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NEW CLASSES OF GENERALIZED CLOSED SETS IN IDEAL BITOPOLOGICAL SPACES

F. H. Khedr

Department of Mathematics
Faculty of Sciences
University of Assiut
Assiut 715161, Egypt
e-mail: khedrfathi@gmail.com
khedrf@aun.edu.eg

Abstract

In this paper, a new type of generalized closed sets in ideal bitopological spaces is introduced and studied. Some applications of these sets in connection with certain separation axioms are given.

1. Introduction and Preliminary

In their paper [1], the authors have studied the concept of generalized closed sets in ideal bitopological spaces. In fact, this study is a study in an ideal topological space. The topology defined is nothing but the topology resulting from the intersection of the two topologies of the bitopological space (X, τ_1, τ_2). Hence, this is not a study in bitopological spaces. The main definition that the authors relied on in formulating the results in their

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paper [1] is the definition of $\tau_1\tau_2$-closed and $\tau_1\tau_2$-open sets. Their definition stated that a subset A of a bitopological space is $\tau_1\tau_2$-closed if $A = \tau_1 - cl(\tau_2 - cl(A))$. In fact this definition means that A is $\tau_1\tau_2$-closed if and only if A is τ_1-closed and τ_2-closed. To show this, if $A = \tau_1 - cl(\tau_2 - cl(A))$, then A is τ_1-closed and $\tau_2 - cl(A) \subset A$ which implies that A is also τ_2-closed. On the other hand if A is τ_1-closed and τ_2-closed, then $A = \tau_1 - cl(\tau_2 - cl(A))$. Therefore if the topology induced by this definition is τ, then $\tau = \tau_1 \cap \tau_2$. So, all the definitions and results is [1] are in the ideal topological space and was studied in [2-4]. Therefore the study in [1] is in the ideal topological space (X, τ, I) not in the ideal bitopological space (X, τ_1, τ_2, I).

In this paper, a new type of generalized closed sets in ideal bitopological spaces is introduced and studied. Some applications of these sets in connection with certain separation axioms are given.

Bitopological spaces were introduced by Kelly [5] in 1963 as an extension of topological spaces. A bitopological space (X, τ_1, τ_2) is a nonempty set X equipped with two topologies τ_1 and τ_2. The concept of ideal topological spaces was initiated by Kuratowski [6] and Vaidyanathaswamy [7]. An Ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies: (i) If $A \in I$ and $B \subseteq A$ then $B \in I$ and (ii) If $A \in I$ and $B \in I$ then $A \cup B \in I$. If $P(X)$ is the set of all subsets of X, in a topological space (X, τ), a set operator $(\cdot)^* : P(X) \to P(X)$ called the local function of A with respect to τ and I and is defined as $A^*(\tau, I) = \{x \in X : U \cap A \notin I, \forall U \in \tau, x \in U\}$. $A^*(\tau, I)$ is written in short A^*. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\tau, I)$, finer than τ, called the *-topology [6], and is defined by $cl^*(A) = A \cup A^*$. τ^* is written for $\tau^*(\tau, I)$. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by $i - cl(A)$ and $i - \text{int}(A)$, respectively, for $i = 1, 2$. A bitopological space (X, τ_1, τ_2) with an ideal I on X is called an ideal bitopological space and is denoted by (X, τ_1, τ_2, I). In an ideal bitopological space (X, τ_1, τ_2, I), the local function of $A \subset X$ with respect to τ_i and I is denoted by A^{*i}, that is, $A^{*i} = \{x \in X : U \cap A \notin I, \forall U \in \tau_i, x \in U\}$. Also, $cl^{*i}(A) = A \cup A^{*i}$ and, in this case, the *-topology is denoted by τ_i^*, for $i = 1, 2$. The members of τ_i^* are called i^*-open. The complement of an i^*-open set is called i^*-closed. The closure and the interior of $A \subset X$ with respect to τ_i^* are denoted by $i - cl^*(A)$ and $i - \text{int}^*(A)$, respectively. Fukutake [8] introduced the notion of generalized closed sets in bitopological spaces.

The aim of this paper is to introduce two new classes of generalized closed sets in ideal bitopological spaces, termed $i^* j - g$-closed and $ij - g_I$-closed. Several characterizations of $i^* j - g$-closed and $ij - g_I$-closed sets are given. Characterizations of pairwise regular and pairwise normal spaces in terms of $i^* j - g$-closed sets are given. Finally, two new kinds of separation axioms in ideal bitopological spaces in terms of $i^* j - g$-closed and $ij - g_I$-closed sets are introduced and investigated. Throughout this paper, $i, j = 1, 2$ and $i \neq j$.

Definition 1.1 [8]**.** A subset A of a bitopological space (X, τ_1, τ_2) is called $ij - g$-closed if $j - cl(A) \subset U$ whenever $A \subset U$ and U is i-open.

Definition 1.2 [5]**.** A bitopological space (X, τ_1, τ_2) is called pairwise regular if for each $x \in X$ and each i-closed set F not containing x, there exist an i-open set U and a j-open sets V such that $x \in U$, $F \subset V$ and $U \cap V = \varnothing$.

Definition 1.3 [5]**.** A bitopological space (X, τ_1, τ_2) is called pairwise normal if for each disjoint pair of an *i*-closed set *F* and a *j*-closed set *H*, there exist an *i*-open set *U* and a *j*-open sets *V* such that $H \subset U$, $F \subset V$ and $U \bigcap V = \varnothing$.

Definition 1.4 [9]**.** A bitopological space (X, τ_1, τ_2) is called a pairwise R_0 space if $x \in U \in \tau_i$ implies that $j - cl(\{x\}) \subset U$.

2. $i^* j$ -Generalized Closed Sets

Definition 2.1. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then A is called $i^* j$ -generalized closed $(i^* j - g$ -closed for short$)$, if $j - cl(A) \subset U$ whenever $A \subset U$ and U is i^* -open. The complement of an $i^* j - g$ -closed set is called $i^* j - g$ -open.

Definition 2.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then A is called $ij - g_I$ -closed if $j - cl^*(A) \subset U$ whenever $A \subset U$ and U is *i*-open. The complement of an $ij - g_I$ -closed set is called $ij - g_I$ -open.

Remark 2.1. Since every *i*-open set is i^* -open in any ideal bitopological space (X, τ_1, τ_2, I), one may deduce the following diagram for different kinds of sets.

$$j - \text{closed} \Rightarrow i^* j - g - \text{closed} \Rightarrow ij - g - \text{closed} \Rightarrow ij - g_I - \text{closed}.$$

But the converses may not be true as can be shown by the following example:

Example 2.1. Consider the ideal bitopological space (X, τ_1, τ_2, I), where $X = \{a, b, c, d\}$, $\tau_1 = \{\varnothing, X, \{a, b\}, \{c, d\}\}$, $\tau_2 = \{\varnothing, X, \{b, c\}, \{a, d\}\}$ and $I = \{\varnothing, \{a\}\}$. Then $\{a, c, d\}$ is $1^* 2 - g$ -closed but not 2-

closed. $\{a, c\}$ is $12 - g$ -closed but not $1^*2 - g$ -closed and $\{a\}$ is $12 - g_I$ -closed but not $12 - g$ -closed.

Theorem 2.1. *In an ideal bitopological space* (X, τ_1, τ_2, I), *the union of a finite number of* $i^* j - g$ *-closed* (*resp.* $ij - g_I$ *-closed*) *sets is* $i^* j - g$ *-closed* (*resp.* $ij - g_I$ *-closed*).

Proof. Let A and B be two $i^* j - g$ -closed sets in (X, τ_1, τ_2, I). Suppose $A \bigcup B \subset U$, where U is an i^* -open set. Then $A \subset U$ and $B \subset U$ which implies, by hypothesis, that $j - cl(A) \subset U$ and $j - cl(B) \subset B$. Therefore $j - cl(A \bigcup B) = j - cl(A) \bigcup j - cl(B) \subset U$ and so $A \bigcup B$ is $i^* j - g$ -closed. The case of $ij - g_I$ -closed is similar. □

Remark 2.2. The intersection of two $i^* j - g$ -closed sets need not be $i^* j - g$ -closed as can shown by the following example.

Example 2.2. Consider the ideal bitopological space (X, τ_1, τ_2, I) as in Example 2.1. Then $\{a, b\}$ and $\{a, c\}$ are $2^*1 - g$ -closed but their intersection $\{a\}$ is not $2^*1 - g$ -closed.

The following result gives a characterization of $i^* j - g$ -closed sets.

Proposition 2.1. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$. *If A is* $i^* j - g$ *-closed, then* $j - cl(A)\backslash A$ *contains no nonempty* i^* *-closed set*.

Proof. Let F be an i^* -closed set such that $F \subset j - cl(A)\backslash A$. Since A is $i^* j - g$ -closed and $A \subset X \backslash F$, then $j - cl(A) \subset X \backslash F$. Thus $F \subset X \backslash j - cl(A)$ and therefore $F \subset (j - cl(A) \bigcap (X \backslash j - cl(A))) = \varnothing$. □

Remark 2.3. The converse of Proposition 2.1 is not true, since in the ideal bitopological space of Example 2.1, If $A = \{a\}$, then $2 - cl(A)\backslash A$

$= \{d\}$ not contains any nonempty i^*-closed set, although A is not $1^*2 - g$-closed.

Proposition 2.2. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space. If* A *is* $ij - g_I$*-closed, then* $j - cl^*(A) \backslash A$ *contains no nonempty* i*-closed set.*

Proposition 2.3. *If* A *is an* $i^* j - g$*-closed set in an ideal bitopological space* (X, τ_1, τ_2, I), *then* $i - cl^*(\{x\}) \bigcap A \neq \varnothing$ *for each* $x \in j - cl(A)$.

Proof. Suppose $x \in j - cl(A)$. If possible, let $i - cl^*(\{x\}) \bigcap A = \varnothing$. Then $A \subset X \backslash i - cl^*(\{x\})$ which implies that $j - cl(A) \subset X \backslash i - cl^*(\{x\})$. Then $j - cl(A) \bigcap i - cl^*(\{x\}) = \varnothing$, a contradiction. □

Proposition 2.4. *If* A *is an* $ij - g_I$*-closed set in an ideal bitopological space* (X, τ_1, τ_2, I), *then* $i - cl(\{x\}) \bigcap A \neq \varnothing$ *for each* $x \in j - cl^*(A)$.

Theorem 2.2. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A, B \subset X$ *such that* $A \subset B \subset j - cl(A)$ *and* A *is* $i^* j - g$*-closed. Then* B *is* $i^* j - g$*-closed.*

Proof. Let $B \subset U$, where U is i^*-open. Then $A \subset U$ and so $j - cl(A) \subset U$. This implies that $j - cl(B) \subset U$ and so B is $i^* j - g$-closed. □

Theorem 2.3. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A, B \subset X$ *such that* $A \subset B \subset j - cl^*(A)$ *and* A *is* $ij - g_I$*-closed. Then* B *is* $ij - g_I$*-closed.*

Theorem 2.4. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$ *be* $i^* j - g$*-closed. Then* A *is* j*-closed if and only if* $j - cl(A) \backslash A$ *is* i^**-closed.*

Proof. If A is j-closed, then $j-cl(A)\backslash A=\varnothing$ and so $j-cl(A)\backslash A$ is i^*-closed. Conversely, Suppose $j-cl(A)\backslash A$ is i^*-closed. Then, by Proposition 2.1, $j-cl(A)\backslash A=\varnothing$ and so A is j-closed. □

Theorem 2.5. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$ *be* $ij-g_I$*-closed. Then A is* j^**-closed if and only if* $j-cl^*(A)\backslash A$ *is i-closed.*

Theorem 2.6. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *Every* i^**-open set is j-closed.*

(b) *Every subset of X is* i^*j-g*-closed.*

Proof. (a) ⇒ (b): Let $A \subset X$ and $A \subset U$, where U is i^*-open. By (a), U is j-closed and so $j-cl(A) \subset j-cl(U)=U$. This shows that A is i^*j-g-closed.

(b) ⇒ (a): Suppose U is an i^*-open set. By (b), U is i^*j-g-closed and hence $j-cl(U) \subset U$. Therefore U is j-closed. □

Corollary 2.1. *In an ideal bitopological space* (X, τ_1, τ_2, I), *if a subset A is* i^**-open and* i^*j-g*-closed, then it is j-closed.*

Theorem 2.7. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *Every i-open set is* j^**-closed.*

(b) *Every subset of X is* $ij-g_I$*-closed.*

Theorem 2.8. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$. *Then A is* $i^* j - g$ *-open if and only if* $F \subset j - \mathrm{int}(A)$ *whenever* $F \subset A$ *and F is an* i^* *-closed set.*

Proof. Suppose A is $i^* j - g$ -open and $F \subset A$, where F is i^* -closed. Then $X \setminus A \subset X \setminus F$. Since $X \setminus A$ is $i^* j - g$ -closed, we have $j - cl(X \setminus A) \subset X \setminus F$ which implies that $F \subset j - \mathrm{int}(A)$. Conversely. suppose the condition holds. Let $X \setminus A \subset U$, where U is i^* -open. Then $X \setminus U \subset A$, where $X \setminus U$ is i^* -closed and so, by hypothesis, $X \setminus U \subset j - \mathrm{int}(A)$. This implies that $j - cl(X \setminus A) \subset U$ and $X \setminus A$ is $i^* j - g$ -closed. Therefore A is $i^* j - g$ -open. □

Corollary 2.2. *If A is a j-open set in an ideal bitopological space* (X, τ_1, τ_2, I), *then A is* $i^* j - g$ *-open.*

Theorem 2.9. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$. *Then A is* $ij - g_I$ *-open if and only if* $F \subset j - \mathrm{int}^*(A)$ *whenever* $F \subset A$ *and F is an i-closed set.*

Theorem 2.10. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A, B \subset X$ *such that* $j - \mathrm{int}(A) \subset B \subset A$. *If A is* $i^* j - g$ *-open, then B is* $i^* j - g$ *-open.*

Proof. Follows from Theorem 2.2. □

Theorem 2.11. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space. For each* $x \in X$, $\{x\}$ *is either* i^* *-closed or* $i^* j - g$ *-open.*

Proof. Let $x \in X$ such that $\{x\}$ is not i^* -closed. Then X is the only i^* -open set containing $X \setminus \{x\}$, and so $X \setminus \{x\}$ is $i^* j - g$ -closed. Therefore $\{x\}$ is $i^* j - g$ -open. □

Theorem 2.12. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space. For each* $x \in X$, $\{x\}$ *is either i-closed or* $ij - g_I$ *-open.*

Theorem 2.13. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$. *If A is* $i^* j - g$ *-closed, then A can be written in the form* $A = F \setminus N$, *where F is j-closed and N contains no nonempty* i^* *-closed set.*

Proof. Suppose A is $i^* j - g$ -closed. By Proposition 2.1, $j - cl(A) \setminus A = N$ contains no nonempty i^*-closed set. Let $F = j - cl(A)$, then $F \setminus N = j - cl(A) \setminus (j - cl(A) \setminus A) = A$. □

Theorem 2.14. *Let* (X, τ_1, τ_2, I) *be an ideal bitopological space and* $A \subset X$. *If A is* $ij - g_I$ *-closed, then A can be written in the form* $A = F \setminus N$, *where F is* j^* *-closed and N contains no nonempty i-closed set.*

Now we prove two results providing characterizations of pairwise regular and pairwise normal spaces in term of $i^* j - g$ -open sets.

Theorem 2.15. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) (X, τ_1, τ_2) *is pairwise regular.*

(b) *For each i-closed set F and* $x \notin F$, *there exist i-open set U and* $i^* j - g$ *-open set V such that* $x \in U$, $F \subset V$ *and* $U \cap V = \varnothing$.

(c) *For each* $A \subset X$ *and each i-closed set F with* $A \cap F = \varnothing$, *there exist an i-open set U and* $i^* j - g$ *-open set V such that* $A \cap U \neq \varnothing$, $F \subset V$ *and* $U \cap V = \varnothing$.

Proof. (a) $\Rightarrow$ (b). Follows from the fact that every *j*-open set is $i^* j - g$ -open.

(b) $\Rightarrow$ (c): Let $A \subset X$ and F an i-closed set with $A \cap F = \varnothing$. By (b), for each $x \in A$, there exist an i-open set U and an $i^* j - g$-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \varnothing$. Thus $A \cap U \neq \varnothing$, $F \subset V$ and $U \cap V = \varnothing$.

(c) $\Rightarrow$ (a): Let F be an i-closed set and $x \notin F$. Then, by (c), there exist an i-open set U and a $i^* j - g$-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \varnothing$. Since $F \subset V$ and V is $i^* j - g$-open, by Theorem 2.7, we have $F \subset j - \mathrm{int}(V)$. Put $j - \mathrm{int}(V) = W$. Then we have $x \in U$, $F \subset W$, where U is i-open, W is j-open and $U \cap V = \varnothing$. This shows that X is pairwise regular. □

Theorem 2.16. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) (X, τ_1, τ_2) *is pairwise normal*.

(b) *For each pair of disjoint an i-closed set F and a j-closed set K, there exist disjoint* $i^* j - g$*-open set U and* $j^* i - g$*-open set V such that* $F \subset U$ *and* $K \subset V$.

(c) *For each i-closed set F and a j-open set V containing F, there exists* $i^* j - g$*-open set U such that* $F \subset U \subset i - cl(U) \subset V$.

(d) *For each i-closed set F and each* $i^* j - g$*-open set V containing F, there exists* $i^* j - g$*-open set U such that* $F \subset U \subset i - cl(U) \subset j - \mathrm{int}(V)$.

(e) *For each* $j^* i - g$*-closed set F and each j-open set V containing F, there exists* $i^* j - g$*-open set U such that* $F \subset i - cl(F) \subset U \subset i - cl(U) \subset V$.

Proof. (a) $\Rightarrow$ (b): Follows from the fact that every j-open set is $i^* j - g$-open.

(b) $\Rightarrow$ (c): Let F be i-closed and V is j-open with $F \subset V$. Then $X \setminus V$ is a j-closed set and $F \cap (X \setminus V) = \varnothing$. Then, by (b), there exist an $i^* j - g$-open set U and a $j^* i - g$-open set W such that $F \subset U$, $X \setminus V \subset W$ and $U \cap W = \varnothing$. Now W is $j^* i - g$-open and $X \setminus V \subset W$ which implies that $X \setminus V \subset i - \text{int}(W)$, by Theorem 2.8. Thus $F \subset U \subset i - cl(U) \subset i - cl(X \setminus W) \subset V$.

(c) $\Rightarrow$ (d): Let F be i-closed and $F \subset V$, where V is $i^* j - g$-open. Then $F \subset j - \text{int}(V)$ and hence, by (c), there exists $i^* j - g$-open set U such that $F \subset U \subset i - cl(U) \subset j - \text{int}(V)$.

(d) $\Rightarrow$ (e): Let F be $j^* i - g$-closed and $F \subset V$, where V is j-open. Then $i - cl(F) \subset V$ and, by (d), there exists $i^* j - g$-open set U such that $F \subset U \subset i - cl(U) \subset j - \text{int}(V) = V$.

(e) $\Rightarrow$ (a): Let F be an i-closed set and K a j-closed set such that $F \cap K = \varnothing$. Then $F \subset X \setminus K$ and F is $j^* i - g$-closed, therefore, by (e), there exists $i^* j - g$-open set U such that $F \subset U \subset i - cl(U) \subset X \setminus K$. Since F is i-closed and $F \subset U$, where U is $i^* j - g$-open, by Theorem 2.8, we have $F \subset j - \text{int}(U)$. Put $j - \text{int}(U) = G$ and $X \setminus i - cl(U) = H$. Then G is j-open, H is i-open, $F \subset G$, $K \subset H$ and $G \cap H = \varnothing$. This means that (X, τ_1, τ_2) is pairwise normal. $\square$

3. Pairwise $* - R_0$ and Pairwise $I - R_0$ Spaces

The notion of pairwise R_0- spaces first has been studied in [9]. In this section we introduce two kinds of separation axioms one being stronger and another weaker than pairwise R_0-spaces.

Definition 3.1. An ideal bitopological space (X, τ_1, τ_2, I) is called

(1) pairwise $*-R_0$- space if for every i^*-open set U and each $x \in U$, we have $j-cl(x) \subset U$. Equivalently every singleton is $i^* j-g$-closed.

(2) pairwise $I-R_0$-space if for every *i*-open set U and each $x \in U$, we have $(x)^{*j} \subset U$. Equivalently every singleton is $ij-g_I$-closed.

Remark 3.1 (a) Since every *i*-open set is i^*-open, it follows that every pairwise $*-R_0$-space is pairwise R_0. But the converse may not be true.

(b) Since every $ij-g$-closed set is $ij-g_I$-closed, it follows that every pairwise R_0 space is pairwise $I-R_0$.

The following three theorems give some characterizations of pairwise $*-R_0$ spaces.

Theorem 3.1. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *X is pairwise* $*-R_0$.

(b) *For each* i^**-closed set F and* $x \notin F$, *there exists a j-open set U such that* $F \subset U$ *and* $x \notin U$.

(c) *For each* i^**-closed set F and* $x \notin F$, *we have* $F \cap j-cl(\{x\}) = \varnothing$.

(d) *For any two distinct points x and y of X, we have* $x \notin i-cl^*(\{y\})$ *implies* $j-cl(\{x\}) \cap i-cl^*(\{y\}) = \varnothing$.

Proof. (a) $\Rightarrow$ (b): Let F be i^*-closed and $x \notin F$. Then $x \in X \setminus F$ and so, by (a), $j-cl(\{x\}) \subset X \setminus F$ which implies that $F \subset X \setminus j-cl(\{x\})$. Put $X \setminus j-cl(\{x\}) = U$. Then U is a *j*-open set such that $F \subset U$ and $x \notin U$.

(b) $\Rightarrow$ (c): Let F be i^*-closed and $x \notin F$. Then, by (b), there exists a j-open set U such that $F \subset U$ and $x \notin U$. Thus $U \cap j - cl(\{x\}) = \varnothing$ and hence $F \cap j - cl(\{x\}) = \varnothing$.

(c) $\Rightarrow$ (d): It is clear.

(d) $\Rightarrow$ (a): Let U be an i^*-open set and $x \in U$. Then for each $y \notin U$, $x \notin i - cl^*(\{y\})$ and hence, by (d), $j - cl(\{x\}) \cap i - cl^*(\{y\}) = \varnothing$ for each $y \notin U$. Then $j - cl(\{x\}) \cap [\bigcup (i - cl^*(\{y\}) : y \in X \setminus U)] = \varnothing$. Now U is i^*-open and $y \in X \setminus U$ which implies that $\{y\} \subset i - cl^*(\{y\}) \subset i - cl^*(X \setminus U)$ $= X \setminus U$. Thus $X \setminus U = \bigcup\{i - cl^*(\{y\}) : y \in X \setminus U\}$. Therefore $j - cl(\{x\}) \cap X \setminus U = \varnothing$, i.e., $j - cl(\{x\}) \subset U$. Hence X is pairwise $* - R_0$. □

Theorem 3.2. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *X is pairwise* $* - R_0$.

(b) *For each nonempty subset A of X and i^*-open set U with $A \cap U \neq \varnothing$, there exists a j-closed set F such that $A \cap F \neq \varnothing$ and $F \subset U$.*

(c) *For each i^*-open set U, we have* $U = \bigcup\{F : F \text{ is } j\text{-closed}, F \subset U\}$.

(d) *For each i^*-closed set F, we have* $F = \bigcap\{U : U \text{ is } j\text{-closed}, F \subset U\}$.

Proof. (a) $\Rightarrow$ (b): Let $A \subset X$ be such that $A \cap U \neq \varnothing$, where U is i^*-open set. Now $x \in U$ implies that $j - cl(\{x\}) \subset U$. Let $j - cl(\{x\}) = F$, then F is j-closed with $A \cap F \neq \varnothing$ and $F \subset U$.

(b) $\Rightarrow$ (c): Let U be i^*-open. Now $\bigcup \{F : F$ is j-closed, $F \subset U\} \subset U$. Let $x \in U$, by (b) there exists a j-closed F containing x and $F \subset U$. Thus

$x \in F \subset \bigcup\{K : K$ is j-closed, $K \subset U\}$. Therefore, for each i^*-open set U, $U = \bigcup\{F : F$ is j-closed, $F \subset U\}$.

(c) $\Rightarrow$ (d): It is clear.

(d) $\Rightarrow$ (a): Let U be i^*-open and $x \in U$. We need to show that $j - cl(\{x\}) \subset U$. If not, let $y \in j - cl(\{x\})$ such that $y \notin U$. Since U is i^*-open neighborhood of each of its points, it follows that $i - cl^*(\{y\}) \cap U = \varnothing$. Now $i - cl^*(\{y\})$ is i^*-closed and hence, by (d), $i - cl^*(\{y\}) = \bigcap \{V : V$ is j-open and $i - cl^*(\{y\}) \subset V\}$. Therefore, $(\bigcap \{V : V$ is j-open and $i - cl^*(\{y\}) \subset V\}) \cap U = \varnothing$ and this means that $x \notin \bigcap \{V : V$ is j-open and $i - cl^*(\{y\}) \subset V\}$. Then there exists a j-open set V such that $x \notin V$ and $i - cl^*(\{y\}) \subset V$. As V is a j-open set containing y such that $x \notin V$, we have $y \notin j - cl(\{x\})$, a contradiction. □

Theorem 3.3. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *X is pairwise* $* -R_0$.

(b) *For any two distinct points* $x, y \in X$, *we have* $x \in i - cl^*(\{y\})$ *if and only if* $y \in j - cl(\{x\})$.

Proof. (a) $\Rightarrow$ (b): Let $x \in i - cl^*(\{y\})$ and U a j-open set containing y. By (a), we have $i - cl(\{y\}) \subset U$. Thus $i - cl^*(\{y\}) \subset i - cl(\{y\}) \subset U$. Therefore $x \in U$ which implies that $y \in j - cl(\{x\})$. Again, let $y \in j - cl(\{x\})$ and U be any i^*-open set containing x. Then, by (a), we have $j - cl(\{x\}) \subset U$ which implies that $y \in U$ and thus $x \in i - cl^*(\{y\})$.

(b) $\Rightarrow$ (a): Let U be an i^*-open set and $x \in U$. If $y \notin U$, then $x \notin i - cl^*(\{y\})$ and hence $y \notin j - cl(\{x\})$. This shows that $j - cl(\{x\}) \subset U$ and therefore X is pairwise $* - R_0$ space. □

The following theorem gives some characterizations of pairwise $I - R_0$ spaces.

Theorem 3.4. *For an ideal bitopological space* (X, τ_1, τ_2, I), *the following statements are equivalent*:

(a) *X is a pairwise $I - R_0$ space.*

(b) *For each i-open set U and $x \in U$, we have $j - cl^*(\{x\}) \subset U$.*

(c) *For each i-closed set F and $x \notin F$, there exists a j^*-open set U such that $F \subset U$ and $x \notin U$.*

(d) *For each i-closed set F and $x \notin F$, we have $F \cap j - cl^*(\{x\}) = \emptyset$.*

(e) *For any two distinct points $x, y \in X$, if $x \notin i - cl(\{y\})$, then $j - cl^*(\{x\}) \cap i - cl(\{y\}) = \emptyset$.*

Proof. (a) $\Rightarrow$ (b): Let U be an i-open set and $x \in U$. Then $(\{x\})^{*j} \subset U$. Therefore $j - cl^*(\{x\}) = x \cup x^{*j} \subset U$.

(b) $\Rightarrow$ (c): Let F be an i-closed set and $x \notin F$. Then $x \in X \setminus F$ and, by (b), $j - cl^*(\{x\}) \subset X \setminus F$ and therefore $F \subset X \setminus j - cl^*(\{x\})$. Put $U = X \setminus j - cl^*(\{x\})$. Then U is j^*-open such that $F \subset U$ and $x \notin U$.

(c) $\Rightarrow$ (d): Let F be an i-closed set and $x \notin F$. By (c), there exists a j^*-open set U such that $F \subset U$ and $x \notin U$. Therefore $U \cap j - cl^*(\{x\}) = \emptyset$ and consequently $F \cap j - cl^*(\{x\}) = \emptyset$.

(d) $\Rightarrow$ (e): it is clear.

(e) $\Rightarrow$ (a): Let U be an i-open set and $x \in U$. Then, for each $y \in X \setminus U$ we have $x \notin i - cl(\{y\})$. Therefore, by (e), $j - cl^*(\{x\}) \cap i - cl(\{y\}) = \phi$ for each $y \in X \setminus U$. This implies that $j - cl^*(\{x\}) \cap (\bigcup \{i - cl(\{y\}) : y \in X \setminus U) = \varnothing$. Therefore $j - cl^*(\{x\}) \cap (X \setminus U) = \varnothing$ which implies that $j - cl^*(\{x\}) \subset U$ and so X is a pairwise $I - R_0$ space. □

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