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*Research Article*

# Some Formulas for Horn's Hypergeometric Function $G_B$ of Three Variables

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Agarwal et al. (2021) established the extension of several fundamental contiguous relations for $\mathbf{G}_B$. Our aim in this work is to investigate several properties of differentiation formulas, differential equations, recursion relations, differential recursion relations, confluence formulas, series representations, integration formulas, and infinite summations for Horn's hypergeometric function $\mathbf{G}_B$ of three variables. Some well-known particular cases have additionally been given.

## 1. Introduction and Notations

The major development of the theory of hypergeometric functions was carried out by Gauss in his famous work of 1812. In [1], Horn's functions of two variables were initiated in the years 1889 and 1931. This paper is considered as a real starting of rigor in mathematics. Several nice papers dealing with hypergeometric functions have been investigated earlier by Euler and others, but Gauss who the first who made an interesting work of the series that defines this function. Hypergeometric series are very useful in mathematics. Note that almost all of the elementary functions of mathematics are either hypergeometric, limiting cases of a hypergeometric series, or ratios of hypergeometric functions. It is well known that there exist several types of hypergeometric functions of three variables. A number of contemporary studies have been devoted to the development of the theory of hypergeometric functions of two or more variables and its important applications, for example, wireless communication theory, nonlinear differential equations, and nonlinear cubic-quintic duffing oscillators, and so on, see [2–6]. Horn introduced four families of hypergeometric functions of three variables each. The Horn functions have several applications, for example in the theory of algebraic equations, statics, operations research, and so on. Finally, the situation completely changed due to the publication of Yang's study, whose book was published by Academic Press [7].

Motivated mainly by these works, Sharma [8] studied properties for Appell functions. Sahin and Agha [9] introduced and studied of recursion formulas for Horn functions. In the recent papers, as Bezrodnykh [2], Dhawan [10], Exton [11], Karlsson [12], Khan and Pathan [13, 14], Padmanabham [15], Sahin [16], Sahai and Verma [17], Saran [18, 19], Srivastava [20, 21], Srivastava [22], Vidunas [23], Wang [24], it has been obtained various results about properties for hypergeometric functions of three variables. In our present work, we establish some new differentiation formulas, differential equations, recursion relations, differential recursion relations, confluence formulas, series representations, integration formulas, and infinite summations for the Horn hypergeometric function $\mathbf{G}_B$ of three variables by using the technique of Ibrahim [25], Ibrahim and Rakha [26], Rakha and Ibrahim [27], Rakha et al. [28], Kim et al. [29] and Brychkov and Savischenko [30–32]. Pathan et al. [33], and Shehata and Shimaa [34, 35] found 7 and 11 hypergeometric series in two variables of order two. We shall derive these $\mathbf{G}_B$ series from our viewpoint. As an application, we shall show that Theorem 2.2 leads to partial differential equations, Theorem 3.1 and Theorem 3.2 lead to differential recursion formulas, Eqs. (42)-(9) lead to series representations, Theorem 6.1 and Theorem 6.2 lead to integral representations of Euler type, and Theorem 7.1 leads to infinite summations for Horn's series $\mathbf{G}_B$ of three variables.

For the purposes of our present study, we begin by recalling here the binomial identities as follows:

$$\binom{\alpha}{\beta} = \frac{\alpha}{\alpha-\beta}\binom{\alpha-1}{\beta}, \tag{1}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2}$$

The Pochhammer symbol $(\mu)_\kappa$ is defined by

$$(\mu)_\kappa = \frac{\Gamma(\mu+\kappa)}{\Gamma(\mu)} = \begin{cases} 1, & (\mu\in\mathbb{C}, \kappa=0); \\ \mu(\mu+1)(\mu+2)\cdots(\mu+\kappa-1), & (\mu\in\mathbb{C}, \kappa\in\mathbb{N}), \end{cases}$$
$$(\mu)_{-\kappa} = \frac{(-1)^\kappa}{(1-\mu)_\kappa}, (\mu\neq 0,\pm1,\pm2,\pm3,\cdots,\kappa\in\mathbb{N}),$$
$$(\mu)_{\ell-\kappa} = \begin{cases} 0, & \kappa>\ell; \\ \dfrac{(-1)^\kappa(\mu)_\ell}{(1-\ell-\mu)_\kappa}, & 0\le\kappa\le\ell, \ell,\kappa\in\mathbb{N} \end{cases} \tag{3}$$

The Horn's function of three variables is defined by [22, 36]:

$$\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}x^n y^k z^\ell;$$
$$(\upsilon \text{ satisfies conditions (1.4) and (1.5)}, \tau\neq 0,-1,-2,\cdots, |x|<1, |y|<1, |z|<1). \tag{4}$$

The following basic lemma (see Srivastava and Manocha ([36], pp. 100)) given below is useful.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\Psi(s,r) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\Psi(s,r-s). \tag{5}$$

The integral operator $\widehat{\mathfrak{I}}$ is given by Abul-Ez and Sayyed, and Sayyed in [12, 37]:

$$\widehat{\mathfrak{I}} = \frac{1}{x}\int_0^x du + \frac{1}{y}\int_0^y dv + \frac{1}{z}\int_0^z dt, \tag{6}$$

where $\widehat{\mathfrak{I}} = \widehat{\mathfrak{I}}_x + \widehat{\mathfrak{I}}_y + \widehat{\mathfrak{I}}_z$, $\widehat{\mathfrak{I}}_x = 1/x\int_0^x du$, $\widehat{\mathfrak{I}}_y = 1/y\int_0^y dv$ and $\widehat{\mathfrak{I}}_z = 1/z\int_0^z dt$.

## 2. Differentiation Formulas

We first give the differentiation formulas and partial differential equations for the function $\mathbf{G}_B$.

**Theorem 2.1.** *The derivative formulas hold: true for $r\in N$, $s\in N$ and $m\in N$:*

$$\frac{\partial^m}{\partial x^m}\frac{\partial^s}{\partial y^s}\frac{\partial^r}{\partial z^r}\mathbf{G}_B = \frac{(\upsilon)_{r+s-m}(\mu)_m(\nu)_s(\omega)_r}{(\tau)_{r+s-m}}\mathbf{G}_B \cdot (\upsilon+r+s-m, \mu+m, \nu+s, \omega + r; \tau+r+s-m; x,y,z). \tag{7}$$

*Proof.* Differentiating $\mathbf{G}_B$ $r$-times with respect to $z$, $s$-times with respect to $y$ and $m$-times with respect to $x$, we obtain (7).

As special cases for $m=r=s$, we have

$$\frac{\partial^r}{\partial x^r}\frac{\partial^r}{\partial y^r}\frac{\partial^r}{\partial z^r}\mathbf{G}_B = \frac{(\upsilon)_r(\mu)_r(\nu)_r(\omega)_r}{(\tau)_r}\mathbf{G}_B \cdot (\upsilon+r, \mu+r, \nu+r, \omega+r; \tau+r; x,y,z). \tag{8}$$

□

**Theorem 2.2.** *Each of the partial differential equations hold: true:*

$$\left[\theta_y\left(\theta_y+\theta_z-\theta_x+\tau-1\right) - y\left(\theta_y+\theta_z-\theta_x+\upsilon\right)\left(\theta_y+\nu\right)\right]\mathbf{G}_B = 0, \tag{9}$$

$$\left[\theta_z\left(\theta_y+\theta_z-\theta_x+\tau-1\right) - z\left(\theta_y+\theta_z-\theta_x+\upsilon\right)\left(\theta_z+\omega\right)\right]\mathbf{G}_B = 0, \tag{10}$$

*where*

$$\theta_x = x\frac{\partial}{\partial x}, \theta_y = y\frac{\partial}{\partial y}, \theta_z = z\frac{\partial}{\partial z}. \tag{11}$$

*Proof.* Applying the differential operators into (4) and simplifying, we obtain (9)-(10).

We note that the results mentioned in Theorem 2.1 and Theorem 2.2 are generalizations of the results in Theorem 3.3 and Theorem 3.5 in the published paper [38]. □

## 3. Contiguous Relations

Here, we establish some recursion formulas and contiguous relations for the function $\mathbf{G}_B$ with respect to parameters. We start with the following theorem. The proofs are straightforward computations.

**Theorem 3.1.** *The Horn hypergeometric function $G_B$ satisfies the following relations:*

$$\mathbf{G}_B(\upsilon + r, \mu, \nu, \omega ; \tau ; x, y, z) = \prod_{i=1}^{r} \left[ 1 + \frac{1}{\upsilon + r - i} \left( \theta_y + \theta_z - \theta_x \right) \right] \mathbf{G}_B(\upsilon, \mu, \nu, \omega ; \tau ; x, y, z) \tag{12}$$

$$\begin{aligned} &\mathbf{G}_B(\upsilon + r, \mu, \nu, \omega ; \tau ; x, y, z) \\ &\quad = \mathbf{G}_B + \sum_{i=1}^{r} \frac{1}{\upsilon + r - i} \left( \theta_y + \theta_z - \theta_x \right) \mathbf{G}_B \\ &\quad \cdot (\upsilon + r - i, \mu, \nu, \omega ; \tau ; x, y, z). \end{aligned} \tag{13}$$

*Proof.* From (4) and (1), we rewrite in the form

$$\mathbf{G}_B(\upsilon, \mu, \nu, \omega ; \tau ; x, y, z) = \sum_{n,k,\ell=0}^{\infty} \frac{\binom{\upsilon + k + \ell - n - 1}{k + \ell - n} \binom{\mu + n - 1}{n} \binom{\nu + k - 1}{k} \binom{\omega + \ell - 1}{\ell}}{\binom{\tau + k + \ell - n - 1}{k + \ell - n}} x^n y^k z^\ell. \tag{14}$$

If we put $\upsilon = \upsilon + \alpha, \mu = \mu + \beta$, $\nu = \nu + \gamma$, $\omega = \omega + \delta$ and $\tau = \tau + \varepsilon$ in (14), we have

$$\mathbf{G}_B(\upsilon + \alpha, \mu + \beta, \nu + \gamma, \omega + \delta ; \tau + \varepsilon ; x, y, z) = \sum_{n,k,\ell=0}^{\infty} \frac{\binom{\upsilon + \alpha + k + \ell - n - 1}{k + \ell - n} \binom{\mu + \beta + n - 1}{n} \binom{\nu + \gamma + k - 1}{k} \binom{\omega + \delta + \ell - 1}{\ell}}{\binom{\tau + \varepsilon + k + \ell - n - 1}{k + \ell - n}} x^n y^k z^\ell. \tag{15}$$

Using (14), we rewrite (15) as follows

$$\begin{aligned} \mathbf{G}_B(\upsilon + \alpha, \mu + \beta, \nu + \gamma, d + \delta ; \tau + \varepsilon ; x, y, z) = &\sum_{n,k,\ell=0}^{\infty} \frac{\Psi_{\upsilon,k+\ell-n}^{\alpha_1,\alpha-1} \Psi_{\mu,n}^{\beta_1,\beta-1} \Psi_{\nu,k}^{\gamma_1,\gamma-1} \Psi_{\omega,\ell}^{\delta_1,\delta-1}}{\Psi_{\tau,k+\ell-n}^{\varepsilon_1,\varepsilon-1}} \\ &\times \frac{\binom{\upsilon + \alpha_1 + k + \ell - n - 1}{k + \ell - n} \binom{\mu + \beta_1 + n - 1}{n} \binom{\nu + \gamma_1 + k - 1}{k} \binom{\omega + \delta_1 + \ell - 1}{\ell}}{\binom{\tau + \varepsilon_1 + k + \ell - n - 1}{k + \ell - n}} x^n y^k z^\ell, \end{aligned} \tag{16}$$

where

$$\Psi_{v,i}^{\alpha_1,\alpha-1} = \Pi_{i=\alpha_1}^{\alpha-1}\left(\frac{v+k+\ell-n+i}{v+i}\right) \tag{17}$$

$$0 \le \alpha_1 \le \alpha, 0 \le \beta_1 \le \beta, 0 \le \gamma_1 \le \gamma, 0 \le \delta_1 \le \delta, \text{and } 0 \le \varepsilon_1 \le \varepsilon. \tag{18}$$

If we put $\alpha = r$, $\alpha_1 = r-1$ and $\beta = \beta_1 = \gamma = \gamma_1 = \delta = \delta_1 = \varepsilon = \varepsilon_1 = 0$ in (16), we get

$$\begin{aligned}\mathbf{G}_B(v+r,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{n,k,\ell=0}^{\infty} \frac{\Psi_v^{r,r-1}\Psi_{\mu,n}^{0,-1}\Psi_{\nu,k}^{0,-1}\Psi_{\omega,\ell}^{0,-1}}{\Psi_{\tau,k+\ell-n}^{0,-1}} \\ &\times \frac{\binom{v+r+k+\ell-n-2}{k+\ell-n}\binom{\mu+n-1}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}} x^n y^k z^\ell.\end{aligned} \tag{19}$$

We observe that

$$\Psi_{v,k+\ell-n}^{r-1,r-1} = \frac{v+k+\ell-n+r-1}{v+r-1}, \Psi_{\mu}^{0,-1} = ,\Psi_{\nu}^{0,-1} = 1, \Psi_{\omega}^{0,-1} = 1, \Psi_{\tau,k+\ell-n}^{0,-1} = 1. \tag{20}$$

Using (20) and (19), we get

$$\begin{aligned}\mathbf{G}_B(v+r,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{n,k,\ell=0}^{\infty}\left[1+\frac{k+\ell-n}{v+r-1}\right] \times \frac{\binom{v+r+k+\ell-n-2}{k+\ell-n}\binom{\mu+n-1}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}} x^n y^k z^\ell \\ &= \sum_{n,k,\ell=0}^{\infty} \frac{\binom{v+r+k+\ell-n-2}{k+\ell-n}\binom{\mu+n-1}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}} x^n y^k z^\ell \\ &+ \sum_{n,k,\ell=0}^{\infty} \frac{k+\ell-n}{v+r-1} \frac{\binom{v+k+\ell-n-2}{k+\ell-n}\binom{\mu+n-1}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}} x^n y^k z^\ell.\end{aligned} \tag{21}$$

Differentiating $\mathbf{G}_B(v+r-1,\mu,\nu,\omega;\tau;x,y,z)$ with respect to $x$ gives

$$\theta_x \mathbf{G}_B(v+r-1,\mu,\nu,\omega;\tau;x,y,z) = \sum_{n,k,\ell=0}^{\infty} \frac{n\binom{v+k+\ell-n-2}{k+\ell-n}\binom{\mu+n-1}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}} x^n y^k z^\ell. \tag{22}$$

Using (21) and (22), we get

$$\begin{aligned}\mathbf{G}_B(\upsilon+r,\mu,\nu,\omega;\tau;x,y,z) &= \mathbf{G}_B(\upsilon+r-1,\mu,\nu,\omega;\tau;x,y,z) \\ &\quad + \frac{1}{\upsilon+r-1}\left[\theta_y+\theta_z-\theta_x\right]\mathbf{G}_B \\ &\quad \cdot(\upsilon+r-1,\mu,\nu,\omega;\tau;x,y,z) \\ &= \left[1+\frac{1}{\upsilon+r-1}\left(\theta_y+\theta_z-\theta_x\right)\right]\mathbf{G}_B \\ &\quad \cdot(\upsilon+r-1,\mu,\nu,\omega;\tau;x,y,z).\end{aligned} \tag{23}$$

As special cases, letting $r=1$ in (23), we get

$$\begin{aligned}&\mathbf{G}_B(\upsilon+1,\mu,\nu,\omega;\tau;x,y,z) \\ &\quad = \left[1+\frac{1}{\upsilon}\left(\theta_y+\theta_z-\theta_x\right)\right]\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z).\end{aligned} \tag{24}$$

In (23) and $r=2$, we obtain

$$\begin{aligned}&\mathbf{G}_B(\upsilon+2,\mu,\nu,\omega;\tau;x,y,z) \\ &\quad = \left[1+\frac{1}{\upsilon+1}\left(\theta_y+\theta_z-\theta_x\right)\right]\mathbf{G}_B(\upsilon+1,\mu,\nu,\omega;\tau;x,y,z).\end{aligned} \tag{25}$$

Combining of the above, we have

$$\begin{aligned}\mathbf{G}_B(\upsilon+2,\mu,\nu,\omega;\tau;x,y,z) &= \left[1+\frac{1}{\upsilon+1}\left(\theta_y+\theta_z-\theta_x\right)\right] \\ &\quad \cdot\left[1+\frac{1}{\upsilon}\left(\theta_y+\theta_z-\theta_x\right)\right]\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z).\end{aligned} \tag{26}$$

This implies for $i \le r$ that

$$\begin{aligned}&\mathbf{G}_B(\upsilon+r,\mu,\nu,\omega;\tau;x,y,z) \\ &\quad = \prod_{i=1}^{r}\left[1+\frac{1}{\upsilon+r-i}\left(\theta_y+\theta_z-\theta_x\right)\right]\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z).\end{aligned} \tag{27}$$

Using (23), we can easily obtain the formula (13). □

Now, we are going to deduce some recurrence relations for $\mathbf{G}_B(\upsilon,\mu+r,c,\omega;\tau;x,y,z)$, $\mathbf{G}_B(\upsilon,\mu,\nu+r,\omega;\tau;x,y,z)$ and $\mathbf{G}_B(\upsilon,\mu,\nu,\omega+r;\tau;x,y,z)$.

**Theorem 3.2.** *The function $G_B$ satisfies the following recurrence relations:*

$$\begin{aligned}&\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z) \\ &\quad = \sum_{s=0}^{r}\frac{\binom{k}{s}}{(\mu)_s}\theta_x^s\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z),\end{aligned} \tag{28}$$

$$\begin{aligned}&\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z) \\ &\quad = \sum_{s=0}^{r}\frac{\binom{r}{s}}{(1-\upsilon)_s}(1-\tau)_s x^s\mathbf{G}_B(\upsilon-s,\mu+s,\nu,\omega;\tau-s;x,y,z),\end{aligned} \tag{29}$$

$$\mathbf{G}_B(\upsilon,\mu,\nu+r,\omega;\tau;x,y,z) = \sum_{s=0}^{r}\frac{\binom{r}{s}}{(\nu)_s}\theta_y^s\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z), \tag{30}$$

$$\begin{aligned}&\mathbf{G}_B(\upsilon,\mu,\nu+r,\omega;\tau;x,y,z) \\ &\quad = \sum_{s=0}^{r}\frac{\binom{r}{s}}{(\nu)_s}(\tau)_s y^s\mathbf{G}_B(\upsilon+s,\mu,\nu+r,\omega;\tau+r;x,y,z),\end{aligned} \tag{31}$$

$$\mathbf{G}_B(\upsilon,\mu,\nu,\omega+r;\tau;x,y,z) = \sum_{s=0}^{r}\frac{\binom{r}{s}}{(\omega)_s}\theta_z^s\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) \tag{32}$$

$$\begin{aligned}&\mathbf{G}_B(\upsilon,\mu,\nu,\omega+r;\tau;x,y,z) \\ &\quad = \sum_{s=0}^{r}\frac{\binom{r}{s}}{(\tau)_s}(\upsilon)_s z^s\mathbf{G}_B(\upsilon+r,\mu,\nu,\omega+r;\tau+r;x,y,z).\end{aligned} \tag{33}$$

*Proof.* If we put $\beta=r$, $\beta_1=r-1$ and $\alpha=\alpha_1=\gamma=\gamma_1=\delta=\delta_1=\varepsilon=\varepsilon_1=0$ in (15), we get

$$\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z) = \sum_{n,k,\ell=0}^{\infty}\frac{\Psi_{\mu,n}^{r,r-1}\Psi_{\upsilon,k+\ell-n}^{0,-1}\Psi_{\nu,k}^{0,-1}\Psi_{\tau,\ell}^{0,-1}}{\Psi_{e,k+\ell-n}^{0,-1}}\frac{\binom{\upsilon+k+\ell-n-1}{k+\ell-n}\binom{\mu+r+n-2}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}}x^n y^k z^\ell. \tag{34}$$

Using (17), we get

$$\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z)=\sum_{n,k,\ell=0}^{\infty}\left[1+\frac{n}{\mu+r-1}\right]\times\frac{\binom{\upsilon+k+\ell-n-1}{k+\ell-n}\binom{\mu+r+n-2}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}}x^n y^k z^\ell$$
$$=\sum_{n,k,\ell=0}^{\infty}\frac{\binom{\upsilon+k+\ell-n-1}{k+\ell-n}\binom{\mu+r+n-2}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}}x^n y^k z^\ell$$
$$+\sum_{n,k,\ell=0}^{\infty}\frac{n}{\mu+r-1}\frac{\binom{\upsilon+k+\ell-n-1}{k+\ell-n}\binom{\mu+r+n-2}{n}\binom{\nu+k-1}{k}\binom{\omega+\ell-1}{\ell}}{\binom{\tau+k+\ell-n-1}{k+\ell-n}}x^n y^k z^\ell. \quad (35)$$

We can rewrite (35) in the form

$$\begin{aligned}&\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z)\\&=\mathbf{G}_B(\upsilon,\mu+r-1,\nu,\omega;\tau;x,y,z)\\&\quad+\frac{1}{\mu+r-1}\theta_x\mathbf{G}_B(\upsilon,\mu+r-1,\nu,\omega;\tau;x,y,z)\\&=\left[1+\frac{1}{\mu+r-1}\theta_x\right]\mathbf{G}_B(\upsilon,\mu+r-1,\nu,\omega;\tau;x,y,z).\end{aligned}\quad (36)$$

This implies for $r\le i$ that

$$\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z)=\sum_{s=0}^{i}\frac{\binom{i}{s}}{(\mu+r-i)_s}\theta_x^s\mathbf{G}_B(\upsilon,\mu+r-i,\nu,\omega;\tau;x,y,z). \quad (37)$$

Putting $r=i$ in (37) implies

$$\mathbf{G}_B(\upsilon,\mu+r,\nu,\omega;\tau;x,y,z)=\sum_{s=0}^{r}\frac{\binom{r}{s}}{(\mu)_s}\theta_x^s\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z). \quad (38)$$

Using (7), we get (29). Similarly, if we take $\gamma=r$, $\gamma_1=r-1$ and $\alpha=\alpha_1=\beta=\beta_1=\delta=\delta_1=\varepsilon=\varepsilon_1=0$ in (16), we get (30) and (31). Again, if we take $\delta=r$, $\delta_1=r-1$ and $\alpha=\alpha_1=\beta=\beta_1=\gamma=\gamma_1=\varepsilon=\varepsilon_1=0$ in (16), we get (32) and (33). □

## 4. Confluence Formulas

The following confluence formulas follow by using the equation (4) via a direct computation.

We recall the relation as follows:

$$\left(\frac{\upsilon}{\alpha}\right)_\ell\sim\left(\frac{\upsilon}{\alpha}\right)^\ell,\ \alpha\longrightarrow 0. \quad (39)$$

Using (4) and (39), we have

$$\begin{aligned}&\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\frac{\upsilon}{\alpha},\mu,\nu,\omega;\frac{\tau}{\alpha};x,y,z\right)\\&=\lim_{\alpha\longrightarrow 0}\sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon/\alpha)^{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau/\alpha)^{k+\ell-n}}x^n y^k z^\ell\\&=\left(1-\frac{\tau}{\upsilon}x\right)^{-\mu}\left(1-\frac{\upsilon}{\tau}y\right)^{-\nu}\left(1-\frac{\upsilon}{\tau}z\right)^{-\omega};\left|\frac{\tau}{\upsilon}x\right|\\&<1,\left|\frac{\upsilon}{\tau}y\right|<1,\left|\frac{\upsilon}{\tau}z\right|<1.\end{aligned}\quad (40)$$

Expanding the expressions $(1-(\tau/\upsilon)x)^{-\mu}$, $(1-(\upsilon/\tau)y)^{-\nu}$ and $(1-(\upsilon/\tau)z)^{-d}$ by applying the binomial theorem (i.e., the expansion of a ${}_1F_0$ function), we get the transformation formula:

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\frac{\upsilon}{\alpha},\mu,\nu,\omega;\frac{\tau}{\alpha};x,y,z\right) = {}_1F_0\left(\mu;-;\frac{\tau}{\upsilon}x\right){}_1F_0\left(\nu;-;\frac{\upsilon}{\tau}y\right){}_1F_0\left(\omega;-;\frac{\upsilon}{\tau}z\right); \cdot\left(\left|\frac{\tau}{\upsilon}x\right|<1,\left|\frac{\upsilon}{\tau}y\right|<1,\left|\frac{\upsilon}{\tau}z\right|<1\right),$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\upsilon,\frac{\mu}{\alpha},\nu,\omega;\tau;\alpha x,y,z\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}\mu^n x^n y^k z^\ell; \cdot(|\mu x|<1,|y|<1,|z|<1),$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\upsilon,\mu,\frac{\nu}{\alpha},\omega;\tau;x,\alpha y,z\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}\nu^k x^n y^k z^\ell; \cdot(|x|<1,|\nu y|<1,|z|<1),$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\upsilon,\mu,\nu,\frac{\omega}{\alpha};\tau;x,y,\alpha z\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k}{n!k!\ell!(\tau)_{k+\ell-n}}\omega^\ell x^n y^k z^\ell; \cdot(|x|<1,|y|<1,|\omega z|<1),$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\upsilon,\frac{\mu}{\alpha},\frac{\nu}{\alpha},\frac{\omega}{\alpha};\tau;\alpha x,\alpha y,\alpha z\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}}{n!k!\ell!(\tau)_{k+\ell-n}}(\mu x)^n(\nu y)^k(\omega z)^\ell; \cdot(|\mu x|<1,|\nu y|<1,|\omega z|<1),$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\frac{\upsilon}{\alpha},\mu,\nu,\omega;\tau;\frac{x}{\alpha},\alpha y,z\alpha\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}\left(\frac{x}{\upsilon}\right)^n(\upsilon y)^k(\upsilon z)^\ell; \cdot\left(\left|\frac{x}{\upsilon}\right|<1,|\upsilon y|<1,|\upsilon z|<1\right)$$

$$\lim_{\alpha\longrightarrow 0}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\frac{\tau}{\alpha};\alpha x,\frac{y}{\alpha},\frac{z}{\alpha}\right) = \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!}(\tau x)^n\left(\frac{y}{\tau}\right)^k\left(\frac{z}{\tau}\right)^\ell; \cdot\left(|\tau x|<1,\left|\frac{y}{\tau}\right|<1,\left|\frac{z}{\tau}\right|<1\right). \quad (41)$$

## 5. Series Representations

Here, we derive the series representations for $\mathbf{G}_B$.

Using (4) and the summation with respect to $n$, $k$ or $\ell$, we get

$$\begin{aligned}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{n,k,\ell=0}^{\infty}\frac{(-1)^n(\upsilon)_{k+\ell}(1-k-\ell-\tau)_n(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(-1)^n(1-k-\ell-\upsilon)_n(\tau)_{k+\ell}}x^n y^k z^\ell \\ &= \sum_{k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell}(\nu)_k(\omega)_\ell}{k!\ell!(\tau)_{k+\ell}}y^k z^\ell{}_2F_1(1-\tau-k-\ell,\mu;1-\upsilon-k-\ell;x) \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\frac{(\upsilon)_k(\nu)_{k-\ell}(\omega)_\ell}{(k-\ell)!\ell!(\tau)_k}y^{k-\ell}z^\ell{}_2F_1(1-\tau-k,\mu;1-\upsilon-k;x) \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\frac{(-k)_\ell(\upsilon)_k(\nu)_k(\omega)_\ell}{k!\ell!(\tau)_k(1-k-\nu)_\ell}y^{k-\ell}z^\ell{}_2F_1(1-\tau-k,\mu;1-\upsilon-k;x) \\ &= \sum_{k=0}^{\infty}\frac{(\upsilon)_k(\nu)_k}{k!(\tau)_k}y^k{}_2F_1(1-\tau-k,\mu;1-\upsilon-k;x){}_2F_1 \\ &\cdot\left(-k,\omega;1-\nu-k;\frac{z}{y}\right);\left(|x|<1,|y|<1,|z|<1,\left|\frac{z}{y}\right|<1\right).\end{aligned} \quad (42)$$

Similarly, after evaluating the finite sums, it follows that

$$\begin{aligned}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{\ell=0}^{\infty}\frac{(\upsilon)_\ell(\omega)_\ell}{\ell!(\tau)_\ell}z^\ell{}_2F_1(1-\tau-\ell,\mu;1-\upsilon-\ell;x){}_2F_1 \\ &\cdot\left(-\ell,\nu;1-\omega-\ell;\frac{y}{z}\right);\left(|x|<1,|y|<1,|z|<1,\left|\frac{y}{z}\right|<1\right),\end{aligned} \quad (43)$$

$$\begin{aligned}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{k,\ell=0}^{\infty}\frac{(\nu)_k(\omega)_\ell}{k!\ell!}{}_2F_1(1-\tau-k-\ell,\mu;1-\upsilon-k-\ell;x)y^k z^\ell; \\ &\cdot(|x|<1,|y|<1,|z|<1).\end{aligned} \quad (44)$$

For special case, by taking $y=z$, we get

$$\begin{aligned}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,y) &= \frac{\Gamma(1-\nu-\omega)}{\Gamma(1-\nu)}\sum_{k=0}^{\infty}\frac{(\upsilon)_k(\nu)_k\Gamma(1-k-\nu)}{k!(\tau)_k\Gamma(1-k-\nu-\omega)}y^k{}_2F_1 \\ &\cdot(\mu,1-k-\tau;1-k-\upsilon;x);(|x|<1,|y|<1).\end{aligned} \quad (45)$$

Thus, we have shown that the above series (42)-(44) converge in the indicated regions.

## 6. Integration Formulas

We present here the integral representations and evaluation of integrals for the function $\mathbf{G}_B$.

**Theorem 6.1.** *For $|t| < 1$, the integration formulas for Horn's function $G_B$:*

$$\widehat{\mathfrak{I}}\mathbf{G}_B = \frac{\upsilon}{x(\mu-1)\tau}\mathbf{G}_B(\upsilon+1,\mu-1;\tau+1;x,y,z) + \frac{\tau-1}{y(\upsilon-1)(\nu-1)}\mathbf{G}_B(\upsilon-1,\nu-1;\tau-1;x,y,z) + \frac{\tau-1}{z(\upsilon-1)(\omega-1)}\mathbf{G}_B(\upsilon-1,\nu-1,\omega-1;\tau-1;x,y,z);x,y,z\neq 0,\mu,\nu,\upsilon,\omega\neq 1,\tau\neq 0, \tag{46}$$

$$\widehat{\mathfrak{I}}_x{}^r\mathbf{G}_B = \frac{(\upsilon)_r(-1)^r}{x^r(\tau)_r(1-\mu)_r}\mathbf{G}_B(\upsilon+r,\mu-r,\nu,\omega;\tau+r;x,y,z);x\neq 0,\tau\neq 0,-1,-2,-3,\cdots,\mu\neq 1,2,3,\cdots, \tag{47}$$

$$\widehat{\mathfrak{I}}_y{}^r\mathbf{G}_B = \frac{(-1)^r(1-\tau)_r}{y^r(1-\upsilon)_r(1-\nu)_r}\mathbf{G}_B(\upsilon-r,\mu,\nu-r,\omega;\tau-r;x,y,z);y\neq 0,\upsilon,\nu\neq 1,2,3,\cdots \tag{48}$$

$$\widehat{\mathfrak{I}}_z{}^r\mathbf{G}_B = \frac{(-1)^r(1-\tau)_r}{z^r(1-\upsilon)_r(1-\omega)_r}\mathbf{G}_B(\upsilon-r,\mu,\nu,\omega-r;\tau-r;x,y,z);z\neq 0,\upsilon,\omega\neq 1,2,3,\cdots. \tag{49}$$

*Proof.* With the help of the integral operator $\widehat{\mathfrak{I}}$, we get the formula

$$\begin{aligned}\widehat{\mathfrak{I}}\mathbf{G}_B &= \sum_{n,k,\ell=1}^{\infty}\left(\frac{1}{n+1}+\frac{1}{k+1}+\frac{1}{\ell+1}\right)\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}x^ny^kz^\ell\\
&= \sum_{n=1,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{(n+1)!k!\ell!(\tau)_{k+\ell-n}}x^ny^kz^\ell + \sum_{n=0,k=1,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!(k+1)!\ell!(\tau)_{k+\ell-n}}x^ny^kz^\ell + \sum_{n,k=0,\ell=1}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!(\ell+1)!(\tau)_{k+\ell-n}}x^ny^kz^\ell\\
&= \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n+1}(\mu)_{n-1}(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n+1}}x^{n-1}y^kz^\ell + \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n-1}(\mu)_n(\nu)_{k-1}(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n-1}}x^ny^{k-1}z^\ell + \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n-1}(\mu)_n(\nu)_k(\omega)_{\ell-1}}{n!k!\ell!(\tau)_{k+\ell-n-1}}x^ny^kz^{\ell-1}\\
&= \sum_{n,k,\ell=0}^{\infty}\frac{\upsilon(\upsilon+1)_{k+\ell-n}(\mu-1)_n(\nu)_k(\omega)_\ell}{n!k!\ell!\tau(\mu-1)(\tau+1)_{k+\ell-n}}x^{n-1}y^kz^\ell + \sum_{n,k,\ell=0}^{\infty}\frac{(\tau-1)(\upsilon-1)_{k+\ell-n}(\mu)_n(\nu-1)_k(\omega)_\ell}{n!k!\ell!(\upsilon-1)(\nu-1)(\tau-1)_{k+\ell-n}}x^ny^{k-1}z^\ell + \sum_{n,k,\ell=0}^{\infty}\frac{(\tau-1)(\upsilon-1)_{k+\ell-n}(\mu)_n(\nu)_k(\omega-1)_\ell}{n!k!\ell!(\upsilon-1)(\omega-1)(\tau-1)_{k+\ell-n}}x^ny^kz^{\ell-1}\\
&= \frac{\upsilon}{x(\mu-1)\tau}\mathbf{G}_B(\upsilon+1,\mu-1;\tau+1;x,y,z) + \frac{\tau-1}{y(\upsilon-1)(\nu-1)}\mathbf{G}_B(\upsilon-1,\nu-1;\tau-1;x,y,z) + \frac{\tau-1}{z(\upsilon-1)(\omega-1)}\mathbf{G}_B(\upsilon-1,\nu-1,\omega-1;\tau-1;x,y,z);x,y,z\neq 0,\mu,\nu,\upsilon,\omega\neq 1,\tau\neq 0.\end{aligned} \tag{50}$$

Applying the integral operator $\widehat{\mathfrak{I}}_x$, we have

$$\begin{aligned}\widehat{\mathfrak{I}}_x\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) &= \sum_{n,k,\ell=0}^{\infty}\left(\frac{1}{n+1}\right)\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n}}x^ny^kz^\ell\\
&= \sum_{n=1,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n}(\mu)_n(\nu)_k(\omega)_\ell}{(n+1)!k!\ell!(\tau)_{k+\ell-n}}x^ny^kz^\ell\\
&= \sum_{n,k,\ell=0}^{\infty}\frac{(\upsilon)_{k+\ell-n+1}(\mu)_{n-1}(\nu)_k(\omega)_\ell}{n!k!\ell!(\tau)_{k+\ell-n+1}}x^{n-1}y^kz^\ell\\
&= \sum_{n,k,\ell=0}^{\infty}\frac{\upsilon(\upsilon+1)_{k+\ell-n}(\mu-1)_n(\nu)_k(\omega)_\ell}{n!k!\ell!e(\mu-1)(\tau+1)_{k+\ell-n}}x^{n-1}y^kz^\ell\\
&= \frac{\upsilon}{x(\mu-1)\tau}\mathbf{G}_B(\upsilon+1,\mu-1;\tau+1;x,y,z), x,\tau\neq 0,\mu\neq 1.\end{aligned} \tag{51}$$

Iterating this integral operator $\widehat{\mathfrak{I}}_x$ on $\mathbf{G}_B$ for $r$-times, we get (47). Similarly, we obtain (48) and (49). □

**Theorem 6.2.** *The integral representations for Horn's function $G_B$ hold:*

$$\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) = \frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)}\int_0^1 t^{\upsilon-1}(1-t)^{\tau-\upsilon-1}{}_3F_0\left(\mu,\nu,\omega;-;\frac{x}{t},yt,zt\right)dt; \cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{x}{t}\right|<1,|yt|<1,|zt|<1\right), \tag{52}$$

*where*

$${}_3F_0(\mu,\nu,\omega;-;x,y,z) = \sum_{n,k,\ell=0}^{\infty}\frac{(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!}x^ny^kz^\ell \tag{53}$$

$$\begin{aligned}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z) &= \frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)\Gamma(\mu)\Gamma(\nu)\Gamma(\omega)\Gamma(1-\mu)\Gamma(1-\nu)\Gamma(1-\omega)}\\
&\times\int_0^1\int_0^1\int_0^1\int_0^1 t_1^{\upsilon-1}t_2^{\mu-1}t_3^{\nu-1}t_4^{\omega-1}(1-t)^{\tau-\upsilon-1}(1-t_2)^{-\mu}(1-t_3)^{-\nu}(1-t_4)^{-\omega}\\
&\times{}_1F_0\left(1;-;\frac{xt_2}{t_1}\right){}_1F_0(1;-;yt_1t_3){}_1F_0(1;-;zt_1t_4)dt_1dt_2dt_3dt_4;\\
&\cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{xt_2}{t_1}\right|<1,|yt_1t_3|<1,|zt_1t_4|<1\right).\end{aligned} \tag{54}$$

*Proof.* If $k+\ell-n$ is a non-negative integer and using the integral definition of the Beta function, we have

$$\frac{(\upsilon)_{k+\ell-n}}{(\tau)_{k+\ell-n}} = \frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)}\int_0^1 t^{\upsilon+k+\ell-n-1}(1-t)^{\tau-\upsilon-1}dt. \tag{55}$$

Therefore,

$$\begin{aligned}
&\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau;x,y,z)\\
&=\frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)}\sum_{n,k,\ell=0}^{\infty}\frac{(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!}x^n y^k z^\ell\\
&\cdot\int_0^1 t^{\upsilon+k+\ell-n-1}(1-t)^{\tau-\upsilon-1}dt\\
&=\frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)}\sum_{n,k,\ell=0}^{\infty}\frac{(\mu)_n(\nu)_k(\omega)_\ell}{n!k!\ell!}\\
&\cdot\int_0^1 t^{\upsilon-1}\left(\frac{x}{t}\right)^n(yt)^k(zt)^\ell(1-t)^{\tau-\upsilon-1}dt\\
&=\frac{\Gamma(\tau)}{\Gamma(\upsilon)\Gamma(\tau-\upsilon)}\int_0^1 t^{\upsilon-1}(1-t)^{\tau-\upsilon-1}{}_3F_0\left(\mu,\nu,\omega;-;\frac{x}{t},yt,zt\right)dt.
\end{aligned}\tag{56}$$

The equation (54) can be proven as the proof of equation (52). □

## 7. Infinite Summations for $\mathrm{G}_B$

Here, we give the infinite summations for Horn's function $\mathbf{G}_B$.

Using the binomial theorem [36], we get

$$(1-\mathrm{t})^{-\alpha}=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}\mathrm{t}^r,\ |\mathrm{t}|<1.\tag{57}$$

**Theorem 7.1.** *For $|t|<1$, the infinite summation for Horn's function $G_B$:*

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\upsilon)_m}{m!}\mathbf{G}_B(\upsilon+m,\mu,\nu,\omega;\tau;x,y,z)t^m\\
&=(1-t)^{-\upsilon}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\tau;x(1-t),\frac{y}{1-t},\frac{z}{1-t}\right);\\
&\cdot\left(|x|<1,|y|<1,|z|<1,|x(1-t)|<1,\left|\frac{y}{1-t}\right|<1,\left|\frac{z}{1-t}\right|<1\right),
\end{aligned}\tag{58}$$

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\mu)_m}{m!}\mathbf{G}_B(\upsilon,\mu+m,\nu,\omega;\tau;x,y,z)t^m\\
&=(1-t)^{-\mu}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\tau;\frac{x}{1-t},y,z\right);\\
&\cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{x}{1-t}\right|<1\right),
\end{aligned}\tag{59}$$

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\nu)_m}{m!}\mathbf{G}_B(\upsilon,\mu,\nu+m,\omega;\tau;x,y,z)t^m\\
&=(1-t)^{-\nu}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\tau;x,\frac{y}{1-t},z\right);\\
&\cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{y}{1-t}\right|<1\right),
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\omega)_m}{m!}\mathbf{G}_B(\upsilon,\mu,\nu,\omega+m;\tau;x,y,z)t^m\\
&=(1-t)^{-\omega}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\tau;x,y,\frac{z}{1-t}\right);\\
&\cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{z}{1-t}\right|<1\right)
\end{aligned}\tag{61}$$

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{(\tau)_m}{m!}\mathbf{G}_B(\upsilon,\mu,\nu,\omega;\tau+m;x,y,z)t^m\\
&=(1-t)^{-\tau}\mathbf{G}_B\left(\upsilon,\mu,\nu,\omega;\tau;\frac{x}{1-t},y(1-t),z(1-t)\right);\\
&\cdot\left(|x|<1,|y|<1,|z|<1,\left|\frac{x}{1-t}\right|\right.\\
&\left.\cdot<1,|y(1-t)|<1,|z(1-t)|<1\right).
\end{aligned}\tag{62}$$

*Proof.* With the aid of (4) and (57), we directly obtain the results (58)-(62). □

## 8. Conclusions and General Remarks

Our paper is generally based on the extension of Horn's hypergeometric functions of two variables. A few specializations relevant to the present discussion have also been derived from results of papers [33–35, 38]. We focused on the generalization of the Horn's function $\mathbf{G}_B$ of three variables and presented some partial differential equations, differential recursion formulas, series representations, integral representations and infinite summations.

In Yang's study, numerous special functions are summarized, and their properties are detected, which can be utilized for applications immediately, and readers can create a new special function for their own practical applications.

To be complemented, Yang's study [7] is very useful for physical and engineering applications. It can attainable to all students, researchers, physicists, and engineers without the necessity of a profound knowledge of mathematics. Any reader can be researched for their beloved shells in the field of special functions.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no competing interests.

## Authors' Contributions

Al authors approve and are agree with the final version. .


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