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Lower interval-valued intuitionistic fuzzy separation axioms

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Abstract

In this paper, some lower separation axioms in interval-valued intuitionistic fuzzy topological spaces are proposed. Furthermore, we pay some attention in determining the corresponding variations of them in intuitionistic fuzzy topological spaces. The four different types of the concepts of $R_\circ$-ness, $T_\circ$-ness and T_1-ness separation axioms are developed and the corresponding R_1-ness and T_2-ness are defined. Also, some conclusions by establishing some results are drawn and several examples for illustration are provided.

Keywords: Interval-valued intuitionistic fuzzy set, topology; fuzzy set, fuzzy topology, Lower separation axioms.
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1. Introduction

After introducing the concept of fuzzy sets [1], researchers have been studying the generalizations of the notion of fuzzy sets. In 1973 and 1975, the notion of a interval-valued fuzzy set has been introduced by Grattan-Guiness [2], Jahn [3] and Zadeh [4] in order to formalize the vagueness. In 1985, the interval representation of language value was discussed by Schwarz [5]. In 1986, interval-valued fuzzy sets which base on the normal forms were studied by Turksen [6]. In 1987, a method about interval-valued fuzzy inference was given by Gorzalczany [7]. The basic research of interval-valued fuzzy sets was also rigorously studied [8, 9, 10, 11]. In 1983, Atanassov proposed the concept of intuitionistic fuzzy set [12], which is a generalization of the notion of fuzzy set. Some basic results on intuitionistic fuzzy sets were published in [13, 14], and the book authored by Atanassov [15] provided a comprehensive coverage of virtually all results in the area of the

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theory and applications of intuitionistic fuzzy sets. Intuitionistic fuzzy sets were further extended by Coskun [16] and Hung and Wu [17]. Çoker and Demirci [18] defined and studied the basic concept of intuitionistic fuzzy point. Later Çoker [19, 20] constructed the fundamental theory on intuitionistic fuzzy topological spaces, and further studies ensued on compactness, connectedness and continuity in intuitionistic fuzzy topological spaces and intuitionistic gradation of neighborhoodness [21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Lupiañez defined the notion of Hausdorffness [31] and obtained some results of nets and filters in intuitionistic fuzzy topological spaces [32]. Lee and Lee [33] showed that the category of fuzzy topological spaces in the sense of Chang [34] (which redefined by Lowen [35] and now known as a stratified fuzzy topology) is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces, while Wang and He [36] showed that every intuitionistic fuzzy set may be regarded as an L-fuzzy set [37] for some appropriate lattice L. Atanassov and Gargov [38] presented the basic preliminaries of interval-valued intuitionistic fuzzy set and some types of operators were defined over interval-valued intuitionistic fuzzy set [39, 40, 41]. Mondal and Samanta [42] defined topology of interval-valued intuitionistic fuzzy sets and studied some of its properties. There are many applications for intuitionistic fuzzy sets and interval-valued fuzzy sets. Rizk-Allah et.al. [43] presented an integrated approach based on a dynamic programming approach and an intuitionistic fuzzy set for solving multi-objective optimization problems. Singh et.al. [44] introduced a new multi-attribute information classification method by employing intuitionistic fuzzy set (IFS) approach. New results on Riesz spaces to fuse uncertain data of Atanassov intuitionistic fuzzy sets were also introduced by Sesma-Sara et.al [45]. Symmetrical intuitionistic fuzzy Bonferroni mean (SIFBM) operator and weighted SIFBM operator were developed and some desirable properties of them were provided by Yang et.al. [46]. Pretopological and topological operators were introduced based on partially continuous linear transformations for intuitionistic fuzzy sets by Marinov and Atanassov [47] They turn out to be a generalization of the topological operators for intuitionistic fuzzy sets. On the other hand, it was a generalization of the fuzzy set pretopological operators introduced by Wenzhong and Kimfung [48]. Bustince at. al. [49] constructed similarity measures between interval-valued fuzzy sets in such a way that a total order for intervals (not only partial) was used and the widths of intervals were considered. Zulkifli et.al [50] proposed the notion of interval-valued intuitionistic fuzzy vague sets (IVIFVS) where membership and non-membership of interval-valued intuitionistic fuzzy sets were combined with truth membership and false membership of vague sets. Wei et.al [51] proposed the novel generalized exponential intuitionistic fuzzy entropy (GIFE) and generalized exponential interval valued intuitionistic fuzzy entropy (GIVIFE) with interval area whereas a complete representation of n-polygonal interval-valued fuzzy sets and numbers was provided and the properties of the topological space of n-polygonal interval-valued fuzzy numbers were studied by Suo et.al. [52]. On the other hand, Du and Yuan [53] proposed some new Bonferroni mean (BM) operators under the interval-valued intuitionistic 2-tuple linguistic environment and applied them to multi-attribute group decision-making (MAGDM) problems. Li et.al. [54] gave a representation and aggregation of multi-source information of modern smart cities based on the intuitionistic polygonal fuzzy set. A characterization of the concept of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets and their set-operations were given by Sayed et.al. [55]. It was proven that for an interval-valued intuitionistic fuzzy topology there exist four fuzzy topologies in the sense of Chang [34]. Also, the concepts of pre-suitable and suitable interval-valued intuitionistic fuzzy topologies were investigated, and some basic notions of these concepts were studied. However, these studies did not discuss the fuzzy separation axioms of the interval-valued intuitionistic fuzzy set and those of intuitionistic fuzzy topological space. Hence we will propose several separation axioms on interval-valued intuitionistic fuzzy topological space to further extend the frontiers of interval-valued intuitionistic fuzzy set.
In our ensuing discussions, several defined terms shall be used such as fuzzy topology (F topology) [34], intuitionistic fuzzy set (IF set) [13], intuitionistic fuzzy topology (IF topology) [19, 20, 21, 22, 23], interval-valued intuitionistic fuzzy set (IIF set) [38, 39, 42] and interval-valued intuitionistic fuzzy topology (IIF topology) [42].

The rest of this paper is organized as follows. The next section contains some necessary concepts and properties. In Section 3, four different types of the concepts of $R_\circ$-ness, $T_\circ$-ness and T_1-ness separation axioms will be introduced in IIF topological spaces. In Section 4, the corresponding R_1-ness and T_2-ness will be defined. In Section 5, more results on lower interval-valued intuitionistic fuzzy separation axioms will be developed and variations of lower separation axioms in IF topological spaces will be formulated. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research. Consequently, we have filled the gap of the separation axioms on IIF topological space by introducing four different types of separation axioms, along with formulating variations of lower separation axioms in IF topological space.

2. Preliminaries

In the following we summarize some definitions and results from Sayed et. al. [55], where $\mathrm{II}(\mathcal{X})$ is the set of all interval-valued intuitionistic fuzzy sets of $\mathcal{X}$ and $\mathrm{IT}(\mathcal{X})$ is the set of all interval-valued intuitionistic fuzzy topologies of $\mathcal{X}$.

Remark 2.1. (See Theorems 2.2 and 2.7 in [55])

(1) It was proved that the concept of intuitionistic fuzzy set of $\mathcal{X}$ can be determined in a uniquely manner as an ordered pair $(\mu,\ \nu)$ of fuzzy sets of $\mathcal{X}$ such that $\mu \leq \nu$.
(2) It was proved that the concept of interval-valued intuitionistic fuzzy set of $\mathcal{X}$ can be determined in a uniquely manner as an ordered quadrable $(\kappa, \lambda, \mu,\ \nu)$ of fuzzy sets of $\mathcal{X}$ such that $\kappa \leq \lambda \leq \nu \leq \mu$.

Theorem 2.2. *(See Theorems 2.8-2.10 in [55]) Let* $f : \mathcal{X} \to \mathcal{Y}$ *and* $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4), \mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4) \in \mathrm{II}(\mathcal{X}),\ \{\mathcal{A}^j : j \in \Lambda\} \subseteq \mathrm{II}(\mathcal{X}),\ \mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4) \in \mathrm{II}(\mathcal{Y})$. *Then:*

(1) $\mathcal{A} \subseteq \mathcal{B}$ *if and only if* $\mathcal{A}_i \leq \mathcal{B}_i$ *for each* $i \in \{1, 2, 3, 4\}$;
(2) $\mathcal{A} = \mathcal{B}$ *if and only if* $\mathcal{A}_i = \mathcal{B}_i$ *for each* $i \in \{1, 2, 3, 4\}$;
(3) $\bigcup_{j\in\Lambda} \mathcal{A}^j = \left(\bigcup_{j\in\Lambda} \mathcal{A}_1^j, \bigcup_{j\in\Lambda} \mathcal{A}_2^j, \bigcup_{j\in\Lambda} \mathcal{A}_3^j, \bigcup_{j\in\Lambda} \mathcal{A}_4^j\right)$;
(4) $\bigcap_{j\in\Lambda} \mathcal{A}^j = \left(\bigcap_{j\in\Lambda} \mathcal{A}_1^j, \bigcap_{j\in\Lambda} \mathcal{A}_2^j, \bigcap_{j\in\Lambda} \mathcal{A}_3^j, \bigcap_{j\in\Lambda} \mathcal{A}_4^j\right)$;
(5) $\mathcal{A}^c = (\mathcal{A}_3^c, \mathcal{A}_4^c, \mathcal{A}_1^c, \mathcal{A}_2^c)$;
(6) $\tilde{1} = (1_\mathcal{X}, 1_\mathcal{X}, 1_\mathcal{X}, 1_\mathcal{X})$;
(7) $\tilde{0} = (1_\phi, 1_\phi, 1_\phi, 1_\phi)$;
(8) $\Box\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_2)$;
(9) $\Diamond\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_4, \mathcal{A}_3, \mathcal{A}_4)$;
(10) $f(\mathcal{A}) = (f(\mathcal{A}_1), f(\mathcal{A}_2), f(\mathcal{A}_3), f(\mathcal{A}_4))$;
(11) $f^{-1}(\mathcal{C}) = (f^{-1}(\mathcal{C}_1), f^{-1}(\mathcal{C}_2), f^{-1}(\mathcal{C}_3), f^{-1}(\mathcal{C}_4))$;
(12) $\bigcup_{j\in\Lambda} (\Box\mathcal{A}^j) = \Box(\bigcup_{j\in\Lambda} \mathcal{A}^j)$;
(13) $\bigcap_{j\in\Lambda} (\Box\mathcal{A}^j) = \Box(\bigcap_{j\in\Lambda} \mathcal{A}^j)$;
(14) $\bigcup_{j\in\Lambda} (\Diamond\mathcal{A}^j) = \Diamond(\bigcup_{j\in\Lambda} \mathcal{A}^j)$;
(15) $\bigcap_{j\in\Lambda} (\Diamond\mathcal{A}^j) = \Diamond(\bigcap_{j\in\Lambda} \mathcal{A}^j)$;
(16) $f(\Box\mathcal{A}) = \Box f(\mathcal{A})$;
(17) $f(\Diamond\mathcal{A}) = \Diamond f(\mathcal{A})$;
(18) $f^{-1}(\Box\mathcal{C}) = \Box f^{-1}(\mathcal{C})$;
(19) $f^{-1}(\Diamond\mathcal{C}) = \Diamond f^{-1}(\mathcal{C})$.

Remark 2.3. (See Theorems 2.3-2.5 in [55]) A similar result of the above theorem in IF setting was given.

Definition 2.4. (See Definition 5.1 in [55]) Let $\tau \in \mathrm{IT}(\mathcal{X}),\ \sigma \in \mathrm{IT}(\mathcal{Y})$ and $f : \mathcal{X} \to \mathcal{Y}$ be a function. Then

(1) f is said to be interval-valued intuitionistic fuzzy continuous (IIF continuous for short) if and only if for each $\mathcal{B} \in \sigma$, $f^{-1}(\mathcal{B}) \in \tau$ [42] .
(2) f is said to be an interval-valued intuitionistic fuzzy homeomorphism (IIF homeomorphism for short) if and only if f is a bijection, and f and f^{-1} are IIF continuous.
(3) A property P of IIF topology is called IIF topological property if it preserved under any IIF homeomorphism.

Definition 2.5. (See Definition 3.1 in [55])

(1) An ordered quadrable $(\tau_1, \tau_2, \tau_3, \tau_4)$ of fuzzy topologies on $\mathcal{X}$ is called a pre-suitable interval-valued intuitionistic fuzzy topology (PIIF topology for short) on $\mathcal{X}$. The family of all pre-suitable interval-valued intuitionistic fuzzy topologies on $\mathcal{X}$ will be denoted by PT($\mathcal{X}$).

(2) A pre-suitable interval-valued intuitionistic fuzzy topology $(\tau_1, \ \tau_2, \tau_3, \tau_4)$ on $\mathcal{X}$ is called suitable interval-valued intuitionistic fuzzy topology (SIIF topology for short) on $\mathcal{X}$ if $\tau_1 \subseteq \ \tau_2 \subseteq \tau_4 \subseteq \tau_3$. The family of all suitable interval-valued intuitionistic fuzzy topologies on $\mathcal{X}$ will be denoted by ST($\mathcal{X}$).

Theorem 2.6. *(See Theorem 3.2 in [55])*

(1) For each $\tau \in \mathrm{IT}(\mathcal{X})$, there exist four fuzzy topologies on $\mathcal{X}$ defined as $\tau_i = \{\mathcal{A}_i : \mathcal{A} \in \tau\}$ for each $i \in \{1, 2, 3, 4\}$ and $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \in \mathrm{II}(X)$, i.e., there exists a function ξ from $\mathrm{IT}(\mathcal{X})$ into $\mathrm{PT}(\mathcal{X})$ such that $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$.
(2) There exists a function $\eta : \mathrm{PT}(\mathcal{X}) \to \mathrm{IT}(\mathcal{X})$ defined as follows:
$\eta((\theta_1, \theta_2, \theta_3, \theta_4)) = \{\mathcal{A} : \mathcal{A} \in \mathrm{II}(\mathcal{X}), \mathcal{A}_i \in \theta_i \ , i = \{1, 2, 3, 4\}\} = \theta.$

Remark 2.7. (See Remark 2.11 in [55])

(1) An IIF set $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ is identified with an IF set if and only if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{A}_3 = \mathcal{A}_4$.
(2) An IIF set $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ is identified with a fuzzy set if and only if $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4$.
(3) An IIF set $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ is identified with an ordinary set if and only if $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4$ and $\mathcal{A}(\mathcal{X}) \subseteq \{0, 1\}$.

Theorem 2.8. *(See Theorem 2.12 in [55]) Let τ be an IIF topology on a nonempty set $\mathcal{X}$ and let τ_{IF}(resp. τ_F, τ_O) $= \{\mathcal{A} : \mathcal{A} \in \tau$ and $\mathcal{A}$ is identified with an intuitionistic fuzzy set (resp. fuzzy set; ordinary set)$\}$. Then τ_{IF} is an IF topology, τ_F is a fuzzy topology and τ_O is a topology on $\mathcal{X}$.*

3. T_0-, T_1- and R_0- IIF topological spaces

In this section we present four different types of the concepts of R_0-ness, T_0- ness and T_1-ness separation axioms in IIF topological spaces and study some of their properties.

First, we introduce a new concept of IIF points.

Definition 3.1. Let $\mathcal{X}$ be a nonempty set.

(1) A fuzzy point of type $\mathcal{M}$ is a fuzzy set denoted and defined by:
$t1_x(y) = t$ when $y = x$ and $t1_x(y) = 0$ when $y \neq x$, where $x \in \mathcal{X}$ (x is called the support of $t1_x$) and $t \in [0, 1]$.
(2) A fuzzy point of type $\mathcal{K}$ is a fuzzy set denoted and defined by:
$t1_x(y) = t$ when $y = x$ and $t1_x(y) = 0$ when $y \neq x$, where $x \in \mathcal{X}$ and $t \in [0, 1)$.
(3) Let $\mathcal{A}$ be a fuzzy set in $\mathcal{X}$. Then:
  (i) $t1_x \tilde{\in} \mathcal{A}$ if and only if $\mathcal{A}(x) \geq t$;
  (ii) $t1_x \dot{\in} \mathcal{A}$ if and only if $\mathcal{A}(x) > t$;

Note: If $t1_x \dot{\in} \mathcal{A}$, then $t1_x \tilde{\in} \mathcal{A}$, where $\mathcal{A}$ is a fuzzy set in $\mathcal{X}$.

Definition 3.2. (1) An interval-valued intuitionistic fuzzy point (IIF point for short) in $\mathcal{X}$ (in the sense of Mondal and Samanta) of type $\mathcal{M}$ can be uniquely determined as an order quadrable $(t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$ of fuzzy points of type $\mathcal{M}$ such that $t_1 \leqslant t_2 \leqslant t_4 \leqslant t_3$ and $t_2 > 0$. In this case, $(t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x) \tilde{\in} \mathcal{A}$ if and only if $t_i 1_x \tilde{\in} \mathcal{A}_i$ for each $i \in \{1, 2, 3, 4\}$, where $\mathcal{A} \in \mathrm{II}(\mathcal{X})$.

(2) An IIF point of type $\mathcal{K}$ in a nonempty set $\mathcal{X}$ is an order quadrable $(t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$ of fuzzy points such that $t_1 1_x, t_2 1_x$ and $t_4 1_x$ are fuzzy points of type $\mathcal{K}$ and $t_3 1_x$ is either of type $\mathcal{M}$ or of type $\mathcal{K}$. In this case, we write $(t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x) \dot{\in} \mathcal{A}$ if and only if $t_i 1_x \tilde{\in} \mathcal{A}_i, i \in \{1, 3\}$ and $t_i 1_x \dot{\in} \mathcal{A}_i, i \in \{2, 4\}$, where $\mathcal{A} \in \mathrm{II}(\mathcal{X})$.

Note: If $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$ is an IIF point in $\mathcal{X}$ and $\mathcal{A} \in \mathrm{II}(\mathcal{X})$, we write $\mathcal{A}(p) = 0$ if and only if $\mathcal{A}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$. Note that $\mathcal{A}(p) = 0$ implies that $p \tilde{\notin} \mathcal{A}$ if p is of type $\mathcal{M}$ and $p \dot{\notin} \mathcal{A}$ if p is of type $\mathcal{K}$, but the converse of this implication may not be true.

Definition 3.3. An IIF topological space $(\mathcal{X}, \tau)$ is said to be:

(1) $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exists $\mathcal{G} \in \tau$ such that either $\mathcal{G}_i(x) \geq t_i$ and $\mathcal{G}_i(y) = 0$ for each $i \in \{1, 2, 3, 4\}$; or $\mathcal{G}_i(y) \geq t_i$ and $\mathcal{G}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$ and $t_2 > 0$;

(2) $\mathrm{IIF}(\mathcal{M}) - T_0$ if and only if $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}) - T_0$

(3) $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exists $\mathcal{G} \in \tau$ such that either $\mathcal{G}_i(x) \geq t_i (i \in \{1, 3\})$, $\mathcal{G}_i(x) > t_i (i \in \{2, 4\})$ and $\mathcal{G}_i(y) = 0$ for each $i \in \{1, 2, 3, 4\}$; or $\mathcal{G}_i(y) \geq t_i (i \in \{1, 3\})$, $\mathcal{G}_i(y) > t_i (i \in \{2, 4\})$ and $\mathcal{G}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$ and $t_4 < 1$;

(4) $\mathrm{IIF}(\mathcal{K}) - T_0$ if and only if for every two distinct points x, y in $\mathcal{X}$, and for every two ordered quadrable (t_1, t_2, t_3, t_4) and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ there exists $\mathcal{G} \in \tau$ such that either $\mathcal{G}_i(x) \geq t_i (i \in \{1, 3\})$, $\mathcal{G}_i(x) > t_i (i \in \{2, 4\})$ and $\mathcal{G}_i(y) = 0$ for each $i \in \{1, 2, 3, 4\}$; or $\mathcal{G}_i(y) \geq \alpha_i (i \in \{1, 3\})$, $\mathcal{G}_i(y) > \alpha_i (i \in \{2, 4\})$ and $\mathcal{G}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i, \alpha_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$, $\alpha_1 \leq \alpha_2 \leq \alpha_4 \leq \alpha_3$ and $t_4 \vee \alpha_4 < 1$.

Proposition 3.4. *Let $(\mathcal{X}, \tau)$ be an IIF topological space.*

(1) $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ if and only if for every two IIF points $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$, $q = (t_1 1_y, t_2 1_y, t_3 1_y, t_4 1_y)$ of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G} \in \tau$ such that either $p \tilde{\in} \mathcal{G}$ and $\mathcal{G}(q) = 0$; or $q \tilde{\in} \mathcal{G}$ and $\mathcal{G}(p) = 0$;

(2) $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M}) - T_0$ if and only if for every two IIF points p, q of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G} \in \tau$ such that either $p \tilde{\in} \mathcal{G}$ and $\mathcal{G}(q) = 0$; or $q \tilde{\in} \mathcal{G}$ and $\mathcal{G}(p) = 0$;

(3) $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$ if and only if for every two IIF points $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$, $q = (t_1 1_y, t_2 1_y, t_3 1_y, t_4 1_y)$ of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G} \in \tau$ such that either $p \dot{\in} \mathcal{G}$ and $\mathcal{G}(q) = 0$; or $q \dot{\in} \mathcal{G}$ and $\mathcal{G}(p) = 0$;

(4) $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{K}) - T_0$ if and only if for every two IIF points p, q of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G} \in \tau$ such that either $p \dot{\in} \mathcal{G}$ and $\mathcal{G}(q) = 0$; or $q \dot{\in} \mathcal{G}$ and $\mathcal{G}(p) = 0$.

Proof. The proofs are straightforward from Definition 3.3. □

Proposition 3.5. *Let $(\mathcal{X}, \tau)$ be an IIF topological space.*

(1) If $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M}) - T_0$ (resp. $\mathrm{IIF}(\mathcal{K}) - T_0$), then $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ (resp. $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$);

(2) If $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M}) - T_0$, then $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{K}) - T_0$;

(3) If $t_i \leq \alpha_i$ for each $i \in \{1, 2, 3, 4\}$, then $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - T_0$ (resp. $\mathrm{IIF}(\mathcal{K})(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - T_0$) implies $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ (resp. $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$);

(4) If $t_4 \neq 1$ and $t_2 \neq 0$, then $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ implies $(\mathcal{X}, \tau)$ is $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$.

Proof. The proofs follow directly from Definition 3.3 and Proposition 3.1. □

Definition 3.6. An IIF topological space $(\mathcal{X}, \tau)$ is said to be be:

(1) IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) \wedge \mathcal{H}_i(x) \geq t_i$ and $\mathcal{G}_i(y) = \mathcal{H}_i(y) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$ and $t_2 > 0$;
(2) IIF$(\mathcal{M}) - T_1$ if and only if $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}) - T_1$
(3) IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) \wedge \mathcal{H}_i(y) \geq t_i (i \in \{1, 3\})$, $\mathcal{G}_i(x) \wedge \mathcal{H}_i(y) > t_i$ $(i \in \{2, 4\})$ and $\mathcal{G}_i(y) = \mathcal{H}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$ and $t_4 < 1$;
(4) IIF$(\mathcal{K}) - T_1$ if and only if for every two distinct points x, y in $\mathcal{X}$, and for every two ordered quadrable (t_1, t_2, t_3, t_4) and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) \geq t_i, \mathcal{H}_i(y) \geq \alpha_i(i \in \{1, 3\})$, $\mathcal{G}_i(x) > t_i, \mathcal{H}_i(y) > \alpha_i(i \in \{2, 4\})$, and $\mathcal{G}_i(y) = \mathcal{H}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$, where $t_i, \alpha_i \in [0, 1]$ for each $i \in \{1, 2, 3, 4\}$, $t_1 \leq t_2 \leq t_4 \leq t_3$, $\alpha_1 \leq \alpha_2 \leq \alpha_4 \leq \alpha_3$ and $t_4 \vee \alpha_4 < 1$.

Proposition 3.7. *Let $(\mathcal{X}, \tau)$ be an* IIF *topological space.*

(1) $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ *if and only if for every two* IIF *points $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$, $q = (t_1 1_y, t_2 1_y, t_3 1_y, t_4 1_y)$ of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $p \tilde{\in} \mathcal{G}$, $q \tilde{\in} \mathcal{H}$, $\mathcal{G}(q) = 0$ and $\mathcal{H}(p) = 0$.*
(2) $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M}) - T_1$ *if and only if for every two* IIF *points p, q of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $p \tilde{\in} \mathcal{G}$, $q \tilde{\in} \mathcal{H}$ $\mathcal{G}(q) = 0$ and $\mathcal{H}(p) = 0$.*
(3) $(\mathcal{X}, \tau)$ is IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$ *if and only if for every two* IIF *points $p = (t_1 1_x, t_2 1_x, t_3 1_x, t_4 1_x)$, $q = (t_1 1_y, t_2 1_y, t_3 1_y, t_4 1_y)$ of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $p \dot{\in} \mathcal{G}$, $q \dot{\in} \mathcal{H}$, $\mathcal{G}(q) = 0$ and $\mathcal{H}(p) = 0$.*
(4) $(\mathcal{X}, \tau)$ is IIF$(\mathcal{K}) - T_1$ *if for every two* IIF *points p, q of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $p \dot{\in} \mathcal{G}$, $q \dot{\in} \mathcal{H}$, $\mathcal{G}(q) = 0$ and $\mathcal{H}(p) = 0$.*

Proof. The proofs are attainable directly from Definition 3.4. □

Proposition 3.8. *Let $(\mathcal{X}, \tau)$ be an* IIF *topological space.*

(1) If $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M}) - T_1$ *(resp.* IIF$(\mathcal{K}) - T_1$*), then $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ *(resp.* IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$*).*
(2) If $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M}) - T_1$*, then $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{K}) - T_1$.
(3) If $t_i \leq \alpha_i$ for each $i \in \{1, 2, 3, 4\}$, then $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - T_1$ *(resp.* IIF$(\mathcal{K})(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - T_1$*) implies $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ *(resp.* IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$*).*
(4) If $t_4 \neq 1$ and $t_2 \neq 0$, then $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ *implies $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$

Proof. The proofs follow from Definition 3.4 and Proposition 3.3. □

Proposition 3.9. *Let $(\mathcal{X}, \tau)$ be an* IIF *topological space. If $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_1$ *(resp.* IIF$(\mathcal{M}) - T_1$, IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_1$ *and* IIF$(\mathcal{K}) - T_1$*), then $(\mathcal{X}, \tau)$ is* IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_0$ *(resp.* IIF$(\mathcal{M}) - T_0$, IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_0$ *and* IIF$(\mathcal{K}) - T_0$*)*

Proof. The proof is straightforward from Propositions 3.1 and 3.3. □

Definition 3.10. An IIF topological space $(\mathcal{X}, \tau)$ is said to be:

(1) IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - R_0$ if and only if for every two distinct points x, y in $\mathcal{X}$, whenever there exists $\mathcal{G} \in \tau$ such that $\mathcal{G}_i(x) \geq t_i$ and $\mathcal{G}_i(y) = 0$ for each $i \in \{1, 2, 3, 4\}$, then there exists $\mathcal{H} \in \tau$ such that $\mathcal{H}_i(y) \geq t_i$ and $\mathcal{H}_i(x) = 0$ for each $i \in \{1, 2, 3, 4\}$;
(2) IIF$(\mathcal{M}) - R_0$ if and only if $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}, 1_{\mathcal{X}}) - R_0$

(3) $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-R_0$ if and only if for every two distinct points x,y in $\mathcal{X}$, whenever there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i(i\in\{1,3\})$, $\mathcal{G}_i(x)>t_i(i\in\{2,4\})$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq t_i(i\in\{1,3\})$, $\mathcal{H}_i(y)>t_i(i\in\{2,4\})$ and $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$;

(4) $\text{IIF}(\mathcal{K})-R_0$ if and only if for every two distinct points x,y in $\mathcal{X}$ and for every two ordered quadrable (t_1,t_2,t_3,t_4) and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, whenever there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i(i\in\{1,3\})$, $\mathcal{G}_i(x)>t_i(i\in\{2,4\})$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq\alpha_i(i\in\{1,3\})$, $\mathcal{H}_i(y)>\alpha_i(i\in\{2,4\})$ and $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$.

Proposition 3.11. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space.*

(1) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_0$ if and only if for every two IIF *points $p=(t_11_x,t_21_x,t_31_x,t_41_x)$, $q=(t_11_y,t_21_y,t_31_y,t_41_y)$ of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\tilde{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\tilde{\in}\mathcal{H}$ and $\mathcal{H}(p)=0$;*

(2) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})-R_0$ if and only if for every two IIF *points p,q of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\tilde{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\tilde{\in}\mathcal{H}$ and $\mathcal{H}(p)=0$;*

(3) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-R_0$ if and only if for every two IIF *points $p=(t_11_x,t_21_x,t_31_x,t_41_x)$, $q=(t_11_y,t_21_y,t_31_y,t_41_y)$ of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\dot{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\dot{\in}\mathcal{H}$ and $\mathcal{H}(p)=0$;*

(4) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{K})-R_0$ if and only if for every two IIF *points p,q of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\dot{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\dot{\in}\mathcal{H}$ and $\mathcal{H}(p)=0$.*

Proof. The proofs are attainable from Definition 3.5. □

Theorem 3.12. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space. $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_1$ (resp. $\text{IIF}(\mathcal{M})-T_1$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_1$ and $\text{IIF}(\mathcal{K})-T_1$) if and only if $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_0$ (resp. $\text{IIF}(\mathcal{M})-T_0$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_0$ and $\text{IIF}(\mathcal{K})-T_0$) and $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_0$ (resp. $\text{IIF}(\mathcal{M})-R_0$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-R_0$ and $\text{IIF}(\mathcal{K})-R_0$).*

Proof. First, suppose that $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_1$. Then, by Proposition 3.5, $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_0$. Further, by Propositions 3.3 and 3.6, $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_0$ Conversely, suppose that $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_0$ and $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_0$. Since $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_0$, then for every $x,y\in\mathcal{X}$ and $x\neq y$ there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$; or $\mathcal{G}_i(y)\geq t_i$ and $\mathcal{G}_i(x)=0$ for each $i\in\{1,2,3,4\}$, where $t_i\in[0,1]$ for each $i\in\{1,2,3,4\}$, $t_1\leq t_2\leq t_4\leq t_3$ and $t_2>0$. Consider the part $\mathcal{G}_i(x)\geq t_i$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$. Since $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_0$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq t_i$ and $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$. Hence, $\mathcal{G},\mathcal{H}\in\tau$, $\mathcal{G}_i(x)\wedge\mathcal{H}_i(x)\geq t_i$ and $\mathcal{G}_i(y)=\mathcal{H}_i(y)=0$ for each $i\in\{1,2,3,4\}$, where $t_i\in[0,1]$ for each $i\in\{1,2,3,4\}$, $t_1\leq t_2\leq t_4\leq t_3$ and $t_2>0$. Therefore,$(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_1$. The proof of other statements are similar. □

4. R_1- and T_2- IIF topological spaces

In this section we define R_1-ness and T_2-ness separation axioms and investigate some of their properties.

Definition 4.1. An IIF topological space $(\mathcal{X},\tau)$ is said to be:

(1) $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$ if and only if for every two distinct points x,y in $\mathcal{X}$, there exist $\mathcal{G},\mathcal{H}\in\tau$ such that $\mathcal{G}_i(x)\wedge\mathcal{H}_i(y)\geq t_i$ and $\mathcal{G}_i(y)=\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$, where $t_i\in[0,1]$ for each $i\in\{1,2,3,4\}$, $t_1\leq t_2\leq t_4\leq t_3$;

(2) $\text{IIF}(\mathcal{M})-T_2$ if and only if $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(1_\mathcal{X},1_\mathcal{X},1_\mathcal{X},1_\mathcal{X})-T_2$

(3) IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-T_2$ if and only if for every two distinct points x,y in $\mathcal{X}$, there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $\mathcal{G}_i(x)\wedge\mathcal{H}_i(y)\geq t_i (i\in\{1,3\})$, $\mathcal{G}_i(x)\wedge\mathcal{H}_i(y)>t_i$ $(i\in\{2,4\})$, $\mathcal{G}_i(y)=\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$, where $t_i\in[0,1]$ for each $i\in\{1,2,3,4\}$, $t_1\leq t_2\leq t_4\leq t_3$ and $t_4<1$;

(4) IIF$(\mathcal{K})-T_2$ if and only if for every two distinct points x,y in $\mathcal{X}$, and for every two ordered quadrable (t_1,t_2,t_3,t_4) and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i,\mathcal{H}_i(y)\geq\alpha_i(i\in\{1,3\})$, $\mathcal{G}_i(x)>t_i,\mathcal{H}_i(y)>\alpha_i(i\in\{2,4\})$, $\mathcal{G}_i(y)=\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$, where $t_i,\alpha_i\in[0,1]$ for each $i\in\{1,2,3,4\}$, $t_1\leq t_2\leq t_4\leq t_3$, $\alpha_1\leq\alpha_2\leq\alpha_4\leq\alpha_3$ and $t_4\vee\alpha_4<1$.

Proposition 4.2. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space.*

(1) $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$ *if and only if for every two* IIF *points $p=(t_11_x,t_21_x,t_31_x,t_41_x)$, $q=(t_11_y,t_21_y,t_31_y,t_41_y)$ of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $p\tilde{\in}\mathcal{G}$, $q\tilde{\in}\mathcal{H}$, $\mathcal{G}(q)=0$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(2) $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})-T_2$ *if and only if for every two* IIF *points p,q of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $p\tilde{\in}\mathcal{G}$, $q\tilde{\in}\mathcal{H}$, $\mathcal{G}(q)=0$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(3) $(\mathcal{X},\tau)$ is IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-T_2$ *if and only if for every two* IIF *points $p=(t_11_x,t_21_x,t_31_x,t_41_x)$, $q=(t_11_y,t_21_y,t_31_y,t_41_y)$ of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $p\dot{\in}\mathcal{G}$, $q\dot{\in}\mathcal{H}$, $\mathcal{G}(q)=0$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(4) $(\mathcal{X},\tau)$ is IIF$(\mathcal{K})-T_2$ *if and only if for every two* IIF *points p,q of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, there exists $\mathcal{G},\mathcal{H}\in\tau$ such that $p\dot{\in}\mathcal{G}$, $q\dot{\in}\mathcal{H}$, $\mathcal{G}(q)=0$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$.*

Proof. The proofs are straightforward from Definition 4.1. □

Proposition 4.3. *(1) If $(\mathcal{X},\tau)$ is* IIF$(\mathcal{M})-T_2$ *(resp.* IIF$(\mathcal{K})-T_2$*), then $(\mathcal{X},\tau)$ is* IIF$(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$) *(resp.* IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-T_2$));

(2) If $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})-T_2$*, then $(\mathcal{X},\tau)$ is* IIF$(\mathcal{K})-T_2$*;*

(3) If $t_i\leq\alpha_i$ for each $i\in\{1,2,3,4\}$ and $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})(\alpha_1,\alpha_2,\alpha_3,\alpha_4)-T_2$ *(resp.* IIF$(\mathcal{K})(\alpha_1,\alpha_2,\alpha_3,\alpha_4)-T_2$*), then $(\mathcal{X},\tau)$ is* IIF$(\mathcal{M})(\mathrm{t}_1,\mathrm{t}_2,\mathrm{t}_3,\mathrm{t}_4)-T_2$ *(rep.* IIF$(\mathcal{K})(\mathrm{t}_1,\mathrm{t}_2,\mathrm{t}_3,\mathrm{t}_4)-T_2$*);*

(4) If $\mathrm{t}_4\neq1$ and $\mathrm{t}_2\neq0$, and $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})(\mathrm{t}_1,\mathrm{t}_2,\mathrm{t}_3,\mathrm{t}_4)-T_2$*, then $(\mathcal{X},\tau)$ is* IIF$(\mathcal{K})(\mathrm{t}_1,\mathrm{t}_2,\mathrm{t}_3,\mathrm{t}_4)-T_2$.

Proof. The proofs are straightforward from Proposition 4.1. □

Proposition 4.4. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space. If $(\mathcal{X},\tau)$ is* IIF$(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$ *(resp.* IIF$(\mathcal{M})-T_2$, IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-T_2$)) *and* IIF$(\mathcal{K})-T_2$*)), then $(\mathcal{X},\tau)$ is* IIF$(\mathcal{M})(t_1,t_2,t_3,t_4)-T_1$ *(resp.* IIF$(\mathcal{M})-T_1$, IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-T_1$)) *and* IIF$(\mathcal{K})-T_1$*)).*

Proof. The proofs are straightforward from Propositions 3.3 and 4.2. □

Definition 4.5. An IIF topological space $(\mathcal{X},\tau)$ is said to be:

(1) IIF$(\mathcal{M})(t_1,t_2,t_3,t_4)-R_1$ if and only if for every two distinct points x,y in $\mathcal{X}$, whenever there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq t_i$, $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;

(2) IIF$(\mathcal{M})-R_1$ if and only if $(\mathcal{X},\tau)$ is IIF$(\mathcal{M})(1_\mathcal{X},1_\mathcal{X},1_\mathcal{X},1_\mathcal{X})-R_1$;

(3) IIF$(\mathcal{K})(t_1,t_2,t_3,t_4)-R_1$ if and only if for every two distinct points x,y in $\mathcal{X}$, whenever there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i(i\in\{1,3\})$, $\mathcal{G}_i(x)>t_i(i\in\{2,4\})$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq t_i(i\in\{1,3\})$, $\mathcal{H}_i(y)>t_i(i\in\{2,4\})$, $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;

(4) IIF$(\mathcal{K})-R_1$ if and only if for every two distinct points x,y in $\mathcal{X}$ and for every two ordered quadrable (t_1,t_2,t_3,t_4) and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, whenever there exists $\mathcal{G}\in\tau$ such that $\mathcal{G}_i(x)\geq t_i(i\in\{1,3\})$, $\mathcal{G}_i(x)>t_i(i\in\{2,4\})$ and $\mathcal{G}_i(y)=0$ for each $i\in\{1,2,3,4\}$, then there exists $\mathcal{H}\in\tau$ such that $\mathcal{H}_i(y)\geq\alpha_i(i\in\{1,3\})$, $\mathcal{H}_i(y)>\alpha_i(i\in\{2,4\})$, $\mathcal{H}_i(x)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$.

Proposition 4.6. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space.*

(1) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_1$ *if and only if for every two* IIF *points $p=(t_1 1_x,t_2 1_x,t_3 1_x,t_4 1_x)$, $q=(t_1 1_y,t_2 1_y,t_3 1_y,t_4 1_y)$ of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\tilde{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\tilde{\in}\mathcal{H}$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(2) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})-R_1$ *if and only if for every two* IIF *points p,q of type $\mathcal{M}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\tilde{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\tilde{\in}\mathcal{H}$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(3) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-R_1$ *if and only if for every two* IIF *points $p=(t_1 1_x,t_2 1_x,t_3 1_x,t_4 1_x)$, $q=(t_1 1_y,t_2 1_y,t_3 1_y,t_4 1_y)$ of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\dot{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\dot{\in}\mathcal{H}$, $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$;*

(4) $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{K})-R_1$ *if and only if for every two* IIF *points p,q of type $\mathcal{K}$ in $\mathcal{X}$ with different supports, whenever there exists $\mathcal{G}\in\tau$ such that $p\dot{\in}\mathcal{G}$ and $\mathcal{G}(q)=0$, then there exists $\mathcal{H}\in\tau$ such that $q\dot{\in}\mathcal{H}$ and $\mathcal{H}(p)=0$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$.*

Proof. The proofs follow from Definition 4.2. □

Theorem 4.7. *Let $(\mathcal{X},\tau)$ be an* IIF *topological space. $(\mathcal{X},\tau)$ is* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$ *(resp.* $\text{IIF}(\mathcal{M})-T_2$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_2$ *and* $\text{IIF}(\mathcal{K})-T_2$*) if and only if $(\mathcal{X},\tau)$ is* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_0$ *(resp.* $\text{IIF}(\mathcal{M})-T_0$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_0$ *and* $\text{IIF}(\mathcal{K})-T_0$*) and* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-R_1$ *(resp.* $\text{IIF}(\mathcal{M})-R_1$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-R_1$ *and* $\text{IIF}(\mathcal{K})-R_1$*).*

Proof. The proof is similar to the proof of Theorem 3.1. □

Theorem 4.8. *Let $(\mathcal{X},\tau)$ and $(\mathcal{Y},\sigma)$ be two* IIF *topological spaces and $f:(\mathcal{X},\tau)\to(\mathcal{Y},\sigma)$ be a function such that f is bijection and f^{-1} is continuous. If $(\mathcal{X},\tau)$ is* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_i$ *(resp.* $\text{IIF}(\mathcal{M})-T_i$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_i$ *and* $\text{IIF}(\mathcal{K})-T_2$*), then $(\mathcal{Y},\sigma)$ is* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_i$ *(resp.* $\text{IIF}(\mathcal{M})-T_i$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_i$ *and* $\text{IIF}(\mathcal{K})-T_2$*), where $i=0,1,2$.*

Proof. Suppose $(\mathcal{X},\tau)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$ and $y_1,y_2\in\mathcal{Y}$ such that $y_1\neq y_2$. Since f is a bijection, then there exist $x_1,x_2\in\mathcal{X}$ such that $x_1\neq x_2$, $f(x_1)=y_1$ and $f(x_2)=y_2$. Since there exist $\mathcal{G},\mathcal{H}\in\tau$ such that $\mathcal{G}_i(x_1)\geq t_i,\mathcal{H}_i(x_2)\geq t_i$ and $\mathcal{G}_i(x_2)=\mathcal{H}_i(x_1)=0$ for each $i\in\{1,2,3,4\}$ and $\mathcal{G}\cap\mathcal{H}=\tilde{0}$, then $f^{-1}(\mathcal{G}),f^{-1}(\mathcal{H})\in\sigma$, $(f^{-1}(\mathcal{G}))_i(y_1)=(f^{-1}(\mathcal{G}_i))(y_1)\geq t_i$, $(f^{-1}(\mathcal{H}))_i(y_2)=(f^{-1}(\mathcal{H}_i))(y_2)\geq t_i$, $(f^{-1}(\mathcal{G}))_i(y_2)=(f^{-1}(\mathcal{G}_i))(y_2)=f^{-1}(\mathcal{H}))_i(y_1)=(f^{-1}(\mathcal{H}_i))(y_1)=0$ and $f^{-1}(\mathcal{G})\cap f^{-1}(\mathcal{H})=f^{-1}(\mathcal{G}\cap\mathcal{H})=f^{-1}(\tilde{0})=\tilde{0}$. Therefore $(\mathcal{Y},\sigma)$ is $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_2$. The proof of other statements are similar. □

Corollary 4.9. *The properties* $\text{IIF}(\mathcal{M})(t_1,t_2,t_3,t_4)-T_j$, $\text{IIF}(\mathcal{M})-T_j$, $\text{IIF}(\mathcal{K})(t_1,t_2,t_3,t_4)-T_j$ *and* $\text{IIF}(\mathcal{K})-T_j$ *are* IIF *topological properties, where $j\in\{0,1,2\}$.*

Proof. The proof follows from Theorem 4.2. □

5. Lower separation axioms in IF and IIF topological spaces

This section is devoted to study separation axioms of IF topology using tools developed for IIF topology. We start this section with the following remark.

Remark 5.1. If the IIF topology τ contrast to an IF topology, i.e., each element $\mathcal{A}$ in τ is identified with an IF set, i.e., $\mathcal{A}_1=\mathcal{A}_2$ and $\mathcal{A}_3=\mathcal{A}_4$; and if $t_1=t_2$ and $t_3=t_4$, then one can have the corresponding lower separation axioms of Definitions 2.2, 2.3, 2.4, 3.1 and 3.2 in IF topological spaces. As an example see the following definitions of IF-T_2-separation axioms:

Definition 5.2. An IF topological space $(\mathcal{X},\tau)$ is said to be:

(1) IF(M)$(t_1, t_2) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) \wedge \mathcal{H}_i(y) \geq t_i$ and $\mathcal{G}_i(y) = \mathcal{H}_i(x) = 0$ for each $i \in \{1, 2\}$ and $\mathcal{G} \cap \mathcal{H} = (1_\phi, 1_\phi)$, where $0 < t_1 \leq t_2 \leq 1$;
(2) IF$(\mathcal{M}) - T_2$ if and only if $(\mathcal{X}, \tau)$ is IF$(\mathcal{M})(1_\mathcal{X}, 1_\mathcal{X}) - T_2$
(3) IF$(\mathcal{K})(t_1, t_2) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) \wedge \mathcal{H}_i(y) > t_i (i \in \{1, 2\})$ and $\mathcal{G} \cap \mathcal{H} = (1_\phi, 1_\phi)$, $0 \leq t_1 \leq t_2 < 1$;
(4) IF$(\mathcal{K}) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$, and for every two ordered pairs (t_1, t_2) and (α_1, α_2) there exists $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}_i(x) > t_i, \mathcal{H}_i(y) > \alpha_i$, $\mathcal{G}_i(y) = \mathcal{H}_i(x) = 0$ for each $i \in \{1, 2\}$ and $\mathcal{G} \cap \mathcal{H} = (1_\phi, 1_\phi)$, where $0 \leq t_1 \leq t_2 < 1$, $0 \leq \alpha_1 \leq \alpha_2 < 1$.

Thus the definitions of IF$(\mathcal{M})(t_1, t_2) - T_i$, IF$(\mathcal{M}) - T_i$, IF$(\mathcal{K})(t_1, t_2) - T_i$, IF$(\mathcal{K}) - T_i$, IF$(\mathcal{M})(t_1, t_2) - R_i$, IF$(\mathcal{M}) - R_i$, IF$(\mathcal{K})(t_1, t_2) - R_i$, and IF$(\mathcal{K}) - R_i$, where $i \in \{0, 1\}$ can be assumed.

Remark 5.3. If the IIF topology τ contrast to a fuzzy topology, i.e., each element $\mathcal{A}$ in τ is identified with a fuzzy set, i.e., $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4$; and if $t_1 = t_2 = t_3 = t_4$, then one can have the corresponding lower separation axioms of Definitions 3.2, 3.3, 3.4, 4.1 and 4.2 in fuzzy topological spaces. As an example see the following definitions of F-T_2-separation axioms:

Definition 5.4. A fuzzy topological space $(\mathcal{X}, \tau)$ is said to be:

(1) F$(\mathcal{M})(t) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}(x) \wedge \mathcal{H}(y) \geq t$ and $\mathcal{G}(y) = \mathcal{H}(x) = 0$ and $\mathcal{G} \cap \mathcal{H} = 1_\phi$, where $0 \leq t < 1$;
(2) F$(\mathcal{M}) - T_2$ if and only if $(\mathcal{X}, \tau)$ is IF$(\mathcal{M})(1_\mathcal{X}) - T_2$
(3) F$(\mathcal{K})(t) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}(x) \wedge \mathcal{H}(y) > t$ and $\mathcal{G} \cap \mathcal{H} = 1_\phi$, where $0 \leq t < 1$;
(4) F$(\mathcal{K}) - T_2$ if and only if for every two distinct points x, y in $\mathcal{X}$ and for every α, β, there exist $\mathcal{G}, \mathcal{H} \in \tau$ such that $\mathcal{G}(x) > \alpha, \mathcal{H}(y) > \beta$, $\mathcal{G}(y) = \mathcal{H}(x) = 0$ and $\mathcal{G} \cap \mathcal{H} = 1_\phi$, where $0 \leq \alpha < 1$, $0 \leq \beta < 1$.

Thus the definitions of F$(\mathcal{M})(t) - T_i$, F$(\mathcal{M}) - T_i$, F$(\mathcal{K})(t) - T_i$, F$(\mathcal{K}) - T_i$, F$(\mathcal{M})(t) - R_i$, F$(\mathcal{M}) - R_i$, F$(\mathcal{K})(t) - R_i$, and F$(\mathcal{K}) - R_i$, where $i \in \{0, 1\}$ can be assumed.

Remark 5.5. If the IIF topology τ contrast to an ordinary topology, i.e., each element $\mathcal{A}$ in τ is identified with an ordinary set, i.e., $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \mathcal{A}_4$ and $\mathcal{A}_1(\mathcal{X}) \subseteq \{0, 1\}$; and if $t_1 = t_2 = t_3 = t_4$, then one can have the corresponding lower separation axioms of Definitions 3.2, 3.3, 3.4, 4.1 and 4.2 in ordinary topological spaces. As an example see the following definitions of F-T_2-separation axioms:

Proposition 5.6. *Let $\tau \in IT(\mathcal{X})$ and $\xi(\tau) = (\tau_1, \tau_2, \tau_3, \tau_4)$. Then:*

(1) If $(\mathcal{X}, \tau)$ is IIF$(\mathcal{M})(t_1, t_2, t_3, t_4) - T_j$ *(resp.* IIF$(\mathcal{M}) - T_j$*), then $(\mathcal{X}, \tau_i)$ is* F$(\mathcal{M})(t_i) - T_j$ *(resp.* F$(\mathcal{M}) - T_j$*) where $i \in \{1, 2, 3, 4\}$ and $j \in \{0, 1, 2\}$;*
(2) If $(\mathcal{X}, \tau)$ is IIF$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_j$ *(resp.* IIF$(\mathcal{K}) - T_j$*), then $(\mathcal{X}, \tau_1)$ is* F$(\mathcal{M})(t_1) - T_j$*, $(\mathcal{X}, \tau_3)$ is* F$(\mathcal{M})(t_3) - T_j$*, $(\mathcal{X}, \tau_2)$ is* F$(\mathcal{K})(t_2) - T_j$ *and $(\mathcal{X}, \tau_4)$ is* F$(\mathcal{K})(t_4) - T_j$*, where $j \in \{0, 1, 2\}$.*

Proof. The proof follows from Theorems 2.2 and 2.3. □

Proposition 5.7. *Let $(\tau_1, \tau_2, \tau_3, \tau_4) \in PT(\mathcal{X})$, $\tau \in IT(\mathcal{X})$, $\tau = \eta(\tau_1, \tau_2, \tau_3, \tau_4)$ and $j \in \{0, 1, 2\}$. Then:*

(1) If $(\mathcal{X}, \tau_i)$ is F$(\mathcal{M})(t_i) - T_j$ *for $i \in \{1, 2, 3, 4\}$ and $t_1 \leq t_2 \leq t_3 \leq t_4$, then $(\mathcal{X}, \tau)$ is* F$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_j$
(2) If $(\mathcal{X}, \tau_i)$ is F$(\mathcal{M})(t_i) - T_j$ *for $i \in \{1, 3\}$ and $(\mathcal{X}, \tau_i)$ is* F$(\mathcal{K})(t_i) - T_j$ *for $i \in \{2, 4\}$ and $t_1 \leq t_2 \leq t_3 \leq t_4$, then $(\mathcal{X}, \tau)$ is* F$(\mathcal{K})(t_1, t_2, t_3, t_4) - T_j$.

Proof. The proof follows from Theorems 2.2 and 2.3. □

Theorem 5.8. *Let $\tau \in IT(\mathcal{X})$ and $j \in \{0, 1, 2\}$. Then:*

(1) $(\mathcal{X}, \tau_O)$ is $T_j \Rightarrow (\mathcal{X}, \tau_F)$ is F$(\mathcal{M}) - T_j \Rightarrow$ *$(\mathcal{X}, \tau_{IF})$ is* IF$(\mathcal{M}) - T_j \Rightarrow$ *$(\mathcal{X}, \tau)$ is* IIF$(\mathcal{M}) - T_j$*;*

(2) $(\mathcal{X}, \tau_F)$ *is* $\mathrm{F}(\mathcal{M})(t_3)T_j \Rightarrow (\mathcal{X}, \tau_{IF})$ *is* $\mathrm{IF}(\mathcal{M})(t_2, t_3) - T_j \Rightarrow (\mathcal{X}, \tau)$ *is* $\mathrm{IIF}(\mathcal{M})(t_1, t_2, t_3, t_4) - T_j$*;*
(3) $(\mathcal{X}, \tau_F)$ *is* $\mathrm{F}(\mathcal{K})(t_3)T_j \Rightarrow (\mathcal{X}, \tau_{IF})$ *is* $\mathrm{IF}(\mathcal{K})(t_2, t_3) - T_j \Rightarrow (\mathcal{X}, \tau)$ *is* $\mathrm{IIF}(\mathcal{K})(t_1, t_2, t_3, t_4) - T_j$*;*
(4) $(\mathcal{X}, \tau_F)$ *is* $\mathrm{F}(\mathcal{K}) - T_j \Rightarrow (\mathcal{X}, \tau_{IF})$ *is* $\mathrm{IF}(\mathcal{K}) - T_j$*.*

Proof. Here we only show the assertion (1), and all other assertions can be proved in a similar way. Therefore, we omit their proofs.
(a) Suppose that $(\mathcal{X}, \tau_0)$ is T_0 space. Then for each distinct points $x, y \in \mathcal{X}$, there exists an interval-valued intuitionistic fuzzy open set $\mathcal{G}$ identified with an ordinary set such that $x \in \mathcal{G}$ and $y \notin \mathcal{G}$. Note that $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$, where $\mathcal{G}_i = \mathcal{G}$ for each $i \in \{1, 2, 3, 4\}$. Then for each $t \in [0, 1)$, the interval-valued intuitionistic fuzzy open set $\mathcal{G}$ identified with a fuzzy set such that $\mathcal{G}(x) \geq t$ and $\mathcal{G}(y) = 0$. Hence $(\mathcal{X}, \tau_F)$ is $\mathrm{F}(\mathcal{M}) - T_0$.
(b) Suppose that $(\mathcal{X}, \tau_F)$ is $\mathrm{F}(\mathcal{M}) - T_0$. Then for each distinct points $x, y \in \mathcal{X}$, there exists an interval-valued intuitionistic fuzzy open set $\mathcal{G}$ identified with a fuzzy set $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$, where $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4$; and $\mathcal{G}_2(x) > t, \mathcal{G}_2(y) = 0 = 1 - (1 - \mathcal{G}_3(y))$ and $1 - \mathcal{G}_3(y) < 1 - t$. Now, the interval-valued intuitionistic fuzzy open set $\mathcal{G}$ identified with an intuitionistic fuzzy set $\tilde{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2)$, $\mathcal{A}_1 = \mathcal{G}_2$ and $\mathcal{A}_2 = 1 - \mathcal{G}_3$. Then $\mathcal{A}_1(x) > t \geq t_1$, $\mathcal{A}_2(x) < 1 - t = t_2$ (i.e.,$\mathcal{A}_2(x) \leq t_2$) and $\mathcal{A}_1(y) = 0 = 1 - \mathcal{A}_2(y)$. Therefore, $(\mathcal{X}, \tau_{IF})$ is $\mathrm{IF}(\mathcal{M}) - T_0$ for every $t_1, t_2 \in [0, 1)$ and $t_1 \leq t_2$ (without loss all of generality).
(c) Suppose that $(\mathcal{X}, \tau_{IF})$ is $\mathrm{IF}(\mathcal{M}) - T_0$ space. Then for each distinct points $x, y \in \mathcal{X}$, there exists an interval-valued intuitionistic fuzzy open set $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$ identified with an intuitionistic fuzzy set $\tilde{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2)$, $\mathcal{A}_1 > t_1$ and $\mathcal{A}_2 \leq 1 - t_2$ and $\mathcal{A}_1(y) = 0 = 1 - \mathcal{A}_2(y)$. Then $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4)$ is an interval-valued intuitionistic fuzzy open set, where $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{A}_1$ and $\mathcal{G}_3 = \mathcal{G}_4 = 1 - \mathcal{A}_2$. Now, for every $t_1, t_2, t_3, t_4 \in [0, 1]$ such that $t_1 \leq t_2 \leq t_4 \leq t_3$ and $0 \leq t_2 \wedge t_4 \leq t_2 \vee t_4 < 1$, we have $\mathcal{G}_1(x) > t_2 \geq t_1, \mathcal{G}_2(x) > t_2, \mathcal{G}_3(x) > t_3, \mathcal{G}_4(x) \geq t_3 > t_4$. □

6. Conclusion

This paper introduces some new results about intuitionistic fuzzy topological spaces by using a characterization of the interval-valued intuitionistic fuzzy sets and two types of interval-valued intuitionistic fuzzy points. It obtains some results about different types of separation axioms. We believe that it would be interesting to extend this approach to other structures such as uniformity, proximity, pre-uniformity, topogenous, syntopogenous, homotopy etc. Also, we intend to extend the suitable and pre-suitable interval-valued intuitionistic fuzzy sets in order to apply it to Pythagorean fuzzy set in future. Furthermore, we may look at the possibility of bridging the suitable or pre-suitable interval-valued intuitionistic fuzzy sets with the vague sets [56, 57], Q-fuzzy sets [58, 59, 60], multi-fuzzy [61, 62] and theoretical numerical analysis [63, 64, 65].

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