(2) If there not exists $\mathcal{C}$ such that $\mathcal{CO}_{12} = \mathcal{CO}_{\mathcal{C}}$, then $\mathcal{CO}_{12}$ is an (L, M)-fuzzy hull operator on X, and $(\mathcal{CO}_{12})_{\mathcal{C}_{\mathcal{CO}_{12}}} = \mathcal{CO}_{12}$ need not be true. So, Proposition 2.6(2) and Corollary 2.7 in Sayed et al. (2019) need not be true;
(3) Let $X = \prod_{i\in J} X_i$, $p_i : X \longrightarrow X_i$ be the i-th projection and let $(\prod_{i\in J} X_i, \prod_{i\in J} \mathcal{C}_i)$ be the product of $\{(X_i, \mathcal{C}_i)\}_{i\in J}$. Define $\mathcal{CO} : L^X \times M_{0_M} \longrightarrow L^X$ by $\forall A \in L^X$ and $r \in M_{0_M}$,

$$\begin{aligned}\mathcal{CO}(A, r) &= \bigwedge_{i\in J} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(A), r)) \\ &= \prod_{i\in J} \mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(A), r).\end{aligned}$$

Then $\mathcal{CO}$ is an (L, M)-fuzzy hull operator on X, and $\mathcal{CO} = \mathcal{CO}_{\prod_{i\in J}\mathcal{C}_i}$ need not be true (here $\mathcal{CO}_{\prod_{i\in J}\mathcal{C}_i}$ is the (L, M)-fuzzy hull operator generated by the product of (L, M)-fuzzy convex structures $\{\mathcal{C}_i : i \in J\}$);
(4) It is not true that all projections p_i are (L, M)-fuzzy convex-to-convex functions, i.e., Theorem 3.14 in Sayed et al. (2019) is incorrect.

By these motivations, the main contributions of the present paper are to give investigations on concave (L, M)-fuzzy interior operators and further results of (L, M)-fuzzy hull operators.

Throughout this paper, let X be a non-empty set, both L and M be two completely distributive lattices with order reversing involution $'$ where $0_M(0_L)$ and $1_M(1_L)$ denote the least and the greatest elements in $M(L)$ respectively, and $M_{0_M} = M - \{0_M\}(L_{0_L} = L - \{0_L\})$. Recall that an order-reversing involution $'$ on L is a map $(-)' : L \longrightarrow L$ such that for any $c, d \in L$, the following conditions hold: (1) $c \leq d$ implies $d' \leq c'$. (2) $c'' = c$. The following properties hold for any subset $\{d_i : i \in J\} \in L$: (1) $\left(\bigvee_{i\in J} d_i\right)' = \bigwedge_{i\in J} d_i'$; (2) $(\bigwedge_{i\in J} d_i)' = \bigvee_{i\in J} d_i'$. An L-fuzzy subset of X in Goguen (1967) is a mapping $A : X \longrightarrow L$ and the family L^X denoted the set of all fuzzy subsets of a given X. For each $\beta \in L$, let $\underline{\beta}$ denote the constant L-fuzzy subset of X with the value β. The greatest and the least elements in L^X are denoted by 1_X and 0_X, respectively. The complementation of a fuzzy subset are defined as $B'(x) = (B(x))'$ for all $x \in X$. An element $e \neq 0_M$ in a lattice is called join-irreducible if $e = d \vee c$ implies $e = d$ or $e = c$ for all $d, c \in M$. Further, e is said to be coprime if $e \leq d \vee c$ implies $e \leq d$ or $e \leq c$ for all $d, c \in M$. The set of all non-zero join-irreducible elements (resp. coprime elements) of M is denoted $J(M)$ (resp. Copr(M)). It can be verified that if M is distributive, then $b \in M$ is join-irreducible iff it is coprime, which means $J(M)$=Copr(M). So, for convenience, we usually use $J(M)$ to stand for the set of all coprime elements of M if M is distributive. If M is a completely distributive lattice and $x \prec \bigvee_{t\in T} y_t$, then there must be $t^\star \in T$ such that $x \prec y_{t^\star}$ (here $x \prec c$ means: $N \subset M, c \leq \bigvee N \Rightarrow \exists y \in N$ such that $x \leq y$), and for each $b \in M$, $b = \bigvee\{a \in M : a \prec b\} = \bigvee\{a \in J(M) : a \prec b\}$. Some more properties of $\prec$ can be found in Liu and Luo (1997). We say $\{A_i : i \in J\}$ is a directed (resp. co-directed) subset of L^X, in symbols $\{A_i : i \in J\} \overset{\text{dir}}{\subseteq} L^X$ (resp. $\{A_i : i \in J\} \overset{\text{cdir}}{\subseteq} L^X$) if for each $A_1, A_2 \in \{A_i : i \in J\}$, there exists $A_3 \in \{A_i : i \in J\}$ such that $A_1, A_2 \leq A_3$ (resp. $A_1, A_2 \geq A_3$).

Definition 1.1 (Shi and Xiu 2017; Zhong et al. 2019) The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex space, where $\mathcal{C} : L^X \longrightarrow M$ satisfies the following axioms:

(LMC1) $\mathcal{C}(0_X) = \mathcal{C}(1_X) = 1_M$.
(LMC2) If $\{A_i : i \in J\} \subseteq L^X$ is nonempty, then $\mathcal{C}\left(\bigwedge_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(A_i)$.
(LMC3) If $\{A_i : i \in J\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then

$$\mathcal{C}\left(\bigvee_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{C}(A_i).$$

The mapping $\mathcal{C}$ is called an (L, M)-fuzzy convex structure on X. The triple $(X, \mathcal{C}_1, \mathcal{C}_2)$ is called an (L, M)-fuzzy biconvex space ((L, M)-fbcs, for short), where $\mathcal{C}_1$ and $\mathcal{C}_2$ are (L, M)-fuzzy convex structures on X.

Definition 1.2 The pair $(X, \mathcal{CO})$ is called an (L, M)-fuzzy hull space, where $\mathcal{CO} : L^X \times M_{0_M} \longrightarrow L^X$ satisfies the following conditions: for any $A, B \in L^X$ and $r, s \in M_{0_M}$,

(1) $\mathcal{CO}(0_X, r) = 0_X$.
(2) $A \leq \mathcal{CO}(A, r)$.
(3) If $A \leq B$, then $\mathcal{CO}(A, r) \leq \mathcal{CO}(B, r)$.
(4) If $r \leq s$, then $\mathcal{CO}(A, r) \leq \mathcal{CO}(A, s)$.
(5) $\mathcal{CO}(\mathcal{CO}(A, r), r) = \mathcal{CO}(A, r)$.
(6) If $\{A_i : i \in J\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then

$$\mathcal{CO}\left(\bigvee_{i\in J} A_i, r\right) = \bigvee_{i\in J} \mathcal{CO}(A_i, r).$$

A mapping $\mathcal{CO}$ is called an (L, M)-fuzzy hull operator on X.

Theorem 1.3 (Sayed et al. 2019) *Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space. Then $\mathcal{CO}_{\mathcal{C}} : L^X \times M_{0_M} \longrightarrow L^X$ defined by $\forall A \in L^X$ and $r \in M_{0_M}$,*

$$\mathcal{CO}_{\mathcal{C}}(A, r) = \bigwedge\{B \in L^X : A \leq B,\ \mathcal{C}(B) \geq r\}$$

is an (L, M)-fuzzy hull operator. The symbol $\mathcal{CO}_{\mathcal{C}}$ is called the (L, M)-fuzzy hull operator generated by an (L, M)-fuzzy convex structure $\mathcal{C}$.

Definition 1.4 (Xiu and Li 2019b) A mapping $\mathcal{A} : L^X \longrightarrow M$ is called an (L, M)-fuzzy concave structure on X if it satisfies the following axioms:

(LMA1) $\mathcal{A}(0_X) = \mathcal{A}(1_X) = 1_M$.
(LMA2) If $\{A_i : i \in J\} \subseteq L^X$ is nonempty, then

$$\mathcal{A}\left(\bigvee_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{A}(A_i).$$

(LMA3) If $\{A_i : i \in J\} \overset{\text{cdir}}{\subseteq} L^X$, then

$$\mathcal{A}\left(\bigwedge_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{A}(A_i).$$

If $\mathcal{A}$ is an (L, M)-fuzzy concave structure, then the pair $(X, \mathcal{A})$ is called an (L, M)-fuzzy concave space.

Proposition 1.5 (Sayed et al. 2019) *Let $(X, \mathcal{C}_1, \mathcal{C}_2)$ be an (L, M)-fbcs. For each $r \in M_{0_M}$ and $A \in L^X$, a mapping $\mathcal{CO}_{12} : L^X \times M_{0_M} \longrightarrow L^X$ is defined as follows*:

$$\mathcal{CO}_{12}(A, r) = \mathcal{CO}_{\mathcal{C}_1}(A, r) \wedge \mathcal{CO}_{\mathcal{C}_2}(A, r).$$

Then, $\mathcal{CO}_{12}$ is an (L, M)-fuzzy hull operator on X.

Proposition 1.6 (Sayed et al. 2019) *For an (L, M)-fuzzy hull operator $\mathcal{CO}_{12}$, $A \in L^X$ and $r \in M_{0_M}$ a mapping $\mathcal{C}_{\mathcal{CO}_{12}} : L^X \longrightarrow M$ is defined as follows*

$$\mathcal{C}_{\mathcal{CO}_{12}}(A) = \bigvee\{r \in M_{0_M} : A = \mathcal{CO}_{12}(A, r)\}.$$

Then:

(1) *$\mathcal{C}_{\mathcal{CO}_{12}}$ is an (L, M)-fuzzy convex structure on X.*
(2) $(\mathcal{CO}_{12})_{\mathcal{C}_{\mathcal{CO}_{12}}} = \mathcal{CO}_{12}$.

Definition 1.7 (Rodabaugh 1997; Zadeh 1965) Let $g : X \longrightarrow Y$. Then the image $g^{\rightarrow}(A)$ of $A \in L^X$ and the preimage $g^{\leftarrow}(B)$ of $B \in L^Y$ are defined by:

$$g^{\rightarrow}(A)(y) = \bigvee\{A(x) : x \in X, g(x) = y\}$$

and $g^{\leftarrow}(B) = B \circ g$, respectively. It can be verified that the pair $(g^{\rightarrow}, g^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$.

Definition 1.8 (Xiu and Li 2019b) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be (L, M)-fuzzy concave spaces, then a function $g : X \longrightarrow Y$ is called an (L, M)-fuzzy concavity preserving function ((L, M)-FCAP, for short) if $\mathcal{A}(g^{\leftarrow}(A)) \geq \mathcal{B}(A)$ for all $A \in L^Y$.

Definition 1.9 (Shi and Xiu 2017) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. A function $g : X \longrightarrow Y$ is called:

(1) An (L, M)-fuzzy convexity preserving function if $\mathcal{C}(g^{\leftarrow}(B)) \geq \mathcal{D}(B)$ for all $B \in L^Y$.
(2) An (L, M)-fuzzy convex-to-convex function if $\mathcal{D}(g^{\rightarrow}(A)) \geq \mathcal{C}(A)$ for all $A \in L^X$.

Theorem 1.10 (Shi and Xiu (2017)) *Let $\{(X_i, \mathcal{C}_i) : i \in J\}$ be a set of (L, M)-fuzzy convex spaces. Let X be the product of the sets X_i for $i \in J$, and let $p_i : X \longrightarrow X_i$ the projection for each $i \in J$. Define a mapping $S : L^X \longrightarrow M$ by*

$$S(A) = \bigvee_{i \in J} \bigvee_{p_i^{\leftarrow}(B)=A} \mathcal{C}_i(B) \quad \text{for each } A \in L^X.$$

Then the product convex structure $\mathcal{C}$ of X is the one generated by subbase S. The resulting (L, M)-fuzzy convex space $(X, \mathcal{C})$ is called the product of $\{(X_i, \mathcal{C}_i) : i \in J\}$ and is denoted by $\prod_{i \in J}(X_i, \mathcal{C}_i)$.

Theorem 1.11 (Shi and Xiu 2017) *Let $(X, \mathcal{C})$ be the product of $\{(X_i, \mathcal{C}_i) : i \in J\}$. Then for all $i \in J$, $p_i : X \longrightarrow X_i$ is an (L, M)-fuzzy convexity preserving function.*

Lemma 1.12 (Sayed et al. 2019) *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. Then $g : X \longrightarrow Y$ is an (L, M)-fuzzy convex-to-convex function if and only if $g^{\rightarrow}(\mathcal{CO}_{\mathcal{C}}(A, r)) \geq \mathcal{CO}_{\mathcal{D}}(g^{\rightarrow}(A), r)$ for all $A \in L^X$ and $r \in M_{0_M}$.*

2 Concave (L, M)-fuzzy interior operators

In this section, the notion of concave (L, M)-fuzzy interior operators will be given, and the relationship between the category of (L, M)-fuzzy concave spaces and that of (concave) (L, M)-fuzzy interior spaces will be studied.

Definition 2.1 A mapping $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ is called an (L, M)-fuzzy interior operator on X if it satisfies the following conditions: for any $A, B \in L^X$ and $r, s \in M_{0_M}$,

(1) $\mathcal{I}(1_X, r) = 1_X$.
(2) $\mathcal{I}(A, r) \leq A$.
(3) If $A \leq B$, then $\mathcal{I}(A, r) \leq \mathcal{I}(B, r)$.
(4) If $r \leq s$, then $\mathcal{I}(A, s) \leq \mathcal{I}(A, r)$.
(5) $\mathcal{I}(\mathcal{I}(A, r), r) = \mathcal{I}(A, r)$.
(6) If $\{A_i : i \in J\} \overset{\text{cdir}}{\subseteq} L^X$, then

$$\mathcal{I}\left(\bigwedge_{i \in J} A_i, r\right) = \bigwedge_{i \in J} \mathcal{I}(A_i, r).$$

The pair $(X, \mathcal{I})$ is called an (L, M)-fuzzy interior space if $\mathcal{I}$ is an (L, M)-fuzzy interior operator on X.

If $\mathcal{I}$ is an (L, M)-fuzzy interior operator on X, and $\mathcal{I}$ satisfies the following condition:

(7) $\mathcal{I}(B, \bigvee\{r \in M_{0_M} : B = \mathcal{I}(B, r)\}) = B$, then we say that $\mathcal{I}$ is a concave (L, M)-fuzzy interior operator on X, and a pair $(X, \mathcal{I})$ is called a concave (L, M)-fuzzy interior space.

Theorem 2.2 *Let $(X, \mathcal{A})$ be an (L, M)-fuzzy concave space. Then $\mathcal{I}^{\mathcal{A}} : L^X \times M_{0_M} \longrightarrow L^X$ defined by $\forall A \in L^X$ and $r \in M_{0_M}$,*

$$\mathcal{I}^{\mathcal{A}}(A, r) = \bigvee\{B \in L^X : B \leq A,\ \mathcal{A}(B) \geq r\},$$

is a concave (L, M)-fuzzy interior operator. The symbol $\mathcal{I}^{\mathcal{A}}$ is called the concave (L, M)-fuzzy interior operator generated by an (L, M)-fuzzy concavity $\mathcal{A}$.

Proof (1) For all $r \in M_{0_M}$, we have $\mathcal{A}(1_X) \geq r$. So, we obtain $\mathcal{I}^{\mathcal{A}}(1_X, r) = 1_X$.

(2) and (3) are satisfied from the definition of $\mathcal{I}^{\mathcal{A}}$.

(4) Suppose that $r \leq s$. Then by (2) and (3) we have $\mathcal{I}^{\mathcal{A}}(\mathcal{I}^{\mathcal{A}}(A, s), r) \leq \mathcal{I}^{\mathcal{A}}(A, s)$ and $\mathcal{I}^{\mathcal{A}}(\mathcal{I}^{\mathcal{A}}(A, s), r) \leq \mathcal{I}^{\mathcal{A}}(A, r)$. By the definition of $\mathcal{I}^{\mathcal{A}}$, we obtain

$$\begin{aligned}\mathcal{A}(\mathcal{I}^{\mathcal{A}}(A, s)) &= \mathcal{A}\left(\bigvee\{B \in L^X : B \leq A,\ \mathcal{A}(B) \geq s\}\right)\\ &\geq s \geq r.\end{aligned}$$

It implies that $\mathcal{I}^{\mathcal{A}}\left(\mathcal{I}^{\mathcal{A}}(A, s), r\right) \geq \mathcal{I}^{\mathcal{A}}(A, s)$. So,

$$\mathcal{I}^{\mathcal{A}}\left(\mathcal{I}^{\mathcal{A}}(A, s), r\right) = \mathcal{I}^{\mathcal{A}}(A, s).$$

Hence $\mathcal{I}^{\mathcal{A}}(A, s) \leq \mathcal{I}^{\mathcal{A}}(A, r)$.

(5) By (2) and (3) we have,

$$\mathcal{I}^{\mathcal{A}}(\mathcal{I}^{\mathcal{A}}(A, r), r) \leq \mathcal{I}^{\mathcal{A}}(A, r). \tag{2.1}$$

On the other hand, by the definition of $\mathcal{I}^{\mathcal{A}}(A, r)$ and (LMA2), we obtain $\mathcal{A}(\mathcal{I}^{\mathcal{A}}(A, r)) \geq r$. This implies,

$$\mathcal{I}^{\mathcal{A}}(A, r) \leq \mathcal{I}^{\mathcal{A}}(\mathcal{I}^{\mathcal{A}}(A, r), r). \tag{2.2}$$

So, from the inequalities (2.1) and (2.2), we obtain $\mathcal{I}^{\mathcal{A}}(\mathcal{I}^{\mathcal{A}}(A, r), r) = \mathcal{I}^{\mathcal{A}}(A, r)$.

(6) Let $\{A_i : i \in J\} \overset{cdir}{\subseteq} L^X$. Then $\{\mathcal{I}^{\mathcal{A}}(A_i, r) : i \in J\} \overset{cdir}{\subseteq} L^X$. By (2) we have,

$$\bigwedge_{i \in J} \mathcal{I}^{\mathcal{A}}(A_i, r) \leq \bigwedge_{i \in J} A_i.$$

By (LMA2) and (LMA3), we have

$$\mathcal{A}\left(\bigwedge_{i \in J} \mathcal{I}^{\mathcal{A}}(A_i, r)\right) \geq \bigwedge_{i \in J} \mathcal{A}(\mathcal{I}^{\mathcal{A}}(A_i, r)) \geq r.$$

So, we have

$$\begin{aligned}&\mathcal{I}^{\mathcal{A}}\left(\bigwedge_{i \in J} A_i, r\right)\\ &\quad= \bigvee\{B \in L^X : B \leq \bigwedge_{i \in J} A_i, \mathcal{A}(B) \geq r\}\\ &\quad\geq \bigwedge_{i \in J} \mathcal{I}^{\mathcal{A}}(A_i, r).\end{aligned} \tag{2.3}$$

On the other hand, for $i \in J$, we have $\bigwedge_{i \in J} A_i \leq A_i$. Therefore by (3), we have

$$\mathcal{I}^{\mathcal{A}}\left(\bigwedge_{i \in J} A_i, r\right) \leq \mathcal{I}^{\mathcal{A}}(A_i, r).$$

Hence,

$$\mathcal{I}^{\mathcal{A}}\left(\bigwedge_{i \in J} A_i, r\right) \leq \bigwedge_{i \in J} \mathcal{I}^{\mathcal{A}}(A_i, r). \tag{2.4}$$

from the inequalities (2.3) and (2.4), we have

$$\mathcal{I}^{\mathcal{A}}\left(\bigwedge_{i\in J} A_i, r\right) = \bigwedge_{i\in J} \mathcal{I}^{\mathcal{A}}(A_i, r).$$

(7) For each $s \in \{r \in M_{0_M} : B = \mathcal{I}^{\mathcal{A}}(B, r)\}$, we have $B = \mathcal{I}^{\mathcal{A}}(B, s)$. So, we obtain

$$\begin{aligned}\mathcal{A}(B) &= \mathcal{A}(\mathcal{I}^{\mathcal{A}}(B, s)) \\ &= \mathcal{A}\left(\bigvee\{C \in L^X : C \leq B,\ \mathcal{A}(C) \geq s\}\right) \geq s.\end{aligned}$$

It follows that $\mathcal{A}(B) \geq \bigvee r \geq s$.

By (4), we have

$$\begin{aligned}B &= \mathcal{I}^{\mathcal{A}}(B, s) \geq \mathcal{I}^{\mathcal{A}}\left(B, \bigvee r\right) \\ &= \bigvee\left\{C \in L^X : C \leq B,\ \mathcal{A}(C) \geq \bigvee r\right\} \geq B.\end{aligned}$$

Hence $\mathcal{I}^{\mathcal{A}}(B, \bigvee\{r \in M_{0_M} : B = \mathcal{I}^{\mathcal{A}}(B, r)\}) = B$. □

Theorem 2.3 *For an (L, M)-fuzzy interior operator $\mathcal{I}$, $A \in L^X$ and $r \in M_{0_M}$, a mapping $\mathcal{A}^{\mathcal{I}} : L^X \longrightarrow M$ is defined as follows:*

$$\mathcal{A}^{\mathcal{I}}(A) = \bigvee\{r \in M_{0_M} : A = \mathcal{I}(A, r)\}.$$

Then:

(1) *$\mathcal{A}^{\mathcal{I}}$ is an (L, M)-fuzzy concave structure on X.*

(2) *$\mathcal{A}^{\mathcal{I}^{\mathcal{A}}} = \mathcal{A}$ and $\mathcal{I}^{\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}} = \mathcal{I}^{\mathcal{A}}$.*

Proof (1) (LMA1) Since for all $r \in M_{0_M}$,

$$\mathcal{I}(1_X, r) = 1_X, \quad \text{and} \quad \mathcal{I}(0_X, r) \leq 0_X,$$

we have $\mathcal{A}^{\mathcal{I}}(0_X) = \mathcal{A}^{\mathcal{I}}(1_X) = 1_M$.

(LMA2) Suppose that $b \in J(M)$ and

$$b \prec \bigwedge_{i\in J} \mathcal{A}^{\mathcal{I}}(A_i).$$

Then, $b \prec \mathcal{A}^{\mathcal{I}}(A_i)$ for all $i \in J$. There exists $r_0^i \in M_{0_M}$ such that $A_i = \mathcal{I}(A_i, r_0^i)$ and $b \prec r_0^i$ (thus $b \leq r_0^i$). Put $r_0 = \bigwedge_{i\in J} r_0^i$, then $b \leq r_0$. Since $\mathcal{I}$ is an (L, M)-fuzzy interior operator, we have

$$\mathcal{I}\left(\bigvee_{i\in J} A_i, r_0\right) \geq \mathcal{I}\left(\bigvee_{i\in J} A_i, r_0^i\right) \geq \mathcal{I}(A_i, r_0^i)$$

for all $i \in J$. Then it follows that

$$\mathcal{I}\left(\bigvee_{i\in J} A_i, r_0\right) \geq \bigvee_{i\in J} \mathcal{I}(A_i, r_0^i) = \bigvee_{i\in J} A_i.$$

On the other hand, by Definition 2.1(2), we have $\mathcal{I}\left(\bigvee_{i\in J} A_i, r_0\right) \leq \bigvee_{i\in J} A_i$. So, we obtain

$$\mathcal{I}\left(\bigvee_{i\in J} A_i, r_0\right) = \bigvee_{i\in J} A_i.$$

Therefore $\mathcal{A}^{\mathcal{I}}\left(\bigvee_{i\in J} A_i\right) \geq r_0 \geq b$. Hence

$$\mathcal{A}^{\mathcal{I}}\left(\bigvee_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{A}^{\mathcal{I}}(A_i).$$

(LMA3) Let $\{A_i : i \in J\} \overset{cdir}{\subseteq} L^X$. Suppose that $b \in J(M)$ and $b \prec \bigwedge_{i\in J} \mathcal{A}^{\mathcal{I}}(A_i)$. Then $b \prec \mathcal{A}^{\mathcal{I}}(A_i)$ for all $i \in J$. There exists $r_0^i \in M_{0_M}$ such that $A_i = \mathcal{I}(A_i, r_0^i)$ and $b \prec r_0^i$ (thus $b \leq r_0^i$). Put $r_0 = \bigwedge_{i\in J} r_0^i$, then $b \leq r_0$. By Definition 2.1(6), we have

$$\begin{aligned}\bigwedge_{i\in J} A_i &\geq \mathcal{I}\left(\bigwedge_{i\in J} A_i, r_0\right) = \bigwedge_{i\in J} \mathcal{I}(A_i, r_0)\\ &\geq \bigwedge_{i\in J} \mathcal{I}(A_i, r_0^i) = \bigwedge_{i\in J} A_i.\end{aligned}$$

So, we obtain

$$\bigwedge_{i\in J} A_i = \mathcal{I}\left(\bigwedge_{i\in J} A_i, r_0\right).$$

Therefore $\mathcal{A}^{\mathcal{I}}(\bigwedge_{i\in J} A_i) \geq r_0 \geq b$. Hence

$$\mathcal{A}^{\mathcal{I}}\left(\bigwedge_{i\in J} A_i\right) \geq \bigwedge_{i\in J} \mathcal{A}^{\mathcal{I}}(A_i).$$

(2) Let $A \in L^X$. Suppose that $b \in M$ and

$$b \prec \mathcal{A}^{\mathcal{I}^{\mathcal{A}}}(A) = \bigvee\{r \in M_{0_M} : A = \mathcal{I}^{\mathcal{A}}(A, r)\},$$

there exists $r_0 \in M_{0_M}$ such that $A = \mathcal{I}^{\mathcal{A}}(A, r_0)$ and $b \prec r_0$ (thus $b \leq r_0$). Therefore, $\mathcal{A}(A) = \mathcal{A}(\mathcal{I}^{\mathcal{A}}(A, r_0)) \geq r_0 \geq b$. Hence, $\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}(A) \leq \mathcal{A}(A)$.

On the other hand, if $\mathcal{A}(A) = 0_M$, we easily obtain $\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}(A) \geq 0_M = \mathcal{A}(A)$. If $\mathcal{A}(A) \in M_{0_M}$, then

$$\begin{aligned}A &\geq \mathcal{I}^{\mathcal{A}}(A, \mathcal{A}(A))\\ &= \bigvee\{B \in L^X : B \leq A,\ \mathcal{A}(B) \geq \mathcal{A}(A)\}\\ &\geq A.\end{aligned}$$

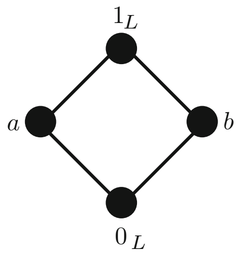
Therefore, $A = \mathcal{I}^{\mathcal{A}}(A, \mathcal{A}(A))$. By the definition of $\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}$, we obtain $\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}(A) \geq \mathcal{A}(A)$. Hence, $\mathcal{A}^{\mathcal{I}^{\mathcal{A}}} = \mathcal{A}$ and $\mathcal{I}^{\mathcal{A}^{\mathcal{I}^{\mathcal{A}}}} = \mathcal{I}^{\mathcal{A}}$. □

Theorem 2.4 *If a mapping $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ is a concave (L, M)-fuzzy interior operator on X, then $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}} = \mathcal{I}$.*

Proof Notice that $\mathcal{I}(\mathcal{I}(A, r), r) = \mathcal{I}(A, r) \leq A$ for each $r \in M_{0_M}$ and $A \in L^X$, we obtain

$$\begin{aligned}\mathcal{I}^{\mathcal{A}^{\mathcal{I}}}(A, r) &= \bigvee\{B \in L^X : B \leq A,\ \mathcal{A}^{\mathcal{I}}(B) \geq r\}\\ &\geq \bigvee\{B \in L^X : A \geq B = \mathcal{I}(B, r)\}\\ &\geq \mathcal{I}(A, r).\end{aligned}$$

Fig. 1 The structure of L



On the other hand, let $B \leq A$, and $\mathcal{A}^{\mathcal{I}}(B) \geq r$. By Definition 2.1(7), we have

$$B \geq \mathcal{I}(B, r) \geq \mathcal{I}(B, \mathcal{A}^{\mathcal{I}}(B)) \\ = \mathcal{I}(B, \bigvee\{s \in M_{0_M} : B = \mathcal{I}(B, s)\}) = B.$$

So, $B = \mathcal{I}(B, r) \leq \mathcal{I}(A, r)$. It follows that $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}}(A, r) = \bigvee\{B \in L^X : B \leq A,\ \mathcal{A}^{\mathcal{I}}(B) \geq r\} \leq \mathcal{I}(A, r)$. Hence, $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}} = \mathcal{I}$. □

Remark 2.5 If a mapping $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ is an (L, M)-fuzzy interior operator, instead of a concave (L, M)-fuzzy interior operator, then $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}} \geq \mathcal{I}$ still established, but there is no $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}} \leq \mathcal{I}$ in general. For example, let $X = \{x\}$ be a single set, let $L = \{0_L, a, b, 1_L\}$ be a diamond-type lattice (see Fig. 1), and let $M = \{0_M, d, e, f, g, 1_M\}$ (see Fig. 2). Then $L^X = \{0_X, \underline{a}, \underline{b}, 1_X\}$. Define the mapping $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ as follows: for all $r \neq \perp_M$, $\mathcal{I}(1_X, r) = 1_X$, $\mathcal{I}(0_X, r) = 0_X$,

$$\mathcal{I}(\underline{a}, r) = \begin{cases} \underline{a}, & \text{if } r = f, \\ \underline{a}, & \text{if } r = d, \\ \underline{a}, & \text{if } r = e, \\ 0_X, & \text{if } r = g, \\ 0_X, & \text{otherwise}, \end{cases}$$

$$\mathcal{I}(\underline{b}, r) = \begin{cases} \underline{b}, & \text{if } r = f, \\ \underline{b}, & \text{if } r = d, \\ \underline{b}, & \text{if } r = e, \\ 0_X, & \text{if } r = g, \\ 0_X, & \text{otherwise}. \end{cases}$$

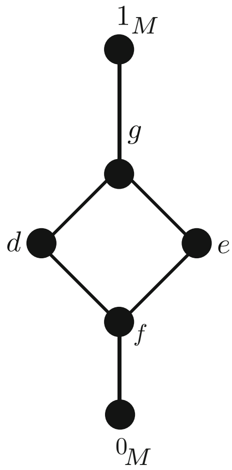
Then we can easily verify that $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ is an (L, M)-fuzzy interior operator, not a concave (L, M)-fuzzy interior operator, and $\mathcal{A}^{\mathcal{I}}$ is an (L, M)-fuzzy concave structure on X. Notice that $\mathcal{A}^{\mathcal{I}}(0_X) = \mathcal{A}^{\mathcal{I}}(1_X) = 1_M$, $\mathcal{A}^{\mathcal{I}}(\underline{a}) = \mathcal{A}^{\mathcal{I}}(\underline{b}) = g$, we obtain

$$\mathcal{I}^{\mathcal{A}^{\mathcal{I}}}(\underline{a}, g) = \bigvee\{B \in L^X : B \leq \underline{a},\ \mathcal{A}^{\mathcal{I}}(B) \geq g\} \\ = \underline{a} \nleq 0_X = \mathcal{I}(\underline{a}, g),$$

i.e., $\mathcal{I}^{\mathcal{A}^{\mathcal{I}}} \nleq \mathcal{I}$.

Definition 2.6 Let $(X, \mathcal{I}_X)$ and $(Y, \mathcal{I}_Y)$ be two (L, M)-fuzzy interior spaces, then a function $g : X \longrightarrow Y$ is called an (L, M)-fuzzy interior preserving function ((L, M)-FINP, for short) if $g^{\leftarrow}(\mathcal{I}_Y(A, r)) \leq \mathcal{I}_X(g^{\leftarrow}(A), r)$ for all $A \in L^Y$ and $r \in M_{0_M}$.

Fig. 2 The structure of M



Proposition 2.7 *If $g : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an (L, M)-FINP, then $g : (X, \mathcal{A}^{\mathcal{I}_X}) \longrightarrow (Y, \mathcal{A}^{\mathcal{I}_Y})$ is an (L, M)-FCAP.*

Proof Suppose that $b \in M$ and

$$b \prec \mathcal{A}^{\mathcal{I}_Y}(A) = \bigvee \{r \in M_{0_M} : A = \mathcal{I}_Y(A, r)\}$$

for each $A \in L^Y$, there exists $r_0 \in M_{0_M}$ such that $A = \mathcal{I}_Y(A, r_0)$, and $b \prec r_0$ (thus $b \leq r_0$). If $g : (X, \mathcal{I}_X) \longrightarrow (Y, \mathcal{I}_Y)$ is an (L, M)-FINP, we obtain

$$\begin{aligned} g^{\leftarrow}(A) &= g^{\leftarrow}(\mathcal{I}_Y(A, r_0)) \\ &\leq \mathcal{I}_X(g^{\leftarrow}(A), r_0) \leq g^{\leftarrow}(A), \end{aligned}$$

i.e., $\mathcal{I}_X(g^{\leftarrow}(A), r_0) = g^{\leftarrow}(A)$. This implies

$$\begin{aligned} &\mathcal{A}^{\mathcal{I}_X}(g^{\leftarrow}(A)) \\ &= \bigvee \{s \in M_{0_M} : g^{\leftarrow}(A) = \mathcal{I}_X(g^{\leftarrow}(A), s)\} \\ &\geq r_0 \geq b. \end{aligned}$$

This shows that

$$\mathcal{A}^{\mathcal{I}_X}(g^{\leftarrow}(A)) \geq \mathcal{A}^{\mathcal{I}_Y}(A).$$

Hence, $g : (X, \mathcal{A}^{\mathcal{I}_X}) \longrightarrow (Y, \mathcal{A}^{\mathcal{I}_Y})$ is an (L, M)-FCAP. □

Proposition 2.8 *If $g : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is an (L, M)-FCAP, then $g : (X, \mathcal{I}^{\mathcal{A}_X}) \longrightarrow (Y, \mathcal{I}^{\mathcal{A}_Y})$ is an (L, M)-FINP.*

Proof If $g : (X, \mathcal{A}_X) \longrightarrow (Y, \mathcal{A}_Y)$ is an (L, M)-FCAP, then $\mathcal{A}_X(g^{\leftarrow}(A)) \geq \mathcal{A}_Y(A)$ for all $A \in L^Y$. Therefore,

$$\begin{aligned}
&g^{\leftarrow}(\mathcal{I}^{\mathcal{A}_Y}(A, r)) \\
&\quad = g^{\leftarrow}\left(\bigvee\{B \in L^Y : B \leq A,\ \mathcal{A}_Y(B) \geq r\}\right) \\
&\quad = \bigvee\{g^{\leftarrow}(B) \in L^X : B \leq A,\ \mathcal{A}_Y(B) \geq r\} \\
&\quad \leq \bigvee\{g^{\leftarrow}(B) \in L^X : g^{\leftarrow}(B) \leq g^{\leftarrow}(A), \mathcal{A}_X(g^{\leftarrow}(B)) \geq r\} \\
&\quad \leq \bigvee\{C \in L^X : C \leq g^{\leftarrow}(A),\ \mathcal{A}_X(C) \geq r\} \\
&\quad = \mathcal{I}^{\mathcal{A}_X}(g^{\leftarrow}(A), r).
\end{aligned}$$

Hence, $g : (X, \mathcal{I}^{\mathcal{A}_X}) \longrightarrow (Y, \mathcal{I}^{\mathcal{A}_Y})$ is an (L, M)-FINP. □

At the end of this section, we will give the relationship between the category of (L, M)-fuzzy concave spaces and that of (concave) (L, M)-fuzzy interior spaces. Some concepts related to category theory can be found in Adáamek et al. (1990).

The category of all (L, M)-fuzzy concave spaces as objects and all (L, M)-FCAPs as morphisms is denoted by **LMFA** (see Xiu and Li 2019b), and the category of all (L, M)-fuzzy interior spaces as objects and all (L, M)-FINPs as morphisms is denoted by **LMFI**. Obviously, from Proposition 2.7, we obtain a concrete functor Ψ : **LMFI** $\longrightarrow$ **LMFA** by

$$\Psi : (X, \mathcal{I}) \mapsto (X, \mathcal{A}^{\mathcal{I}}) \quad \text{and} \quad g \mapsto g.$$

From Proposition 2.8, we obtain a concrete functor Θ : **LMFA** $\longrightarrow$ **LMFI** by

$$\Theta : (X, \mathcal{A}) \mapsto (X, \mathcal{I}^{\mathcal{A}}) \quad \text{and} \quad g \mapsto g.$$

Theorem 2.9 (Θ, Ψ) *is a Galois correspondence between the category of* (L, M)*-fuzzy concave spaces and that of* (L, M)*-fuzzy interior spaces, and* Ψ *is a left inverse of* Θ.

Proof By Theorem 2.3, if $\mathcal{A} : L^X \longrightarrow M$ is an (L, M)-fuzzy concave structure on X, then the identity map $id_X : (X, \mathcal{A}) \longrightarrow (X, \Psi(\Theta(\mathcal{A}))) = (X, \mathcal{A}^{\mathcal{I}^{\mathcal{A}}}) = (X, \mathcal{A})$ is an **LMFA**-morphism. Moreover, by Remark 2.5, if $\mathcal{I} : L^X \times M_{0_M} \longrightarrow L^X$ is an (L, M)-fuzzy interior operator on X, then the identity map $id_Y : (Y, \Theta(\Psi(\mathcal{I}))) = (Y, \mathcal{I}^{\mathcal{A}^{\mathcal{I}}}) \longrightarrow (Y, \mathcal{I})$ is an **LMFI**-morphism. Therefore, (Θ, Ψ) is a Galois correspondence. Furthermore, by $\Psi(\Theta(X, \mathcal{A})) = (X, \mathcal{A}^{\mathcal{I}^{\mathcal{A}}}) = (X, \mathcal{A})$ for all $(X, \mathcal{A}) \in$ **LMFA**-objects, Ψ is a left inverse of Θ. □

The category of all concave (L, M)-fuzzy interior spaces as objects and their concave interior preserving functions as morphisms is denoted by **C-LMFI**. Obviously, **C-LMFI** is subcategory of **LMFI**.

By Theorems 2.3 and 2.4, and Propositions 2.7 and 2.8, we immediately obtain Theorem 2.10 as follows:

Theorem 2.10 **C-LMFI** *is isomorphic to* **LMFA**.

3 Further results of (L, M)-fuzzy hull operators

In this section, we will give a further study on Sayed's spaces. Particularly, we will correct some results in Sayed et al. (2019).

Remark 3.1 (1) Notice that L (M) is a completely distributive lattice, not a unit interval [0, 1]. So, if $a \not\leq b$, it doesn't imply $a > b$. Because there exists another case that a and b may are not comparable, i.e., $a \parallel b$. Thus, the proof of Theorem 2.4(5) in Sayed et al. (2019) is not true. And, we can prove it using the method of Theorem 2.2(5) (another proving method, please see Zhao et al. 2021b).

(2) Similar to Theorem 2.3 and Remark 2.5, if there not exists $\mathcal{C}$ such that $\mathcal{CO}_{12} = \mathcal{CO}_{\mathcal{C}}$, then $\mathcal{CO}_{12}$ is an (L, M)-fuzzy hull operator on X, and $(\mathcal{CO}_{12})_{\mathcal{C}_{\mathcal{CO}_{12}}} = \mathcal{CO}_{12}$ need not be true. So, Proposition 2.6 and Corollary 2.7 in Sayed et al. (2019) need not be true.

Definition 3.2 Let $(X, \mathcal{CO}_X)$ and $(Y, \mathcal{CO}_Y)$ be two (L, M)-fuzzy hull spaces, then a function $g : X \longrightarrow Y$ is called an (L, M)-fuzzy hull preserving function ((L, M)-FHLP, for short) if $g^{\rightarrow}(\mathcal{CO}_X(A, r)) \leq \mathcal{CO}_Y(g^{\rightarrow}(A), r)$ for all $A \in L^X$ and $r \in M_{0_M}$.

Notice that L (M) is a completely distributive lattice, not a unit interval [0, 1]. So, the proof of Proposition 2.8(1) in Sayed et al. (2019) does not true, and a correct proof is as follows (also see Zhao et al. 2021b).

Proposition 3.3 (Sayed et al. 2019, Proposition 2.8(1)) *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convex spaces. Then $g : X \longrightarrow Y$ is an (L, M)-fuzzy convexity preserving function iff $g : (X, \mathcal{CO}_{\mathcal{C}}) \longrightarrow (Y, \mathcal{CO}_{\mathcal{D}})$ is an (L, M)-FHLP.*

Proof ($\Longrightarrow$) Since $g : X \longrightarrow Y$ is an (L, M)-fuzzy convexity preserving function, we obtain $\mathcal{C}(g^{\leftarrow}(\varpi)) \geq \mathcal{D}(\varpi)$ for all $\varpi \in L^Y$. So, for each $r \in M_{0_M}$ and $A \in L^X$, we obtain

$$\begin{aligned}
&g^{\leftarrow}[\mathcal{CO}_{\mathcal{D}}(g^{\rightarrow}(A), r)] \\
&\quad = g^{\leftarrow}\left[\bigwedge \left\{\varpi \in L^Y : g^{\rightarrow}(A) \leq \varpi,\ \mathcal{D}(\varpi) \geq r\right\}\right] \\
&\quad = \bigwedge \left\{g^{\leftarrow}(\varpi) \in L^X : g^{\rightarrow}(A) \leq \varpi,\ \mathcal{D}(\varpi) \geq r\right\} \\
&\quad \geq \bigwedge \left\{g^{\leftarrow}(\varpi) \in L^X : A \leq g^{\leftarrow}(\varpi),\ \mathcal{C}(g^{\leftarrow}(\varpi)) \geq r\right\} \\
&\quad \geq \bigwedge \left\{B \in L^X : A \leq B,\ \mathcal{C}(B) \geq r\right\} = \mathcal{CO}_{\mathcal{C}}(A, r).
\end{aligned}$$

Therefore,

$$\begin{aligned}
g^{\rightarrow}(\mathcal{CO}_{\mathcal{C}}(A, r)) &\leq g^{\rightarrow}g^{\leftarrow}[\mathcal{CO}_{\mathcal{D}}(g^{\rightarrow}(A), r)] \\
&\leq \mathcal{CO}_{\mathcal{D}}(g^{\rightarrow}(A), r).
\end{aligned}$$

Hence, $g : (X, \mathcal{CO}_{\mathcal{C}}) \longrightarrow (Y, \mathcal{CO}_{\mathcal{D}})$ is an (L, M)-FHLP.

($\Longleftarrow$) Suppose that $b \in J(M)$ and $b \prec \mathcal{D}(\varpi)$ for all $\varpi \in L^Y$, then $b \leq \mathcal{D}(\varpi)$. So, $g^{\rightarrow}(\mathcal{CO}_{\mathcal{C}}(g^{\leftarrow}(\varpi), b)) \leq \mathcal{CO}_{\mathcal{D}}(g^{\rightarrow}(g^{\leftarrow}(\varpi)), b) \leq \mathcal{CO}_{\mathcal{D}}(\varpi, b) = \varpi$. It follows that

$$g^{\leftarrow}(\varpi) \leq \mathcal{CO}_{\mathcal{C}}(g^{\leftarrow}(\varpi), b) \leq g^{\leftarrow}(\varpi).$$

Therefore, $\mathcal{CO}_{\mathcal{C}}(g^{\leftarrow}(\varpi), b) = g^{\leftarrow}(\varpi)$. Furthermore,

$$\begin{aligned}
&\mathcal{C}(g^{\leftarrow}(\varpi)) \\
&\quad = \mathcal{C}(\mathcal{CO}_{\mathcal{C}}(g^{\leftarrow}(\varpi), b)) \\
&\quad = \mathcal{C}\left(\bigwedge \left\{B \in L^X : g^{\leftarrow}(\varpi) \leq \nu,\ \mathcal{C}(B) \geq b\right\}\right) \\
&\quad \geq \bigwedge_{g^{\leftarrow}(\varpi) \leq \nu,\ \mathcal{C}(B) \geq b} \mathcal{C}(B) \geq b.
\end{aligned}$$

Hence, $\mathcal{C}(g^{\leftarrow}(\varpi)) \geq \mathcal{D}(\varpi)$ and $g : X \longrightarrow Y$ is an (L, M)-fuzzy convexity preserving function. □

Theorem 3.4 *Let X be any set, let $\{f_i : (X, \mathcal{C}) \longrightarrow (X_i, \mathcal{C}_i)\}_{i\in J}$ be any family of (L, M)-fuzzy convexity preserving functions. and let $\{(X_i, \mathcal{CO}_{\mathcal{C}_i})\}_{i\in J}$ be any family of (L, M)-fuzzy hull spaces indexed by a class J. Define $\mathcal{CO} : L^X \times M_{0_M} \longrightarrow L^X$ as follows: for each $r \in M_{0_M}$ and $A \in L^X$,*

$$\mathcal{CO}(A, r) = \bigwedge_{i\in J} f_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r)).$$

Then,

(1) *$\mathcal{CO}$ is an (L, M)-fuzzy hull operator on X, and $f_i : (X, \mathcal{CO}) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP for each $i \in J$.*
(2) *Let $(Y, \mathcal{CO}^\star)$ is an (L, M)-fuzzy hull spaces, then $g : (Y, \mathcal{CO}^\star) \longrightarrow (X, \mathcal{CO})$ is an (L, M)-FHLP iff $f_i \circ g : (Y, \mathcal{CO}^\star) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP for each $i \in J$.*
(3) *$\mathcal{CO}_{\mathcal{C}} \le \mathcal{CO}$.*

Proof (1) We can easily check that $\mathcal{CO}$ is an (L, M)-fuzzy hull operator on X, and for each $i \in J$,

$$\begin{aligned} f_i^{\rightarrow}(\mathcal{CO}(A, r)) &= f_i^{\rightarrow}\left(\bigwedge_{i\in I} f_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r))\right) \\ &\le \bigwedge_{i\in I} f_i^{\rightarrow}(f_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r))) \\ &\le \bigwedge_{i\in I} \mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r) \\ &\le \mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r), \end{aligned}$$

i.e., $f_i^{\rightarrow}(\mathcal{CO}(A, r)) \le \mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(A), r)$. Hence, $f_i : (X, \mathcal{CO}) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP for each $i \in J$.

(2) Let $(Y, \mathcal{CO}^\star)$ is an (L, M)-fuzzy hull space, if $g : (Y, \mathcal{CO}^\star) \longrightarrow (X, \mathcal{CO})$ is an (L, M)-FHLP, then, for each $i \in J$,

$$\begin{aligned} (f_i \circ g)^{\rightarrow}(\mathcal{CO}^\star(B, r)) &= f_i^{\rightarrow}(g^{\rightarrow}(\mathcal{CO}^\star(B, r)) \\ &\le f_i^{\rightarrow}(\mathcal{CO}(g^{\rightarrow}(B), r)) \\ &\le \mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(g^{\rightarrow}(B)), r) \\ &= \mathcal{CO}_{\mathcal{C}_i}((f_i \circ g)^{\rightarrow}(B), r). \end{aligned}$$

Hence, $f_i \circ g : (Y, \mathcal{CO}^\star) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP for each $i \in J$.

On the other hand, for each $r \in M_{0_M}$ and $B \in L^Y$, if $f_i \circ g : (Y, \mathcal{CO}^\star) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP for each $i \in J$. Then,

$$\begin{aligned} g^{\rightarrow}(\mathcal{CO}^\star(B, r)) &\le \bigwedge_{i\in J} \left(f_i^{\leftarrow}[f_i^{\rightarrow}(g^{\rightarrow}(\mathcal{CO}^\star(B, r))]\right) \\ &= \bigwedge_{i\in J} (f_i^{\leftarrow}[(f_i \circ g)^{\rightarrow}(\mathcal{CO}^\star(B, r))]) \\ &\le \bigwedge_{i\in J} f_i^{\leftarrow}[\mathcal{CO}_{\mathcal{C}_i}((f_i \circ g)^{\rightarrow}(B), r)]) \\ &= \bigwedge_{i\in J} (f_i^{\leftarrow}[\mathcal{CO}_{\mathcal{C}_i}(f_i^{\rightarrow}(g^{\rightarrow}(B)), r)]) \end{aligned}$$

$$= \mathcal{CO}(g^{\rightarrow}(B), r)$$

Hence, $g : (Y, \mathcal{CO}^{\star}) \longrightarrow (X, \mathcal{CO})$ is an (L, M)-FHLP.

(3) By Theorem 1.3, we easily know that $\mathcal{CO}_{\mathcal{C}}$ is an (L, M)-fuzzy hull operator on X. Let $(Y, \mathcal{CO}^{\star}) = (X, \mathcal{CO}_{\mathcal{C}})$ and $g = id_X : (X, \mathcal{CO}_{\mathcal{C}}) \longrightarrow (X, \mathcal{CO})$, then, by Proposition 3.3, we obtain $f_i = f_i \circ id_X : (X, \mathcal{CO}_{\mathcal{C}}) \longrightarrow (X_i, \mathcal{CO}_{\mathcal{C}_i})$ is an (L, M)-FHLP. By (2), we obtain $id_X : (X, \mathcal{CO}_{\mathcal{C}}) \longrightarrow (X, \mathcal{CO})$ is an (L, M)-FHLP. Hence,

$$id_X^{\rightarrow}(\mathcal{CO}_{\mathcal{C}}(A, r)) \leq \mathcal{CO}(id_X^{\rightarrow}(A), r),$$

i.e., $\mathcal{CO}_{\mathcal{C}} \leq \mathcal{CO}$, □

In particular, according to Theorem 1.11, when we take f_i in Theorem 3.4 equals to each projection p_i, by Proposition 3.3 and Theorem 3.4, we easily obtain the following conclusion.

Corollary 3.5 *Let $X = \prod_{i\in J} X_i$, $p_i : X \longrightarrow X_i$ be the i-th projection and let $(\prod_{i\in J} X_i, \prod_{i\in J} \mathcal{C}_i)$ be the product of $\{(X_i, \mathcal{C}_i)\}_{i\in J}$. Then for each $r \in M_{0_M}$ and $A \in L^X$,*

$$\begin{aligned} \mathcal{CO}_{\prod_{i\in J} \mathcal{C}_i}(A, r) &\leq \bigwedge_{i\in J} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(A), r)) \\ &= \prod_{i\in J} \mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(A), r). \end{aligned}$$

Finally, we give a counterexample to show that not all projections p_i $(i \in J)$ are (L, M)-fuzzy convex-to-convex functions, i.e., Theorem 3.14 in Sayed et al. (2019) is incorrect. Meanwhile, for each $r \in M_{0_M}$ and $A \in L^X$, there is no

$$\mathcal{CO}_{\prod\limits_{i\in J} \mathcal{C}_i}(A, r) = \bigwedge_{i\in J} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(A), r))$$

in general.

Example 3.6 Let $Y = X = \{x\}$ be a single set, let $M = L = \{0_L, a, b, 1_L\}$ be a diamond-type lattice (see Fig. 1). Then $L^X = \{0_X, \underline{a}, \underline{b}, 1_X\}$ and $L^{X\times X} = \{0_{X\times X}, \underline{a}, \underline{b}, 1_{X\times X}\}$. Define two mappings $\mathcal{C}_1, \mathcal{C}_2 : L^X \longrightarrow M$ as follows:

$$\mathcal{C}_1(A) = \begin{cases} 1_L, & \text{if } A = 0_X, 1_X, \\ b, & \text{if } A = \underline{a}, \underline{b}, \end{cases}$$

$$\mathcal{C}_2(A) = \begin{cases} 1_L, & \text{if } A = 0_X, 1_X, \\ a, & \text{if } A = \underline{a}, \underline{b}. \end{cases}$$

Then both $\mathcal{C}_1$ and $\mathcal{C}_2$ are (L, M)-fuzzy convexities on X, and

$$\mathcal{CO}_{\mathcal{C}_1}(\underline{a}, r) = \begin{cases} 1_X, & \text{if } r = a, 1_L, \\ \underline{a}, & \text{if } r = b. \end{cases}$$

$$\mathcal{CO}_{\mathcal{C}_2}(\underline{a}, r) = \begin{cases} 1_X, & \text{if } r = b, 1_L, \\ \underline{a}, & \text{if } r = a. \end{cases}$$

By Definition 1.2(2) and Corollary 3.5, we have

$$\begin{aligned} a &= p_1^{\rightarrow}(\underline{a})(x) \leq p_1^{\rightarrow}(\mathcal{CO}_{\mathcal{C}_1\times\mathcal{C}_2}(\underline{a}, r))(x) \\ &= \mathcal{CO}_{\mathcal{C}_1\times\mathcal{C}_2}(\underline{a}, r)(x, x) \\ &\leq \bigwedge_{i\in\{1,2\}} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(\underline{a}), r))(x, x) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{CO}_{\mathcal{C}_1}(\underline{a}, r)(x) \wedge \mathcal{CO}_{\mathcal{C}_2}(\underline{a}, r)(x) \\
&= \begin{cases} 1_L, & \text{if } r = 1_L, \\ a, & \text{if } r = a, b. \end{cases}
\end{aligned}
$$

Therefore,

$$
\underline{a} \leq \mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, r) \leq \begin{cases} 1_{X \times X}, & \text{if } r = 1_L, \\ \underline{a}, & \text{if } r = a, b. \end{cases}
$$

and

$$
\begin{aligned}
&\bigwedge_{i \in \{1,2\}} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(\underline{a}), r)) \\
&\quad = \begin{cases} 1_{X \times X}, & \text{if } r = 1_L, \\ \underline{a}, & \text{if } r = a, b. \end{cases}
\end{aligned}
$$

Hence,

$$
\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, r) = \underline{a}, \quad \forall r \in \{a, b\}.
$$

Moreover, by the definition of $\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, r)$, there must be

$$
\begin{aligned}
&\mathcal{C}_1 \times \mathcal{C}_2(\underline{a}) \\
&\quad = \mathcal{C}_1 \times \mathcal{C}_2(\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, r)) \\
&\quad = \mathcal{C}_1 \times \mathcal{C}_2\left(\bigwedge\{\alpha \in L^{X \times X} : \underline{a} \leq \alpha, \mathcal{C}_1 \times \mathcal{C}_2(\alpha) \geq r\}\right) \\
&\quad \geq \bigwedge\{\mathcal{C}_1 \times \mathcal{C}_2(\alpha) : \underline{a} \leq \alpha, \mathcal{C}_1 \times \mathcal{C}_2(\alpha) \geq r\} \\
&\quad \geq r
\end{aligned}
$$

for each $r \in \{a, b\}$. i.e., $\mathcal{C}_1 \times \mathcal{C}_2(\underline{a}) \geq a$ and $\mathcal{C}_1 \times \mathcal{C}_2(\underline{a}) \geq b$. It implies that $\mathcal{C}_1 \times \mathcal{C}_2(\underline{a}) = 1_L$. So, $\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, 1_L) = \underline{a}$. Therefore,

$$
\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, 1_L) \ngeq \bigwedge_{i \in \{1,2\}} p_i^{\leftarrow}(\mathcal{CO}_{\mathcal{C}_i}(p_i^{\rightarrow}(\underline{a}), 1_L)),
$$

and $p_1^{\rightarrow}(\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, a)) = \underline{a}$. Notice that

$$
\begin{aligned}
\mathcal{CO}_{\mathcal{C}_1}(p_1^{\rightarrow}(\underline{a}), r) &= \mathcal{CO}_{\mathcal{C}_1}(\underline{a}, r) \\
&= \begin{cases} 1_X, & \text{if } r = a, 1_L, \\ \underline{a}, & \text{if } r = b. \end{cases}
\end{aligned}
$$

Hence,

$$
p_1^{\rightarrow}(\mathcal{CO}_{\mathcal{C}_1 \times \mathcal{C}_2}(\underline{a}, a)) \ngeq \mathcal{CO}_{\mathcal{C}_1}(p_1^{\rightarrow}(\underline{a}), a).
$$

By Lemma 1.12, we immediately know that p_1 is not an (L, M)-fuzzy convex-to-convex function.

4 Conclusions

Following the notion of (L, M)-fuzzy hull operators in (L, M)-fuzzy convex structures introduced by Sayed et al. (2019), we gave some investigations on (concave) (L, M)-fuzzy interior operators, and we pointed out some existing problems in paper (Sayed et al. 2019), and we did a further research on related problems.

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