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On generalized fuzzy soft compact spaces

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Abstract. In the present paper, we continue studying generalized fuzzy soft topological spaces. We first introduce generalized fuzzy soft $p-$cover and utilize it to define a new type of generalized fuzzy soft compact topological spaces so-called a generalized fuzzy soft p^*-compact topological spaces which is a generalization of compactness in fuzzy soft topological spaces in [27]. In fact, a generalized fuzzy soft p^*-compact topological space is more general than generalized fuzzy soft compact spaces in [17]. In general, we investigate some basic results, relations and properties of generalized fuzzy soft p^*-compact space and provide some illustrative examples.

Keywords: generalized fuzzy soft set, generalized fuzzy soft topology, generalized fuzzy soft $p-$cover, generalized fuzzy soft $p-$compact, generalized fuzzy soft p^*-compact.

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1. Introduction

In 1999, Molodtsov [23] introduced the concept of soft sets as a new approach to model uncertain objects. Shabir and Naz [32] introduced and studied the topological structures of soft sets. Moreover, many authors studied soft topology and its applications [1, 6, 7, 8, 9, 10, 14, 21, 28, 29, 30, 31]. The concept of fuzzy soft set introduced by Maji et. al. [19]. Later, Tanay et al. [33] and Roy et. al. [26] introduced fuzzy soft topological space independently. In 2010, Majumdar and Samanta [20] introduced the notion of generalized fuzzy soft set as a generalization of fuzzy soft sets and some of its basic properties. Chakraborty et. al. [11] gave the topological structure of generalized fuzzy soft sets.

The concept of compactness is one of the most important concepts in topological spaces. In fuzzy topology, compactness was first introduced by Chang [12] and in soft topology, compactness was introduced by Zorlutuna et. al. [31]. Then, Al-shami et al studied soft compactness in [2, 3, 4, 5] and revised the relationships etween soft compact set and soft Hausdorff in [6, 7, 8]. In fuzzy soft topology, compactness was introduced by Osmanoglu et al. [25] and Gain et al. [13] which is extension of Chang's fuzzy compactness [12]. Also, Mishra et. al. [22] introduced the compactness in fuzzy soft topology as extension of Lown's fuzzy compactness [18].

In this paper, we introduce and study a new type of cover and compactness in generalized fuzzy soft topology so-called generalized fuzzy soft $p-$cover and generalized fuzzy soft p^*-compact which is a generalization of compactness in [27], development the compactness in [13, 25], and is more general than that which are presented by Khedr et. al. [17] and give some basic definitions, results, relations and theorems related to the p^*-compactness are studied.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1 ([34]). *Let X be a non-empty set. A fuzzy set A in X is defined by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the 'grade of membership' of x in A for x in X. The set of all fuzzy sets in a set X is denotes by I^X, where I is the closed unit interval $[0,1]$. The support of $A \in I^X$ is the crisp set $S(A) = \{x \in X : \mu(x) > 0\}$.*

Definition 2.2 ([19, 26]). *Let X be an initial universe set and E be a set of parameters. Let $A \subseteq E$. A fuzzy soft set f_A over X is a mapping from E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq \bar{0}$ if $e \in A \subset E$, and $f_A(e) = \bar{0}$ if $e \notin A$, where $\bar{0}$ denotes empty fuzzy set in X. The support of f_A denoted by $S(f_A(e))$ is the set, $S(f_A(e)) = \{x \in X : f_A(e)(x) > 0\}$.*

Definition 2.3 ([20]). *Let X be a universal set of elements and E be a universal set of parameters for X. Let $F : E \longrightarrow I^X$ and μ be a fuzzy subset of E, i.e., $\mu : E \longrightarrow I$. Let F_μ be the mapping $F_\mu : E \longrightarrow I^X \times I$ defined as follows: $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in I^X$ and $\mu(e) \in I$. Then F_μ is called a generalised fuzzy soft set (GFSS in short) over (X, E). The family of all generalized fuzzy soft sets (GFSSs in short) over (X, E) is denotes by $GFSS(X, E)$.*

Definition 2.4 ([17]). *Let $F_\mu \in GFSS(X, E)$. The support of F_μ denoted by $S(F_\mu(e))$ is the set $S(F_\mu(e)) = \{x \in X : F(e)(x) > 0$ and $\mu(e) > 0, e \in E\}$.*

Definition 2.5 ([20]). *Let F_μ and G_δ be two GFSSs over (X, E). Then we have:*

1) *F_μ is called a generalized null fuzzy soft set, denoted by $\widetilde{0}_\theta$, if $\widetilde{0}_\theta : E \longrightarrow I^X \times I$ such that $\widetilde{0}_\theta(e) = (\widetilde{0}(e), \theta(e))$ where $\widetilde{0}(e) = \overline{0}$ $\forall e \in E$ and $\theta(e) = 0$ $\forall e \in E$ (Where $\overline{0}(x) = 0$, $\forall x \in X$).*

2) *F_μ is called a generalized absolute fuzzy soft set, denoted by $\widetilde{1}_\triangle$, if $\widetilde{1}_\triangle : E \longrightarrow I^X \times I$, where $\widetilde{1}_\triangle(e) = (\widetilde{1}(e), \triangle(e))$ is defined by $\widetilde{1}(e) = \overline{1}, \forall e \in E$ and $\triangle(e) = 1, \forall e \in E$ (Where $\overline{1}(x) = 1, \forall x \in X$).*

3) *F_μ is a generalized fuzzy soft subset of G_δ denoted by $F_\mu \sqsubseteq G_\delta$ if μ is a fuzzy subset of δ and $F(e)$ is also a fuzzy subset of $G(e)$, $\forall e \in E$.*

4) *The generalized fuzzy soft complement of F_μ, denoted by F^c_μ, is defined by F^c_μ=G_δ, where $\delta(e) = \mu^c(e)$ and $G(e) = F^c(e), \forall e \in E$. Obviously $(F^c_\mu)^c = F_\mu$.*

Definition 2.6 ([11]). *Let F_μ and G_δ be two GFSSs over (X, E).*

1) *The union of F_μ and G_δ, denoted by $F_\mu \sqcup G_\delta$, is The GFSS H_ν, defined as $H_\nu : E \longrightarrow I^X \times I$ such that $H_\nu(e) = (H(e), \nu(e))$, where $H(e) = F(e) \vee G(e)$ and $\nu(e) = \mu(e) \vee \delta(e), \forall e \in E$.*

2) *The Intersection of F_μ and G_δ, denoted by $F_\mu \sqcap G_\delta$, is the GFSS M_σ, defined as $M_\sigma : E \longrightarrow I^X \times I$ such that $M_\sigma(e) = (M(e), \sigma(e))$, where $M(e) = F(e) \wedge G(e)$ and $\sigma(e) = \mu(e) \wedge \delta(e)$, $\forall e \in E$.*

Definition 2.7 ([15]). *The generalized fuzzy soft set $F_\mu \in GFSS(X, E)$ is called a generalized fuzzy soft point (GFS point for short) over (X, E) if there exist $e \in E$ and $x \in X$ such that*

(1) *$F(e)(x) = \alpha (0 < \alpha \leq 1)$ and $F(e)(y) = 0$ for all $y \in X - \{x\}$,*

(2) *$\mu(e) = \lambda (0 < \lambda \leq 1)$ and $\mu(e') = 0$ for all $e' \in E - \{e\}$. We denote this generalized fuzzy soft point $F_\mu = (e_\lambda, x_\alpha)$.*

(e, x) and (λ, α) are called respectively, the support and the value of (e_λ, x_α). The class of all GFS points in (X, E), denoted by $GFSP(X, E)$.

We say that $(e_\lambda, x_\alpha) \widetilde{\in} F_\mu$ read as (e_λ, x_α) belongs to the GFSS F_μ if for the element $e \in E$, $\alpha \leq F(e)(x)$ and $\lambda \leq \mu(e)$.

The complement of a GFS point (e_λ, x_α) is a GFSS denoted by $(e_\lambda, x_\alpha)^c$, is defined by $(e_\lambda, x_\alpha)^c = G_\delta$, where $G(e) = 1 - F(e)(x)$ and $\delta(e) = 1 - \mu(e)$, $\forall x \in X, e \in E$.

Definition 2.8 ([24])**.** *Let* $F_\mu, G_\delta \in GFSS(X,E)$ *over* (X,E). F_μ *is said to be a generalised soft quasi-coincident with [GFS quasi−coincident in short]* G_δ, *denoted by* $F_\mu q G_\delta$, *if there exist* $e \in E$ *and* $x \in X$ *such that* $F(e)(x) + G(e)(x) > 1$ *and* $\mu(e) + \delta(e) > 1$.

If F_μ *is not GFS quasi−coincident with* G_δ, *then we write* $F_\mu \bar{q} G_\delta$ *i.e, For every* $e \in E$ *and* $x \in X$, $F(e)(x) + G(e)(x) \leq 1$ *or for every* $e \in E$ *and* $x \in X$, $\mu(e) + \delta(e) \leq 1$.

A GFS point (e_λ, x_α) *is said to be GFS quasi−coincident with* F_μ, *denoted by* $(x_\alpha, e_\lambda) q F_\mu$, *if and only if there exists an element* $e \in E$ *such that* $\alpha + F(e)(x) > 1$ *and* $\lambda + \mu(e) > 1$.

Theorem 2.9 ([16, 24])**.** *Let* $F_\mu, G_\delta \in GFSS(X,E)$ *and* $(e_\lambda, x_\alpha) \in GFSP(X,E)$. *Then:*

(1) $F_\mu \bar{q} G_\delta \Leftrightarrow F_\mu \sqsubseteq G_\delta^c$,
(2) $F_\mu \sqcap G_\delta = \widetilde{0}_\theta \Rightarrow F_\mu \bar{q} G_\delta$,
(3) $F_\mu q G_\delta \Rightarrow F_\mu \sqcap G_\delta \neq \widetilde{0}_\theta$,
(4) $F_\mu \bar{q} F_\mu^c$,
(5) $(x_\alpha, e_\lambda) \bar{q} F_\mu \Leftrightarrow (x_\alpha, e_\lambda) \widetilde{\in} F_\mu^c$.

Definition 2.10 ([11])**.** *Let* T *be a collection of generalized fuzzy soft sets over* (X,E). *Then* T *is said to be a generalized fuzzy soft topology (GFST, in short) over* (X,E) *if the following conditions are satisfied:*

(1) $\widetilde{0}_\theta$ *and* $\widetilde{1}_\triangle$ *are in* T.
(2) *Arbitrary unions of members of* T *belong to* T.
(3) *Finite intersections of members of* T *belong to* T.

The triple (X,T,E) *is called a generalized fuzzy soft topological space (GFST-space, in short) over* (X,E).

The member of T *are called generalized fuzzy soft open set [GFS open for short] in* (X,T,E) *and their generalized fuzzy soft complements are called GFS closed sets in* (X,T,E). *The family of all GFS closed sets in* (X,T,E) *is denoted by* T^c.

Definition 2.11 ([15])**.** *Let* (X,T,E) *be a GFST-space and* $Y \subseteq X$. *Let* H_ν^Y *be a GFSS over* (Y,E) *where* $H_\nu^Y : E \longrightarrow I^X \times I$ *such that* $\forall e \in E, H_\nu^Y(e) = (H^Y(e), \nu(e))$,

$$H^Y(e)(x) = \begin{cases} 1, & x \in Y \\ 0, & x \notin Y \end{cases}, \nu(e) = 1$$

i.e., $H^Y(e) = Y$, $\forall e \in E$, $\nu(e) = 1$

Let $T_Y = \{H_\nu^Y \sqcap G_\delta : G_\delta \in T\}$, *then* T_Y *is a GFS topology over* (X,E) *and* $H_\nu^Y \in T$ *is called a GFS subspace of* (Y, T_Y, E). *If* $H_\nu^Y \in T$ *(resp.* $H_\nu^Y \in T^c$*) then* (Y, T_Y, E) *is called GFS open (resp. closed) subspace of* (X,T,E).

Definition 2.12 ([11, 15])**.** *Let* (X,T,E) *be a GFST−space. A GFSS* F_μ *in* $GFSS(X,E)$ *is called a generalized fuzzy soft neighborhood (briefly, GFS−nbd)* *of* H_ν *[resp.* (e_λ, x_α)*] if there exists* $G_\delta \in T$ *such that* $H_\nu \sqsubseteq G_\delta \sqsubseteq F_\mu$ *[resp.* $(e_\lambda, x_\alpha)\widetilde{\in} \sqsubseteq G_\delta \sqsubseteq F_\mu$*]. The family of all* $GFS-$*nbds of* H_ν *[resp.* (e_λ, x_α)*], is denoted by* $N(H_\nu)$ *[resp.* $N(e_\lambda, x_\alpha)$*].*

Notation. The notation O_{H_ν} [resp. $O_{(e_\lambda,x_\alpha)}$] refers to a GFS open set contains H_ν [resp. (e_λ, x_α)] and called a $GFS-$nbd of H_ν [resp. (e_λ, x_α)].

Definition 2.13 ([17]). *Let* $F_\mu \in GFSS(X, E)$*. The* $GFSS$ F_μ *is called the* $\alpha - (X, E)-$*universal* $GFSS$*, denoted by* $\widetilde{\alpha}_{(X,E)}$*, if* $F_\mu(e) = (\widetilde{\alpha}_E, \alpha_X)$*, where* $\widetilde{\alpha}_E$ *the constant fuzzy soft set on* (X, E) *and* α_X *constant fuzzy set on* X*, i.e.,* $\widetilde{\alpha}_E(e) = \alpha_X$ *and* $\alpha_X(x) = \alpha$ *for each* $e \in E$*. Clearly,* $(\widetilde{\alpha}_{(X,E)})^c = (\widetilde{(1-\alpha)}_E, (1-\alpha)_X)$*.*

Definition 2.14 ([16]). *A* $GFST-$*space* (X, T, E) *is said to be:*

(1) *Generalized fuzzy soft quasi* R_0-*space (*$GFS\ Q - R_0-$*space for short) if for every* $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ *with* $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta) \Longrightarrow cl(e_\lambda, x_\alpha)$ $\bar{q}(e'_\gamma, y_\beta)$*.*

(2) *Generalized fuzzy soft quasi* R_1-*space (*$GFS\ Q - R_1-$*space for short) if for every* $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ *with* $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ *implies* $\exists O_{(e_\lambda,x_\alpha)} \in N_{(e_\lambda,x_\alpha)}$ *and* $O_{(e'_\gamma,y_\beta)} \in N_{(e'_\gamma,y_\beta)}$ *such that* $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$*.*

Definition 2.15 ([16]). *A* $GFST-$*space* (X, T, E) *is said to be:*

(1) Generalized fuzzy soft quasi T_1-*space (*$GFS\ Q - T_1-$*space for short) if for every* $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ *with* $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ *implies there exist* $O_{(e_\lambda,x_\alpha)} \in N_{(e_\lambda,x_\alpha)}$ *such that* $O_{(e_\lambda,x_\alpha)}\bar{q}(e'_\gamma, y_\beta)$ *and there exist* $O_{(e'_\gamma,y_\beta)} \in N_{(e'_\gamma,y_\beta)}$ *such that* $O_{(e'_\gamma,y_\beta)}\bar{q}(e_\lambda, x_\alpha)$*.*

(2) *Generalized fuzzy soft quasi* T_2-*space (*$GFS\ Q - T_2-$*space for short) if for every* $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ *with* $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ *implies there exist* $O_{(e_\lambda,x_\alpha)} \in N_{(e_\lambda,x_\alpha)}$ *and* $O_{(e'_\gamma,y_\beta)} \in N_q(e'_\gamma, y_\beta)$ *such that* $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$*.*

(3) *Generalized fuzzy soft quasi regular*$-$*space (*$GFS\ Q$ *regular*$-$*space for short) if for every* $(e_\lambda, x_\alpha) \in GFSP(X, E)$ *and* $G_\delta \in T^c$ *with* $(e_\lambda, x_\alpha)\bar{q}G_\delta$ *implies* $\exists O_{(e_\lambda,x_\alpha)} \in N_{(e_\lambda,x_\alpha)}$ *and* $O_{G_\delta} \in N_{G_\delta}$ *such that* $O_{(e_\lambda,x_\alpha)}\bar{q}O_{G_\delta}$*.*

(4) *Generalized fuzzy soft quasi normal*$-$*space (*$GFS\ Q$ *normal*$-$*space for short) if for every* $F_\mu, G_\delta \in T^c$ *with* $F_\mu\bar{q}G_\delta$ *implies* $\exists O_{F_\mu} \in N_{(F_\mu)}$ *and* $O_{G_\delta} \in N_{(G_\delta)}$ *such that* $O_{F_\mu}\bar{q}O_{G_\delta}$*.*

(5) *Generalized fuzzy soft quasi* T_3-*space (*$GFS\ Q - T_3-$*space for short) if* $GFS\ Q$ *regular and* $GFS\ Q - T_1-$*space.*

(6) *Generalized fuzzy soft quasi* T_4-*space (*$GFS\ Q - T_4-$*space for short) if* $GFS\ Q$ *normal and* $GFS\ Q - T_1-$*space.*

Definition 2.16 ([17]). *A family* β *of* $GFSS$*s is a generalized fuzzy soft cover (*GFS *cover for short) of a* $GFSS$ F_μ *if* $F_\mu \sqsubseteq \sqcup\{(F_\mu)_i : i \in I, (F_\mu)_i \in \beta\}$*. It is a* GFS *open cover if each member of* β *is a* GFS *open set. A subcover of* β *is a subfamily of* β *which is also a cover.*

Definition 2.17 ([17]). *Let (X, T, E) be $GFST-$space and $F_\mu \in GFSS(X, E)$. A $GFSS$ F_μ is called generalized fuzzy soft compact ($GFS-$compact for short) if each GFS open cover of F_μ has a finite GFS open subcover. A $GFST-$space (X, T, E) is called $GFS-$compact if each GFS open cover of $\widetilde{1}_\triangle$ has a finite GFS open subcover.*

3. Generalized fuzzy soft p^*-compact topological spaces

Definition 3.1. *A family $\psi = \{(F_\mu)_i : i \in I, (F_\mu)_i \in GFSS(X, E)\}$ is called a generalized fuzzy soft cover $p-$cover (GFS $p-$cover for short) of a $GFSS$ G_δ if for all $(e_\lambda, x_\alpha)\widetilde{\in}G_\delta$ there exists $i_0 \in I$ such that $(e_\lambda, x_\alpha)\widetilde{\in}(F_\mu)_{i_0}$. It is a GFS open $p-$cover if every member of ψ is a GFS open set. A GFS $p-$ subcover of ψ is a subfamily of ψ which is also a GFS $p-$cover.*

Remark 3.2. Every GFS $p-$cover is a GFS cover in the sense of Definition 2.16. But the converse may not be true in general as shown by the following example.

Example 3.3. Let X be an infinite set and $\psi = \{(F_\mu)_n : n \in N\}$ be a family of $GFSSs$ over (X, E) defined $F_n(e)(x) = 1-\frac{1}{n}, \mu_n(e) = 1-\frac{1}{n}$ where $e \in E, x \in X$ and $n \in N$. Then ψ is a GFS cover of $\widetilde{1}_\triangle$. But ψ is not a GFS $p-$cover of $\widetilde{1}_\triangle$. For the GFS point $(e_1, x_1)\widetilde{\in}\widetilde{1}_\triangle$ there no exists any element in ψ containing $(e_1, x_1) = \{(e = \{\frac{x}{1}\}, 1)\}$.

Definition 3.4. *Let (X, T, E) be $GFST-$space and $F_\mu \in GFSS(X, E)$. A $GFSS$ F_μ is called a generalized fuzzy soft $p-$compact (GFS $p-$compact for short) if every GFS open $p-$cover of F_μ has a finite GFS open $p-$subcover. A $GFST-$space (X, T, E) is called a GFS $p-$compact if each GFS open $p-$cover of $\widetilde{1}_\triangle$ has a finite GFS open $p-$subcover.*

Definition 3.5. *Let $F_\mu \in GFSS(X, E)$. The soft support of F_μ, denoted by $Ssup(F_\mu)$, is a soft set given by, $Ssup(F_\mu) = \{(e, S(F(e)) : e \in E\}$, where $S(F(e))$ is the support of fuzzy set $F(e)$, which is given by the set $F(e) = \{x \in X : F(e)(x) > 0\} \subseteq X$.*

Remark 3.6. Every finite $GFSS$ F_μ (i.e., the support of $F(e), e \in E$ is finite) is GFS $p-$compact set. Also (X, T, E) is GFS $p-$compact if X is finite.

Definition 3.7. *A $GFST-$space (X, T, E) is called a GFS p^*-compact if every GFS closed set over (X, E) is a GFS $p-$compact set.*

Theorem 3.8. *Let X be an infinite set. A cofinite $GFST-$space (X, T_∞, E) is GFS p^*-compact space, where $T_\infty = \{\widetilde{0}_\theta, F_\mu \in GFSS(X, E) : S(F^c(e))$ is a finite subset of X and $e \in E\}$.*

Proof. Let F_μ be GFS close set in (X, T_∞, E), then F_μ is finite or $F_\mu = \widetilde{1}_\triangle$. Now we have two cases: If F_μ is finite, then the result holds. If $F_\mu = \widetilde{1}_\triangle$.

Suppose ψ is a GFS open $p-$cover of $F_\mu = \widetilde{1}_\triangle$. Choose $(e_1, x_1)\widetilde{\in}\widetilde{1}_\triangle$, then there is $O_{(e_1,x_1)} \in \psi$ and so, $O_{(e_1,x_1)^c}$ is finite. Now take $G_\delta = \{(F(e), \mu(e)) : F(e) = y_1^i, e \in E$ and $y^i \in Ssup(O^c_{(e_1,x_1)}), i = 1, 2, ...n\}$ which is finite, thus for all $(e_1, y_1^i)\widetilde{\in}G_\delta$ there exist $O_{(e_1,y_1^i)} \in \psi, i = 1, 2, ..., n$ and so the family $\{O_{(e_1,y_1^i)} : i = 1, 2, ..., n\}\sqcup\{O_{(e_1,x_1)}\}$ is a finite GFS open $p-$subcover of $F_\mu = \widetilde{1}_\triangle$, then $\widetilde{1}_\triangle$ is GFS $p-$compact. Hence (X, T_∞, E) is GFS p^*-compact. □

Remark 3.9. Every GFS p^*-compact space is GFS compact in the sense of Definition 2.17. But the converse is not necessary true, as shown by the following example.

Example 3.10. Let X be an infinite set and $T = \{\widetilde{1}_\triangle, F_\mu \in GFSS(X, E) : F_\mu \sqsubseteq (0.5)_{(X,E)}\}$, then it is easy to check that (X, T, E) is a $GFS-$compact space. but it is not GFS p^*-compact. Indeed the $GFSS$ $F_\mu = \widetilde{0.5}_{(X,E)} \in T^c$ and the family $\psi = \{(e_{0.5}, x_{0.5}) : x \in X, e \in E\}$ is a GFS open $p-$cover of F_μ which has no a finite GFS open $p-$subcover. Hence the result holds.

Remark 3.11. Every GFS p^*-compact space is GFS $p-$compact. But the converse may not be true in general, this fact can be shown by the pervious example.

From Remark 3.2, Remark 3.9, and Remark 3.11, relations between GFS-compact, GFS $p-$compact and GFS p^*-compact spaces can be described by the following diagram:

$$\begin{tikzcd}
& GFS\ p^* -\text{Compact space} \arrow[dl] \arrow[dr] & \\
GFS -\text{Compact space} & & GFS\ p -\text{Compact space} \arrow[ll]
\end{tikzcd}$$

Proposition 3.12. *Let (X, T_1, E) and (X, T_2, E) be two $GFST-$space and $T_1 \sqsubseteq T_2$, then (X, T_1, E) is GFS $p-$compact (GFS p^*-compact) if (Y, T_2, E) is GFS $p-$ compact (GFS p^*-compact).*

Proof. For the first case, let $\{(F_\mu)_i : (F_\mu)_i \in T_1, i \in J\}$ be a GFS open $p-$cover of $\widetilde{1}_\triangle$. Since $T_1 \sqsubseteq T_2$, then $\{(F_\mu)_i : (F_\mu)_i \in T_2, i \in J\}$ is a GFS open $p-$cover of $\widetilde{1}_\triangle$. But (X, T_2, E) is GFS $p-$compact. So that, for all $(e_\lambda, x_\alpha)\widetilde{\in}\widetilde{1}_\triangle$ there exists $i = 1, 2, 3, ..., n : i \in J$ such that $(e_\lambda, x_\alpha)\widetilde{\in}(F_\mu)_i$ then $\{(F_\mu)_i : i = 1, 2, 3, ..., n : i \in J\}$ is a finite GFS open $p-$subcover of $\widetilde{1}_\triangle$. Hence (X, T_1, E) a is GFS $p-$compact space. The proof of the rest case is obvious. □

Remark 3.13. If X be a finite set, then (X, T, E) is GFS p^*-compact. But the convers may not be true as shown by the following example.

Example 3.14. Let X be an infinite set and $T_c = \{\widetilde{\alpha}_{(X,E)} : \alpha \in [0, 1]\}$, then (X, T_c, E) is GFS p^*-compact space.

Theorem 3.15. *Let (X, T, E) be the discrete $GFST-$space, then (X, T, E) is GFS p^*-compact if and only if X is finite.*

Proof. Let (X, T, E) be the discrete GFS p^*-compact space. Suppose that X an infinite set. Since (X, T, E) is GFS p^*-compact, then for every GFS closed set over (X, E) is a GFS $p-$compact set. Let F_μ be a GFS closed set, then for all GFS open $p-$cover of F_μ has a finite GFS open $p-$subcover. Take $\psi = \{(e_\lambda, x_\alpha) : x \in X, e \in E\}$, then ψ is a GFS open $p-$cover of F_μ which has no a finite GFS open $p-$subcover. This is contradiction. Hence X is a finite set. Conversely, the proof follows direct from Remark 3.13. $\square$

Definition 3.16. *Let $\zeta = \{(F_\mu)_i : i \in J\}$ be a family of $GFSSs$ and $G_\delta \in GFSS(X, E)$. Then:*
1) ζ is said to be have GFS $q-$intersection with respect to (w.r.t., for short) G_δ if and only if there exists $(e_\lambda, x_\alpha)\widetilde{\in} G_\delta$ such that $(e_\lambda, x_\alpha)\bar{q}(F_\mu)_i$ for all $i \in J$.
2) ζ is called has the GFS finite intersection property ($GFSFIP$, for short) w.r.t. G_δ if and only if every finite subfamily of ζ has GFS $q-$intersection w.r.t. G_δ.

Theorem 3.17. *Let (X, T, E) be a $GFST-$space. A $GFSS$ G_δ is GFS $p-$compac if and only if each family of GFS closed sets over (X, E) having the GFSFIP w.r.t. G_δ has GFS $q-$intersection w.r.t. G_δ.*

Proof. Let $G_\delta \in GFSS(X, E)$ be GFS $p-$compact and let $\zeta = \{(F_\mu)_i : i \in J\}$ be the family of GFS closed sets over (X, E) which has the $GFSFIP$ w.r.t. G_δ. Now suppose ζ has no GFS $q-$intersection w.r.t. G_δ. Then for all $(e_\lambda, x_\alpha)\widetilde{\in} G_\delta, \exists i \in J$ such that $(e_\lambda, x_\alpha)\bar{q}(F_\mu)_i$ and so, $\zeta^c = \{(F_\mu)_i^c : i \in J\}$ is a GFS open $p-$cover of G_δ. Since G_δ is GFS $p-$compac, then there is a finite GFS open $p-$subcover of ζ^c say, $\{(F_\mu)_s^c : s = 1, 2, ..., n \in J\}$. So, $\{(F_\mu)_s : s = 1, 2, ..., n \in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Contradiction that ζ has the $GFSFIP$ w.r.t.G_δ. Hence, we obtain the result.

Conversely, let the family $\zeta = \{O^i_{(e_\lambda, x_\alpha)} : i \in J\}$ be a GFS open $p-$cover of G_δ. Then $\zeta^c = \{(O^i_{(e_\lambda, x_\alpha)})^c : i \in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Thus ζ^c has no $GFSFIP$ w.r.t. G_δ. So there are $i_1, i_2, ..., i_n \in J$ such that $\{(O^{i_s}_{(e_\lambda, x_\alpha)})^c : i_1, i_2, ..., i_n \in J\}$ has no GFS $q-$intersection w.r.t. G_δ. Then $\{(O^{i_s}_{(e_\lambda, x_\alpha)}) : i_1, i_2, ..., i_n \in J\}$ is a finite GFS open $p-$subcover of G_δ. Hence G_δ is GFS $p-$compact. $\square$

Theorem 3.18. *Every GFS closed subspace (X, T_Y, E) of a GFS p^*-compact space (X, T, E) is a GFS p^*-compact space.*

Proof. Obvious. $\square$

4. Generalized fuzzy soft p^*-compactness and generalized fuzzy soft quasi separation axioms

Theorem 4.1. *Let* (X,T,E) *be a* $GFS\ Q-T_3-$*space and* $G_\delta \in GFSS(X,E)$ *be a* $GFS\ p-$*compact set, then for all* $F_\mu \in T^c$ *with* $F_\mu \bar{q} G_\delta$ *there are* $O_{F_\mu}, O_{G_\delta} \in T$ *such that* $O_{F_\mu} \bar{q} O_{G_\delta}$.

Proof. Let (X,T,E) be a $GFS\ Q-T_3-$space, $F_\mu \in T^c$ and $G_\delta \in GFSS(X,E)$ be $GFS\ p-$compac, then for every $(e_\lambda, x_\alpha) \widetilde{\in} G_\delta$ there exist $O_{(e_\lambda,x_\alpha)}, O_{F_\mu} \in T$ such that $O_{(e_\lambda,x_\alpha)} \bar{q} O_{F_\mu}$. Clearly, $\{(e_\lambda, x_\alpha) : (e_\lambda, x_\alpha) \widetilde{\in} G_\delta\}$ is a GFS open $p-$cover of G_δ. Since G_δ is $GFS\ p-$compac, then there exists a finite GFS open $p-$subcover of G_δ. say, $\{O^i_{(e_\lambda,x_\alpha)} : i = 1, 2, ..., n\}$. One readily verifies that $O_{G_\delta} = \sqcup^n_{i=1} O^i_{(e_\lambda,x_\alpha)}$ and $O_{F_\mu} = \sqcap^n_{i=1} O^i_{F_\mu}$ have the required property. □

Theorem 4.2. *Let* (X,T,E) *be a* $GFS\ Q-T_2-$*space,* $(e_\lambda, x_\alpha) \in GFSP(X,E)$ *and* G_δ *be* $GFS\ p-$*compact with* $(e_\lambda, x_\alpha) \bar{q} G_\delta$*, then there are* $O_{(e_\lambda,x_\alpha)} \in T$ *and* $O_{G_\delta} \in T$ *such that* $O_{(e_\lambda,x_\alpha)} \bar{q} O_{G_\delta}$*. Moreover, if* F_μ *and* G_δ *are be* $GFS\ p-$*compact with* $F_\mu \bar{q} G_\delta$*, then there are* $O_{F_\mu} \in T$ *and* $O_{G_\delta} \in T$ *such that* $O_{F_\mu} \bar{q} O_{G_\delta}$.

Proof. It is similar to that of the above theorem. □

Theorem 4.3. *Every* $GFS\ p-$*compact set in* $GFS\ Q-T_2-$*space is a* GFS *closed set.*

Proof. Let G_δ be a $GFS\ p-$compact set in $GFS\ Q-T_2-$space (X,T,E), suppose $(e_\lambda, x_\alpha) \widetilde{\in} G^c_\delta$, this implies $(e_\lambda, x_\alpha) \bar{q} G_\delta$ then from the above theorem we have for every $(e_\lambda, x_\alpha) \widetilde{\in} G_\delta$ there exist $O_{(e_\lambda,x_\alpha)} \in T$ and $O_{G_\delta} \in T$ such that $O_{(e_\lambda,x_\alpha)} \bar{q} O_{G_\delta}$ i.e., $O_{(e_\lambda,x_\alpha)} \bar{q} G_\delta$ that is, for all $(e_\lambda, x_\alpha) \widetilde{\in} G^c_\delta$ there exists $O_{(e_\lambda,x_\alpha)} \in T$ such that $O_{(e_\lambda,x_\alpha)} \sqsubseteq G^c_\delta$. Therefore, G_δ is GFS open. Hence the result holds. □

Theorem 4.4. *If* (X,T,E) *is* $GFS\ p^*-$*compact* $GFS\ Q-T_2-$*space, then* (X,T,E) *is a* $GFS\ Q-T_4-$*space.*

Proof. Let (X,T,E) be a $GFS\ p^*-$compact $GFS\ Q-T_2-$space. Let $G_\delta, F_\mu \in T^c$ with $F_\mu \bar{q} G_\delta$. Since (X,T,E) is $GFS\ p^*-$compact, then G_δ, F_μ are $GFS\ p$-compact and so, from Theorem 4.2, there exist $O_{F_\mu} \in T$ and $O_{G_\delta} \in T$ such that $O_{F_\mu} \bar{q} O_{G_\delta}$. Hence (X,T,E) is a $GFS\ Q-T_4-$space. □

Corollary 4.5. *If* (X,T,E) *is* $GFS\ p^*-$*compact* $GFS\ Q-T_2-$*space, then* (X,T,E) *is a* $GFS\ Q-T_3-$*space.*

Theorem 4.6. *Let* (X,T,E) *be a* $GFS\ Q-R_1-$*space. Then* (X,T,E) *is a* $GFS\ Q-T_2-$*space if and only if every* $GFS\ p-$*compact set is* GFS *closed.*

Proof. The necessity follows from Theorem 4.3

Conversely, let every $GFS\ p-$compact set is GFS closed, first we prove that (X,T,E) is a $GFS\ Q-T_1-$space. Let $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X,E)$

with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$, then $(e_\lambda, x_\alpha) \sqsubseteq (e'_\gamma, y_\beta)^c = O_{(e_\lambda,x_\alpha)} \in T$ and $(e'_\gamma, y_\beta) \sqsubseteq (e_\lambda, x_\alpha)^c = O_{(e'_\gamma,y_\beta)} \in T$ (since $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta)$ are closed).

So, that there exist $O_{(e_\lambda,x_\alpha)}$ and $O_{(e'_\gamma,y_\beta)}$ such that $(e_\lambda, x_\alpha)\bar{q}(e_\lambda, x_\alpha)^c = O_{(e'_\gamma,y_\beta)}$ and $(e'_\gamma, y_\beta)\bar{q}(e'_\gamma, y_\beta)^c = O_{(e_\lambda,x_\alpha)}$. Hence (X, T, E) is a $GFS\ Q - T_1$. Now, (X, T, E) is $GFS\ Q - R_1$ and $GFS\ Q - T_1-$space, $(e_\lambda, x_\alpha), (e'_\gamma, y_\beta) \in GFSP(X, E)$ with $(e_\lambda, x_\alpha)\bar{q}(e'_\gamma, y_\beta)$ by $GFS\ Q-T_1$ implies that $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$ [as $(e'_\gamma, y_\beta) = cl(e'_\gamma, y_\beta)$ see[16]. Theorem 4.3] and by $GFS\ Q - R_1$, there exist $O_{(e_\lambda,x_\alpha)}, O_{(e'_\gamma,y_\beta)} \in T$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. Hence (X, T, E) is a $GFS\ Q - T_2$. □

Theorem 4.7. *Every* GFS p^*-*compact* $GFS\ Q - R_1-$*space* (X, T, E) *is a* $GFS\ Q$ *regular* ($GFS\ Q$ *normal*)$-$*space.*

Proof. We prove the theorem for $GFS\ Q$ regular, the proof of the rest case is similar.

Let (X, T, E) be a GFS p^*-compact $GFS\ Q - R_1-$space and $F_\mu \in T^c$ with $(e_\lambda, x_\alpha)\bar{q}F_\mu$. Then for all GFS point $(e'_\gamma, y_\beta)\tilde{\in}F_\mu$ we have, $(e_\lambda, x_\alpha)\bar{q}cl(e'_\gamma, y_\beta)$. Since (X, T, E) is $GFS\ Q - R_1$, then there exist $O_{(e_\lambda,x_\alpha)} \in T$ and $O_{(e'_\gamma,y_\beta)} \in T$ such that $O_{(e_\lambda,x_\alpha)}\bar{q}O_{(e'_\gamma,y_\beta)}$. Then the family $\{O_{(e'_\gamma,y_\beta)} : (e'_\gamma, y_\beta)\tilde{\in}F_\mu\}$ is a GFS open $p-$cover of F_μ. Since (X, T, E) is GFS p^*-compact, then F_μ is GFS $p-$compact, and so there exists $\{O^i_{(e'_\gamma,y_\beta)} : (e'_\gamma, y_\beta)\tilde{\in}F_\mu, i = 1, 2, ..., n\}$ is a finite GFS open $p-$subcover of F_μ. Now take $O^*_{(e_\lambda,x_\alpha)} = \sqcap_{i=1}^n O^i_{(e_\lambda,x_\alpha)}$ and $O_{F_\mu} = \sqcup_{i=1}^n O^i_{(e'_\gamma,y_\beta)}$, then $O^*_{(e_\lambda,x_\alpha)}, O_{F_\mu} \in T$ and $O^*_{(e_\lambda,x_\alpha)}\bar{q}O_{F_\mu}$. Hence (X, T, E) is a $GFS\ Q$ regular$-$space. □

Corollary 4.8. *Let* (X, T, E) *be* GFS p^*-*compact, then the following statements are equivalent:*

1) (X, T, E) *is* $GFS\ Q - R_1$,
2) (X, T, E) *is* $GFS\ Q$ *regular,*
3) (X, T, E) *is* $GFS\ Q - R_0$ *and* $GFS\ Q$ *normal.*

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