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Soft Theta-Topology Based on Many-Valued Logic

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Abstract In the present paper we introduce fuzzifying soft topology which is defined over an initial universe with a fixed set of parameters. The notions of fuzzifying soft closed sets, fuzzifying soft neighborhood system of a soft point, fuzzifying soft derived, fuzzifying soft closure and fuzzifying soft interior are introduced and their basic properties are investigated. Furthermore, we define the notions of fuzzifying soft θ-neighborhood system of a soft point, fuzzifying soft θ-derived, fuzzifying soft θ-closure, fuzzifying soft θ-interior and investigate their properties. Finally, a fuzzifying soft continuity, a fuzzifying soft strong θ-continuity and fuzzifying soft θ-continuity are given and some of their basic properties are studied

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1. Introduction

Since the theory of fuzzy sets was introduced by Zadeh [1] in 1965, fuzzy ordered structures on a universal set had become a useful tool to model fuzziness and/or uncertainty in the real world. Chang [2] Wong [3], and others have discussed various aspects of fuzzy topology with crisp methods. Ying [4–6] proposed fuzzifying topology and developed (elementary) fuzzy topology from a new direction with the semantic method of continuous valued logic. So far, there has been significant research on fuzzifying topologies [7–14]. At the present time, the theory of fuzzy sets (i.e., fuzzy logic) is progressing rapidly. But there exists a difficulty: how to set the membership function in each particular case. We should not impose only one way to set the membership function. The nature of the membership function is extremely individual. Everyone may understand the notation $\mu(x) = 0.7$ in his own manner. So, the fuzzy set (i.e., fuzzy logic) operations based on the arithmetic operations with membership functions do not look natural. It may occur that these operations are similar to the addition of weights and lengths. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory. The fuzzifying topology is depending on fuzzy logic as special case from fuzzy topology,

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so there is several difficulties as refer above. Therefore, in this paper we will depend on a new mathematical tool for dealing with uncertainties which is free of the difficulties mentioned above, which its consider the combining fuzzifying topology and soft topology. In 1999, Molodtsov [15] introduced soft sets as a general mathematical tool for dealing with uncertainties. Shabir and Naz [16] introduced the notion of soft topological spaces and their basic properties were investigated. Cağman et al. [17] defined a soft topology on a soft set which is more general than the soft topology by Shabir and Naz [16]. Therefore, we follows their notations and mathematical formalism. Maji et al. [18] combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. These results were further revised and improved by Ahmad and Kharal [19]. In 2011, Tanay and Kandemir [20] gave the topological structure of fuzzy soft sets. In the present paper we introduce the concept of a fuzzifing soft topology. The organization of this paper is as follows: In section 2, known basic notions of łukasiewicz logic and results concerning the theory of soft sets and fuzzy soft sets are given. In section 3, definitions of the fuzzifying soft topology, fuzzifying soft closed sets, fuzzifying soft neighborhood system of a soft point, fuzzifying soft derived, fuzzifying soft closure and fuzzifying soft interior are given and some of their properties have been investigated. In section 4, the notions of fuzzifying soft θ-neighborhood system of a soft point, fuzzifying soft θ-derived, fuzzifying soft θ-closure, fuzzifying soft θ-interior are defined and their properties are investigated. In section 5, the notions of fuzzifying soft continuity, fuzzifying soft strong θ-continuity and fuzzifying soft θ-continuity have been introduced and studied.

2. Preliminaries

The reader is assumed to be familiar with Ying's papers [4–6].

First, we display the Łukasiewicz logic and corresponding set-theoretical notations used in this paper. For any formulas φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0, 1]$. A formula φ is valid, we write $\models \varphi$ if and only if $[\varphi] = 1$ for every interpretation. The truth valuation rules for primary fuzzy logical formulas and corresponding set theoretical rotations are:

(1) $[\alpha] := \alpha(\alpha \in [0, 1])$, $[\varphi \wedge \psi] := \min([\varphi], [\psi])$, $[\varphi \to \psi] := \min(1, 1 - [\varphi] + [\psi])$.

(2) If $\widetilde{A} \in \Im(X)$, where $\Im(X)$ is the family of all fuzzy subsets of X, then $[x \in \widetilde{A}] := \widetilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X}[\varphi(x)]$.

In addition, the truth valuation rules for some derived formulae are

(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$;

(2) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;

(3) $[\varphi \otimes \psi] := [\neg(\varphi \to \neg\psi)] = \max(0, [\varphi] + [\psi] - 1)$;

(4) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] = \sup_{x \in X}[\varphi(x)]$;

(5) If $\widetilde{A}, \widetilde{B} \in \Im(X)$, then

(a) $[\widetilde{A} \subseteq \widetilde{B}] := [\forall x(x \in \widetilde{A} \to x \in \widetilde{B})] = \inf_{x \in X} \min(1, 1 - \widetilde{A}(x) + \widetilde{B}(x))$;

(b) $[\widetilde{A} \equiv \widetilde{B}] := [(\widetilde{A} \subseteq \widetilde{B}) \wedge (\widetilde{B} \subseteq \widetilde{A})]$.

Second, we present the basic definitions and results of soft set theory and fuzzy soft set theory which may be found in earlier studies [15, 18, 19, 21, 22]. Throughout this work, X refers to an initial universe, E is a set of parameters, $P(X)$ is the power set of X, and $A \subseteq E$.

Definition 2.1 ([17, 21])**.** A soft set F_A on the universe X is a mapping $F_A : E \to P(X)$, where $F_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $F_A(e) = \emptyset$ if $e \notin A$. The subscript A in the notation F_A indicates where the image of F_A is non-empty.
A soft set can be defined by the set of ordered pairs $F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in P(X)\}$. The value $F_A(e)$ is a set called the $e-$element of the soft set for all $e \in E$. The set of all soft sets over X will be denoted by $S(X)$.

Definition 2.2 ([21])**.** Let $F_A \in S(X)$. If $F_A(e) = \emptyset$ for all $e \in E$, then F_A is called an empty set, denoted by $\mathbf{0}_A$. $F_A(e) = \emptyset$ means that there is no element in X related to the parameter $e \in E$. Therefore, we do not display such elements in the soft sets as it is meaningless to consider such parameters.

Definition 2.3 ([21])**.** Let $F_A \in S(X)$. If $F_A(e) = X$ for all $e \in A$, then F_A is called an A-universal soft set, denoted by $\mathbf{1}_A$. If $A = E$, then the A-universal soft set is called a universal soft set, denoted by $\mathbf{1}_E$.

Definition 2.4 ([21])**.** Let $F_A, F_B \in S(X)$. Then, F_A and F_B are soft equal, denoted by $F_A = F_B$, if $F_A(e) = F_B(e)$ for all $e \in E$ and F_A is a soft subset of F_B, denoted by $F_A \widetilde{\subseteq} F_B$, if $F_A(e) \subseteq F_B(e)$ for all $e \in E$.

Definition 2.5 ([21])**.** Let $F_A, F_B \in S(X)$. Then, the soft union $F_A \widetilde{\cup} F_B$, the soft intersection $F_A \widetilde{\cap} F_B$, and the soft difference $F_A \widetilde{\setminus} F_B$, of F_A and F_B are defined by the approximate functions
$F_{A\widetilde{\cup}B}(e) = F_A(e) \cup F_B(e)$, $F_{A\widetilde{\cap}B}(e) = F_A(e) \cap F_B(e)$, $F_{A\widetilde{\setminus}B}(e) = F_A(e) \setminus F_B(e)$,
respectively, and the soft complement $F_A^{\widetilde{c}}$ of F_A is defined by the approximate function $F_{A^{\widetilde{c}}}(e) = F_A^c(e)$, where $F_A^{\widetilde{c}}(e)$ is the complement of the set $F_A(e)$; that is, $F_A^{\widetilde{c}}(e) = X \widetilde{\setminus} F_A(e)$ for all $e \in E$.

Definition 2.6 ([23])**.** Let $F_A \in S(X)$. A soft set F_A is called a soft point in X, denoted by e_M, if for the element $e \in E, F(e) \neq \emptyset$ and $F(e') = \emptyset$ for all $e' \in A \setminus \{e\}$. The set of all soft points of X is denoted by $\mathbf{SP}(X)$. The soft point e_M is said to be in the soft set G_B, denoted by $e_M \widetilde{\in} G_B$, if for the element $e \in A$ and $F_A(e) \subseteq G_B(e)$.

Definition 2.7 ([24])**.** Let $S(X, A)$ and $S(Y, B)$ be the families of all soft sets over X and Y, respectively. The mapping f_{pu} is called a soft mapping from X to Y, denoted by $f_{pu} : S(X, A) \to S(Y, B)$, where $u : X \to Y$ and $p : A \to B$ are two mappings.

(1) Let $F_A \in S(X, A)$. The image of F_A under f_{pu}, written as $f_{pu}(F_A) = (f_{pu}(F), p(A))$, is a soft set in $S(Y, B)$ such that

$$f_{pu}(F)(y) = \begin{cases} \bigcup_{e \in p^{-1}(y) \cap A} u(F(e)), & if p^{-1}(y) \cap A \neq \emptyset; \\ \emptyset, & otherwise, \end{cases}$$

for all $y \in B$.

(2) Let $G_B \in S(Y, B)$, then the inverse image of G_B under f_{pu}, written as $f_{pu}^{-1}(G_B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is the soft set in $S(X, A)$ such that

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))), & if p(e) \in B; \\ \emptyset, & otherwise, \end{cases}$$

for all $e \in A$.

Proposition 2.8 ([21]). *Let If $F_A, F_B \in S(X)$. Then De Morgan's laws are valid*
(1) $(F_A \widetilde{\cup} F_B)^{\widetilde{c}} = F_A^{\widetilde{c}} \widetilde{\cap} F_B^{\widetilde{c}}$;
(2) $(F_A \widetilde{\cap} F_B)^{\widetilde{c}} = F_A^{\widetilde{c}} \widetilde{\cup} F_B^{\widetilde{c}}$.

Definition 2.9 ([18]). A fuzzy soft set f_A on the universe X is a mapping $f_A : E \to \Im(X)$, where $f_A(e) \neq 0_X$ if $e \in A \subseteq E$ and $f_A(e) = 0_X$ if $e \notin A$, where 0_X is empty fuzzy set on X. The set of all fuzzy soft sets over X will be denoted by $\Im_E(S(X))$.

3. Basic Properties on Fuzzifying Soft Topology

The purpose of this section is to introduce and study the concepts of fuzzifying soft topology, fuzzifying soft closed sets, fuzzifying soft neighborhood system of a soft point, fuzzifying soft derived, fuzzifying soft closure and fuzzifying soft interior of a soft set.

Definition 3.1. Let X be an initial universe, $\widetilde{\tau} \in \Im(S(X))$ satisfy the following conditions:
(1) $\widetilde{\tau}(\mathbf{0}_A) = \widetilde{\tau}(\mathbf{1}_A) = 1$;
(2) For any $F_A, F_B \in S(X), \widetilde{\tau}(F_A \widetilde{\cap} F_B) \geq \widetilde{\tau}(F_A) \wedge \widetilde{\tau}(F_B)$;
(3) For any $\{F_{A_\lambda} : \lambda \in \Lambda\} \widetilde{\subseteq} S(X), \widetilde{\tau}(\widetilde{\bigcup}_{\lambda \in \Lambda} F_{A_\lambda}) \geq \bigwedge_{\lambda \in \Lambda} \widetilde{\tau}(F_{A_\lambda})$.

Then, $\widetilde{\tau}$ is a fuzzifying soft topology and $(X, \widetilde{\tau}, A)$ is a fuzzifying soft topological space.

Example 3.2. Let $X = \{x_1, x_2, x_3\}$ and $A = \{e\}$. Then, $\mathbf{S(X)} = \{\mathbf{0}_A, \mathbf{1}_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}, F_{A_5}, F_{A_6}\}$, where $F_{A_1} = \{(e, \{x_1\})\}, F_{A_2} = \{(e, \{x_2\})\}, F_{A_3} = \{(e, \{x_3\})\}, F_{A_4} = \{(e, \{x_1, x_2\})\}, F_{A_5} = \{(e, \{x_1, x_3\})\}, F_{A_6} = \{(e, \{x_2, x_3\})\}$. Define a mapping $\widetilde{\tau} \in \Im(S(X))$ as follows: $\widetilde{\tau}(\mathbf{0}_A) = \widetilde{\tau}(\mathbf{1}_A) = \widetilde{\tau}(F_{A_4}) = 1, \widetilde{\tau}(F_{A_1}) = \widetilde{\tau}(F_{A_5}) = \frac{1}{2}, \widetilde{\tau}(F_{A_2}) = \widetilde{\tau}(F_{A_6}) = \frac{1}{8}$ and $\widetilde{\tau}(F_{A_3}) = \frac{1}{4}$. Then, $\widetilde{\tau}$ is a fuzzifying soft topology and $(X, \widetilde{\tau}, A)$ is a fuzzifying soft topological space.

Now, we will give new definition of fuzzifying soft closed sets as follows:

Definition 3.3. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A \in S(X)$. The family of all fuzzifying soft closed sets, denoted by $\widetilde{\digamma} \in \Im(S(X))$, is defined as $F_A \in \widetilde{\digamma} := (\mathbf{1}_A \setminus F_A) \in \widetilde{\tau}$, where $\mathbf{1}_A \setminus F_A$ is the complement of F_A.

Example 3.4. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2, we have $\widetilde{\digamma}(\mathbf{1}_A) = \widetilde{\digamma}(\mathbf{0}_A) = \widetilde{\digamma}(F_{A_3}) = 1; \widetilde{\digamma}(F_{A_1}) = \widetilde{\digamma}(F_{A_5}) = \frac{1}{8}; \widetilde{\digamma}(F_{A_2}) = \widetilde{\digamma}(F_{A_6}) = \frac{1}{2}$ and $\widetilde{\digamma}(F_{A_4}) = \frac{1}{4}$.

Theorem 3.5. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space. Then, the following conditions hold.*
(1) $\widetilde{\digamma}(\mathbf{0}_A) = \widetilde{\digamma}(\mathbf{1}_A) = 1$;
(2) *For any* $F_C, F_D \in S(X), \widetilde{\digamma}(F_C \widetilde{\cup} F_D) \geq \widetilde{\digamma}(F_C) \wedge \widetilde{\digamma}(F_D)$;
(3) *For any* $\{F_{C_\lambda} : \lambda \in \Lambda\} \widetilde{\subseteq} S(X), \widetilde{\digamma}(\widetilde{\bigcap}_{\lambda \in \Lambda} F_{C_\lambda}) \geq \bigwedge_{\lambda \in \Lambda} \widetilde{\digamma}(F_{C_\lambda})$.

Proof. (1) $\widetilde{\digamma}(\mathbf{0}_A) = \widetilde{\tau}(\mathbf{1}_A \setminus \mathbf{0}_A) = \widetilde{\tau}(\mathbf{1}_A) = 1$ and $\widetilde{\digamma}(\mathbf{1}_A) = \widetilde{\tau}(\mathbf{1}_A \setminus \mathbf{1}_A) = \widetilde{\tau}(\mathbf{0}_A) = 1$.

(2) For any $F_C, F_D \in S(X)$, we have $\widetilde{\digamma}(F_C \widetilde{\cup} F_D) = \widetilde{\tau}((\mathbf{1}_A \setminus F_C) \widetilde{\cap} (\mathbf{1}_A \setminus F_D)) \geq \widetilde{\tau}(\mathbf{1}_A \setminus F_C) \wedge \widetilde{\tau}(\mathbf{1}_A \setminus F_D) = \widetilde{\digamma}(F_C) \wedge \widetilde{\digamma}(F_D)$.

(3) For any $\{F_{C_\lambda} : \lambda \in \Lambda\}\widetilde{\subseteq} S(X)$, we have $\widetilde{\mathcal{F}}(\widetilde{\bigcap}_{\lambda\in\Lambda} F_{C_\lambda}) = \widetilde{\tau}(\widetilde{\bigcup}_{\lambda\in\Lambda}(\mathbf{1}_A \setminus F_{C_\lambda})) \geq \bigwedge_{\lambda\in\Lambda} \widetilde{\tau}((\mathbf{1}_A \setminus F_{C_\lambda})) = \bigwedge_{\lambda\in\Lambda} \widetilde{\mathcal{F}}(F_{C_\lambda})$. ∎

We will propose novel concept of fuzzifying soft neighborhood system of soft point e_M, as follows:

Definition 3.6. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space, $e_M \in \mathbf{SP}(X)$ and $F_A, F_B \in S(X)$. The fuzzifying soft neighborhood system of e_M, is denoted by $N_{e_M} \in \Im(S(X))$, is defined as

$$F_A \in N_{e_M} := \exists F_B\big((F_B \in \widetilde{\tau}) \wedge (e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A)\big), \text{ i.e., } \quad N_{e_M}(F_A) = \bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A} \widetilde{\tau}(F_B).$$

Example 3.7. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2. Let $e_M = \{(e, \{x_3\})\}$ be a soft point. The fuzzifying soft neighborhood systems of e_M are: $N_{e_M}(\mathbf{1}_A) = 1; N_{e_M}(\mathbf{0}_A) = N_{e_M}(F_{A_1}) = N_{e_M}(F_{A_2}) = N_{e_M}(F_{A_4}) = 0; N_{e_M}(F_{A_3}) = N_{e_M}(F_{A_6}) = \frac{1}{4}$ and $N_{e_M}(F_{A_5}) = \frac{1}{2}$.

The main results (i.e., Theorem 3.8, Corollary 3.9, and Theorem 3.10) on fuzzifying soft neighborhood system are presented as follows:

Theorem 3.8. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B, F_C \in S(X)$. Then*

(1) $\models F_A \in \widetilde{\tau} \leftrightarrow \forall e_M\big(e_M \widetilde{\in} F_A \to \exists F_B\big(F_B \in \widetilde{\tau} \wedge e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A\big)\big)$;

(2) $\models F_A \in \widetilde{\tau} \leftrightarrow \forall e_M\big(e_M \widetilde{\in} F_A \to \exists F_B\big(F_B \in N_{e_M} \wedge F_B \widetilde{\subseteq} F_A\big)\big)$.

Proof. (1) $\big[\forall e_M\big(e_M \widetilde{\in} F_A \to \exists F_B\big(F_B \in \widetilde{\tau} \wedge e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A\big)\big)\big] = \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A} \widetilde{\tau}(F_B)$.

It is clear that $\bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A} \widetilde{\tau}(F_B) \geq \widetilde{\tau}(F_A)$. In the other hand, let $\mathfrak{B}_{e_M} = \{F_B : e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A\}$. Then, for any $f \in \prod_{e_M \widetilde{\in} F_A} \mathfrak{B}_{e_M}$, we have $\widetilde{\bigcup}_{e_M \widetilde{\in} F_A} f(e_M) = F_A$ and so

$\widetilde{\tau}(F_A) = \widetilde{\tau}(\widetilde{\bigcup}_{e_M \widetilde{\in} F_A} f(e_M)) \geq \bigwedge_{e_M \widetilde{\in} F_A} \widetilde{\tau}\big(f(e_M)\big)$.

Thus, $\widetilde{\tau}(F_A) \geq \bigvee_{f \in \prod_{e_M \widetilde{\in} F_A} \mathfrak{B}_{e_M}} \bigwedge_{e_M \widetilde{\in} F_A} \widetilde{\tau}\big(f(e_M)\big) = \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A} \widetilde{\tau}(F_B)$.

(2) From (1) we have,

$$\begin{aligned}
\big[\forall e_M\big(e_M \widetilde{\in} F_A \to \exists F_B\big(F_B \in N_{e_M} \wedge F_B \widetilde{\subseteq} F_A\big)\big)\big] &= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \widetilde{\subseteq} F_A} N_{e_M}(F_B) \\
&= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \widetilde{\subseteq} F_A} \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} F_B} \widetilde{\tau}(F_C) \\
&= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} F_B} \widetilde{\tau}(F_C) \\
&= \big[F_A \in \widetilde{\tau}\big].
\end{aligned}$$ ∎

Corollary 3.9. $\bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_A) = \widetilde{\tau}(F_A)$.

Theorem 3.10. *The mapping $N : \mathrm{SP}(X) \to \Im^N_E(S(X))$, $e_M \mapsto N_{e_M}$ where $\Im^N_E(S(X))$ is the set of all normal fuzzy soft subsets of $S(X)$ has the following properties:*

(1) *For any* $e_M, F_A, \models F_A \in N_{e_M} \to e_M \widetilde{\in} F_A$;
(2) *For any* $e_M, F_A, F_B, \models (F_A \in N_{e_M}) \wedge (F_B \in N_{e_M}) \to F_A \widetilde{\cap} F_B \in N_{e_M}$;
(3) *For any* $e_M, F_A, F_B, \models F_A \widetilde{\subseteq} F_B \to (F_A \in N_{e_M} \to F_B \in N_{e_M})$.

Proof. One can easily have that for each $e_F \in \mathbf{SP}(X), N_{e_F} = 1$, i.e. N_{e_F} is normal.

(1) If $N_{e_M}(F_A) = 0$, then the result holds. If $N_{e_M}(F_A) > 0$, then $\bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} F_A} \widetilde{\tau}(F_B) > 0$ and so there exists $F_C \in S(X)$ such that $e_M \widetilde{\in} F_C \widetilde{\subseteq} F_A$. Thus $[e_M \in F_A] = 1 \geq N_{e_M}(F_A)$.

$$
\begin{aligned}
(2)\ [F_A \widetilde{\cap} F_B \in N_{e_M}] &= \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} F_A \widetilde{\cap} F_B} \widetilde{\tau}(F_C) \\
&= \bigvee_{e_M \widetilde{\in} F_{C_1} \widetilde{\subseteq} F_A, e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} F_B} \widetilde{\tau}(F_{C_1} \widetilde{\cap} F_{C_2}), \text{ where } F_C = F_{C_1} \widetilde{\cap} F_{C_2} \\
&\geq \bigvee_{e_M \widetilde{\in} F_{C_1} \widetilde{\subseteq} F_A, e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} F_B} \big(\widetilde{\tau}(F_{C_1}) \wedge \widetilde{\tau}(F_{C_2})\big) \\
&= \bigvee_{e_M \widetilde{\in} F_{C_1} \widetilde{\subseteq} F_A} \widetilde{\tau}(F_{C_1}) \wedge \bigvee_{e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} F_B} \widetilde{\tau}(F_{C_2}) \\
&= \big[(F_A \in N_{e_M}) \wedge (F_B \in N_{e_M})\big].
\end{aligned}
$$

(3) If $[F_A \widetilde{\subseteq} F_B] = 0$, then the result holds. If $[F_A \widetilde{\subseteq} F_B] = 1$, then $N_{e_M}(F_B) = \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} F_B} \widetilde{\tau}(F_C) \geq \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} F_A} \widetilde{\tau}(F_C) = N_{e_M}(F_A)$. ■

Next, we will introduce the notion of fuzzifying derived in fuzzifying soft topological space as follows:

Definition 3.11. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. The fuzzifying soft derived of F_A, denoted by $D \in \Im(S(X))$, is defined as

$$e_M \widetilde{\in} D(F_A) := \forall F_B \big(F_B \in N_{e_M} \to (F_B \widetilde{\cap} (F_A \backslash \{e_M\}) \neq \mathbf{0}_A)\big),$$

i.e.,

$$D(F_A)(e_M) = \bigwedge_{F_B \widetilde{\cap} (F_A \backslash \{e_M\}) = \mathbf{0}_A} \big(1 - N_{e_M}(F_B)\big).$$

Example 3.12. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Examples 3.2, we have $D(\mathbf{1}_A)(e_M) = D(F_{A_1})(e_M) = D(F_{A_4})(e_M) = D(F_{A_5})(e_M) = \frac{3}{4}$; $D(\mathbf{0}_A)(e_M) = D(F_{A_3})(e_M) = 0$ and $D(F_{A_2})(e_M) = D(F_{A_6})(e_M) = \frac{1}{2}$.

Theorem 3.13. *Let* $(X, \widetilde{\tau}, A)$ *be a fuzzifying soft topological space and* $F_A, F_B \in S(X)$. *Then*
(1) $\models D(F_A)(e_M) = 1 - N_{e_M}\big((1_A \setminus F_A) \widetilde{\cup} \{e_M\}\big)$;
(2) $\models D(0_A) \equiv 0_A$;
(3) $\models F_A \widetilde{\subseteq} F_B \to D(F_A) \widetilde{\subseteq} D(F_B)$;
(4) $\models F_A \in \widetilde{F} \leftrightarrow D(F_A) \widetilde{\subseteq} F_A$.

Proof. (1) $D(F_A)(e_M) = \bigwedge_{F_B \widetilde{\cap} (F_A \backslash \{e_M\}) = \mathbf{0}_A} \big(1 - N_{e_M}(F_B)\big) = 1 - \bigvee_{F_B \widetilde{\cap} (F_A \backslash \{e_M\}) = \mathbf{0}_A} N_{e_M}(F_B) = 1 - \bigvee_{F_B \widetilde{\subseteq} (\mathbf{1}_A \backslash F_A) \widetilde{\cup} \{e_M\}} N_{e_M}(F_B) = 1 - N_{e_M}\big((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}\big)$.

(2) From (1) above and since N_{e_M} is normal, we have $D(\mathbf{0}_A)(e_M) = 1 - N_{e_M}\big((\mathbf{1}_A \setminus \mathbf{0}_A) \widetilde{\cup}\{e_M\}\big) = 1 - N_{e_M}(\mathbf{1}_A) = 1 - 1 = 0$.

(3) If $[F_A \widetilde{\subseteq} F_B] = 0$, then the result holds. If $[F_A \widetilde{\subseteq} F_B] = 1$, then $D(F_A)(e_M) = 1 - N_{e_M}\big((\mathbf{1}_A \setminus F_A)\widetilde{\cup}\{e_M\}\big) \leq 1 - N_{e_M}\big((\mathbf{1}_A \setminus F_B)\widetilde{\cup}\{e_M\}\big) = D(F_B)(e_M)$.

(4)
$$\begin{aligned}
[D(F_A)\widetilde{\subseteq}F_A] &= \forall e_M\big(e_M\widetilde{\in}D(F_A) \to e_M\widetilde{\in}F_A\big)\\
&= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - D(F_A)(e_M) + [e_M\widetilde{\in}F_A]\big)\\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \big(1 - D(F_A)(e_M)\big)\\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \Big(1 - \big(1 - N_{e_M}\big((\mathbf{1}_A \setminus F_A)\ \widetilde{\cup}\{e_M\}\big)\big)\Big)\\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} N_{e_M}\big((\mathbf{1}_A \setminus F_A)\ \widetilde{\cup}\{e_M\}\big)\\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} N_{e_M}(\mathbf{1}_A \setminus F_A)\\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \ \bigvee_{e_M \widetilde{\in} F_B \widetilde{\subseteq} \mathbf{1}_A \setminus F_A} \widetilde{\tau}(F_B)\\
&= \widetilde{\tau}(\mathbf{1}_A \setminus F_A) = [F_A \in \widetilde{\mathcal{F}}].
\end{aligned}$$
■

Next, we will introduce the notion of fuzzifying closure in fuzzifying soft topological space as follows:

Definition 3.14. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. The fuzzifying soft closure of F_A, denoted by $Cl \in \Im(S(X))$, is defined as

$$e_M\widetilde{\in}Cl(F_A) := \forall F_B\big(\big(F_A\widetilde{\subseteq}F_B\big) \wedge \big(F_B \in \widetilde{\mathcal{F}}\big) \to e_M\widetilde{\in}F_B\big),$$

i.e.,

$$Cl(F_A)(e_M) = \bigwedge_{e_M \widetilde{\notin} F_B \widetilde{\supseteq} F_A} \big(1 - \widetilde{\mathcal{F}}(F_B)\big).$$

Example 3.15. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2, we have $Cl(\mathbf{1}_A)(e_M) = Cl(F_{A_3})(e_M) = Cl(F_{A_5})(e_M) = Cl(F_{A_6})(e_M) = 1; Cl(\mathbf{0}_A)(e_M) = 0; Cl(F_{A_1})(e_M) = Cl(F_{A_4})(e_M) = \frac{3}{4}$ and $Cl(F_{A_2})(e_M) = \frac{1}{2}$.

Theorem 3.16. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B, F_C \in S(X)$. Then*

(1) $\models Cl(F_A)(e_M) = 1 - N_{e_M}(1_A \setminus F_A)$;
(2) $\models Cl(0_A) \equiv 0_A$;
(3) $\models F_A\widetilde{\subseteq}Cl(F_A)$;
(4) $\models F_A\widetilde{\subseteq}F_B \to Cl(F_A)\widetilde{\subseteq}Cl(F_B)$;
(5) $\models Cl(F_A\widetilde{\cup}F_B) \equiv Cl(F_A)\widetilde{\cup}Cl(F_B)$;
(6) $\models e_M\widetilde{\in}Cl(F_A) \leftrightarrow \forall F_B(F_B \in N_{e_M} \to F_A\widetilde{\cap}F_B \neq 0_A)$;
(7) $\models Cl(F_A) \equiv F_A\widetilde{\cup}D(F_A)$;
(8) $\models F_A \equiv Cl(F_A) \leftrightarrow F_A \in \widetilde{\mathcal{F}}$.

Proof. (1) $Cl(F_A)(e_M) = \bigwedge_{e_M \widetilde{\notin} F_B \widetilde{\supseteq} F_A} \left(1 - \widetilde{F}(F_B)\right) = \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_B \widetilde{\subseteq} \mathbf{1}_A \setminus F_A} \left(1 - \widetilde{\tau}(\mathbf{1}_A \setminus F_B)\right) = 1 - \bigvee_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_B \widetilde{\subseteq} \mathbf{1}_A \setminus F_A} \widetilde{\tau}(\mathbf{1}_A \setminus F_B) = 1 - N_{e_M}(\mathbf{1}_A \setminus F_A)$.

(2) From (1) above and since N_{e_M} is normal, we have $Cl(\mathbf{0}_A)(e_M) = 1 - N_{e_M}(\mathbf{1}_A \setminus \mathbf{0}_A) = 1 - N_{e_M}(\mathbf{1}_A) = 1 - 1 = 0$.

(3) It is clear that for any $F_A \in S(X)$ and any $e_M \in \mathbf{SP}(X)$, if $e_M \widetilde{\notin} F_A$, then $N_{e_M}(F_A) = 0$. If $e_M \widetilde{\in} F_A$, then $Cl(F_A)(e_M) = 1 - N_{e_M}(\mathbf{1}_A \setminus F_A) = 1 - 0 = 1$. Then $[F_A \widetilde{\subseteq} Cl(F_A)] = 1$.

(4) It is similar to the proof of Theorem 3.13 (3).

(5) First by (4). We can easy get $Cl(F_A) \widetilde{\cup} Cl(F_B) \widetilde{\subseteq} Cl(F_A \widetilde{\cup} F_B)$. Conversely, for every $e_M \in \mathbf{SP}(X)$,

$$\begin{aligned}
Cl(F_A \widetilde{\cup} F_B)(e_M) &= 1 - N_{e_M}\left(\mathbf{1}_A \setminus (F_A \widetilde{\cup} F_B)\right) \\
&= 1 - N_{e_M}\left((\mathbf{1}_A \setminus F_A) \widetilde{\cap} (\mathbf{1}_A \setminus F_B)\right) \\
&= 1 - \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} (\mathbf{1}_A \setminus F_A) \widetilde{\cap} (\mathbf{1}_A \setminus F_B)} \widetilde{\tau}(F_C) \\
&= 1 - \bigvee_{e_M \widetilde{\in} F_{C_1} \widetilde{\subseteq} (\mathbf{1}_A \setminus F_A), e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} (\mathbf{1}_A \setminus F_B)} \widetilde{\tau}(F_{C_1} \widetilde{\cap} F_{C_2}) \\
&\leq 1 - \bigvee_{e_M \widetilde{\in} F_{C_1} \widetilde{\subseteq} (\mathbf{1}_A \setminus F_A), e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} (\mathbf{1}_A \setminus F_B)} \left(\widetilde{\tau}(F_{C_1}) \wedge \widetilde{\tau}(F_{C_2})\right) \\
&= \left(1 - \bigvee_{e_M \widetilde{\in} F_C \widetilde{\subseteq} (\mathbf{1}_A \setminus F_A)} \widetilde{\tau}(F_{C_1})\right) \vee \left(1 - \bigvee_{e_M \widetilde{\in} F_{C_2} \widetilde{\subseteq} (\mathbf{1}_A \setminus F_B)} \widetilde{\tau}(F_{C_2})\right) \\
&= \left(1 - N_{e_M}(\mathbf{1}_A \setminus F_A)\right) \vee \left(1 - N_{e_M}(\mathbf{1}_A \setminus F_B)\right) \\
&= Cl(F_A)(e_M) \vee Cl(F_B)(e_M).
\end{aligned}$$

(6) $[\forall F_B(F_B \in N_{e_M} \to F_A \widetilde{\cap} F_B \neq \mathbf{0}_A)] = \bigwedge_{F_B \widetilde{\subseteq} \mathbf{1}_A \setminus F_A} \left(1 - N_{e_M}(F_B)\right) = 1 - N_{e_M}(\mathbf{1}_A \setminus F_A) = [e_M \widetilde{\in} Cl(F_A)]$.

(7) If $e_M \widetilde{\in} F_A$, then the result holds. If $e_M \widetilde{\notin} F_A$, then $\left(F_A \widetilde{\cup} D(F_A)\right)(e_M) = \max \left(F_A(e_M), D(F_A)(e_M)\right) = \max \left(F_A(e_M), 1 - N_{e_M}\left((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}\right)\right) = 1 - N_{e_M}\left((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}\right) = 1 - N_{e_M}(\mathbf{1}_A \setminus F_A) = Cl(F_A)(e_M)$.

(8) From Theorem 3.13 (4) and (7) above we have, $F_A \in \widetilde{F} \leftrightarrow D(F_A) \widetilde{\subseteq} F_A \leftrightarrow F_A \equiv F_A \widetilde{\cup} D(F_A) \equiv Cl(F_A)$. ■

Next, we will introduce the notion of fuzzifying interior in fuzzifying soft topological space as follows:

Definition 3.17. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A \in S(X)$. The fuzzifying soft interior of F_A, denoted by $Int \in \Im(S(X))$, is defined as

$$e_M \widetilde{\in} Int(F_A) := F_A \in N_{e_M}, \text{ i.e., } \quad Int(F_A)(e_M) = N_{e_M}(F_A).$$

Theorem 3.18. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. Then*

(1) $\models Int(F_A) \equiv 1_A \setminus Cl(1_A \setminus F_A)$;
(2) $\models Int(1_A) \equiv 1_A$;
(3) $\models Int(F_A) \widetilde{\subseteq} F_A$;
(4) $\models (F_A \in \widetilde{\tau}) \wedge (F_A \widetilde{\subseteq} F_B) \to F_A \widetilde{\subseteq} Int(F_B)$;
(5) $\models e_M \widetilde{\in} Int(F_A) \leftrightarrow (e_M \widetilde{\in} F_A) \wedge \big(e_M \widetilde{\in} (1_A \setminus D(1_A \setminus F_A))\big)$;
(6) $\models F_A \equiv Int(F_A) \leftrightarrow F_A \in \widetilde{\tau}$.

Proof. (1) From Theorem 3.16 (1), we have $Cl(\mathbf{1}_A \setminus F_A)(e_M) = 1 - N_{e_M}(F_A) = 1 - Int(F_A)(e_M)$. Then, $[Int(F_A) \equiv \mathbf{1}_A \setminus Cl(\mathbf{1}_A \setminus F_A)] = 1$.

(2) From (1), we have $Int(\mathbf{1}_A) = \mathbf{1}_A \setminus Cl(\mathbf{1}_A \setminus \mathbf{1}_A) = \mathbf{1}_A \setminus Cl(\mathbf{0}_A) = \mathbf{1}_A \setminus \mathbf{0}_A = \mathbf{1}_A$.

(3) $Int(F_A) \equiv \mathbf{1}_A \setminus Cl(\mathbf{1}_A \setminus F_A) \widetilde{\subseteq} \mathbf{1}_A \setminus (\mathbf{1}_A \setminus F_A) \equiv F_A$.

(4) If $[F_A \widetilde{\subseteq} F_B] = 0$, then the result holds. If $[F_A \widetilde{\subseteq} F_B] = 1$, then Theorem 3.10 (3) and Corollary 3.9, we have
$[F_A \widetilde{\subseteq} Int(F_B)] = \bigwedge_{e_M \widetilde{\in} F_A} Int(F_B)(e_M) = \bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_B) \geq \bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_A) = \widetilde{\tau}(F_A) = [(F_A \in \widetilde{\tau}) \wedge (F_A \widetilde{\subseteq} F_B)]$.

(5) If $e_M \widetilde{\notin} F_A$, then by Theorem 3.10 (1) $N_{e_M}(F_A) = 0$. Hence $[e_M \widetilde{\in} Int(F_A)] = 0 = \big[(e_M \widetilde{\in} F_A) \wedge \big(e_M \widetilde{\in} (\mathbf{1}_A \setminus D(\mathbf{1}_A \setminus F_A))\big)$. If $e_M \widetilde{\in} F_A$, then $\big[\big(e_M \widetilde{\in} (\mathbf{1}_A \setminus D(\mathbf{1}_A \setminus F_A))\big)\big] = [1 - D(\mathbf{1}_A \setminus F_A)(e_M)] = [1 - (1 - N_{e_M}(F_A \widetilde{\cup} \{e_M\}))] = [N_{e_M}(F_A)] = [e_M \in Int(F_A)]$.

(6) From Corollary 3.9, we have

$$\begin{aligned}[F_A \equiv Int(F_A)] &= \min\Big(\bigwedge_{e_M \widetilde{\in} F_A} Int(F_A)(e_M), \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} (1 - Int(F_A)(e_M))\Big) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} Int(F_A)(e_M) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_A) = \widetilde{\tau}(F_A) = [F_A \in \widetilde{\tau}]. \qquad \blacksquare\end{aligned}$$

4. Fuzzifying Soft θ-Open Sets and Fuzzifying Soft θ-Closed Sets

In this section, we will introduce and study the concepts of fuzzifying soft θ-neighborhood system, fuzzifying soft θ-closure, fuzzifying soft θ-closed sets, fuzzifying soft θ-closed sets, and fuzzifying soft θ-interior of a soft set.

Definition 4.1. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space, $e_M \in \mathbf{SP}(X)$ and $F_A, F_B \in S(X)$. The fuzzifying soft θ-neighborhood system of e_M, denoted by $N^{\theta}_{e_M} \in \Im(S(X))$, is defined as

$$F_A \in N^{\theta}_{e_M} := \exists F_B \big((F_B \in N_{e_M}) \otimes (Cl(F_B) \widetilde{\subseteq} F_A)\big),$$

i.e., $$N^{\theta}_{e_M}(F_A) = \bigvee_{F_B \in S(X)} \max\big(0, N_{e_M}(F_B) + \bigwedge_{e_H \widetilde{\notin} F_A} N_{e_H}(\mathbf{1}_A \setminus F_B) - 1\big).$$

Example 4.2. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2, we have $N^{\theta}_{e_M}(\mathbf{1}_A) = N^{\theta}_{e_M}(F_{A_2}) = N^{\theta}_{e_M}(F_{A_4}) = N^{\theta}_{e_M}(F_{A_6}) = 1$ and $N^{\theta}_{e_M}(\mathbf{0}_A) = N^{\theta}_{e_M}(F_{A_1}) = N^{\theta}_{e_M}(F_{A_3}) = N^{\theta}_{e_M}(F_{A_5}) = \frac{1}{4}$.

Theorem 4.3. *The mapping $N^{\theta} : \mathrm{SP}(X) \to \Im^N_E(S(X))$, $e_M \mapsto N^{\theta}_{e_M}$ where $\Im^N_E(S(X))$ is the set of all normal fuzzy soft subsets of $S(X)$ has the following properties:*

(1) *For any $e_M, F_A, \models F_A \in N^{\theta}_{e_M} \to e_M \widetilde{\in} F_A$;*

(2) *For any $e_M, F_A, F_B, \models (F_A \in N^{\theta}_{e_M}) \wedge (F_B \in N^{\theta}_{e_M}) \leftrightarrow F_A \widetilde{\cap} F_B \in N^{\theta}_{e_M}$;*

(3) *For any $e_M, F_A, F_B, \models F_A \widetilde{\subseteq} F_B \to (F_A \in N^{\theta}_{e_M} \to F_B \in N^{\theta}_{e_M})$.*

Proof. (1) If $[F_A \in N^{\theta}_{e_M}] = 0$, then the result holds. If $[F_A \in N^{\theta}_{e_M}] > 0$, then $\bigvee_{F_B \in S(X)} \max \big(0, N_{e_M}(F_B) + [Cl(F_B) \widetilde{\subseteq} F_A] - 1\big) > 0$. There exists $F_C \in S(X)$ such that $N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1 > 0$. From Theorems 3.10 (1), 3.16 (3) we have, $[Cl(F_C) \widetilde{\subseteq} F_A] > 1 - N_{e_M}(F_C) \geq 1 - [e_M \widetilde{\in} F_C] \geq 1 - Cl(F_C)(e_M)$. Therefore, $[Cl(F_C) \widetilde{\subseteq} F_A] = \bigwedge_{e_H \widetilde{\notin} F_A} \big(1 - Cl(F_C)(e_H)\big) > 1 - Cl(F_C)(e_M)$, and so $e_M \widetilde{\in} F_A$. Otherwise, if $e_M \widetilde{\notin} F_A$, then $\bigwedge_{e_H \widetilde{\notin} F_A} \big(1 - Cl(F_C)(e_H)\big) > 1 - Cl(F_C)(e_M)$, contradiction with the definition of the infimum. So $[e_M \in F_A] = 1$. Then $[F_A \in N^{\theta}_{e_M}] \leq [e_M \widetilde{\in} F_A]$.

(2) Given $F_C \in S(X)$,

$$\begin{aligned}
[Cl(F_C) \widetilde{\subseteq} F_A \widetilde{\cap} F_B] &= \forall e_M \big(e_M \widetilde{\in} Cl(F_C) \to e_M \widetilde{\in} F_A \widetilde{\cap} F_B\big) \\
&= \bigwedge_{e_M \in SP(X)} \min\,(1, 1 - Cl(F_C)(e_M) + [(F_A \widetilde{\cap} F_B)(e_M)]) \\
&= \bigwedge_{e_M \widetilde{\notin} F_A \widetilde{\cap} F_B} \big(1 - Cl(F_C)(e_M)\big) \\
&= \bigwedge_{e_M \widetilde{\in} (\mathbf{1}_A \setminus F_A) \widetilde{\cup} (\mathbf{1}_A \setminus F_B)} \big(1 - Cl(F_C)(e_M)\big) \\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \big(1 - Cl(F_C)(e_M)\big) \wedge \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_B} \big(1 - Cl(F_C)(e_M)\big) \\
&= [Cl(F_C) \widetilde{\subseteq} F_A] \wedge [Cl(F_C) \widetilde{\subseteq} F_B],
\end{aligned}$$

$$\begin{aligned}
N^{\theta}_{e_M}(F_A \widetilde{\cap} F_B) &= \bigvee_{F_C \in S(X)} \max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A \widetilde{\cap} F_B] - 1) \\
&= \bigvee_{F_C \in S(X)} \max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] \wedge [Cl(F_C) \widetilde{\subseteq} F_B] - 1) \\
&= \bigvee_{F_C \in S(X)} \max\,(0, (N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1) \wedge (N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_B] - 1)) \\
&= \bigvee_{F_C \in S(X)} (\max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1) \wedge \max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_B] - 1)) \\
&= (\bigvee_{F_C \in S(X)} \max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1)) \wedge \\
&\qquad (\bigvee_{F_C \in S(X)} \max\,(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_B] - 1)) \\
&= N^{\theta}_{e_M}(F_A) \wedge N^{\theta}_{e_M}(F_B).
\end{aligned}$$

(3) If $[F_A \widetilde{\subseteq} F_B] = 0$, then the result holds. If $[F_A \widetilde{\subseteq} F_B] = 1$ and $F_C \in S(X)$, then $[Cl(F_C) \widetilde{\subseteq} F_B] = \bigwedge_{e_M \widetilde{\notin} \mathbf{1}_A \setminus F_B} \big(1 - Cl(F_C)(e_M)\big) \geq \bigwedge_{e_M \widetilde{\notin} \mathbf{1}_A \setminus F_A} \big(1 - Cl(F_C)(e_M)\big) = [Cl(F_C) \widetilde{\subseteq} F_A]$, and so $N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_B] - 1 \geq N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1$. Thus, $\bigvee_{F_C \in S(X)} \max \big(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_B] - 1\big) \geq \bigvee_{F_C \in S(X)} \max \big(0, N_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} F_A] - 1\big)$. Hence, $N^{\theta}_{e_M}(F_A) \leq N^{\theta}_{e_M}(F_B)$. ∎

Definition 4.4. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. The fuzzifying soft θ-closure of F_A, denoted by $Cl_\theta \in \Im(S(X))$, is defined as

$$e_M \widetilde{\in} Cl_\theta(F_A) := \forall F_B (F_B \in N_{e_M} \to \neg(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A)).$$

Example 4.5. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2. Let $e_H = \{(e, \{x_2\})\}$ be a soft point. Then, $Cl_\theta(\mathbf{1}_A)(e_M) = Cl_\theta(F_{A_2})(e_M) = Cl_\theta(F_{A_4})(e_M) = Cl_\theta(F_{A_6})(e_M) = \frac{3}{4}$ and $Cl_\theta(\mathbf{0}_A)(e_M) = Cl_\theta(F_{A_1})(e_M) = Cl_\theta(F_{A_3})(e_M) = Cl_\theta(F_{A_5})(e_M) = 0$.

Based on Definition 4.4, we will propose the following Lemma 4.6.

Lemma 4.6. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. Then* $Cl_\theta(F_A)(e_M) := \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + \bigvee_{e_H \in F_A} Cl(F_B)(e_H)\big)$.

Proof. $[\forall F_B (F_B \in N_{e_M} \to \neg(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A))]$

$$\begin{aligned}
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + [\neg(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A)]\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + 1 - [F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A]\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + 1 - ([F_A \widetilde{\cap} Cl(F_B) \widetilde{\subseteq} \mathbf{0}_A] \wedge [\mathbf{0}_A \widetilde{\subseteq} F_A \widetilde{\cap} Cl(F_B)])\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + 1 - \bigwedge_{e_H \widetilde{\in} F_A} \min \big(1, 1 - (F_A \widetilde{\cap} Cl(F_B)(e_H) + 0)\big)\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + \bigvee_{e_H \widetilde{\in} F_A} \max \big(0, (F_A \widetilde{\cap} Cl(F_B)(e_H))\big)\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + \bigvee_{e_H \widetilde{\in} F_A} \min \big((F_A)(e_H), Cl(F_B)(e_H)\big)\big)\\
&= \bigwedge_{F_B \in S(X)} \min \big(1, 1 - N_{e_M}(F_B) + \bigvee_{e_H \widetilde{\in} F_A} Cl(F_B)(e_H)\big).
\end{aligned}$$

In the following, we show the definitions of fuzzifying soft θ-closed sets and fuzzifying soft θ-open sets.

Definition 4.7. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A \in S(X)$. The family of all fuzzifying soft θ-closed sets, denoted by $\widetilde{F}_\theta \in \Im(S(X))$, is defined as

$$F_A \in \widetilde{F}_\theta := F_A \equiv Cl_\theta(F_A),$$

The family of all fuzzifying soft θ-open sets, denoted by $\widetilde{\tau}_\theta \in \Im(S(X))$, is defined as

$$F_A \in \widetilde{\tau}_\theta := \mathbf{1}_A \setminus F_A \in \widetilde{F}_\theta,$$

Intuitively, the degree to which F_A is fuzzifying soft θ-closed is

$$\widetilde{F}_\theta(F_A) = \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \big(1 - Cl_\theta(F_A)(e_M)\big).$$

Example 4.8. The fuzzifying soft topological space $(X,\widetilde{\tau},A)$ is the same as in Example 3.2, we have $\widetilde{F}_\theta(\mathbf{1}_A)=\widetilde{F}_\theta(F_{A_3})=\widetilde{F}_\theta(F_{A_5})=\widetilde{F}_\theta(F_{A_6})=0;\widetilde{F}_\theta(\mathbf{0}_A)=\widetilde{F}_\theta(F_{A_1})=1$ and $\widetilde{F}_\theta(F_{A_2})=\widetilde{F}_\theta(F_{A_4})=\frac{1}{4}$. And $\widetilde{\tau}_\theta(\mathbf{1}_A)=\widetilde{\tau}_\theta(F_{A_6})=1;\widetilde{\tau}_\theta(\mathbf{0}_A)=\widetilde{\tau}_\theta(F_{A_1})=\widetilde{\tau}_\theta(F_{A_2})=\widetilde{\tau}_\theta(F_{A_4})=0$ and $\widetilde{\tau}_\theta(F_{A_3})=\widetilde{\tau}_\theta(F_{A_5})=\frac{1}{4}$.

Theorem 4.9. *Let $(X,\widetilde{\tau},A)$ be a fuzzifying soft topological space and $F_A,F_B\in S(X)$. Then*

(1) $Cl_\theta(F_A)(e_M)=1-N^\theta_{e_M}(1_A\setminus F_A)$;

(2) $\models Cl(F_A)\widetilde{\subseteq}Cl_\theta(F_A)$;

(3) $\models F_A\in N^\theta_{e_M}\to F_A\in N_{e_M}$;

(4) $\models Cl_\theta(0_A)\equiv 0_A$;

(5) $\models F_A\widetilde{\subseteq}Cl_\theta(F_A)$;

(6) $\models F_A\widetilde{\subseteq}F_B\to Cl_\theta(F_A)\widetilde{\subseteq}Cl_\theta(F_B)$;

(7) $\models Cl_\theta(F_A\widetilde{\cup}F_B)\equiv Cl_\theta(F_A)\widetilde{\cup}Cl_\theta(F_B)$.

Proof. (1)
$$\begin{aligned}Cl_\theta(F_A)(e_M)&=\bigwedge_{F_B\in S(X)}\min\Big(1,1-N_{e_M}(F_B)+\bigvee_{e_H\widetilde{\in}F_A}Cl(F_B)(e_H)\Big)\\&=1-\bigvee_{F_B\in S(X)}\max\Big(0,N_{e_M}(F_B)-\bigvee_{e_H\widetilde{\in}F_A}Cl(F_B)(e_H)\Big)\\&=1-\bigvee_{F_B\in S(X)}\max\Big(0,N_{e_M}(F_B)+1-\bigvee_{e_H\widetilde{\in}F_A}Cl(F_B)(e_H)-1\Big)\\&=1-\bigvee_{F_B\in S(X)}\max\Big(0,N_{e_M}(F_B)+\bigwedge_{e_H\widetilde{\in}F_A}\big(1-Cl(F_B)(e_H)\big)-1\Big)\\&=1-\Big[(\exists F_B)\Big((F_B\in N_{e_M})\otimes\big(Cl(F_B)\widetilde{\subseteq}\mathbf{1}_A\setminus F_A\big)\Big)\Big]\\&=1-N^\theta_{e_M}(\mathbf{1}_A\setminus F_A).\end{aligned}$$

(2)
$$\begin{aligned}Cl(F_A)(e_M)&=\bigwedge_{F_B\in S(X)}\min\Big(1,1-N_{e_M}(F_B)+\bigvee_{e_H\widetilde{\in}F_A}(F_B)(e_H)\Big)\\&\le\bigwedge_{F_B\in S(X)}\min\Big(1,1-N_{e_M}(F_B)+\bigvee_{e_H\widetilde{\in}F_A}Cl(F_B)(e_H)\Big)\\&=Cl_\theta(F_A)(e_M).\end{aligned}$$

(3) Follows from (2) above.

(4), (5) and (6) are similar to the proof of Theorem 3.16 (2), (3) and (4) respectively.

(7) It is easy to get $Cl_\theta(F_A)\widetilde{\cup}Cl_\theta(F_B)\widetilde{\subseteq}Cl_\theta(F_A\widetilde{\cup}F_B)$.

Conversely, for every $e_M\in\mathbf{SP}(X)$,
$$\begin{aligned}&Cl_\theta(F_A\widetilde{\cup}F_B)(e_M)\\&=\bigwedge_{F_C\in S(X)}\min\Big(1,1-N_{e_M}(F_C)+\bigvee_{e_H\widetilde{\in}(F_A\widetilde{\cup}F_B)}Cl(F_C)(e_H)\Big)\\&=\bigwedge_{F_C\in S(X)}\min\Big(1,1-N_{e_M}(F_C)+\Big(\bigvee_{e_H\widetilde{\in}F_A}Cl(F_C)(e_H)\vee\bigvee_{e_H\widetilde{\in}F_B}Cl(F_C)(e_H)\Big)\Big)\\&\le\bigwedge_{F_C\in S(X)}\min\Big(1,1-N_{e_M}(F_C)+\bigvee_{e_H\widetilde{\in}F_A}Cl(F_C)(e_H)\Big)\vee\bigwedge_{F_C\in S(X)}\min\Big(1,1-N_{e_M}(F_C)\\&\qquad+\bigvee_{e_H\widetilde{\in}F_B}Cl(F_C)(e_H)\Big)\\&=Cl_\theta(F_A)(e_M)\vee Cl_\theta(F_B)(e_M).\end{aligned}$$
∎

The equality in Theorem 4.9 (2) and (3) does not hold as shown in the following example.

Example 4.10. Let us consider the fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ in Example 3.2. Then, $Cl_\theta(F_{A_6})(e_F) = \frac{3}{4} \le Cl(F_{A_6})(e_F) = 1$ and $N^\theta_{e_F}(F_{A_6}) = 1 \ge N_{e_F}(F_{A_6}) = \frac{1}{4}$.

The connect between the fuzzifying soft θ-closure and fuzzifying soft closure is presented as follows:

Theorem 4.11. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. Then*

(1) $\models F_A \in \widetilde{\tau} \to \big(Cl_\theta(F_A) \equiv Cl(F_A)\big)$;

(2) $\models F_A \in \widetilde{F_\theta} \to \big(Cl_\theta(F_A) \equiv Cl(F_A)\big)$.

Proof. (1) First we prove that $\models F_A \in \widetilde{\tau} \to \neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big) \to \neg\big(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A\big)$. In fact, from Corollary 3.9, we have

$$\begin{aligned}
&\widetilde{\tau}(F_A) \otimes \big[\neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big)\big] \\
&\quad = \max\Big(0, \bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_A) + \bigvee_{e_H \widetilde{\in} F_A} Cl(F_B)(e_H) - 1\Big) \\
&\quad = \max\Big(0, \bigvee_{e_H \widetilde{\in} F_A} Cl(F_B)(e_H) - \bigvee_{e_M \widetilde{\in} F_A} \big(1 - N_{e_M}(F_A)\big)\Big) \\
&\quad \le \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big(Cl(F_B)(e_M) - 1 + N_{e_M}(F_A)\big)\Big) \\
&\quad = \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big((F_B \widetilde{\cup} D(F_B))(e_M) - 1 + N_{e_M}(F_A)\big)\Big) \\
&\quad = \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big(\max\big((F_B)(e_M), D(F_B)(e_M)\big) - 1 + N_{e_M}(F_A)\big)\Big).
\end{aligned}$$

If $e_M \widetilde{\in} F_B$, then

$$\begin{aligned}
\widetilde{\tau}(F_A) \otimes \big[\neg\big(F_A \widetilde{\cap} Cl(F_B)\big) \equiv \mathbf{0}_A\big] &= \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big((F_B)(e_M) - 1 + N_{e_M}(F_A)\big)\Big) \\
&\le \bigvee_{e_M \widetilde{\in} F_A} N_{e_M}(F_A)(e_M) \\
&\le \bigvee_{e_M \widetilde{\in} F_A} F_A(e_M) \\
&\le \bigvee_{e_M \widetilde{\in} F_A} \big(F_A(e_M) \wedge F_B(e_M)\big) \\
&= \neg\big(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A\big).
\end{aligned}$$

If $e_M \widetilde{\notin} F_B$, then

$$\begin{aligned}
&\widetilde{\tau}(F_A) \otimes \big[\neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big)\big] \\
&\quad = \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big(D(F_B)(e_M) - 1 + N_{e_M}(F_A)\big)\Big) \\
&\quad = \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \Big(\bigwedge_{F_C \widetilde{\in} S(X)} \min\Big(1, 1 - N_{e_M}(F_C) + \bigvee_{e_H \widetilde{\in} F_C} \big(F_C \setminus \{e_M\}(e_H)\big) - 1 \\
&\qquad + N_{e_M}(F_A)\Big)\Big)\Big)
\end{aligned}$$

$$\leq \max\Big(0, \bigvee_{e_M \widetilde{\in} F_A} \big(1 - N_{e_M}(F_C) + \bigvee_{e_H \widetilde{\in} F_A} (F_B \setminus \{e_M\}(e_H) - 1 + N_{e_M}(F_A))\big)\Big)$$
$$= \bigvee_{e_H \widetilde{\in} F_A} (F_B \setminus \{e_M\})(e_H)$$
$$= \bigvee_{e_H \widetilde{\in} F_A} (F_B)(e_H) = [\neg(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A)].$$

So, $\big[\neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big)\big] \to \big[\neg\big(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A\big)\big] \geq \widetilde{\tau}(F_A)$.

Therefore,

$$\big[Cl_\theta(F_A) \equiv Cl(F_A)\big] = \forall F_B\Big(F_B \in N_{e_M} \to \neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big)\Big) \to \forall F_C\Big(F_C \in N_{e_M} \to \neg\big(F_A \widetilde{\cap} F_C \equiv \mathbf{0}_A\big)\Big)$$
$$= \forall F_B\Big(F_B \in N_{e_M} \to \neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big)\big) \to \big(F_B \in N_{e_M} \to \neg\big(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A\big)\Big)$$
$$\geq \forall F_B\Big(\neg\big(F_A \widetilde{\cap} Cl(F_B) \equiv \mathbf{0}_A\big) \to \neg\big(F_A \widetilde{\cap} F_B \equiv \mathbf{0}_A\big)\Big) \geq \bigwedge_{F_B \in S(X)} \widetilde{\tau}(F_A) = \widetilde{\tau}(F_A).$$

(2)
$$\begin{aligned}
[Cl_\theta(F_A) \equiv Cl(F_A)] &= [Cl_\theta(F_A) \widetilde{\subseteq} Cl(F_A)] \wedge [Cl(F_A) \widetilde{\subseteq} Cl_\theta(F_A)] \\
&= [Cl_\theta(F_A) \widetilde{\subseteq} Cl(F_A)] \\
&= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - Cl_\theta(F_A)(e_M) + Cl(F_A)(e_M)\big) \\
&= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - Cl_\theta(F_A)(e_M) + 1 - N_{e_M}(\mathbf{1}_A \setminus F_A)\big) \\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \min\big(1, 1 - Cl_\theta(F_A)(e_M) + 1 - N_{e_M}(\mathbf{1}_A \setminus F_A)\big) \\
&\geq \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \min\big(1, 1 - Cl_\theta(F_A)(e_M)\big) \\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus F_A} \big(1 - Cl_\theta(F_A)(e_M)\big) \\
&= [\mathbf{1}_A \setminus F_A \in \widetilde{\tau}_\theta] = [F_A \in \widetilde{F}_\theta].
\end{aligned}$$
■

Next, we will introduce the notion of fuzzifying soft θ-derived of soft set F_A.

Definition 4.12. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. The fuzzifying soft θ-derived of F_A, denoted by $D_\theta \in \Im(S(X))$, is defined as

$$e_M \widetilde{\in} D_\theta(F_A) := \forall F_B\big(F_B \in N^\theta_{e_M} \to F_B \widetilde{\cap} (F_A \setminus \{e_M\}) \neq \mathbf{0}_A\big),$$

i.e.,

$$D_\theta(F_A)(e_M) = \bigwedge_{F_B \widetilde{\cap} (F_A \setminus \{e_M\}) = \mathbf{0}_A} \big(1 - N^\theta_{e_M}(F_B)\big).$$

Example 4.13. The fuzzifying soft topological space $(X, \widetilde{\tau}, A)$ is the same as in Example 3.2, we have $D_\theta(\mathbf{1}_A)(e_M) = D_\theta(F_{A_2})(e_M) = D_\theta(F_{A_4})(e_M) = D_\theta(F_{A_6})(e_M) = \frac{3}{4}$ and $D_\theta(\mathbf{0}_A)(e_M) = D_\theta(F_{A_1})(e_M) = D_\theta(F_{A_3})(e_M) = D_\theta(F_{A_5})(e_M) = 0$.

Theorem 4.14. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. Then*

(1) $\models D_\theta(F_A)(e_M) = 1 - N^\theta_{e_M}\big((1_A \setminus F_A) \widetilde{\cup} \{e_M\}\big)$;
(2) $\models D_\theta(0_A) \equiv 0_A$;
(3) $\models F_A \widetilde{\subseteq} F_B \to D_\theta(F_A) \widetilde{\subseteq} D_\theta(F_B)$;

(4) $\models Cl_\theta(F_A) \equiv F_A \widetilde{\cup} D_\theta(F_B)$;
(5) $\models D_\theta(F_A) \widetilde{\cup} D_\theta(F_B) \equiv D_\theta(F_A \widetilde{\cup} F_B)$.

Proof. (1), (2) and (3) are similar to the proof of Theorem 3.13 (1), (2) and (3) respectively.
(4) It is similar to the proof of Theorem 3.16 (7).
(5) From Theorem 4.3 (2), we have

$$\begin{aligned} D_\theta(F_A \widetilde{\cup} F_B)(e_M) &= 1 - N^\theta_{e_M}\left((\mathbf{1}_A \setminus (F_A \widetilde{\cup} F_B)) \widetilde{\cup} \{e_M\}\right) \\ &= 1 - N^\theta_{e_M}\left(((\mathbf{1}_A \setminus F_A) \widetilde{\cap} (\mathbf{1}_A \setminus F_B)) \widetilde{\cup} \{e_M\}\right) \\ &= 1 - N^\theta_{e_M}\Big(((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}) \widetilde{\cap} ((\mathbf{1}_A \setminus F_B) \widetilde{\cup} \{e_M\})\Big) \\ &= 1 - \Big(N^\theta_{e_M}\left((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}\right) \wedge N^\theta_{e_M}\left((\mathbf{1}_A \setminus F_B) \widetilde{\cup} \{e_M\}\right)\Big) \\ &= \Big(1 - N^\theta_{e_M}\left((\mathbf{1}_A \setminus F_A) \widetilde{\cup} \{e_M\}\right)\Big) \vee \Big(1 - N^\theta_{e_M}\left((\mathbf{1}_A \setminus F_B) \widetilde{\cup} \{e_M\}\right)\Big) \\ &= D_\theta(F_A)(e_M) \vee D_\theta(F_B)(e_M). \end{aligned}$$

(6) Follows from Theorem 4.9 (3). ■

Next, we will introduce the notion of fuzzifying soft θ-interior of soft set F_A.

Definition 4.15. Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A \in S(X)$. The fuzzifying soft θ-interior of F_A, denoted by $Int_\theta \in \Im(S(X))$, is defined as $e_M \widetilde{\in} Int_\theta(F_A) := F_A \in N^\theta_{e_M}$, i.e., $Int_\theta(F_A)(e_M) = N^\theta_{e_M}(F_A)$.

Theorem 4.16. *Let $(X, \widetilde{\tau}, A)$ be a fuzzifying soft topological space and $F_A, F_B \in S(X)$. Then*
(1) $\models Int_\theta(F_A) \equiv 1_A \setminus Cl_\theta(1_A \setminus F_A)$;
(2) $\models Int_\theta(1_A) \equiv 1_A$;
(3) $\models Int_\theta(F_A) \widetilde{\subseteq} F_A$;
(4) $\models F_A \in \widetilde{\tau}_\theta \leftrightarrow \forall e_M \left(e_M \widetilde{\in} F_A \rightarrow F_A \in N^\theta_{e_M}\right)$;
(5) $\models (F_A \in \widetilde{\tau}_\theta) \wedge (F_A \widetilde{\subseteq} F_B) \rightarrow F_A \widetilde{\subseteq} Int_\theta(F_B)$;
(6) $\models F_A \equiv Int_\theta(F_A) \leftrightarrow F_A \in \widetilde{\tau}_\theta$;
(7) $\models F_A \in \widetilde{\tau}_\theta \rightarrow F_A \in \widetilde{\tau}$;
(8) $\models F_A \in \widetilde{F}_\theta \rightarrow F_A \in \widetilde{F}$.

Proof. (1), (2) and (3) are similar to the proof of Theorem 3.18 (1), (2) and (3) respectively.

$$\begin{aligned} (4)\ [F_A \in \widetilde{\tau}_\theta] &= \bigwedge_{e_M \widetilde{\in} F_A} \left(1 - Cl_\theta(\mathbf{1}_A \setminus F_A)(e_M)\right) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} \Big(1 - \bigwedge_{F_B \in S(X)} \min\big(1, 1 - N_{e_M}(F_B) + \bigvee_{e_H \widetilde{\in} \mathbf{1}_A \setminus F_A} Cl(F_B)(e_H)\big)\Big) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \in S(X)} \max\big(0, N_{e_M}(F_B) - \bigvee_{e_H \widetilde{\in} \mathbf{1}_A \setminus F_A} Cl(F_B)(e_H)\big) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \in S(X)} \max\big(0, N_{e_M}(F_B) + 1 - \bigvee_{e_H \widetilde{\in} \mathbf{1}_A \setminus F_A} Cl(F_B)(e_H) - 1\big) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \in S(X)} \max\big(0, N_{e_M}(F_B) + \bigwedge_{e_H \widetilde{\in} \mathbf{1}_A \setminus F_A} (1 - Cl(F_B)(e_H)) - 1\big) \\ &= \bigwedge_{e_M \widetilde{\in} F_A} \bigvee_{F_B \in S(X)} \big(N_{e_M}(F_B) \otimes \bigwedge_{e_H \widetilde{\in} \mathbf{1}_A \setminus F_A} (1 - Cl(F_B)(e_H))\big) \end{aligned}$$

$$= \big[\forall e_M \big(e_M \in F_A \to \exists F_B \big((F_B \in N_{e_M}) \otimes \big(Cl(F_B) \widetilde{\subseteq} F_A\big)\big)\big)\big]$$
$$= \big[\forall e_M \big(e_M \widetilde{\in} F_A \to F_A \in N^{\theta}_{e_M}\big)\big].$$

(5) Follows from (4) above.

(6)
$$\begin{aligned}
[F_A \in \widetilde{\tau}_\theta] &= [\mathbf{1}_A \setminus F_A \in \widetilde{F}_\theta] \\
&= [\mathbf{1}_A \setminus F_A \equiv Cl_\theta(\mathbf{1}_A \setminus F_A)] \\
&= \big([\mathbf{1}_A \setminus F_A \widetilde{\subseteq} Cl_\theta(\mathbf{1}_A \setminus F_A)] \wedge [Cl_\theta(\mathbf{1}_A \setminus F_A) \widetilde{\subseteq} \mathbf{1}_A \setminus F_A]\big) \\
&= [Cl_\theta(\mathbf{1}_A \setminus F_A) \widetilde{\subseteq} \mathbf{1}_A \setminus F_A] \\
&= \forall e_M \big(e_M \widetilde{\in} Cl_\theta(\mathbf{1}_A \setminus F_A) \to e_M \widetilde{\in} (\mathbf{1}_A \setminus F_A)\big) \\
&= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - Cl_\theta(\mathbf{1}_A \setminus F_A)(e_M) + [e_M \widetilde{\in} (\mathbf{1}_A \setminus F_A)]\big) \\
&= \bigwedge_{e_M \widetilde{\in} \mathbf{1}_A \setminus (\mathbf{1}_A \setminus F_A)} \big(1 - Cl_\theta(\mathbf{1}_A \setminus F_A)(e_M)\big) \\
&= \bigwedge_{e_M \widetilde{\in} F_A} \big(Int_\theta(F_A)(e_M)\big) \\
&= [F_A \widetilde{\subseteq} Int_\theta(F_A)] \\
&= \big([F_A \widetilde{\subseteq} Int_\theta(F_A)] \wedge [Int_\theta(F_A) \widetilde{\subseteq} F_A]\big) \\
&= [F_A \equiv Int_\theta(F_A)].
\end{aligned}$$

(7) From Corollary 3.9 and Theorem 4.9 (2), we have
$$\begin{aligned}
[F_A \in \widetilde{\tau}_\theta] &= \bigwedge_{e_M \widetilde{\in} F_A} \big(1 - Cl_\theta(\mathbf{1}_A \setminus F_A))(e_M)\big) \\
&\le \bigwedge_{e_M \widetilde{\in} F_A} \big(1 - Cl(\mathbf{1}_A \setminus F_A)(e_M)\big) \\
&= \bigwedge_{e_M \widetilde{\in} F_A} \big(1 - 1 + N_{e_M}(F_A)\big) \\
&= \bigwedge_{e_M \widetilde{\in} F_A} N_{e_M}(F_A) \\
&= \widetilde{\tau}(F_A) = [F_A \in \widetilde{\tau}].
\end{aligned}$$

(8) Follows from (7) above. ∎

The equality in Theorem 4.16 (7) and (8) does not hold as shown in the following example.

Example 4.17. Let us consider consider the fuzzifying soft topological $(X, \widetilde{\tau}, A)$ in Example 3.2. Then, $\widetilde{\tau}_\theta(F_{A_6}) = 1 \ge \widetilde{\tau}(F_{A_6}) = \frac{1}{8}$ and $\widetilde{F}_\theta(F_{A_1}) = 1 \ge \widetilde{F}(F_{A_1}) = \frac{1}{8}$.

5. Fuzzifying Soft θ-Continuous Mappings

In this section, we will present the concepts of fuzzifying soft continuous, fuzzifying soft θ-continuous and fuzzifying soft strong θ-continuous between two fuzzifying soft topological spaces $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$.

Definition 5.1. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological spaces, $u : X \to Y$ and $p : A \to B$ be mappings. The fuzzifying soft continuity, denoted by $C \in \Im(S(Y, B)^{S(X,A)})$, is defined as
$$C(f_{pu}) := \big(\forall F_B\big)\big((F_B \in \widetilde{\sigma}) \to (f^{-1}_{pu}(F_B) \in \widetilde{\tau})\big).$$

Intuitively, the degree to which f_{pu} is a fuzzifying soft continuous is

$$[C(f_{pu})] = \bigwedge_{F_B \in S(Y)} min\big(1, 1 - \widetilde{\sigma}(F_B) + \widetilde{\tau}(f_{pu}^{-1}(F_B))\big).$$

Example 5.2. Let $(X, \widetilde{\tau}, A)$ be the fuzzifying soft topological space defined in Example 3.2; consider $Y = \{y_1, y_2\}$ and $B = \{m\}$. Then $F_{B_1} = \{(m, \{y_1\})\}, F_{B_2} = \{(m, \{y_2\})\}, F_{B_3} = \mathbf{1}_B, F_{B_4} = \mathbf{0}_B$ are all soft subsets of $\mathbf{1}_B$. Define a mapping $\widetilde{\sigma} \in \Im(S(X))$ as follows: $\widetilde{\sigma}(\mathbf{0}_B) = \widetilde{\sigma}(\mathbf{1}_B) = 1, \widetilde{\sigma}(F_{B_1}) = \frac{1}{2}$ and $\widetilde{\sigma}(F_{B_2}) = \frac{1}{4}$. Then, $\widetilde{\sigma}$ is a fuzzifying soft topology. Let $u : X \to Y$ be the map such that $u(x_1) = u(x_2) = u(x_3) = y_1$ and $p : A \to B$ be the map such that $p(e) = m$. Then, the degree to which f_{pu} is a fuzzifying soft continuous is $[C(f_{pu})] = 1$.

Theorem 5.3. *Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space and $e_M \in \mathbf{SP}(X)$. We set*

(1) $\alpha_1\big(f_{pu}\big) := \big(\forall F_B\big)\big(\big(F_B \in \widetilde{F}_Y\big) \to \big(f_{pu}^{-1}(F_B) \in \widetilde{F}_X\big)\big)$,
where $\widetilde{F}_X, \widetilde{F}_Y$ are the families of fuzzifying soft closed sets soft subsets of X and Y, respectively;

(2) $\alpha_2\big(f_{pu}\big) := \big(\forall e_M\big)\big(\forall F_C\big)\big(\big(F_C \in N^Y_{f_{pu}(e_M)}\big) \to \big(f_{pu}^{-1}(F_C) \in N^X_{e_M}\big)\big)$,
where N^X, N^Y are fuzzifying soft neighborhood systems of X and Y, respectively;

(3) $\alpha_3\big(f_{pu}\big) := \big(\forall e_M\big)\big(\forall F_C\big)\big(F_C \in N^Y_{f_{pu}(e_M)} \to \exists F_D\big(F_D \in N^X_{e_M} \to f_{pu}(F_D)\widetilde{\subseteq}F_C\big)\big)$;

(4) $\alpha_4\big(f_{pu}\big) := \big(\forall F_A\big)\big(f_{pu}(Cl_X(F_A))\widetilde{\subseteq}Cl_Y(f_{pu}(F_A))\big)$,
where Cl_X, Cl_Y are fuzzifying soft closure of X and Y, respectively;

(5) $\alpha_5\big(f_{pu}\big) := \big(\forall F_B\big)\big(Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq}f_{pu}^{-1}(Cl_Y(F_B))\big)$.

Then, $\models C\big(f_{pu}\big) \leftrightarrow \alpha_i\big(f_{pu}\big), i = 1, 2, 3, 4, 5$.

Proof. (1) We prove that $[C\big(f_{pu}\big)] = [\alpha_1\big(f_{pu}\big)]$.

$$\begin{aligned}
[\alpha_1\big(f_{pu}\big)] &= \bigwedge_{F_B \in S(Y)} \min\big(1, 1 - \widetilde{F}_Y(F_B) + \widetilde{F}_X\big(f_{pu}^{-1}(F_B)\big)\big)\\
&= \bigwedge_{F_B \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(\mathbf{1}_B \setminus F_B) + \widetilde{\tau}\big(\mathbf{1}_A \setminus f_{pu}^{-1}(F_B)\big)\big)\\
&= \bigwedge_{F_B \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(\mathbf{1}_B \setminus F_B) + \widetilde{\tau}\big(f_{pu}^{-1}\big(\mathbf{1}_B \setminus F_B\big)\big)\big)\\
&= \bigwedge_{F_U \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(F_U) + \widetilde{\tau}\big(f_{pu}^{-1}(F_U)\big)\big) = [C\big(f_{pu}\big)].
\end{aligned}$$

(2) First, we want prove that $[C\big(f_{pu}\big)] \leq [\alpha_2\big(f_{pu}\big)]$. Since $[\alpha_2\big(f_{pu}\big)] = \bigwedge_{e_M \in SP(X)} \bigwedge_{F_C \in S(Y)}$ $\min\big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\big)$, it suffices to show that for any $e_M \in \mathbf{SP}(X)$ and $F_C \in S(Y)$, $\min\big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\big) \geq [C\big(f_{pu}\big)]$.

If $N^Y_{f_{pu}(e_M)}(F_C) \leq N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)$, it holds obviously. Suppose $N^Y_{f_{pu}(e_M)}(F_C) > N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)$. It is clear that, if $f_{pu}(e_M)\widetilde{\in}F_B\widetilde{\subseteq}F_C$, then $e_M\widetilde{\in}f_{pu}^{-1}(F_B)\widetilde{\subseteq}f_{pu}^{-1}(F_C)$. Then,

$$\begin{aligned}
N^Y_{f_{pu}(e_M)}(F_C) - N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big) &= \bigvee_{f_{pu}(e_M)\widetilde{\in}F_B\widetilde{\subseteq}F_C} \widetilde{\sigma}(F_B) - \bigvee_{e_M\widetilde{\in}F_D\widetilde{\subseteq}f_{pu}^{-1}(F_C)} \widetilde{\tau}(F_D)\\
&\leq \bigvee_{f_{pu}(e_M)\widetilde{\in}F_B\widetilde{\subseteq}F_C} \widetilde{\sigma}(F_B) - \bigvee_{f_{pu}(e_M)\widetilde{\in}F_B\widetilde{\subseteq}F_C} \widetilde{\tau}(f_{pu}^{-1}(F_B))
\end{aligned}$$

$$\leq \bigvee_{f_{pu}(e_M)\widetilde{\in} F_B \widetilde{\subseteq} F_C} \big(\widetilde{\sigma}(F_B) - \widetilde{\tau}(f_{pu}^{-1}(F_B))\big).$$

So, $1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big) \geq \bigwedge_{f_{pu}(e_M)\widetilde{\in} F_B \widetilde{\subseteq} F_C} \big(1 - \widetilde{\sigma}(F_B) + \widetilde{\tau}(f_{pu}^{-1}(F_B))\big)$,

and thus, $\min\big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\big) \geq \bigwedge_{f_{pu}(e_M)\widetilde{\in} F_B \widetilde{\subseteq} F_C} \min\big(1, 1 - \widetilde{\sigma}(F_B) + \widetilde{\tau}(f_{pu}^{-1}(F_B))\big) \geq \bigwedge_{F_L \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(F_L) + \widetilde{\tau}(f_{pu}^{-1}(F_L))\big) = [C\big(f_{pu}\big)]$.

Hence,

$$\bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_C \in S(Y)} \min\big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\big) \geq [C\big(f_{pu}\big)].$$

Secondly, we prove that $[C\big(f_{pu}\big)] \geq [\alpha_2\big(f_{pu}\big)]$. From Corollary 3.9, we have

$$\begin{aligned}
[C\big(f_{pu}\big)] &= \bigwedge_{F_C \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(F_C) + \widetilde{\tau}(f_{pu}^{-1}(F_C))\big) \\
&= \bigwedge_{F_C \in S(Y)} \min\Bigg(1, 1 - \bigwedge_{f_{pu}(e_M)\widetilde{\in} F_C} N^Y_{f_{pu}(e_M)}(F_C) + \bigwedge_{e_M \widetilde{\in} f_{pu}^{-1}(F_C)} N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\Bigg) \\
&\geq \bigwedge_{F_C \in S(Y)} \min\Bigg(1, 1 - \bigwedge_{e_M \widetilde{\in} f_{pu}^{-1}(F_C)} N^Y_{f_{pu}(e_M)}(F_C) + \bigwedge_{e_M \widetilde{\in} f_{pu}^{-1}(F_C)} N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\Bigg) \\
&\geq \bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_C \in S(Y)} \min\big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}\big(f_{pu}^{-1}(F_C)\big)\big) \\
&= [\alpha_2\big(f_{pu}\big)].
\end{aligned}$$

(3) We prove that $[\alpha_2\big(f_{pu}\big)] = [\alpha_3\big(f_{pu}\big)]$. Since $F_D \widetilde{\subseteq} f_{pu}^{-1}(F_C)$ if and only if $f_{pu}^{-1}(F_D) \widetilde{\subseteq} F_C$ we have

$$\begin{aligned}
[\alpha_3\big(f_{pu}\big)] &= \bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_C \in S(Y)} \min\Bigg(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + \bigvee_{F_D \in S(X), f_{pu}(F_D) \widetilde{\subseteq} F_C} N^X_{e_M}(F_D)\Bigg) \\
&= \bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_C \in S(Y)} \min\Big(1, 1 - N^Y_{f_{pu}(e_M)}(F_C) + N^X_{e_M}(f_{pu}^{-1}(F_C))\Big) \\
&= [\alpha_2\big(f_{pu}\big)].
\end{aligned}$$

(4) We prove that $[\alpha_4\big(f_{pu}\big)] = [\alpha_5\big(f_{pu}\big)]$. First, for $F_B \in S(Y)$ one can deduce that

$$[f_{pu}^{-1}\big(f_{pu}(Cl_X(f_{pu}^{-1}(F_B)))\big) \widetilde{\supseteq} Cl_X(f_{pu}^{-1}(F_B))] = 1, \quad [Cl_Y(f_{pu}(f_{pu}^{-1}(F_B))) \widetilde{\subseteq} Cl_Y(F_B)] = 1$$

and

$$[f_{pu}^{-1}(Cl_Y(f_{pu}(f_{pu}^{-1}(F_B)))) \widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))] = 1.$$

$$\begin{aligned}
[Cl_X(f_{pu}^{-1}(F_B)) \widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))] &\geq [f_{pu}^{-1}(f_{pu}(Cl_X(f_{pu}^{-1}(F_B)))) \widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))] \\
&\geq [f_{pu}^{-1}(f_{pu}(Cl_X(f_{pu}^{-1}(F_B)))) \widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(f_{pu}(f_{pu}^{-1}(F_B))))] \\
&\geq [f_{pu}\big(Cl_X(f_{pu}^{-1}(F_B))\big) \widetilde{\subseteq} Cl_Y(f_{pu}(f_{pu}^{-1}(F_B)))].
\end{aligned}$$

Therefore,

$$\begin{aligned}\big[\alpha_5\big(f_{pu}\big)\big] &= \bigwedge_{F_B\in S(Y)} \big[Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))\big]\\ &\geq \bigwedge_{F_B\in S(Y)} \big[f_{pu}(Cl_X(f_{pu}^{-1}(F_B)))\widetilde{\subseteq} Cl_Y(f_{pu}(f_{pu}^{-1}(F_B)))\big]\\ &\geq \bigwedge_{F_A\in S(X)} \big[f_{pu}(Cl_X(F_A))\widetilde{\subseteq} Cl_Y(f_{pu}(F_A))\big] = \big[\alpha_4\big(f_{pu}\big)\big].\end{aligned}$$

Secondly, for $F_A \in S(X)$, there exists $F_B \in S(Y)$ such that $f_{pu}(F_A) = F_B$ and $F_A\widetilde{\subseteq} f_{pu}^{-1}(F_B)$. Hence,

$$\begin{aligned}\big[Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))\big] &\leq \big[Cl_X(F_A)\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(f_{pu}(F_A)))\big]\\ &\leq \big[f_{pu}(Cl_X(F_A))\widetilde{\subseteq} f_{pu}(f_{pu}^{-1}(Cl_Y(f_{pu}(F_A))))\big]\\ &\leq \big[f_{pu}(Cl_X(F_A))\widetilde{\subseteq} Cl_Y(f_{pu}(F_A))\big].\end{aligned}$$

Thus,

$$\begin{aligned}\big[\alpha_4\big(f_{pu}\big)\big] &= \bigwedge_{F_A\in S(X)} \big[Cl_X(F_A)\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(f_{pu}(F_A)))\big]\\ &\geq \bigwedge_{F_B\in S(Y), F_B=f_{pu}(F_A)} \big[Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))\big]\\ &\geq \bigwedge_{F_B\in S(Y)} \big[Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B))\big] = \big[\alpha_5\big(f_{pu}\big)\big].\end{aligned}$$

(5) Finally, We prove that $\big[\alpha_5\big(f_{pu}\big)\big] = \big[\alpha_2\big(f_{pu}\big)\big]$.

$$\begin{aligned}\big[\alpha_5\big(f_{pu}\big)\big] &= \big[\forall F_B(Cl_X(f_{pu}^{-1}(F_B))\widetilde{\subseteq} f_{pu}^{-1}(Cl_Y(F_B)))\big]\\ &= \bigwedge_{F_B\in S(Y)} \bigwedge_{e_M\in \mathbf{SP}(X)} \min\big(1, 1-(1-N_{e_M}(\mathbf{1}_A\backslash f_{pu}^{-1}(F_B)))+1-N_{f_{pu}(e_M)}(\mathbf{1}_B\backslash F_B)\big)\\ &= \bigwedge_{F_B\in S(Y)} \bigwedge_{e_M\in \mathbf{SP}(X)} \min\big(1, 1-N_{f_{pu}(e_M)}(\mathbf{1}_B\backslash F_B)+N_{e_M}(f_{pu}^{-1}(\mathbf{1}_B\backslash(F_B)))\big)\\ &= \bigwedge_{F_C\in S(Y)} \bigwedge_{e_M\in \mathbf{SP}(X)} \min\big(1, 1-N_{f_{pu}(e_M)}(F_C)+N_{e_M}(f_{pu}^{-1}(F_C))\big)\\ &= \big[\alpha_2\big(f_{pu}\big)\big].\end{aligned}$$

■

Definition 5.4. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space, $u : X \to Y$ and $p : A \to B$ be mappings. The fuzzifying soft strong θ-continuity, denoted by $C_{S\theta} \in \Im(S(Y, B)^{S(X,A)})$, is defined as

$$C_{S\theta}(f_{pu}) := \big(\forall F_B\big)\big((F_B \in \widetilde{\sigma}) \to (f_{pu}^{-1}(F_B) \in \widetilde{\tau}_\theta\big),$$

where $\widetilde{\tau}_\theta$ is the family of fuzzifying soft θ-open sets soft subsets of X.

Intuitively, the degree to which f_{pu} is a fuzzifying soft strong θ-continuous is

$$[C_{S\theta}(f_{pu})] = \bigwedge_{F_B\in S(Y)} min\big(1, 1-\widetilde{\sigma}(F_B)+\widetilde{\tau}_\theta(f_{pu}^{-1}(F_B))\big).$$

Example 5.5. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space defined in Example 3.2. Let $u : X \to Y$ be the map such that $u(x_1) = u(x_2) = u(x_3) = y_1$ and

$p : A \to B$ be the map such that $p(e) = m$. Then, the degree to which f_{pu} is a fuzzifying soft strong θ-continuous is $[C_{S\theta}(f_{pu})] = 1$.

Theorem 5.6. *Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space and $e_M \in \mathbf{SP}(X)$. We set*

(1) $\beta_1(f_{pu}) := (\forall F_C)\big((F_C \in \widetilde{F}_Y) \to (f_{pu}^{-1}(F_C) \in \widetilde{F}_\theta)\big)$,
where $\widetilde{F}_\theta$ is the family of fuzzifying soft θ-closed sets soft subset of X;
(2) $\beta_2(f_{pu}) := (\forall\, e_M)(\forall\, F_B)\big(F_B \in N^Y_{f_{pu}(e_M)} \to (\exists F_C)\big((F_C \in N^X_{e_M}) \otimes (f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B)\big)\big)$;
(3) $\beta_3(f_{pu}) := (\forall F_B)\big(f_{pu}(Cl_\theta(F_B)) \widetilde{\subseteq} Cl(f_{pu}(F_B))\big)$;
(4) $\beta_4(f_{pu}) := (\forall F_C)\big(Cl_\theta(f_{pu}^{-1}(F_C)) \widetilde{\subseteq} f_{pu}^{-1}(Cl(F_C))\big)$.
Then, $\models C_{S\theta}(f_{pu}) \leftrightarrow \beta_i(f_{pu}), i = 1, 2, 3, 4$.

Proof. (1)
$$\begin{aligned}[\beta_1(f_{pu})] &= \bigwedge_{F_C \in S(X)} \min\big(1, 1 - \widetilde{F}_Y(F_C) + \widetilde{F}_\theta(f_{pu}^{-1}(F_C))\big)\\ &= \bigwedge_{F_C \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(\mathbf{1}_B \setminus F_C) + \widetilde{\tau}_\theta(\mathbf{1}_A \setminus f_{pu}^{-1}(F_C))\big)\\ &= \bigwedge_{F_C \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(\mathbf{1}_B \setminus F_C) + \widetilde{\tau}_\theta(f_{pu}^{-1}(\mathbf{1}_B \setminus F_C))\big)\\ &= \bigwedge_{F_U \in S(Y)} \min\big(1, 1 - \widetilde{\sigma}(F_U) + \widetilde{\tau}_\theta(f_{pu}^{-1}(F_U))\big) = [C_{S\theta}(f_{pu})].\end{aligned}$$

(2) First, we prove that $[C_{S\theta}(f_{pu})] \le [\beta_2(f_{pu})]$. Since $[\beta_2(f_{pu})] = \bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_B \in S(Y)}$ $\min\Big(1, 1 - N^Y_{f_{pu}(e_M)}(F_B) + \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B] - 1\big)\Big)$, it suffices to show that for any $e_M \in \mathbf{SP}(X)$ and $F_B \in S(Y)$,
$$\min\Big(1, 1 - N^Y_{f_{pu}(e_M)}(F_B) + \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B] - 1\big)\Big) \ge$$
$[C_{S\theta}(f_{pu})]$. If $N^Y_{f_{pu}(e_M)}(F_B) \le \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B] - 1\big)$, it holds obviously.

Suppose $N^Y_{f_{pu}(e_M)}(F_B) > \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B] - 1\big)$. It is clear that, if $Cl(F_C) \widetilde{\subseteq} f_{pu}^{-1}(F_B)$, then $f_{pu}(Cl(F_C)) \widetilde{\subseteq} (F_B)$. Therefore,

$$\begin{aligned}&N^Y_{f_{pu}(e_M)}(F_B) - \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [f_{pu}(Cl(F_C)) \widetilde{\subseteq} F_B] - 1\big)\\ &\le N^Y_{f_{pu}(e_M)}(F_B) - \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + [Cl(F_C) \widetilde{\subseteq} f_{pu}^{-1}(F_B)] - 1\big)\\ &= N^Y_{f_{pu}(e_M)}(F_B) - \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) + \bigwedge_{e_H \widetilde{\notin} f_{pu}^{-1}(F_B)} (1 - Cl(F_C)(e_H)) - 1\big)\\ &= \bigvee_{f_{pu}(e_M) \widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) - \bigvee_{e_H \widetilde{\notin} f_{pu}^{-1}(F_B)} Cl(F_C)(e_H)\big)\\ &\le \bigvee_{f_{pu}(e_M) \widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \bigvee_{F_C \in S(X)} \max\big(0, N^X_{e_M}(F_C) - \bigvee_{e_H \widetilde{\notin} f_{pu}^{-1}(F_A)} Cl(F_C)(e_H)\big)\\ &= \bigvee_{f_{pu}(e_M) \widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \big(1 - \bigwedge_{F_C \in S(X)} \min\big(1, 1 - N^X_{e_M}(F_C) + \bigvee_{e_H \widetilde{\notin} f_{pu}^{-1}(F_A)} Cl(F_C)(e_H)\big)\big)\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{f_{pu}(e_M)\widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \big(1 - Cl_\theta(f_{pu}^{-1}(\mathbf{1}_A \setminus F_A))(e_H)\big) \\
&\leq \bigvee_{f_{pu}(e_M)\widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \bigwedge_{e_H \widetilde{\in} f_{pu}^{-1}(F_A)} \big(1 - Cl_\theta(f_{pu}^{-1}(\mathbf{1}_A \setminus F_A))(e_H)\big) \\
&= \bigvee_{f_{pu}(e_M)\widetilde{\in} F_A \widetilde{\subseteq} F_B} \widetilde{\sigma}(F_A) - \widetilde{\tau}_\theta\big(f_{pu}^{-1}(F_A)\big) \\
&\leq \bigvee_{f_{pu}(e_M)\widetilde{\in} F_A \widetilde{\subseteq} F_B} \big(\widetilde{\sigma}(F_A) - \widetilde{\tau}_\theta(f_{pu}^{-1}(F_A))\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\min\left(1, 1 - N^{Y}_{f_{pu}(e_M)}(F_B) + \bigvee_{F_C \in S(X)} \max\big(0, N^{X}_{e_M}(F_C) + \big[f_{pu}(Cl(F_C))\widetilde{\subseteq} F_B\big] - 1\big)\right) \\
&\geq \bigvee_{f_{pu}(e_M)\widetilde{\in} F_A \widetilde{\subseteq} F_B} \min\left(1, 1 - \widetilde{\sigma}(F_A) + \widetilde{\tau}_\theta\big(f_{pu}^{-1}(F_A)\big)\right) \\
&\geq \bigvee_{F_B \in S(Y)} \min\left(1, 1 - \widetilde{\sigma}(F_B) + \widetilde{\tau}_\theta\big(f_{pu}^{-1}(F_B)\big)\right) = [C_{S\theta}(f_{pu})].
\end{aligned}
$$

(3) We prove $\big[\beta_2\big(f_{pu}\big)\big] \leq \big[\beta_3\big(f_{pu}\big)\big]$. Since

$$
\big[\beta_3\big(f_{pu}\big)\big] = \bigwedge_{F_B \in S(X)} \bigwedge_{e_H \in \mathbf{SP}(Y)} min\left(1, 1 - f_{pu}\big(Cl_\theta(F_B)\big)(e_H) + Cl(f_{pu}(F_B))(e_H)\right).
$$

It is sufficient to show that for any $F_B \in S(X)$ and $e_H \in \mathbf{SP}(Y)$,

$$
min\big(1, 1 - f_{pu}\big(Cl_\theta(F_B)\big)(e_H) + Cl(f_{pu}(F_B))(e_H)\big) \geq \big[\beta_2\big(f_{pu}\big)\big].
$$

If $f_{pu}\big(Cl_\theta(F_B)\big)(e_H) \leq Cl(f_{pu}(F_B))(e_H)$, it holds obviously.
Suppose $f_{pu}\big(Cl_\theta(F_B)\big)(e_H) > Cl(f_{pu}(F_B))(e_H)$.

Therefore,

$$
\begin{aligned}
&f_{pu}\big(Cl_\theta(F_B)\big)(e_H) - Cl(f_{pu}(F_B))(e_H) \\
&= \bigvee_{f_{pu}(e_M)=e_H} \bigwedge_{F_C \in S(X)} \min\left(1, 1 - N_{e_M}(F_C) + \bigvee_{e_Z \widetilde{\in} F_B} Cl(F_C)(e_Z)\big) - \big(1 - N_{e_H}(\mathbf{1}_A \setminus f_{pu}(F_B))\right) \\
&\leq \bigvee_{e_M \in \mathbf{SP}(X)} \left(\bigwedge_{F_C \in S(X)} \min\big(1, 1 - N_{e_M}(F_C) + \bigvee_{e_Z \widetilde{\in} f_{pu}(F_B)} f_{pu}(Cl(F_C)(e_Z))\big) - 1 + N_{f_{pu}(e_M)}(\mathbf{1}_A \setminus f_{pu}(F_B))\right) \\
&= \bigvee_{e_M \in \mathbf{SP}(X)} \left(N_{f_{pu}(e_M)}(\mathbf{1}_A \setminus f_{pu}(F_B)) - \bigvee_{F_C \in S(X)} \max\big(0, N_{e_M}(F_C) - \bigvee_{e_Z \widetilde{\in} f_{pu}(F_B)} f_{pu}(Cl(F_C)(e_Z))\big)\right) \\
&= \bigvee_{e_M \in \mathbf{SP}(X)} \left(N_{f_{pu}(e_M)}(\mathbf{1}_A \setminus f_{pu}(F_B)) - \bigvee_{F_C \in S(X)} \max\big(0, N_{e_M}(F_C) + \bigwedge_{e_Z \widetilde{\in} f_{pu}(F_B)} \big(1 - f_{pu}(Cl(F_C)(e_Z)\big) - 1\big)\right) \\
&= \bigvee_{e_M \in \mathbf{SP}(X)} \left(N_{f_{pu}(e_M)}(\mathbf{1}_A \setminus f_{pu}(F_B)) - \bigvee_{F_C \in S(X)} \max\big(0, N_{e_M}(F_C) + [f_{pu}(Cl(F_C))\widetilde{\subseteq}(\mathbf{1}_A \setminus f_{pu}(F_B))] - 1\big)\right) \\
&\leq \bigvee_{e_M \in \mathbf{SP}(X)} \bigvee_{F_D \in S(X)} \left(N_{f_{pu}(e_M)}(F_D) - \bigvee_{F_C \in S(X)} \max\big(0, N_{e_M}(F_C) + \big[f_{pu}(Cl(F_C))\widetilde{\subseteq} F_D\big] - 1\big)\right), \\
&\qquad \min\left(1, 1 - f_{pu}\big(Cl_\theta(F_B)\big)(e_H) + Cl(f_{pu}(F_B))(e_H)\right) \\
&\geq \bigwedge_{e_M \in \mathbf{SP}(X)} \bigwedge_{F_D \in S(X)} \min\left(1, 1 - N_{f_{pu}(e_M)}(F_D) - \bigvee_{F_C \in S(X)} \max\big(0, N_{e_M}(F_C) + \big[f_{pu}(Cl(F_C))\widetilde{\subseteq} F_D\big] - 1\big)\right)
\end{aligned}
$$

$= \big[\beta_2(f_{pu})\big]$.

(4) We prove that $\big[\beta_3\big(f_{pu}\big)\big] \leq \big[\beta_4\big(f_{pu}\big)\big]$. For any $F_C \in S(Y)$, with $f_{pu}^{-1}\big(f_{pu}(Cl_\theta(f_{pu}^{-1}(F_C)))\big)\widetilde{\supseteq}Cl_\theta(f_{pu}^{-1}(F_C)), f_{pu}(f_{pu}^{-1}(F_C))\widetilde{\subseteq}F_C, Cl(f_{pu}(f_{pu}^{-1}(F_C)))$ $\widetilde{\subseteq}Cl(F_C), f_{pu}^{-1}(Cl(f_{pu}(f_{pu}^{-1}(F_C))))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))$, we have

$$\begin{aligned}
\big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))\big] &\geq \big[f_{pu}^{-1}(f_{pu}(Cl_\theta(f_{pu}^{-1}(F_C))))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))\big] \\
&\geq \big[f_{pu}^{-1}(f_{pu}(Cl_\theta(f_{pu}^{-1}(F_C))))\widetilde{\subseteq}f_{pu}^{-1}(Cl(f_{pu}(f_{pu}^{-1}(F_C))))\big] \\
&\geq \big[f_{pu}\big(Cl_\theta(f_{pu}^{-1}(F_C))\big)\widetilde{\subseteq}Cl(f_{pu}(f_{pu}^{-1}(F_C)))\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big[\beta_4\big(f_{pu}\big)\big] &= \bigwedge_{F_C \in S(Y)} \big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))\big] \\
&\geq \bigwedge_{F_C \in S(Y)} \big[f_{pu}(Cl_\theta(f_{pu}^{-1}(F_C)))\widetilde{\subseteq}Cl(f_{pu}(f_{pu}^{-1}(F_C)))\big] \\
&\geq \bigwedge_{F_B \in S(X)} \big[f_{pu}(Cl_\theta(F_B))\widetilde{\subseteq}Cl(f_{pu}(F_B))\big] = \big[\beta_3\big(f_{pu}\big)\big].
\end{aligned}$$

(5) We want to show that $\big[\beta_4\big(f_{pu}\big)\big] \leq \big[\beta_1\big(f_{pu}\big)\big]$. In fact for any $F_C \in S(Y)$,

$$\begin{aligned}
&\big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))\big] \otimes \widetilde{\digamma}_Y(F_C) \\
&\quad= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - Cl_\theta(f_{pu}^{-1}(F_C))(e_M) + f_{pu}^{-1}(Cl(F_C))(e_M)\big) \otimes \widetilde{\digamma}_Y(F_C) \\
&\quad= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\Big(\widetilde{\digamma}_Y(F_C), \max\big(0, \widetilde{\digamma}_Y(F_C) + f_{pu}^{-1}(Cl(F_C))(e_M) - Cl_\theta(f_{pu}^{-1}(F_C))(e_M)\big)\Big), \\
&\qquad\qquad \widetilde{\digamma}_Y(F_C) + f_{pu}^{-1}(Cl(F_C))(e_M) - 1 \\
&\quad= \big[Cl(F_C)\widetilde{\subseteq}F_C\big] + f_{pu}^{-1}((Cl(F_C)))(e_M) - 1 \\
&\quad\leq \big[f_{pu}^{-1}(Cl(F_C))\widetilde{\subseteq}f_{pu}^{-1}(F_C)\big] + f_{pu}^{-1}(Cl(F_C))(e_M) - 1 \\
&\quad= \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - f_{pu}^{-1}(Cl(F_C))(e_M) + f_{pu}^{-1}(F_C)(e_M)\big) + f_{pu}^{-1}(Cl(F_C))(e_M) - 1 \\
&\quad\leq 1 - f_{pu}^{-1}(Cl(F_C))(e_M) + f_{pu}^{-1}(F_C)(e_M) + f_{pu}^{-1}(Cl(F_C))(e_M) - 1 \\
&\quad= f_{pu}^{-1}(F_C)(e_M),
\end{aligned}$$

$$\begin{aligned}
&\big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(Cl(F_C))\big] \otimes \widetilde{\digamma}_Y(F_C) \\
&\quad\leq \bigwedge_{e_M \in \mathbf{SP}(X)} \min\big(1, 1 - Cl_\theta(f_{pu}^{-1}(F_C))(e_M) + f_{pu}^{-1}(F_C)(e_M)\big) \\
&\quad= \big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(F_C)\big] = \big[f_{pu}^{-1}(F_C) \in \widetilde{\digamma}_\theta\big],
\end{aligned}$$

$$\big[Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq}f_{pu}^{-1}(F_C)\big] \leq \big(\widetilde{\digamma}_Y(F_C) \to \big[f_{pu}^{-1}(F_C) \in \widetilde{\digamma}_\theta\big]\big).$$

Therefore,

$$\big[\beta_4(f_{pu})\big] = \big[\forall F_C(Cl_\theta(f_{pu}^{-1}(F_C)))\widetilde{\subseteq} f_{pu}^{-1}(Cl(F_C))\big]$$
$$\leq \big[\forall F_C((F_C \in \widetilde{F}_Y) \to f_{pu}^{-1}(F_C) \in \widetilde{F}_\theta)\big] = \big[\beta_1(f_{pu})\big]. \qquad \blacksquare$$

Definition 5.7. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space, $u : X \to Y$ and $p : A \to B$ be mappings. The fuzzifying soft θ-continuity, denoted by $C_\theta \in \Im(S(Y,B)^{S(X,A)})$, is defined as

$$C_\theta(f_{pu}) := \big(\forall F_B\big)\big((F_B \in \widetilde{\sigma}_\theta) \to (f_{pu}^{-1}(F_B) \in \widetilde{\tau}_\theta\big),$$

where $\widetilde{\sigma}_\theta$ is the family of fuzzifying soft θ-open sets soft subsets of Y.

Intuitively, the degree to which f_{pu} is a fuzzifying soft θ-continuous is

$$[C_\theta(f_{pu})] = \bigwedge_{F_B \in S(Y)} min\big(1, 1 - \widetilde{\sigma}_\theta(F_B) + \widetilde{\tau}_\theta(f_{pu}^{-1}(F_B))\big).$$

Example 5.8. Let us consider Examples 3.2 and 5.2. Then, we can calculate $\widetilde{\sigma}_\theta$. First, let $e_V = \{(e, \{y_1\})\}$ be a soft point. The fuzzifying soft neighborhood system of e_V are: $N_{e_V}(\mathbf{1}_B) = 1; N_{e_V}(\mathbf{0}_B) = N_{e_V}(F_{B_2}) = 0$ and $N_{e_V}(F_{B_1}) = \frac{1}{4}$. Second, let $e_z = \{(e, \{y_2\})\}$ be another soft point. The fuzzifying soft neighborhood system of e_Z are: $N_{e_Z}(\mathbf{1}_B) = 1; N_{e_Z}(\mathbf{0}_B) = N_{e_Z}(F_{B_1}) = 0$ and $N_{e_Z}(F_{B_2}) = \frac{1}{8}$ and we have the fuzzifying soft closure of e_Z are: $Cl(\mathbf{1}_B)(e_Z) = Cl(F_{B_2})(e_Z) = 1; Cl(\mathbf{0}_B)(e_Z) = 0$ and $Cl(F_{B_1})(e_Z) = \frac{7}{8}$. The fuzzifying soft θ-closure of e_V are: $Cl_\theta(\mathbf{1}_B)(e_V) = Cl_\theta(F_{B_1})(e_V) = 1$ and $Cl_\theta(\mathbf{0}_B)(e_V) = Cl_\theta(F_{B_2})(e_V) = 0$. Now, the family of all fuzzifying soft θ-open sets of Y are: $\widetilde{\sigma}_\theta(\mathbf{1}_B) = \widetilde{\sigma}_\theta(F_{B_1}) = 1$ and $\widetilde{\sigma}_\theta(\mathbf{0}_B) = \widetilde{\sigma}_\theta(F_{B_2}) = 0$.

Example 5.9. Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space defined in Examples 3.2 and 5.2, respectively and $\widetilde{\tau}_\theta$ and $\widetilde{\sigma}_\theta$ defined in Examples 4.8 and 5.8, respectively. Let $u : X \to Y$ be the map such that $u(x_1) = u(x_2) = u(x_3) = y_2$ and $p : A \to B$ be the map such that $p(e) = m$. Then, the degree to which f_{pu} is a fuzzifying soft continuous is $[C_\theta(f_{pu})] = 1$.

Theorem 5.10. *Let $(X, \widetilde{\tau}, A)$ and $(Y, \widetilde{\sigma}, B)$ be two fuzzifying soft topological space and $e_M \in \mathbf{SP}(X)$. We set*

(1) $\gamma_1\big(f_{pu}\big) := \big(\forall F_C\big)\big(\big(F_C \in \widetilde{F}_\theta^Y\big) \to \big(f_{pu}^{-1}(F_C) \in \widetilde{F}_\theta^X\big)\big)$, *where $\widetilde{F}_\theta^X, \widetilde{F}_\theta^Y$ are the families of fuzzifying soft θ-closed sets soft subsets of X and Y, respectively;*

(2) $\gamma_2(f_{pu}) := (\forall\, e_M)(\forall\, F_B)(F_B \in N^Y_{f_{pu}(e_M)} \to (\exists F_C)((F_C \in N^Y_{e_M}) \otimes (f_{pu}(Cl(F_C))\widetilde{\subseteq} Cl(F_B))))$;

(3) $\gamma_3\big(f_{pu}\big) := \big(\forall F_B\big)\big(f_{pu}\big(Cl_\theta(F_B)\big)\widetilde{\subseteq} Cl_\theta(f_{pu}(F_B))\big)$;

(4) $\gamma_4\big(f_{pu}\big) := \big(\forall F_C\big)\big(Cl_\theta(f_{pu}^{-1}(F_C))\widetilde{\subseteq} f_{pu}^{-1}(Cl_\theta(F_C))\big)$.

Then, $\models C_\theta(f_{pu}) \leftrightarrow \gamma_i\big(f_{pu}\big), i = 1, 2, 3, 4.$

Proof. The proof is similar to that of Theorem 5.6. ■

6. Conclusion

The present paper investigates soft topological notions in semantic method of continuous valued-logic. It continue various investigations into fuzzy soft topology in a legitimate way and extend some fundamental results in soft topology to fuzzifying soft topology. An important virtue of our approach is that we define fuzzifying soft topological notions by formulae of Łukasiewicz logic. Unlike the (more wide-spread) style of defining notions in fuzzy soft mathematics as crisp predicates of fuzzy soft sets, fuzzy soft predicates of fuzzy soft sets provide a more genuine fuzzification; furthermore the theorems in the form of valid fuzzy soft implications are more general than the corresponding theorems on crisp predicates of fuzzy soft sets. The main contributions of the present paper are to give the definitions of fuzzifying soft topological spaces and fuzzifying soft continuity and obtain some of their basic properties.

An obvious problem for further study is:

Our results are derived in Łukasiewicz continuous logic. It is possible to generalize them to more general logic setting, like BL-Algebra or residuated lattice-valued logic considered in [25, 26].

In the future, we expect that the fuzzifying soft topology will be of great importance to researchers in more than one field interested in the decision-making process and problem-solving. Further, we will provide a real applications with a real data set or we will apply the fuzzifying soft topology to lung cancer disease [27] and coronary artery disease [28]. Further, we will extent the fuzzifying soft topology to fuzzifying b-θ-open sets [29], fuzzifying semipre-θ-open sets [30], and fuzzifying pre-θ-open sets [31].

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