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Uniform spaces based on a way below relation

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Abstract
This paper fosters both uniform spaces and way below relations with an innovative analysis of their mutual relationships. A new concept of uniform spaces based on a way below relation (*LB*-fuzzifying uniform space, or LBFU space, for short) will be introduced and investigated. With this aim, first some fundamental concepts in *L*-fuzzifying topological spaces will be studied. Then, we shall explore some *L*-fuzzifying topological spaces induced by an LBFU space. Furthermore, new concepts of interior, closure, bases and subbases relative to an LBFU topology will be established. Finally, the continuity of functions between LBFU spaces will be introduced and investigated.

Keywords Uniform spaces · *L*-fuzzifying topology · *LB*-fuzzifying uniform continuity · Way below relation

Mathematics Subject Classification 54A40 · 54E15

1 Introduction and preliminary concepts

Zadeh produced the pathbreaking concept of fuzzy set in his acclaimed (Zadeh 1965). Since that milestone, mathematicians have struggled to extend fundamental mathematical structures such as groups, rings, vector spaces, topologies, uniformities, and proximities to a fuzzy framework. Particularly, one of the fuzzy extensions of the notion of a topology was studied in a sequence of articles and books (Höhle et al. 1995; Höhle and Rodabaugh 1999; Höhle 2001; Liu and Luo 1998; Lowen 1982; Ying 1991). Relatedly, Höhle (1980b) and Zhang

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(2002) put forward the idea of an L-fuzzy topology, which consists of an L-valued mapping on the standard power set of $\mathcal{X}$, namely, $\mathcal{P}(\mathcal{X})$. According to Höhle and Rodabaugh (Höhle and Rodabaugh 1999, Chapter 3), an L-fuzzy topology is an L-valued function on the L-power set $\mathcal{L}^{\mathcal{X}}$ of $\mathcal{X}$. Lowen Lowen (1979) generalized the well-developed topological theory of convergence to the context of stratified I-topologies. In Höhle (1978), Höhle (1982a), a comparable notion received the name of probabilistic topology. These two structures have been contrasted in (1999, Chapter 5).

Uniform spaces are mathematical objects that lie in between topological and metric spaces. In particular, every uniform space originates a natural topology, and every metric space can be endowed with a canonical uniform structure. We can trace uniform spaces back to the late 1930s. Many textbooks on general topology (like Willard Willard 1970) adhere to the Weil approach Weil (1937), which is often known as the "surrounding" or "entourage" approach. Uniform coverings yield an alternative perspective Tukey (1940), which was followed by textbooks like Isbell Isbell (1964). L-fuzzy uniform spaces were considered by Hutton Hutton (1977), Hutton (1983). The fuzzification of the structure of a uniform space was established by Lowen Lowen (1981), Lowen and Wuyts (1982), Lowen and Wuyts (1983). Another fuzzification of the concept was introduced by Höhle Höhle (1980a), Höhle (1982b). Kotzé (1999, Chapter 8) introduced equivalent perspectives on uniform spaces and presented an approach to L-uniformities extending Höhle's. Some authors (e.g., 1999, Chapter 9) outlined how some of the features of the theory of uniform spaces can be generalized to become a fuzzy notion. When $\mathcal{L} = [0, 1]$, Katsaras Katsaras (1984) proved that the [0, 1]-topology produced from a uniformizable point-set topology must be uniformizable in the sense of Hutton. Extensions of well-known theorems to L-topological spaces appeared in Artico and Moresco (1988). Nowadays the theory of fuzzy uniform spaces is under further development and over 100 papers on the topic have been produced, see Ahsanullah (1992), Artico and Moresco (1987), Artico and Moresco (1989), Badard et al. (1993), Burton (1993a), Burton (1993b), Burton (1993c), Burton (1993d), Kandil et al. (1994), Soetens and Wuyts (1993), Srivastava (1989), Wuyts and Lowen (1983) as a short sample. Burton et al. Burton et al. (1996) produced generalized uniform spaces. They proved that the categories of Lowen fuzzy uniform spaces and generalized uniform spaces are isomorphic, and that the category of generalized uniform spaces is a good extension of the category of uniform spaces. Also, they showed that the category of super uniform spaces defined in Gutiérrez García and de Prada Vicente (1997) includes these categories, a fact that is detailedly studied in Gutiérrez et al. (1997). The semantic method of continuous valued logic allowed Ying Ying (1993b) to define the fuzzifying uniform space in a completely different direction, and he studied some of its properties. Khedr et al. Khedr et al. (2003) introduced the notion of the strong fuzzifying uniformity and established its relationship with the fuzzifying proximity. The way below relation was defined in Gierz et al. (1980), and in this article some of its properties were studied too. Also, Bancerek Bancerek (1997) introduced the way below relation and stated several propositions in topics such as continuous lattices, directed powers, and topological spaces.

However, as far as we are aware of there exists no analysis of the relationships between the fuzzy structure of a uniform space and relations such as the way below relation. So far they have remained as two divergent fields of research. Here we shall conduct a substantial analysis of their mutual relationships. This achievement produces a basically theoretical article which is nonetheless necessary to provide a strong foundation of this novel aspect of topological fuzzy set theory. Both disciplines should be promoted with this pioneering analysis, which may also foster the inspection of other relationships among different types of topological structures.

The rest of this paper is organized as follows. This section contains some necessary concepts and properties. In section 2, we shall introduce the new concepts of L-fuzzifying derived, L-fuzzifying interior, and L-fuzzifying closure operators. In the next section, the concepts of base and subbase in the framework of L-fuzzifying topological spaces are defined, and with their help we shall investigate L-fuzzifying continuous mapping and L-fuzzifying open mapping. In section 4, we put forward the structure of an LBFU space in the framework of L-fuzzifying topology. Then we prove some of their fundamental properties. In section 5, we investigate some L-fuzzifying topological spaces which are induced by LBFU spaces. Furthermore, the concepts of interior and closure relative to the LBFU topology are investigated. In section 6, the concept of LBFU continuity is given and some results are discussed. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research.

In this paper we adopt the standard terminology from lattice theory (which can be consulted in monographs like Birkhoff (1967), Gierz et al. (1980), Gratzer (1978)). We assume that $(\mathcal{L}, \leq)$ is a complete lattice whose smallest element is $\perp$ and whose largest element is $\top$. All other additional requirements on $\mathcal{L}$ will be made explicit when required.

In this context, a first fundamental concept is given in our next definition:

Definition 1.1 (Bancerek 1997; Gierz et al. 1980). Let $x, y \in \mathcal{L}$. Then x is way below y, represented by $x \ll y$, when for any directed subset $\mathcal{D} \subseteq \mathcal{L}$, the relation $y \leq \bigvee \mathcal{D}$ always implies that an element $d \in \mathcal{D}$ exists with $x \leq d$.

Some immediate facts ensue from this notion:

Proposition 1.1 *(Bancerek 1997; Gierz et al. 1980). For all $u, x, y, z \in \mathcal{L}$, the following statements hold:*

(i) $x \ll y$ implies $x \leq y$;
(ii) $u \leq x \ll y \leq z$ implies $u \ll z$;
(iii) If $x \ll z$ and $y \ll z$ then $x \vee y \ll z$;
(iv) $\perp \ll x$;
(v) If $x \ll y$ and $z \leq y$ then $x \ll z$;
(vi) If L is a complete chain (Birkhoff 1967), then $x \ll y$ if and only if $x \leq y$.

A second fundamental notion is given in the next definition:

Definition 1.2 (Höhle 1980b, (1999, Chapter 5), (Ying 1993a; Zhang 2002)). Let $\mathcal{X}$ be the universe of discourse and $\tau \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, where $\mathcal{P}(\mathcal{X})$ is the power set of $\mathcal{X}$. Suppose that the following conditions hold true:

(1) $\tau(\mathcal{X}) = \tau(\phi) = \top$;
(2) For all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{X})$, $\tau(\mathcal{A}) \wedge \tau(\mathcal{B}) \leq \tau(\mathcal{A} \cap \mathcal{B})$;
(3) For all $\{\mathcal{A}_i : i \in \Lambda\} \subseteq \mathcal{P}(\mathcal{X})$, $\bigwedge_{i \in \Lambda} \tau(\mathcal{A}_i) \leq \tau\left(\bigcup_{i \in \Lambda} \mathcal{A}_i\right)$.

Then τ is called an L-fuzzifying topology, and $(\mathcal{X}, \tau)$ is called an L-fuzzifying topological space.
If $(\mathcal{X}, \tau_1)$ and $(\mathcal{Y}, \tau_2)$ are two L-fuzzifying topological spaces, then we say that the function $f : (\mathcal{X}, \tau_1) \longrightarrow (\mathcal{Y}, \tau_2)$ is L-fuzzifying continuous if for all $\mathcal{B} \in \mathcal{P}(\mathcal{Y})$, $\tau_2(\mathcal{B}) \leq \tau_1(f^{-1}(\mathcal{B}))$.

Henceforth, $(\mathcal{X}, \tau)$ will denote an L-fuzzifying topological space, with $\mathcal{X}$ being the universe of discourse.

Associated with Definition 1.2 a concept exists whose properties are stated below in Proposition 1.2, under the assumption that the lattice is completely distributive:

Definition 1.3 ((Höhle and Rodabaugh 1999, Chapter 5), Liu and Zhang (2000), Zhang (2002)). The L-fuzzifying neighborhood system of a point $x \in \mathcal{X}$, denoted by $\mathfrak{N}_x \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is defined as follows: $\mathfrak{N}_x(\mathcal{A}) = \bigvee_{x \in \mathcal{B} \subseteq \mathcal{A}} \tau(\mathcal{B}), \forall \mathcal{A} \in \mathcal{P}(\mathcal{X})$.

Proposition 1.2 *(Höhle and Rodabaugh 1999). Let $\mathcal{L}$ be a completely distributive lattice. Then for all $x \in \mathcal{X}$, the following statements are true:*

(1) $\mathfrak{N}_x(\mathcal{X}) = \top$, $\mathfrak{N}_x(\phi) = \bot$;
(2) $\mathfrak{N}_x(\mathcal{A} \cap \mathcal{B}) = \mathfrak{N}_x(\mathcal{A}) \wedge \mathfrak{N}_x(\mathcal{B})$;
(3) $\mathfrak{N}_x(\mathcal{A}) = \bot$ *whenever* $x \notin \mathcal{A}$;
(4) $\mathfrak{N}_x(\mathcal{A}) \leq \bigvee_{y \notin \mathcal{B}} (\mathfrak{N}_y(\mathcal{A}) \vee \mathfrak{N}_x(\mathcal{B}))$.

Moreover $\tau(\mathcal{A}) = \bigwedge_{x \in \mathcal{A}} \mathfrak{N}_x(\mathcal{A})$, for each $\mathcal{A} \in \mathcal{P}(\mathcal{X})$.

2 Fundamental concepts: Derived set, closure, interior

Along this section, $\mathcal{L}$ represents a completely distributive lattice with order reversing involution denoted by $'$. Under this condition, we can define:

Definition 2.1 Let $x \in \mathcal{X}$ and $\mathcal{A} \in \mathcal{P}(\mathcal{X})$.

(1) The family of all L-fuzzifying closed sets, represented by $\mathfrak{F}_\tau \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is given by $\mathfrak{F}_\tau(\mathcal{A}) = \tau(\mathcal{X} \sim \mathcal{A})$, where $\mathcal{X} \sim \mathcal{A}$ is the complement of $\mathcal{A}$;
(2) The L-fuzzifying derived set of $\mathcal{A}$, represented by $\mathfrak{D}_\tau(\mathcal{A}) \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is given as follows: $\mathfrak{D}_\tau(\mathcal{A})(x) = \bigwedge_{\mathcal{B} \in \mathcal{P}(\mathcal{X}), \mathcal{B} \cap (\mathcal{A} \sim \{x\}) = \phi} (\mathfrak{N}_x(\mathcal{B}))'$;
(3) The L-fuzzifying interior set of $\mathcal{A}$, represented by $Int_\tau(\mathcal{A}) \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is given as follows: $Int_\tau(\mathcal{A})(x) = \mathfrak{N}_x(\mathcal{A})$;
(4) The L-fuzzifying closure set of $\mathcal{A}$, represented by $Cl_\tau(\mathcal{A}) \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is given as follows: $Cl_\tau(\mathcal{A})(x) = (\mathfrak{N}_x(\mathcal{X} \sim \mathcal{A}))'$.

The next technical result will help us give some important facts about Definition 2.1. We omit the proofs since they are routine:

Lemma 2.1 *For all $\mathcal{A} \in \mathcal{P}(\mathcal{X})$ and $x \in \mathcal{X}$, $\mathfrak{D}_\tau(\mathcal{A})(x) = (\mathfrak{N}_x((\mathcal{X} \sim \mathcal{A}) \cup \{x\}))'$.*

Theorem 2.1 *For all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{X})$ and $x \in \mathcal{X}$ we have:*

(1) $\mathfrak{D}_\tau(\phi)(x) = \bot$;
(2) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathfrak{D}_\tau(\mathcal{A}) \leq \mathfrak{D}_\tau(\mathcal{B})$;
(3) $\mathfrak{D}_\tau(\mathcal{A}) \vee \mathfrak{D}_\tau(\mathcal{B}) = \mathfrak{D}_\tau(\mathcal{A} \cup \mathcal{B})$;
(4) $\mathfrak{F}_\tau(\mathcal{A}) = \bigwedge_{x \notin \mathcal{A}} (\mathfrak{D}_\tau(\mathcal{A})(x))'$.

Theorem 2.2 *For all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{X})$ and $x \in \mathcal{X}$ we have*

(1) $Int_\tau(\mathcal{X})(x) = \top$;
(2) $Int_\tau(\mathcal{A})(x) \leq \mathcal{A}(x)$;
(3) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $Int_\tau(\mathcal{A})(x) \leq Int_\tau(\mathcal{B})(x)$;
(4) $Int_\tau(\mathcal{A} \cap \mathcal{B}) = Int_\tau(\mathcal{A}) \wedge Int_\tau(\mathcal{B})$;

(5) $Int_\tau(\mathcal{A})(x) = \mathcal{A}(x) \wedge (\mathfrak{D}_\tau(\mathcal{X} \sim \mathcal{A})(x))^{'}$.

Theorem 2.3 *For all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\mathcal{X})$ and $x \in \mathcal{X}$ we have:*

(1) $Cl_\tau(\phi)(x) = \bot$;
(2) $\mathcal{A}(x) \leq Cl_\tau(\mathcal{A})(x)$;
(3) *If* $\mathcal{A} \subseteq \mathcal{B}$, *then* $Cl_\tau(\mathcal{A})(x) \leq Cl_\tau(\mathcal{B})(x)$;
(4) $Cl_\tau(\mathcal{A} \cup \mathcal{B}) = Cl_\tau(\mathcal{A}) \vee Cl_\tau(\mathcal{B})$;
(5) $Cl_\tau(\mathcal{A})(x) = \bigwedge_{x \notin \mathcal{B} \supseteq \mathcal{A}} (\mathfrak{F}_\tau(\mathcal{B}))^{'}$;
(6) $Cl_\tau(\mathcal{A})(x) = \bigwedge_{\mathcal{A} \cap \mathcal{B} = \phi} (\mathfrak{N}_x(\mathcal{B}))^{'}$;
(7) $Cl_\tau(\mathcal{A})(x) = \mathcal{A}(x) \vee \mathfrak{D}_\tau(\mathcal{A})(x)$;
(8) $\mathfrak{F}_\tau(\mathcal{A}) = \bigwedge_{x \notin \mathcal{A}} (Cl_\tau(\mathcal{A})(x))^{'}$.

3 Bases and subbases

First in this section we introduce the next two related notions:

Definition 3.1 A lattice $\mathcal{L}$ is

(1) S-compact, if $a \leq \bigvee_{j \in \Lambda} \alpha_j$ for all $\alpha_j, a \in \mathcal{L}$, implies the existence of $j_\circ \in \Lambda$ such that $a \leq \alpha_{j_\circ}$.
(2) I-compact, if $\bigwedge_{j \in \Lambda} \alpha_j \leq a$ for all $\alpha_j, a \in \mathcal{L}$, implies the existence of $j_\circ \in \Lambda$ such that $\alpha_{j_\circ} \leq a$.

Our next Lemma shows that in the case of complemented lattices, both concepts given in Definition 3.1 are equivalent:

Lemma 3.1 *Let $\mathcal{L}$ be a complemented lattice. Then $\mathcal{L}$ is S-compact if and only if it is I-compact.*

Proof Suppose that $\mathcal{L}$ is S-compact and $\bigwedge_{j \in \Lambda} \alpha_j \leq a$. Then $a^{'} \leq \left(\bigwedge_{j \in \Lambda} \alpha_j\right)^{'}$. Hence $a^{'} \leq \bigvee_{j \in \Lambda} \alpha_j^{'}$. So, there exists $j_\circ \in \Lambda$ such that $a^{'} \leq \alpha_{j_\circ}^{'}$. Thus, $\alpha_{j_\circ} \leq a$. Therefore $\mathcal{L}$ is I-compact. Similarly, the converse implication can be proven. □

We now introduce a concept of base of τ, an L-fuzzifying topology on $\mathcal{X}$. This name will be justified by the subsequent Theorem 3.1:

Definition 3.2 A map $\mathfrak{B} : \mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{L}$ is a base of τ, an L-fuzzifying topology on $\mathcal{X}$, if and only if $\mathfrak{B}$ fulfills the following conditions:

(1) $\mathfrak{B} \leq \tau$, and
(2) $\mathfrak{N}_x(\mathcal{A}) \leq \bigvee_{x \in \mathcal{B} \subseteq \mathcal{A}} \mathfrak{B}(\mathcal{B})$.

The following two theorems provide necessary and sufficient condition for a base of an L-fuzzifying topology. The proof is similar to the proof of Theorem 4.1 (Ying 1991).

Theorem 3.1 *Let $\mathfrak{B}$ be a base of τ. Define for each $\mathcal{A} \subseteq \mathcal{X}$,*

$$\mathfrak{B}^{(\cup)}(\mathcal{A}) = \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda = \mathcal{A}} \bigwedge_{\lambda\in\Lambda} \mathfrak{B}(\mathcal{B}_\lambda).$$

Then:

(1) $\tau \geq \mathfrak{B}^{(\cup)}$, and
(2) When $\mathcal{L}$ is a completely distributive lattice, it must be the case that $\tau = \mathfrak{B}^{(\cup)}$.

Theorem 3.2 *If $\tau = \mathfrak{B}^{(\cup)}$ and $\mathfrak{B}$ is monotone, then $\mathfrak{B}$ is a base of τ.*

The next theorem identifies when an element from $\mathcal{L}^{\mathcal{P}(\mathcal{X})}$ behaves as a base for some L-fuzzifying topology, provided that $\mathcal{L}$ is a completely distributive lattice. The sketch of proof is similar to Theorem 3.3 (Liang and Yan 2014).

Theorem 3.3 *Let $\mathcal{L}$ be a completely distributive lattice and $\mathfrak{B} \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$.*

(1) If $\mathfrak{B}$ is a base of τ, an L-fuzzifying topology on $\mathcal{X}$, then
   (a) $\mathfrak{B}^{(\cup)}(\mathcal{X}) = \top$,
   (b) $\mathfrak{B}(\mathcal{A}) \wedge \mathfrak{B}(\mathcal{B}) \leq \bigvee_{x\in\mathcal{C}\subseteq\mathcal{A}\cap\mathcal{B}} \mathfrak{B}(\mathcal{C})$

(2) If $\mathfrak{B}$ satisfies conditions (a) and (b) in (1), then $\mathfrak{B}$ is a base for some L-fuzzifying topology.

We complement the investigation of bases of L-fuzzifying topologies with the related notion of subbase, defined in the following terms:

Definition 3.3 $\mathfrak{S} : \mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{L}$ is a subbase of τ if $\mathfrak{S}^{(\sqcap)} : \mathcal{P}(\mathcal{X}) \longrightarrow \mathcal{L}$ is a base, where $\mathfrak{S}^{(\sqcap)}(\mathcal{A}) = \bigvee_{\sqcap_{\lambda\in\Lambda} \mathcal{B}_\lambda = \mathcal{A}} \bigwedge_{\lambda\in\Lambda} \mathfrak{S}(\mathcal{B}_\lambda)$, for all $\mathcal{A} \in \mathcal{P}(\mathcal{X})$, where $(\sqcap)$ stands for "finite intersection".

The restriction to infinitely distributive lattices allows us to give a full characterization of the elements from $\mathcal{L}^{\mathcal{P}(\mathcal{X})}$ that behave as a subbase of some L-fuzzifying topology:

Theorem 3.4 *Let $\mathcal{L}$ be an infinitely distributive lattice. Then $\mathfrak{S} \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$ is a subbase of some L-fuzzifying topology if and only if $\mathfrak{S}^{(\cup)}(\mathcal{X}) = \top$.*

Proof Necessity: From the definition of subbase and Theorem 3.3, we have $(\mathfrak{S}^{(\sqcap)})^{(\cup)}(\mathcal{X}) = \top$. Thus $\forall\alpha \neq \top$ one has

$$\alpha \leq (\mathfrak{S}^{(\sqcap)})^{(\cup)}(\mathcal{X}) = \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{X}} \bigwedge_{\lambda\in\Lambda} \mathfrak{S}^{(\sqcap)}(\mathcal{A}_\lambda) = \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{X}} \bigwedge_{\lambda\in\Lambda} \bigvee_{\sqcap_{\rho\in\Lambda_\lambda} \mathcal{B}_\rho = \mathcal{A}_\lambda} \bigwedge_{\rho\in\Lambda_\lambda} \mathfrak{S}(\mathcal{B}_\rho).$$

Then there exists $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ with $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{X}$ and $\left\{\mathcal{B}_\rho : \rho \in \Lambda_\lambda\right\}$ with $\sqcap_{\rho\in\Lambda_\lambda} \mathcal{B}_\rho = \mathcal{A}_\lambda$ for each $\lambda \in \Lambda$ such that $\alpha \leq \mathfrak{S}(\mathcal{B}_\rho)$. Thus $\bigcup_{\rho\in\Lambda_\lambda,\lambda\in\Lambda} \mathcal{B}_\rho = \mathcal{X}$ and $\alpha \leq \bigwedge_{\rho\in\Lambda_\lambda,\lambda\in\Lambda} \mathfrak{S}(\mathcal{B}_\rho)$. So $\alpha \leq \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{X}} \bigwedge_{\lambda\in\Lambda} \mathfrak{S}(\mathcal{B}_\lambda) = \mathfrak{S}^{(\cup)}(\mathcal{X})$. From the arbitrariness of α, we have $\mathfrak{S}^{(\cup)}(\mathcal{X}) = \top$.
Sufficiency: The proof is similar to the proof of Theorem 4.3 (Ying 1991). □

Our next goal is to show that subbases are helpful to simplify the verification of certain properties of mappings between L-fuzzifying topological spaces. The proof is similar to the proof of Theorem 3.4 (Liang and Yan 2014).

Theorem 3.5 *Let $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ be two L-fuzzifying topological spaces, and let σ be generated by its subbase $\mathfrak{S}$. Suppose that the mapping $f : (\mathcal{X}, \tau) \longrightarrow (\mathcal{Y}, \sigma)$ satisfies $\mathfrak{S}(\mathcal{U}) \leq \tau(f^{-1}(\mathcal{U}))$, for all $\mathcal{U} \in \mathcal{P}(\mathcal{Y})$. Then f is L-fuzzifying continuous.*

Theorem 3.6 *Let $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ be two L-fuzzifying topological spaces, and let τ generated by its base $\mathfrak{B}$. If the mapping $f : (\mathcal{X}, \tau) \longrightarrow (\mathcal{Y}, \sigma)$ satisfies $\mathfrak{B}(\mathcal{U}) \leq \sigma(f(\mathcal{U}))$, for all $\mathcal{U} \in \mathcal{P}(\mathcal{X})$, then f is L-fuzzifying open, i.e., for all $\mathcal{W} \in \mathcal{P}(\mathcal{X})$, $\tau(\mathcal{W}) \leq \sigma(f(\mathcal{W}))$.*

Proof The proof is similar to the proof of Theorem 3.5 (Liang and Yan 2014). □

Corollary 3.1 *If $f : (\mathcal{X}, \tau) \to (\mathcal{Y}, \sigma)$ is a bijection and τ is generated by its subbase $\mathfrak{S}$, and $\mathfrak{S}(\mathcal{U}) \leq \sigma(f(\mathcal{U}))$ for every $\mathcal{U} \in \mathcal{P}(\mathcal{X})$, then f is L-fuzzifying open.*

Furthermore, the broad idea of relative topologies helps us establish some additional results. When $(\mathcal{X}, \tau)$ is an L-fuzzifying topological space, and $\mathcal{Z} \subseteq \mathcal{X}$, we define $(\mathcal{Z}, \tau|_{\mathcal{Z}})$ by the expression $(\tau|_{\mathcal{Z}})(\mathcal{A}) = \bigvee_{\mathcal{A}=\mathcal{U}\cap\mathcal{Z}} \tau(\mathcal{U})$, for all $x \in \mathcal{Z}$, $\mathcal{A} \in \mathcal{P}(\mathcal{Z})$. Then one has:

Theorem 3.7 *Let $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ be two L-fuzzifying topological spaces, and $\mathcal{Z} \subseteq \mathcal{X}$. The mapping $f|_{\mathcal{Z}} : (\mathcal{Z}, \tau|_{\mathcal{Z}}) \longrightarrow (\mathcal{Y}, \sigma)$ is L-fuzzifying continuous, where $(f|_{\mathcal{Z}}(x)) = f(x)$.*

The next result shows that under certain circumstances, bases are inherited by relative topologies:

Theorem 3.8 *Let τ be generated by its base $\mathfrak{B}$ and define $\mathfrak{B}|_{\mathcal{Y}}(\mathcal{U}) = \bigvee_{\mathcal{W}\cap\mathcal{Y}=\mathcal{U}} \mathfrak{B}(\mathcal{W})$, for $\mathcal{Y} \subseteq \mathcal{X}$, $\mathcal{U} \in \mathcal{P}(\mathcal{Y})$. If $\mathcal{L}$ is S-compact, then $\mathfrak{B}|_{\mathcal{Y}}$ is a base of $\tau|_{\mathcal{Y}}$.*

Proof The proof is similar to the proof of Theorem 3.6 (Liang and Yan 2014). □

Now we proceed to investigate the product of L-fuzzifying topologies.

Theorem 3.9 *Let $\{(\mathcal{X}_\lambda, \tau_\lambda) : \lambda \in \Lambda\}$ be a family of L-fuzzifying topological spaces and let $P_\alpha : \prod_{\lambda\in\Lambda} \mathcal{X}_\lambda \longrightarrow \mathcal{X}_\alpha$ be the projection on the α component. For any $\mathcal{W} \in \mathcal{P}\left(\prod_{\lambda\in\Lambda} \mathcal{X}_\lambda\right)$, define $\mathfrak{S}(\mathcal{W}) = \bigvee_{\lambda\in\Lambda} \bigvee_{P_\lambda^{-1}(\mathcal{U})=\mathcal{W}} \tau_\lambda(\mathcal{U})$. If $\mathcal{L}$ is an infinitely distributive lattice, then $\mathfrak{S}$ is a subbase of some L-fuzzifying topology τ which is called the product of the L-fuzzifying topologies $\{\tau_\lambda : \lambda \in \Lambda\}$. We write $\tau = \prod_{\lambda\in\Lambda} \tau_\lambda$, and then we say that $\left(\prod_{\lambda\in\Lambda} \mathcal{X}_\lambda, \prod_{\lambda\in\Lambda} \tau_\lambda\right)$ is the product space.*

Proof The proof is similar to the proof of Theorem 3.7 (Liang and Yan 2014). □

Note 1 From Theorems 3.3 and 3.9, we have

$$\tau(\mathcal{A}) = \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda=\mathcal{A}} \bigwedge_{\lambda\in\Lambda} \mathfrak{S}^{(\sqcap)}(\mathcal{B}_\lambda) = \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda=\mathcal{A}} \bigwedge_{\lambda\in\Lambda} \bigvee_{\sqcap_{\alpha\in\Gamma} \mathcal{C}_\alpha=\mathcal{B}_\lambda} \bigwedge_{\alpha\in\Gamma} \mathfrak{S}(\mathcal{C}_\alpha)$$
$$= \bigvee_{\bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda=\mathcal{A}} \bigwedge_{\lambda\in\Lambda} \bigvee_{\sqcap_{\alpha\in\Gamma} \mathcal{C}_\alpha=\mathcal{B}_\lambda} \bigwedge_{\alpha\in\Gamma} \bigvee_{j\in J} \bigvee_{P_j^{-1}(\mathcal{D})=\mathcal{C}_\alpha} \tau_j(\mathcal{D}).$$

Therefore, we have the following consequence:

Corollary 3.2 *Let* $\left(\prod_{\alpha\in J}\mathcal{X}_\alpha, \prod_{\alpha\in J}\tau_\alpha\right)$ *be the product space of a family of L-fuzzifying topological spaces* $\{(\mathcal{X}_\alpha,\tau_\alpha):\alpha\in J\}$. *Then* $P_j:\left(\prod_{\alpha\in J}\mathcal{X}_\alpha, \prod_{\alpha\in J}\tau_\alpha\right)\longrightarrow(\mathcal{X}_j,\tau_j)$ *is L-fuzzifying continuous, for all* $j\in J$.

Proposition 3.1 *Let* $\mathcal{L}$ *be a completely distributive lattice and* $\left(\prod_{\lambda\in\Lambda}\mathcal{X}_\lambda, \prod_{\lambda\in\Lambda}\tau_\lambda\right)$ *be the product space of a family of L-fuzzifying topological spaces* $\{(\mathcal{X}_\lambda,\tau_\lambda)\mid\lambda\in\Lambda\}$. *Then*

$$\tau_\lambda(\mathcal{A}_\lambda)\leq\left(\prod_{\lambda\in\Lambda}\tau_\lambda\right)\left(P_\lambda^{-1}(\mathcal{A}_\lambda)\right),\ \text{for all } \lambda\in\Lambda,\ \mathcal{A}_\lambda\subseteq\mathcal{X}_\lambda$$

Proof Suppose that $\tau=\prod_{\lambda\in\Lambda}\tau_\lambda$. Then for all $\lambda\in\Lambda$, $\mathcal{A}_\lambda\subseteq\mathcal{X}_\lambda$, we have $\tau\left(P_\lambda^{-1}(\mathcal{A}_\lambda)\right)=\bigwedge_{x_\lambda\in P_\lambda^{-1}(\mathcal{A}_\lambda)}\mathfrak{N}^\tau_{x_\lambda}\left(P_\lambda^{-1}(\mathcal{A}_\lambda)\right)\geq\bigwedge_{x_\lambda\in P_\lambda^{-1}(\mathcal{A}_\lambda)}\mathfrak{N}^{\tau_\lambda}_{P_\lambda(x_\lambda)}(\mathcal{A}_\lambda)=\bigwedge_{P(x_\lambda)\in\mathcal{A}_\lambda}\mathfrak{N}^{\tau_\lambda}_{P_\lambda(x_\lambda)}(\mathcal{A}_\lambda)\geq\bigwedge_{y_\lambda\in\mathcal{A}_\lambda}\mathfrak{N}^{\tau_\lambda}_{y_\lambda}(\mathcal{A}_\lambda)=\tau_\lambda(\mathcal{A}_\lambda)$, where $x_\lambda\in P_\lambda^{-1}(\mathcal{A}_\lambda)\Leftrightarrow P(x_\lambda)\in\mathcal{A}_\lambda$, and $\mathfrak{N}^\tau$, $\mathfrak{N}^{\tau_\lambda}$ express τ, τ_λ-neighborhood system in $\prod_{\lambda\in\Lambda}\mathcal{X}_\lambda$ and $\mathcal{X}_\lambda$, respectively. □

4 LBFU space

First, we recall the following notations. Suppose $x\in\mathcal{X}$, $\mathcal{A}\subseteq\mathcal{X}$ and $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{X}\times\mathcal{X})$. Then we define

(1) $\Delta=\Delta_{\mathcal{X}}:=\{(x,x)\mid x\in\mathcal{X}\}$.
(2) $\mathcal{U}^{-1}:=\{(x,y)\mid(y,x)\in\mathcal{U}\}$.
(3) $\mathcal{U}\circ\mathcal{V}:=\{(x,y)\in\mathcal{X}\times\mathcal{X}\mid\exists z\in\mathcal{X},(x,z)\in\mathcal{V}\text{ and }(z,y)\in\mathcal{U}\}$.
(4) $\mathcal{U}[x]:=\{y\in\mathcal{X}\mid(x,y)\in\mathcal{U}\}$.
(5) $\mathcal{U}[\mathcal{A}]:=\bigcup_{x\in\mathcal{A}}\mathcal{U}[x]=\{y\in\mathcal{X}\mid x\in\mathcal{A},(x,y)\in\mathcal{U}\}$.

Definition 4.1 A normal function $\mathfrak{U}:\mathcal{P}(\mathcal{X}\times\mathcal{X})\longrightarrow\mathcal{L}$ (see Remark 4.1 below) is called an *LB*-fuzzifying uniformity on $\mathcal{X}$, if the following axioms are satisfied for any $\mathcal{U},\mathcal{V}\subseteq\mathcal{X}\times\mathcal{X}$:
(LBFU1): If $\Delta\not\subseteq\mathcal{U}$, then $\mathfrak{U}(\mathcal{U})=\bot$;
(LBFU2): $\mathfrak{U}(\mathcal{U})\ll\mathfrak{U}(\mathcal{U}^{-1})$;
(LBFU3): If $\mathfrak{U}(\mathcal{U})\neq\bot$, then $\mathfrak{U}(\mathcal{U})\ll\bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}}\mathfrak{U}(\mathcal{V})$;
(LBFU4): $\mathfrak{U}(\mathcal{U})\wedge\mathfrak{U}(\mathcal{V})\ll\mathfrak{U}(\mathcal{U}\cap\mathcal{V})$;
(LBFU5): If $\mathfrak{U}(\mathcal{U})\neq\bot$ and $\mathcal{U}\subseteq\mathcal{V}$, then $\mathfrak{U}(\mathcal{U})\ll\mathfrak{U}(\mathcal{V})$.
The pair $(\mathcal{X},\mathfrak{U})$ is called an LBFU space.

Remark 4.1 Let $(\mathcal{X},\mathfrak{U})$ be an LBFU space.

(1) Since $\mathfrak{U}$ is normal, then there exists $\mathcal{U}\subseteq\mathcal{X}\times\mathcal{X}$ such that $\mathfrak{U}(\mathcal{U})=\top$. From (LBFU5) we have $\mathfrak{U}(\mathcal{X}\times\mathcal{X})=\top$.
(2) From (LBFU1), if $\mathfrak{U}(\mathcal{U})\neq\bot$, then $\Delta\subseteq\mathcal{U}$.
(3) If $\mathcal{L}=[0,1]$, then the *LB*-fuzzifying uniformity coincides with the fuzzifying uniformity due to Ying Ying (1993b).

(4) If $\mathcal{L} = \{\bot, \top\}$, then we obtain the diagonal uniformity (Kotzé 1999, Chapter 8).

Example 4.1 Let $\mathcal{X} = \{a, b\}$ be a finite set. Define a map $\mathfrak{U} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$, where $\mathcal{L} = \mathcal{L}(\mathcal{X}, \subseteq, \cap, \cup, \top, \bot, ^c)$, as follows:

$$\mathfrak{U}(\mathcal{U}) = \begin{cases} \top, & \text{if } \Delta \subseteq \mathcal{U}, \\ \bot, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{X}, \mathfrak{U})$ is an LBFU space.

Example 4.2 Let $\mathcal{X}$ be a finite set. Define a map $\mathfrak{U} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow [0, 1]$, as follows:

$$\mathfrak{U}(\mathcal{U}) = \begin{cases} 1, & \text{if } \mathcal{U} = \mathcal{X} \times \mathcal{X}, \\ 0.5, & \text{if } \Delta \subset \mathcal{U} \neq \mathcal{X} \times \mathcal{X}, \\ 0.4, & \text{if } \Delta = \mathcal{U}, \\ 0, & \text{Otherwise.} \end{cases}$$

Then $(\mathcal{X}, \mathfrak{U})$ is an LBFU space.

Note that in the unit interval $(0, 1]$, the way below relation becomes the strictly-less-than relation.

Note 2 Extensions of standard mathematical notions abound, and the value of any resulting theory should be judged by the strength of its link with firmly established theories. The following proposition gives the link between the concept of LBFU space and the diagonal uniformity (1999, Definition 1.1.1, Chapter 8). The later element is arguably the most important concept in the study of uniformities. Thus, a comprehensive research of LBFU spaces is important for future work in the area of canonical examples in lattice-valued topology.

Proposition 4.1 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space and suppose that $\mathcal{L}$ is S-compact. Then for each $\alpha \in \mathcal{L} - \{\bot\}$, $\mathfrak{U}_\alpha = \{\mathcal{U} : \mathfrak{U}(\mathcal{U}) \geq \alpha\}$ is the diagonal uniformity.*

Proof Since $\mathfrak{U}$ is normal, then there exists $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$ such that $\mathfrak{U}(\mathcal{U}) = \top \geq \alpha$. Hence $\mathcal{U} \in \mathfrak{U}_\alpha$ and $\mathfrak{U}_\alpha \neq \phi$. Also, suppose $\phi \in \mathfrak{U}_\alpha$. Then $\mathfrak{U}(\phi) \geq \alpha$. So $\mathfrak{U}(\phi) \neq \bot$. Hence $\Delta \subseteq \phi$, a contradiction. Therefore $\phi \notin \mathfrak{U}_\alpha$;

(DU1): Suppose $\mathcal{U} \in \mathfrak{U}_\alpha$. Then $\mathfrak{U}(\mathcal{U}) \geq \alpha$ or $\mathfrak{U}(\mathcal{U}) \neq \bot$. Hence from (LBFU1) we have $\Delta \subseteq \mathcal{U}$.

(DU2): Suppose $\mathcal{U} \in \mathfrak{U}_\alpha$. Then $\mathfrak{U}(\mathcal{U}) \geq \alpha$. From (LBFU2), we have $\alpha \leq \mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}^{-1})$. Hence $\alpha \leq \mathfrak{U}(\mathcal{U}) \leq \mathfrak{U}(\mathcal{U}^{-1})$ or $\mathcal{U}^{-1} \in \mathfrak{U}_\alpha$.

(DU3): Suppose $\mathcal{U} \in \mathfrak{U}_\alpha$. Then $\mathfrak{U}(\mathcal{U}) \neq \bot$ and by (LBFU3) we have $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{U}(\mathcal{V})$, $\mathcal{V} \subseteq \mathcal{X} \times \mathcal{X}$. Hence $\mathfrak{U}(\mathcal{U}) \leq \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{U}(\mathcal{V})$. Since $\mathcal{L}$ is S-compact, then there exists $\mathcal{V}_o \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ such that $\mathcal{V}_o \circ \mathcal{V}_o \subseteq \mathcal{U}$ and $\mathfrak{U}(\mathcal{U}) \leq \mathfrak{U}(\mathcal{V}_o)$. Therefore, $\mathfrak{U}(\mathcal{V}_o) \geq \alpha$ and $\mathcal{V}_o \in \mathfrak{U}_\alpha$.

(DU4): Suppose that $\mathcal{U}, \mathcal{V} \in \mathfrak{U}_\alpha$. Then $\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) \geq \alpha$. From (LBFU4) we have $\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) \ll \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$ which implies $\alpha \leq \mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) \leq \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$. Hence $\mathcal{U} \cap \mathcal{V} \in \mathfrak{U}_\alpha$.

(DU5): Suppose $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ are such that $\mathcal{U} \in \mathfrak{U}_\alpha$ and $\mathcal{U} \subseteq \mathcal{V}$. Then, from (LBFU5), $\mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{V})$. Thus, $\alpha \leq \mathfrak{U}(\mathcal{U}) \leq \mathfrak{U}(\mathcal{V})$. Therefore $\mathfrak{U}(\mathcal{V}) \geq \alpha$ and $\mathcal{V} \in \mathfrak{U}_\alpha$. □

Definition 4.2 Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and let $\mathfrak{B} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ be a normal function such that $\mathfrak{B}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U})$ for any $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$ and $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \subseteq \mathcal{U}} \mathfrak{B}(\mathcal{V})$. Then $\mathfrak{B}$ is called a base of $\mathfrak{U}$.

Remark 4.2 It is obvious that if $\mathfrak{B}$ is a base of $\mathfrak{U}$, then $\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V})$.

Lemma 4.1 *Let $\alpha_i, \beta_i \in \mathcal{L}$ and $\alpha_i \ll \beta_i$, for each $i \in \Lambda$. Then*

(1) $\bigvee_{i\in\Lambda} \alpha_i \ll \bigvee_{i\in\Lambda} \beta_i$;
(2) If $\mathcal{L}$ is I-compact, then $\bigwedge_{i\in\Lambda} \alpha_i \ll \bigwedge_{i\in\Lambda} \beta_i$.

Proof (1) Suppose that for every directed set $\mathcal{D} \subseteq \mathcal{L}$, $\bigvee_{i\in\Lambda} \beta_i \leq \bigvee \mathcal{D}$ is true. Then $\beta_i \leq \bigvee \mathcal{D}$ for every $i \in \Lambda$. Since $\alpha_i \ll \beta_i$, then there exists $d \in \mathcal{D}$ such that $\alpha_i \leq d$ for each $i \in \Lambda$. Hence $\bigvee_{i\in\Lambda} \alpha_i \leq d$. We conclude $\bigvee_{i\in\Lambda} \alpha_i \ll \bigvee_{i\in\Lambda} \beta_i$.

(2) Suppose that for every directed set $\mathcal{D} \subseteq \mathcal{L}$, $\bigwedge_{i\in\Lambda} \beta_i \leq \bigvee \mathcal{D}$ is true. Since $\mathcal{L}$ is I-compact, then there exists $i_\circ \in \Lambda$ such that $\beta_{i_\circ} \leq \bigvee \mathcal{D}$. Since $\alpha_{i_\circ} \ll \beta_{i_\bullet}$, then there exists $d \in \mathcal{D}$ such that $\alpha_{i_\circ} \leq d$. Hence $\bigwedge_{i\in\Lambda} \alpha_i \leq d$. We conclude $\bigwedge_{i\in\Lambda} \alpha_i \ll \bigwedge_{i\in\Lambda} \beta_i$. □

Our next theorem identifies some properties of the bases of an *LB*-fuzzifying uniformity. It also gives conditions that ensure that we are in the presence of a base of some *LB*-fuzzifying uniformity:

Theorem 4.1 *If $\mathfrak{B} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ is a base of some LB-fuzzifying uniformity, then $\mathfrak{B}$ satisfies the following conditions:*

(LBB1): If $\Delta \nsubseteq \mathcal{U}$, then $\mathfrak{B}(\mathcal{U}) = \perp$;
(LBB2): $\mathfrak{B}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\subseteq\mathcal{U}^{-1}} \mathfrak{B}(\mathcal{V})$;
(LBB3): $\mathfrak{B}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V})$;
(LBB4): $\mathfrak{B}(\mathcal{U}) \wedge \mathfrak{B}(\mathcal{V}) \ll \bigvee_{\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}} \mathfrak{B}(\mathcal{W})$.

Conversely, if $\mathfrak{B} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ satisfies (LBB1-LBB4) and $\mathcal{L}$ is infinitely meet distributive, then $\mathfrak{B}$ is a base of some LB-fuzzifying uniformity $\mathfrak{U}$ on $\mathcal{X}$.

Proof Suppose $\mathfrak{B}$ is a base of some *LB*-fuzzifying uniformity $\mathfrak{U}$ on $\mathcal{X}$.

(LBB1): If $\mathfrak{B}(\mathcal{U}) \neq \perp$, then $\mathfrak{U}(\mathcal{U}) \neq \perp$ and from (LBFU1), we have $\Delta \subseteq \mathcal{U}$.

(LBB2): From (LBFU2) we have $\mathfrak{B}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}^{-1}) \ll \bigvee_{\mathcal{V}\subseteq\mathcal{U}^{-1}} \mathfrak{B}(\mathcal{V})$.

(LBB3): From (LBFU3) and Lemma 4.1,

$\mathfrak{B}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{V}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \bigvee_{\mathcal{W}\subseteq\mathcal{V}} \mathfrak{B}(\mathcal{W}) \leq \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{V}\circ\mathcal{V}} \mathfrak{B}(\mathcal{W}) = \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{W})$.

Therefore, $\mathfrak{B}(\mathcal{U}) \ll \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{W})$.

(LBB4): Since $\mathfrak{B}(U) \ll \mathfrak{U}(\mathcal{U})$ and $\mathfrak{B}(\mathcal{V}) \ll \mathfrak{U}(\mathcal{V})$, then from condition (LBFU4) we deduce $\mathfrak{B}(\mathcal{U}) \wedge \mathfrak{B}(\mathcal{V}) \leq \mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) \ll \mathfrak{U}(\mathcal{U}\cap\mathcal{V}) \ll \bigvee_{\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}} \mathfrak{B}(\mathcal{W})$. Thus $\mathfrak{B}(\mathcal{U}) \wedge \mathfrak{B}(\mathcal{V}) \ll \bigvee_{\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}} \mathfrak{B}(\mathcal{W})$.

Conversely, for any $\mathfrak{B}$ satisfying (LBB1)-(LBB4), we set $\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V})$. We shall check that it is an *LB*-fuzzifying uniformity on $\mathcal{X}$. Since $\mathfrak{B}$ is normal, then there exists $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$ such that $\mathfrak{B}(\mathcal{U}) \neq \perp$. Hence $\mathfrak{U}(\mathcal{U}) \neq \perp$ and $\mathfrak{U}$ is normal.

(LBFU1): Suppose $\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V}) \neq \perp$. Then, there exists $\mathcal{V}_o \subseteq \mathcal{U}$ such that $\mathfrak{B}(\mathcal{V}_o) \neq \perp$. Thus, from (LBB1), we have $\Delta \subseteq \mathcal{V}_o \subseteq \mathcal{U}$.

(LBFU2): From (LBB2) and Lemma 4.1, we have $\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V}) \ll \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \bigvee_{\mathcal{W}\subseteq\mathcal{V}^{-1}} \mathfrak{B}(\mathcal{W}) = \bigvee_{\mathcal{W}\subseteq\mathcal{U}^{-1}} \mathfrak{B}(\mathcal{W}) = \mathfrak{U}(\mathcal{U}^{-1})$. Hence $\mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}^{-1})$.

(LBFU3): Suppose that $\mathfrak{U}(\mathcal{U}) \neq \perp$, for any $\mathcal{U} \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$. Then, from (LBB3) and Lemma 4.1, we have

$$\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{V}) \ll \bigvee_{\mathcal{V}\subseteq\mathcal{U}} \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{V}} \mathfrak{B}(\mathcal{W}) = \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{W}) \leq \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \bigvee_{\mathcal{Z}\subseteq\mathcal{W}} \mathfrak{B}(\mathcal{Z}) = \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{W}).$$

Hence $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{W}\circ\mathcal{W}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{W})$.

(LBFU4): Since $\mathcal{L}$ is infinitely meet distributive, then from (LBB4) and Lemma 4.1,

$$\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) = \left(\bigvee_{\mathcal{W}_1\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{W}_1)\right) \wedge \left(\bigvee_{\mathcal{W}_2\subseteq\mathcal{V}} \mathfrak{B}(\mathcal{W}_2)\right) = \bigvee_{\mathcal{W}_1\subseteq\mathcal{U},\ \mathcal{W}_2\subseteq\mathcal{V}} (\mathfrak{B}(\mathcal{W}_1) \wedge \mathfrak{B}(\mathcal{W}_2))$$
$$\ll \bigvee_{\mathcal{W}_1\cap\mathcal{W}_2\subseteq\mathcal{U}\cap\mathcal{V}} \bigvee_{\mathcal{W}\subseteq\mathcal{W}_1\cap\mathcal{W}_2} \mathfrak{B}(\mathcal{W}) = \bigvee_{\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}} \mathfrak{B}(\mathcal{W}) = \mathfrak{U}(\mathcal{U}\cap\mathcal{V}).$$

Hence $\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) \ll \mathfrak{U}(\mathcal{U}\cap\mathcal{V})$.

(LBFU5): Suppose that $\mathfrak{U}(\mathcal{U}) \neq \perp$ and $\mathcal{U} \subseteq \mathcal{V}$. Then $\mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{W}\subseteq\mathcal{U}} \mathfrak{B}(\mathcal{W}) \ll \bigvee_{\mathcal{W}\subseteq\mathcal{U}} \bigvee_{\mathcal{Z}\subseteq\mathcal{W}^{-1}} \mathfrak{B}(\mathcal{Z})$

$$= \bigvee_{\mathcal{Z}\subseteq\mathcal{U}^{-1}} \mathfrak{B}(\mathcal{Z}) \leq \bigvee_{\mathcal{Z}\subseteq\mathcal{V}^{-1}} \mathfrak{B}(\mathcal{Z}) = \mathfrak{U}(\mathcal{V}^{-1}) \ll \mathfrak{U}(\mathcal{V}). \qquad \square$$

Subbases of LBFU spaces are also worth considering. They are defined as follows:

Definition 4.3 Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and let $\mathfrak{S} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ be a normal function. If for any $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$, $\mathfrak{S}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U})$ and $\mathfrak{S}^{(\cap)}(\mathcal{U}) = \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i=\mathcal{U}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i)$, $n \in \mathbb{N}$, and $\mathcal{U}_i \subseteq \mathcal{X} \times \mathcal{X}$ is a base of $\mathfrak{U}$, then $\mathfrak{S}$ is said to be a subbase of $\mathfrak{U}$.

Theorem 4.2 *Let $\mathcal{L}$ be an I-compact completely distributive lattice. Let us suppose that $\mathfrak{S} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ is a normal function that satisfies the following conditions for any $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X} \times \mathcal{X}$:*
(LBS1): If $\Delta \nsubseteq \mathcal{U}$, then $\mathfrak{S}(\mathcal{U}) = \perp$;
(LBS2): $\mathfrak{S}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\subseteq\mathcal{U}^{-1}} \mathfrak{S}(\mathcal{U})$;
(LBS3): $\mathfrak{S}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{S}(\mathcal{V})$.
Then $\mathfrak{S}$ is a subbase of some LB-fuzzifying uniformity on $\mathcal{X}$.

Proof Since $\mathfrak{S}$ is normal, then $\mathfrak{S}(\mathcal{X} \times \mathcal{X}) = \top$. Hence $\mathfrak{S}^{(\cap)}(\mathcal{X} \times \mathcal{X}) = \top$. Therefore, $\mathfrak{S}^{(\cap)}$ is normal. Now, it suffices to check (LBB1-LBB4) for $\mathfrak{S}^{(\cap)}$.

(LBB1): Suppose that $\mathfrak{S}^{(\cap)}(\mathcal{U}) \neq \perp$. Then $\bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i=\mathcal{U}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) \neq \perp$. Therefore, there exists a finite set $\Lambda = \{1, 2, .., n\}$ and $\mathcal{U}_\lambda \in \mathcal{P}(\mathcal{X}\times\mathcal{X})$ $(\lambda \in \Lambda)$ such that $\bigcap_{\lambda\in\Lambda} \mathcal{U}_\lambda = \mathcal{U}$ and for any $\lambda \in \Lambda$, $\mathfrak{S}(\mathcal{U}_\lambda) \neq \perp$. From (LBS1), $\Delta \subseteq \mathcal{U}_\lambda$ for any $\lambda \in \Lambda$. Hence $\Delta \subseteq \bigcap_{\lambda\in\Lambda} \mathcal{U}_\lambda = \mathcal{U}$.

(LBB2): Suppose that $\mathcal{M}_i = \left\{\mathcal{V}_i \,\middle|\, \mathcal{V}_i \subseteq \mathcal{U}_i^{-1}\right\}$, $(i = 1, 2, ..., n)$. If $f \in \prod_{i=1}^{n} \mathcal{M}_i$, then $\mathcal{V} = \bigcap_{i=1}^{n} f(i) \subseteq \bigcap_{i=1}^{n} \mathcal{U}_i^{-1} = (\bigcap_{i=1}^{n} \mathcal{U}_i)^{-1} = \mathcal{U}^{-1}$. Then from (LBS2), we obtain

$$\mathfrak{S}^{(\cap)}(\mathcal{U}) = \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) \ll \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigwedge_{i=1}^{n} \bigvee_{\mathcal{V}_i \subseteq \mathcal{U}_i^{-1}} \mathfrak{S}(\mathcal{V}_i)$$
$$= \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigvee_{f(i) \in \prod_{i=1}^{n} \mathcal{M}_i} \bigwedge_{i=1}^{n} \mathfrak{S}(f(i)) \leq \bigvee_{\bigcap_{i=1}^{n} \mathcal{V}_i \subseteq \mathcal{U}^{-1}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{V}_i) = \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \mathfrak{S}^{(\cap)}(\mathcal{V}).$$

Therefore, $\mathfrak{S}^{(\cap)}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \mathfrak{S}^{(\cap)}(\mathcal{V})$.

(LBB3): Suppose that $\mathcal{M}_i = \{\mathcal{V}_i : \mathcal{V}_i \circ \mathcal{V}_i \subseteq \mathcal{U}_i\}$. If $f \in \prod_{i=1}^{n} \mathcal{M}_i$, then $\mathcal{V} \circ \mathcal{V} = (\bigcap_{i=1}^{n} f(i)) \circ (\bigcap_{i=1}^{n} f(i)) = \bigcap_{i=1}^{n} (f(i) \circ f(i)) = \bigcap_{i=1}^{n} (\mathcal{V}_i \circ \mathcal{V}_i) \subseteq \bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}$. Since $\mathcal{L}$ is I-compact, then from (LBS3), we have

$$\mathfrak{S}^{(\cap)}(\mathcal{U}) = \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) \ll \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigwedge_{i=1}^{n} \bigvee_{\mathcal{V}_i \circ \mathcal{V}_i \subseteq \mathcal{U}_i} \mathfrak{S}(\mathcal{V}_i)$$
$$= \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigvee_{f \in \prod_{i=1}^{n} \mathcal{M}_i} \bigwedge_{i=1}^{n} \mathfrak{S}(f(i)) \leq \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{S}^{(\cap)}(\mathcal{V}).$$

Hence $\mathfrak{S}^{(\cap)}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{S}^{(\cap)}(\mathcal{V})$.

(LBB4): From (LBB2) above, we have

$$\mathfrak{S}^{(\cap)}(\mathcal{U}) \wedge \mathfrak{S}^{(\cap)}(\mathcal{V}) = \left(\bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) \right) \wedge \left(\bigvee_{\bigcap_{i=1}^{n} \mathcal{V}_i = \mathcal{V}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{V}_i) \right)$$
$$= \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{U}} \bigvee_{\bigcap_{i=1}^{n} \mathcal{V}_i = \mathcal{V}} \left(\left(\bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) \right) \wedge \left(\bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{V}_i) \right) \right)$$
$$\leq \bigvee_{\bigcap_{\lambda=1}^{n} \mathcal{W}_\lambda = \mathcal{U} \cap \mathcal{V}} \bigwedge_{\lambda=1}^{n} \mathfrak{S}(\mathcal{W}_\lambda) = \mathfrak{S}^{(\cap)}(\mathcal{U} \cap \mathcal{V})$$
$$\ll \bigvee_{\mathcal{W}^{-1} \subseteq \mathcal{U} \cap \mathcal{V}} \mathfrak{S}^{(\cap)}(\mathcal{W}^{-1}).$$

Therefore $\mathfrak{S}^{(\cap)}(\mathcal{U}) \wedge \mathfrak{S}^{(\cap)}(\mathcal{V}) \ll \bigvee_{\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}} \mathfrak{S}^{(\cap)}(\mathcal{W})$. □

Proposition 4.2 *The following two statements hold true:*

(1) If $\mathfrak{U}_i$ is an LB-fuzzifying uniformity on $\mathcal{X}$ for any $i \in \Lambda$, then $\mathfrak{U} = \bigvee_{i \in \Lambda} \mathfrak{U}_i$ is a subbase for some LB-fuzzifying uniformity on $\mathcal{X}$.
(2) If $\mathcal{L}$ is I-compact and $\mathfrak{U}_i$ is an LB-fuzzifying uniformity on $\mathcal{X}$ for any $i \in \Lambda$, then $\mathfrak{U} = \bigwedge_{i \in \Lambda} \mathfrak{U}_i$ is an LB-fuzzifying uniformity on $\mathcal{X}$.

Proof (1) Since $\mathfrak{U}_i$ is normal for each $i \in \Lambda$, then there exists $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$ such that $\mathfrak{U}_i(\mathcal{U}) = \top$. Hence $\mathfrak{U}(\mathcal{U}) = \left(\bigvee \mathfrak{U}_i\right)(\mathcal{U}) = \bigvee(\mathfrak{U}_i(\mathcal{U})) = \top$.
To prove that $\mathfrak{U}$ is a subbase of some *LB*-fuzzifying uniformity on $\mathcal{X}$, it suffices to check (LBS1-LBS3) for $\mathfrak{U}$.

(LBS1): Suppose that $\mathfrak{U}(\mathcal{U}) = \left(\bigvee_{i \in \Lambda} \mathfrak{U}_i\right)(\mathcal{U}) = \bigvee_{i \in \Lambda} (\mathfrak{U}_i(\mathcal{U})) \neq \bot, \mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$. Then there exists $\lambda \in \Lambda$ such that $\mathfrak{U}_\lambda(\mathcal{U}) \neq \bot$. Hence $\Delta \subseteq \mathcal{U}$.

(LBS2): $\mathfrak{U}(\mathcal{U}) = \left(\bigvee_{i \in \Lambda} \mathfrak{U}_i\right)(\mathcal{U}) = \bigvee_{i \in \Lambda} (\mathfrak{U}_i(\mathcal{U})) \ll \bigvee_{i \in \Lambda} \left(\mathfrak{U}_i(\mathcal{U}^{-1})\right) \ll \bigvee_{i \in \Lambda} \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \bigvee_{i \in \Lambda} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \left(\bigvee_{i \in \Lambda} \mathfrak{U}_i\right)(\mathcal{V}) = \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \mathfrak{U}(\mathcal{V})$. Hence $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \subseteq \mathcal{U}^{-1}} \mathfrak{U}(\mathcal{V})$.

(LBS3): $\mathfrak{U}(\mathcal{U}) = \left(\bigvee_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{U}) = \bigvee_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \ll \bigvee_{i\in\Lambda} \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \bigvee_{i\in\Lambda} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \left(\bigvee_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{V})$.

Therefore $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{V})$.

Then $\bigvee_{i\in\Lambda} \mathfrak{U}_i$ is a subbase of some *LB*-fuzzifying uniformity on $\mathcal{X}$.

(2) Since $\mathfrak{U}_i$ is normal for each $i \in \Lambda$, then there exists $\mathcal{U} \subseteq \mathcal{X} \times \mathcal{X}$, $\mathfrak{U}_i(\mathcal{U}) = \top$ for each $i \in \Lambda$. Hence $\mathfrak{U}(\mathcal{U}) = (\bigwedge \mathfrak{U}_i)(\mathcal{U}) = \bigwedge(\mathfrak{U}_i(\mathcal{U})) = \top$.

To prove that $\mathfrak{U}$ is an *LB*-fuzzifying uniformity on $\mathcal{X}$, we check (LBFU1)-(LBFU5) for $\mathfrak{U}$. Since $\mathcal{L}$ is I-compact, then we can argue as follows.

(LBFU1): Suppose that $\mathfrak{U}(\mathcal{U}) = \left(\bigwedge_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{U}) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \neq \bot, \mathcal{U} \subseteq X \times X$. Then for each $i \in \Lambda, \mathfrak{U}_i(\mathcal{U}) \neq \bot$. Hence $\Delta \subseteq \mathcal{U}$.

(LBFU2): $\mathfrak{U}(\mathcal{U}) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \ll \bigwedge_{i\in\Lambda} \left(\mathfrak{U}_i(\mathcal{U}^{-1})\right) = \left(\bigwedge_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{U}^{-1}) = \mathfrak{U}(\mathcal{U}^{-1})$.

(LBFU3): $\mathfrak{U}(\mathcal{U}) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \ll \bigwedge_{i\in\Lambda} \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \bigwedge_{i\in\Lambda} \mathfrak{U}_i(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \left(\bigwedge_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{U}(\mathcal{V})$.

(LBFU4): $\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \wedge \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{V})) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U}) \wedge \mathfrak{U}_i(\mathcal{V})) \ll \bigwedge_{i\in\Lambda} \mathfrak{U}_i(\mathcal{U} \cap \mathcal{V}) = \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$.

(LBFU5): Suppose $\mathcal{U} \subseteq \mathcal{V}$ and $\mathfrak{U}(\mathcal{U}) \neq \bot$. Then $\mathfrak{U}(\mathcal{U}) = \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{U})) \ll \bigwedge_{i\in\Lambda} (\mathfrak{U}_i(\mathcal{V})) = \left(\bigwedge_{i\in\Lambda} \mathfrak{U}_i\right)(\mathcal{V}) = \mathfrak{U}(\mathcal{V})$.

Therefore $\mathfrak{U}$ is an *LB*-fuzzifying uniformity on $\mathcal{X}$. □

5 *L*-fuzzifying topological spaces induced by LBFU spaces

This section has two related targets. First we introduce three constructions of L-fuzzifying topologies induced by an LB-fuzzifying uniformity. This is done in Theorem 5.1, and also in Theorems 5.2 and 5.3. Then we investigate the main design that is given in Theorem 5.1. Its behavior when $\mathcal{L}$ belongs to different classes will be investigated afterwards.

Let us begin with the main construction in this section:

Theorem 5.1 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and assume that $\mathcal{L}$ is an I-compact infinitely distributive lattice. Then $\tau_{\mathfrak{U}} \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$ defined as $\tau_{\mathfrak{U}}(\mathcal{A}) = \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{U}[x]\subseteq\mathcal{A}} \mathfrak{U}(\mathcal{U})$, $\mathcal{A} \subseteq \mathcal{X}$, is an L-fuzzifying topology on $\mathcal{X}$, which is called the L-fuzzifying (uniform) topology of $\mathfrak{U}$.*

Proof The proof is similar to the proof of Lemma 3.1 (Ying 1993b). □

The next result produces a natural property in relation with coarser L-fuzzifying (uniform) topologies of a common $\mathfrak{U}$:

Proposition 5.1 *Let $\mathfrak{U}_1$, $\mathfrak{U}_2$ be two LB-fuzzifying uniformities on $\mathcal{X}$. If $\mathfrak{U}_1 \leq \mathfrak{U}_2$, then $\tau_{\mathfrak{U}_1} \leq \tau_{\mathfrak{U}_2}$.*

Proof $\tau_{\mathfrak{U}_1}(\mathcal{A}) = \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{U}[x]\subseteq\mathcal{A}} \mathfrak{U}_1(\mathcal{U}) \le \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{U}[x]\subseteq\mathcal{A}} \mathfrak{U}_2(\mathcal{U}) = \tau_{\mathfrak{U}_2}(\mathcal{A})$.
Therefore $\tau_{\mathfrak{U}_1} \le \tau_{\mathfrak{U}_2}$. □

In the next three theorems we produce other particular expressions for L-fuzzifying (uniform) topologies on $\mathcal{X}$ generated from certain LB-fuzzifying uniformities.

Theorem 5.2 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and let $\mathcal{L}$ be an infinitely distributive lattice. Then $\tau^{(1)}_{\mathfrak{U}} \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$ defined as $\tau^{(1)}_{\mathfrak{U}}(\mathcal{A}) = \bigvee_{\mathcal{U}[\mathcal{A}^c]\cap\mathcal{U}[\mathcal{A}]=\phi} \mathfrak{U}(\mathcal{U})$, $\mathcal{A} \subseteq \mathcal{X}, \mathcal{U} \subseteq \mathcal{X}\times\mathcal{X}$, is an L-fuzzifying (uniform) topology on $\mathcal{X}$.*

Proof (1) Since $\mathcal{U}[\mathcal{X}] = \mathcal{X}$ and $\mathcal{U}[\phi] = \phi$, then $\tau^{(1)}_{\mathfrak{U}}(\mathcal{X}) = \tau^{(1)}_{\mathfrak{U}}(\phi) = \top$.

(2) Now, set $\mathcal{B} = ((\mathcal{U}_1\cap\mathcal{U}_2)[(\mathcal{A}_1\cap\mathcal{A}_2)^c])\cap((\mathcal{U}_1\cap\mathcal{U}_2)[\mathcal{A}_1\cap\mathcal{A}_2]) \subseteq \big(\mathcal{U}_1[\mathcal{A}_1^c]\cup\mathcal{U}_2[\mathcal{A}_2^c]\big)\cap(\mathcal{U}_1[\mathcal{A}_1]\cap\mathcal{U}_2[\mathcal{A}_2]) \subseteq \big(\mathcal{U}_1[\mathcal{A}_1^c]\cap\mathcal{U}_1[\mathcal{A}_1]\big)\cup\big(\mathcal{U}_2[\mathcal{A}_2^c]\cap\mathcal{U}_2[\mathcal{A}_2]\big)$.

$$\begin{aligned}
\tau^{(1)}_{\mathfrak{U}}(\mathcal{A}_1)\wedge\tau^{(1)}_{\mathfrak{U}}(\mathcal{A}_2) &= \left(\bigvee_{\mathcal{U}_1[\mathcal{A}_1^c]\cap\mathcal{U}_1[\mathcal{A}_1]=\phi}\mathfrak{U}(\mathcal{U}_1)\right)\wedge\left(\bigvee_{\mathcal{U}_2[\mathcal{A}_2^c]\cap\mathcal{U}_2[\mathcal{A}_2]=\phi}\mathfrak{U}(\mathcal{U}_2)\right)\\
&= \bigvee_{(\mathcal{U}_1[\mathcal{A}_1^c]\cap\mathcal{U}_1[\mathcal{A}_1])\cup(\mathcal{U}_2[\mathcal{A}_2^c]\cap\mathcal{U}_2[\mathcal{A}_2])=\phi}(\mathfrak{U}(\mathcal{U}_1)\wedge\mathfrak{U}(\mathcal{U}_2))\\
&\le \bigvee_{\mathcal{B}=\phi}(\mathfrak{U}(\mathcal{U}_1)\wedge\mathfrak{U}(\mathcal{U}_2)) \ll \bigvee_{\mathcal{B}=\phi}\mathfrak{U}(\mathcal{U}_1\cap\mathcal{U}_2)\\
&\le \bigvee_{\mathcal{B}=\mathcal{U}[(\mathcal{A}_1\cap\mathcal{A}_2)^c]\cap\mathcal{U}[\mathcal{A}_1\cap\mathcal{A}_2]=\phi}\mathfrak{U}(\mathcal{U}) = \tau^{(1)}_{\mathfrak{U}}(\mathcal{A}_1\cap\mathcal{A}_2).
\end{aligned}$$

$$\begin{aligned}
(3)\ \tau^{(1)}_{\mathfrak{U}}\left(\bigcup_{j\in\Lambda}\mathcal{A}_j\right) &= \bigvee_{\mathcal{U}\left[\bigcap_{j\in\Lambda}\mathcal{A}_j^c\right]\cap\mathcal{U}\left[\bigcup_{j\in\Lambda}\mathcal{A}_j\right]=\phi}\mathfrak{U}(\mathcal{U}) \ge \bigvee_{\bigcup_{j\in\Lambda}\left(\mathcal{U}\left[\mathcal{A}_j^c\right]\cap\mathcal{U}[\mathcal{A}_j]\right)=\phi}\mathfrak{U}(\mathcal{U})\\
&= \bigwedge_{j\in\Lambda}\bigvee_{\mathcal{U}\left[\mathcal{A}_j^c\right]\cap\mathcal{U}[\mathcal{A}_j]=\phi}\mathfrak{U}(\mathcal{U}) = \bigwedge_{j\in\Lambda}\tau^{(1)}_{\mathfrak{U}}(\mathcal{A}_j).
\end{aligned}$$

Hence $\tau^{(1)}_{\mathfrak{U}}$ is an L-fuzzifying uniform topology. □

Theorem 5.3 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space and*

$$\tau^{(2)}_{\mathfrak{U}}(\mathcal{A}) = \begin{cases}\top, & \text{if } \mathcal{A}=\phi,\\ \mathfrak{U}(\mathcal{U}\langle\mathcal{A}\rangle), & \text{if } \phi\neq\mathcal{A}\in\mathcal{P}(\mathcal{X}),\end{cases} \quad \textit{where } \mathcal{U}\langle\mathcal{A}\rangle = \begin{cases}\Delta, & \text{if } \mathcal{A}=\{x\},\\ \{(x,y)\,|x,y\in\mathcal{A}\}, & \text{otherwise.}\end{cases}$$

Then $\tau^{(2)}_{\mathfrak{U}}$ is an L-fuzzifying (uniform) topology on $\mathcal{X}$.

Proof (1) $\tau^{(2)}_{\mathfrak{U}}(\mathcal{X}) = \mathfrak{U}(\mathcal{U}\langle\mathcal{X}\rangle) = \mathfrak{U}(\mathcal{X}\times\mathcal{X}) = \top$, and $\tau^{(2)}_{\mathfrak{U}}(\phi) = \top$.

(2) Since $\mathcal{U}\langle\mathcal{A}_1\rangle\cap\mathcal{U}\langle\mathcal{A}_2\rangle = \mathcal{U}\langle\mathcal{A}_1\cap\mathcal{A}_2\rangle$, then
$\tau^{(2)}_{\mathfrak{U}}(\mathcal{A}_1)\wedge\tau^{2}_{\mathfrak{U}}(\mathcal{A}_2) = \mathfrak{U}(\mathcal{U}\langle\mathcal{A}_1\rangle)\wedge\mathfrak{U}(\mathcal{U}\langle\mathcal{A}_2\rangle) \ll \mathfrak{U}(\mathcal{U}\langle\mathcal{A}_1\rangle \cap \mathcal{U}\langle\mathcal{A}_2\rangle) = \mathfrak{U}(\mathcal{U}\langle\mathcal{A}_1\cap\mathcal{A}_2\rangle) = \tau^{(2)}_{\mathfrak{U}}(\mathcal{A}_1\cap\mathcal{A}_2)$.

(3) Since $\mathcal{U}\left(\langle\mathcal{A}_j\rangle\right)\subseteq\mathcal{U}\left(\langle\bigcup_{j\in\Lambda}\mathcal{A}_j\rangle\right)$, then we have

$$\bigwedge_{j\in\Lambda}\tau^{(2)}_{\mathfrak{U}}(\mathcal{A}_j) \le \tau^{(2)}_{\mathfrak{U}}(\mathcal{A}_j) = \mathfrak{U}(\mathcal{U}\langle\mathcal{A}_j\rangle) \ll \mathfrak{U}(\mathcal{U}\left\langle\bigcup_{j\in\Lambda}\mathcal{A}_j\right\rangle) = \tau^{(2)}_{\mathfrak{U}}\left(\bigcup_{j\in\Lambda}\mathcal{A}_j\right).$$

Then $\tau^{(2)}_{\mathfrak{U}}$ is an L-fuzzifying uniform topology. □

Theorem 5.4 *Let $\mathcal{L}$ be an S-compact infinitely meet distributive lattice, and let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space. Then $\tau^{(3)}_{\mathfrak{U}} \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$ defined as $\tau^{(3)}_{\mathfrak{U}}(\mathcal{A}) = \bigvee_{\alpha\in\mathcal{L}-\{\bot\},\ \mathcal{A}\in\tau_{\mathfrak{U}_\alpha}} \alpha$ is an L-fuzzifying* *topology, where $\tau_{\mathfrak{U}_\alpha}$ is the topology generated by the diagonal uniformity (see Höhle and Rodabaugh 1999, Theorem 1.2.1, Chapter 8).*

Proof (1) Since $\mathcal{X}, \phi \in \tau_{\mathfrak{U}_\alpha}$ for each $\alpha \in \mathcal{L} - \{\bot\}$, then we have $\tau^{(3)}_{\mathfrak{U}}(\mathcal{X}) = \tau^{(3)}_{\mathfrak{U}}(\phi) = \top$.

(2) For any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$, we have

$$\tau^{(3)}_{\mathfrak{U}}(\mathcal{A}) \wedge \tau^{(3)}_{\mathfrak{U}}(\mathcal{B}) = \left(\bigvee_{\alpha \in \mathcal{L}-\{\bot\},\ \mathcal{A} \in \tau_{\mathfrak{U}_\alpha}} \alpha \right) \wedge \left(\bigvee_{\beta \in \mathcal{L}-\{\bot\},\ \mathcal{B} \in \tau_{\mathfrak{U}_\beta}} \beta \right)$$

$$= \bigvee_{\alpha\wedge\beta \in \mathcal{L}-\{\bot\},\ \mathcal{A} \in \tau_{\mathfrak{U}_\alpha},\ \mathcal{B} \in \tau_{\mathfrak{U}_\beta}} (\alpha \wedge \beta)$$

$$\leq \bigvee_{\gamma \in \mathcal{L}-\{\bot\},\ \mathcal{A}\cap\mathcal{B} \in \tau_{\mathfrak{U}_\gamma}} \gamma = \tau^{(3)}_{\mathfrak{U}}(\mathcal{A} \cap \mathcal{B}).$$

(3) For any $\{\mathcal{A}_\lambda | \lambda \in \Lambda\} \subseteq \mathcal{P}(\mathcal{X})$, we have

$$\tau^{(3)}_{\mathfrak{U}}(\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda) = \bigvee_{\alpha \in \mathcal{L}-\{\bot\},\ \bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \in \tau_{\mathfrak{U}_\alpha}} \alpha \geq \bigvee_{\alpha \in \mathcal{L}-\{\bot\},\ \mathcal{A}_\lambda \in \tau_{\mathfrak{U}_\alpha},\ \lambda\in\Lambda} \alpha = \bigvee_{f \in \prod_{\lambda\in\Lambda} \mathcal{M}_\lambda} \bigwedge_{\lambda\in\Lambda} f(\lambda)$$

$$= \bigwedge_{\lambda\in\Lambda} \bigvee_{\alpha \in \mathcal{M}_\lambda} \alpha = \bigwedge_{\lambda\in\Lambda} \bigvee_{\alpha \in \mathcal{L}-\{\bot\},\ \mathcal{A}_\lambda \in \tau_{\mathfrak{U}_\alpha}} \alpha = \bigwedge_{\lambda\in\Lambda} \tau^{(3)}_{\mathfrak{U}}(\mathcal{A}_\lambda).$$

where $\mathcal{M}_\lambda = \{\alpha \in \mathcal{L} - \{\bot\} \mid \mathcal{A}_\lambda \in \tau_\lambda \forall \lambda \in \Lambda\}$. □

In the rest of this section we shall give some results concerning the uniform topology $\tau_{\mathfrak{U}}$ which was defined in Theorem 5.1. For the other topologies defined in this section, similar results can be obtained. Firstly we investigate the computation of interiors in case that $\mathcal{L}$ is an S-compact completely distributive lattice.

Theorem 5.5 *Suppose that $\mathcal{L}$ is an S-compact completely distributive lattice. Let $(\mathcal{X}, \mathfrak{U})$ be an LB-fuzzifying uniform space and let $\tau_{\mathfrak{U}}$ be the L-fuzzifying topology of $\mathfrak{U}$. Then for any $x \in \mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}$:*

$$Int(\mathcal{A})(x) = \bigvee_{\mathcal{U}[x]\subseteq\mathcal{A}} \mathfrak{U}(\mathcal{U}).$$

Proof The proof is similar to the proof of Theorem 3.2 (Ying 1993b). □

The conditions of the previous theorem allow us to further enhance the knowledge about the construction in Theorem 5.1:

Theorem 5.6 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and let $\tau_{\mathfrak{U}}$ be the fuzzifying topology of $\mathfrak{U}$. If $\mathcal{L}$ is an S-compact completely distributive lattice, then for any $x \in \mathcal{X}$ and $\mathcal{A} \subseteq \mathcal{X}$:*

(1) The neighborhood system of x, denoted by $\mathfrak{N}_x \in \mathcal{L}^{\mathcal{P}(\mathcal{X})}$, is given by $\mathfrak{N}_x(\mathcal{A}) = \bigvee_{\mathcal{U}[x]=\mathcal{A}} \mathfrak{U}(\mathcal{U})$.
(2) If $\mathfrak{B}$ is a base of $\mathfrak{U}$, then $\mathfrak{B}_x(\mathcal{A}) = \bigvee_{\mathcal{U}[x]=\mathcal{A}} \mathfrak{B}(\mathcal{U})$ is a base of $\mathfrak{N}_x$.
(3) If $\mathfrak{S}$ is a subbase of $\mathfrak{U}$, then $\mathfrak{S}_x(\mathcal{A}) = \bigvee_{\mathcal{U}[x]=\mathcal{A}} \mathfrak{S}(\mathcal{U})$ is a subbase of $\mathfrak{N}_x$.

Proof The proof is similar to the proof of Theorem 3.3 (Ying 1993b). □

Our last result in this section concerns the construction in Theorem 5.1 too. But now we investigate its behavior when $\mathcal{L}$ is a completely distributive lattice with order reversing involution $'$. Some technical results preceed that result, namely, Theorem 5.7 below.

Lemma 5.1 *Let $(\mathcal{X}_1, \tau_1)$ and $(\mathcal{X}_2, \tau_2)$ be two L-fuzzifying topological spaces on $\mathfrak{U}$, $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$. If $\mathcal{L}$ is an S-compact completely distributive lattice, then for any $\mathcal{A}_1 \subseteq \mathcal{X}_1$, $\mathcal{A}_2 \subseteq \mathcal{X}_2$, we have $\mathfrak{N}^1_{x_1}(\mathcal{A}_1) \wedge \mathfrak{N}^2_{x_2}(\mathcal{A}_2) \ll \mathfrak{N}_{(x_1,x_2)}(\mathcal{A}_1 \times \mathcal{A}_2)$.*

Proof $\mathfrak{N}^1_{x_1}(\mathcal{A}_1) \wedge \mathfrak{N}^2_{x_2}(\mathcal{A}_2) = \bigvee_{\mathcal{U}[x_1]=\mathcal{A}_1} \mathfrak{U}(\mathcal{U}) \wedge \bigvee_{\mathcal{V}[x_2]=\mathcal{A}_2} \mathfrak{U}(\mathcal{V}) = \bigvee_{\mathcal{U}[x_1]=\mathcal{A}_1, \mathcal{V}[x_2]=\mathcal{A}_2} (\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}))$

$$\ll \bigvee_{\mathcal{U}[x_1]=\mathcal{A}_1, \mathcal{V}[x_2]=\mathcal{A}_2} \mathfrak{U}(\mathcal{U} \cap \mathcal{V}) \leq \bigvee_{(\mathcal{U}\cap\mathcal{V})[x_1]\times(\mathcal{U}\cap\mathcal{V})[x_2]\subseteq\mathcal{A}_1\times\mathcal{A}_2} \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$$
$$\leq \bigvee_{\mathcal{W}[x_1]\times\mathcal{W}[x_2]\subseteq\mathcal{A}_1\times\mathcal{A}_2} \mathfrak{U}(\mathcal{W}) = \mathfrak{N}_{(x_1,x_2)}(\mathcal{A}_1 \times \mathcal{A}_2).$$ □

Lemma 5.2 *Let $(\mathcal{X}_1, \tau_1)$, and let $(\mathcal{X}_2, \tau_2)$ be two L-fuzzifying topological spaces on $\mathfrak{U}$, $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$. If $\mathcal{L}$ is an S-compact completely distributive lattice and $\mathfrak{B}_{x_i}$ is a base of $\mathfrak{N}_{x_i}$ $(i = 1, 2)$, then $\mathfrak{B}_{(x_1,x_2)}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathfrak{B}_{x_1}(\mathcal{A}_1) \wedge \mathfrak{B}_{x_2}(\mathcal{A}_2)$ is a base of $\mathfrak{N}_{(x_1,x_2)}$.*

Proof $\mathfrak{B}_{(x_1,x_2)}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathfrak{B}_{x_1}(\mathcal{A}_1) \wedge \mathfrak{B}_{x_2}(\mathcal{A}_2) \leq \mathfrak{N}_{x_1}(\mathcal{A}_1) \wedge \mathfrak{N}_{x_2}(\mathcal{A}_2) \ll \mathfrak{N}_{(x_1,x_2)}(\mathcal{A}_1 \times \mathcal{A}_2)$.
In addition,

$$\mathfrak{N}_{(x_1,x_2)}(W) \leq \bigvee_{\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}} (\mathfrak{N}_{x_1}(\mathcal{U}) \wedge \mathfrak{N}_{x_2}(\mathcal{V})) = \bigvee_{\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}} \left(\bigvee_{\mathcal{A}\subseteq\mathcal{U}} \mathfrak{B}_{x_1}(\mathcal{A}) \wedge \bigvee_{\mathcal{B}\subseteq\mathcal{V}} \mathfrak{B}_{x_2}(\mathcal{B}) \right)$$
$$= \bigvee_{\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}} \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{U}\times\mathcal{V}} \big(\mathfrak{B}_{x_1}(\mathcal{A}) \wedge \mathfrak{B}_{x_2}(\mathcal{B})\big) = \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \big(\mathfrak{B}_{x_1}(\mathcal{A}) \wedge \mathfrak{B}_{x_2}(\mathcal{B})\big)$$
$$= \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \mathfrak{B}_{(x_1,x_2)}(\mathcal{A} \times \mathcal{B}).$$ □

Lemma 5.3 *Let $(\mathcal{X}, \mathfrak{U})$ be an LBFU space, and let $\tau_{\mathfrak{U}}$ be the L-fuzzifying topology on $\mathfrak{U}$. If $\mathcal{L}$ is an S-compact completely distributive lattice, then for any $x, y \in \mathcal{X}$,*

$$\mathfrak{B}_{(x,y)}(\mathcal{A} \times \mathcal{B}) = \bigvee_{\mathcal{U}[x]=\mathcal{A}, \mathcal{U}[y]=\mathcal{B}, \mathcal{U}=\mathcal{U}^{-1}} \mathfrak{U}(\mathcal{U})$$

is a base of $\mathfrak{N}_{(x,y)}$, where $\mathfrak{N}_{(x,y)}$ is the L-fuzzifying neighborhood system of (x, y) with respect to $\tau_{\mathfrak{U}} \times \tau_{\mathfrak{U}}$.

Proof $\mathfrak{B}_{(x,y)}(\mathcal{A} \times \mathcal{B}) = \bigvee_{\mathcal{U}[x]=\mathcal{A}, \mathcal{U}[y]=\mathcal{B}, \mathcal{U}=\mathcal{U}^{-1}} \mathfrak{U}(\mathcal{U}) \leq \bigvee_{\mathcal{U}[x]=\mathcal{A}} \mathfrak{U}(\mathcal{U}) \wedge \bigvee_{\mathcal{U}[y]=\mathcal{B}} \mathfrak{U}(\mathcal{U})$
$= \mathfrak{N}_x(\mathcal{A}) \wedge \mathfrak{N}_y(\mathcal{B}) \ll \mathfrak{N}_{(x,y)}(\mathcal{A} \times \mathcal{B})$.
Therefore $\mathfrak{B}_{(x,y)}(\mathcal{A} \times \mathcal{B}) \ll \mathfrak{N}_{(x,y)}(\mathcal{A} \times \mathcal{B})$.
In addition, for any $\mathcal{W} \subseteq \mathcal{X} \times \mathcal{X}$,

$$\mathfrak{N}_{(x,y)}(\mathcal{W}) = \bigwedge_{(x,y)\in\mathcal{W}} \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \big(\mathfrak{N}_x(\mathcal{A}) \wedge \mathfrak{N}_y(\mathcal{B})\big) \leq \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \big(\mathfrak{N}_x(\mathcal{A}) \wedge \mathfrak{N}_y(\mathcal{B})\big)$$
$$= \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \left(\bigvee_{\mathcal{U}[x]=\mathcal{A}} \mathfrak{U}(\mathcal{U}) \wedge \bigvee_{\mathcal{V}[y]=\mathcal{B}} \mathfrak{U}(\mathcal{V}) \right)$$
$$= \bigvee_{\mathcal{A}\times\mathcal{B}\subseteq\mathcal{W}} \bigvee_{\mathcal{U}[x]\times\mathcal{V}[y]=\mathcal{A}\times\mathcal{B}} (\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}))$$
$$= \bigvee_{\mathcal{U}[x]\times\mathcal{V}[y]\subseteq\mathcal{W}} (\mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V})) \ll \bigvee_{\mathcal{U}[x]\times\mathcal{V}[y]\subseteq\mathcal{W}} \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$$
$$\leq \bigvee_{(\mathcal{U}\cap\mathcal{V})[x]\times(\mathcal{U}\cap\mathcal{V})[y]\subseteq\mathcal{W}} \mathfrak{U}(\mathcal{U} \cap \mathcal{V})$$
$$\ll \bigvee_{\mathcal{U}[x]\times\mathcal{U}[y]\subseteq\mathcal{W}} \mathfrak{U}(\mathcal{U}) \leq \bigvee_{(\mathcal{U}\cap\mathcal{U}^{-1})[x]\times(\mathcal{U}\cap\mathcal{U}^{-1})[y]\subseteq\mathcal{W}} \mathfrak{U}(\mathcal{U} \cap \mathcal{U}^{-1})$$
$$= \bigvee_{\mathcal{U}[x]\times\mathcal{U}[y]\subseteq\mathcal{W}, \mathcal{U}=\mathcal{U}^{-1}} \mathfrak{U}(\mathcal{U}) = \mathfrak{B}_{(x,y)}(\mathcal{W}) \leq \bigvee_{\mathcal{Z}\subseteq\mathcal{W}} \mathfrak{B}_{(x,y)}(\mathcal{Z}).$$ □

Theorem 5.7 *Let* $(\mathcal{X}, \mathfrak{U})$ *be an LBFU space, and let* $\tau_{\mathfrak{U}}$ *be the fuzzifying uniform topology on* $\mathfrak{U}$. *If* $\mathcal{L}$ *is a completely distributive lattice with order reversing involution* $'$, *then for any* $x, y \in \mathcal{X}$, $\mathcal{A} \subseteq \mathcal{X}$ *and* $\mathcal{M} \subseteq \mathcal{X} \times \mathcal{X}$:

(1) $Cl_{\tau_{\mathfrak{U}}}(\mathcal{A})(x) = \bigwedge_{x \notin \mathcal{U}[\mathcal{A}]} (\mathfrak{U}(\mathcal{U}))'$;

(2) $Cl_{\tau_{\mathfrak{U}} \times \tau_{\mathfrak{U}}}(\mathcal{M})(x, y) = \bigwedge_{(x,y) \in \mathcal{U} \circ \mathcal{M} \circ \mathcal{U}} (\mathfrak{U}(\mathcal{U}))'$.

Proof (1) From Theorems 2.3 (6) and 5.6 (1), the proof is similar to the proof of Theorem 3.8 (1) (Ying 1993b).

(2) From Lemma 5.3, the proof is similar to the proof of Theorem 3.8 (2) (Ying 1993b). □

6 LBFU continuity

To conclude the theoretical contribution of this paper, in this section we define and investigate the concept of LBFU continuity. This notion is formalized as follows:

Definition 6.1 Let $(\mathcal{X}, \mathfrak{U})$ and $(\mathcal{Y}, \mathfrak{V})$ be two LBFU spaces. A function $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ is called *LB*-fuzzifying uniformly continuous if and only if $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$, $\forall \mathcal{V} \in \mathcal{P}(\mathcal{Y} \times \mathcal{Y})$.

A technical characterization gives an alternative view of the concept above:

Lemma 6.1 *Suppose that* $(\mathcal{X}, \mathfrak{U})$ *and* $(\mathcal{Y}, \mathfrak{V})$ *are two LBFU spaces. Then the function* $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *is LB-fuzzifying uniformly continuous if and only if* $\mathfrak{V}(\mathcal{V}) \ll \bigvee_{(f \times f)(\mathcal{U}) \subseteq \mathcal{V}} \mathfrak{U}(\mathcal{U})$, *for each* $\mathcal{V} \in \mathcal{P}(\mathcal{Y} \times \mathcal{Y})$.

Proof Suppose that f is *LB*-fuzzifying uniformly continuous. Then
$\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V})) \leq \bigvee_{\mathcal{U} \subseteq (f \times f)^{-1}(\mathcal{V})} \mathfrak{U}(\mathcal{U}) = \bigvee_{(f \times f)(\mathcal{U}) \subseteq \mathcal{V}} \mathfrak{U}(\mathcal{U})$.
Therefore $\mathfrak{V}(\mathcal{V}) \ll \bigvee_{(f \times f)(\mathcal{U}) \subseteq \mathcal{V}} \mathfrak{U}(\mathcal{U})$.
Conversely, $\mathfrak{V}(\mathcal{V}) \ll \bigvee_{(f \times f)(\mathcal{U}) \subseteq \mathcal{V}} \mathfrak{U}(\mathcal{U}) = \bigvee_{\mathcal{U} \subseteq (f \times f)^{-1}(\mathcal{V})} \mathfrak{U}(\mathcal{U}) = \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$.
Hence $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$ and f is *LB*-fuzzifying uniformly continuous. □

Subbases are helpful for the verification of the axiom of LBFU continuity:

Lemma 6.2 *Let* $(\mathcal{X}, \mathfrak{U})$ *and* $(\mathcal{Y}, \mathfrak{V})$ *be two LBFU spaces and suppose that* $\mathfrak{S}$ *is a subbase of* $\mathfrak{V}$. *Then* $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *is LB-fuzzifying uniformly continuous if and only if* $\mathfrak{S}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$, *for each* $\mathcal{V} \in \mathcal{P}(\mathcal{Y} \times \mathcal{Y})$.

Proof Suppose that f is *LB*-fuzzifying uniformly continuous. Then $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$. Since $\mathfrak{S}$ is a subbase of $\mathfrak{V}$, then $\mathfrak{S}(\mathcal{V}) \ll \mathfrak{V}(\mathcal{V})$. Hence $\mathfrak{S}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$.
Conversely, suppose that $\mathfrak{S}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$.

$$\mathfrak{V}(\mathcal{V}) = \bigvee_{\mathcal{W} \subseteq \mathcal{V}} \mathfrak{S}^{(\cap)}(\mathcal{W}) = \bigvee_{\mathcal{W} \subseteq \mathcal{V}} \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i = \mathcal{W}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i) = \bigvee_{\bigcap_{i=1}^{n} \mathcal{U}_i \subseteq \mathcal{V}} \bigwedge_{i=1}^{n} \mathfrak{S}(\mathcal{U}_i)$$
$$\leq \mathfrak{S}(\mathcal{V})$$
$$\ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$$

Then $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$. □

Another natural property that holds true concerns the composition of *LB*-fuzzifying uniformly continuous functions:

Lemma 6.3 *Let* $(\mathcal{X}, \mathfrak{U})$, $(\mathcal{Y}, \mathfrak{V})$ *and* $(\mathcal{Z}, \mathfrak{W})$ *be three LBFU spaces. If* $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *and* $g : (\mathcal{Y}, \mathfrak{V}) \longrightarrow (\mathcal{Z}, \mathfrak{W})$ *are LB-fuzzifying uniformly continuous, then* $g \circ f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Z}, \mathfrak{W})$ *is LB-fuzzifying uniformly continuous.*

Proof For all $\mathcal{W} \in \mathcal{P}(\mathcal{Z} \times \mathcal{Z})$, we have $\mathfrak{W}(\mathcal{W}) \ll \mathfrak{V}((g \times g)^{-1}(\mathcal{W}))$ as g is *LB*-fuzzifying uniformly continuous. Since f is *LB*-fuzzifying uniformly continuous, then

$$\mathfrak{V}((g \times g)^{-1}(\mathcal{W})) \ll \mathfrak{U}((f \times f)^{-1}((g \times g)^{-1}(\mathcal{W}))) = \mathfrak{U}(((g \circ f) \times (g \circ f))^{-1}(\mathcal{W})).$$

Therefore $\mathfrak{W}(\mathcal{W}) \ll \mathfrak{U}(((g \circ f) \times (g \circ f))^{-1}(\mathcal{W}))$. So $g \circ f$ is *LB*-fuzzifying uniformly continuous. □

Theorem 6.1 *Let* $\mathcal{L}$ *be an I-compact infinitely distributive lattice, let* $(\mathcal{X}, \mathfrak{U})$ *and* $(\mathcal{Y}, \mathfrak{V})$ *be two LBFU spaces, and suppose that* $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *is LB-fuzzifying uniformly continuous. Then:*

(1) $f : (\mathcal{X}, \tau_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau_{\mathfrak{V}})$ *is L-fuzzifying continuous.*
(2) $f : (\mathcal{X}, \tau^{(1)}_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau^{(1)}_{\mathfrak{V}})$ *is L-fuzzifying continuous.*
(3) If $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *is injective, then* $f : (\mathcal{X}, \tau^{(2)}_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau^{(2)}_{\mathfrak{V}})$ *is L-fuzzifying continuous.*
(4) If $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ *is surjective, then* $f : (\mathcal{X}, \tau^{(3)}_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau^{(3)}_{\mathfrak{V}})$ *is L-fuzzifying continuous.*

Proof (1)
$$\begin{aligned}
\tau_{\mathfrak{V}}(\mathcal{B}) &= \bigwedge_{y \in \mathcal{B}} \bigvee_{\mathcal{V}[y] \subseteq \mathcal{B}} \mathfrak{V}(\mathcal{V}),\ \mathcal{B} \subseteq \mathcal{Y} \\
&\leq \bigwedge_{x \in f^{-1}(\mathcal{B})} \bigvee_{\mathcal{V}[f(x)] \subseteq \mathcal{B}} \mathfrak{V}(\mathcal{V}) \\
&\leq \bigwedge_{x \in f^{-1}(\mathcal{B})} \bigvee_{f^{-1}(\mathcal{V}[f(x)]) \subseteq f^{-1}(\mathcal{B})} \mathfrak{V}(\mathcal{V}) \\
&\ll \bigwedge_{x \in f^{-1}(\mathcal{B})} \bigvee_{(((f \times f)^{-1}(\mathcal{V}))[x]) \subseteq f^{-1}(\mathcal{B})} \mathfrak{V}((f \times f)^{-1}(\mathcal{V})) \\
&\leq \bigwedge_{x \in f^{-1}(\mathcal{B})} \bigvee_{\mathcal{U}[x] \subseteq f^{-1}(\mathcal{B})} \mathfrak{U}(\mathcal{U}) \\
&= \tau_{\mathfrak{U}}(f^{-1}(\mathcal{B})).
\end{aligned}$$
Then $f : (\mathcal{X}, \tau_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau_{\mathfrak{V}})$ is an L-fuzzifying continuous function.
Note that we used the fact that $f^{-1}(\mathcal{V}[f(x)]) = ((f \times f)^{-1}(\mathcal{V}))[x]$. In fact,
$\forall z \in f^{-1}(\mathcal{V}[f(x)]) \Leftrightarrow f(z) \in \mathcal{V}[f(x)] \Leftrightarrow (f(z), f(x)) \in \mathcal{V} \Leftrightarrow (z, x) \in (f \times f)^{-1}(\mathcal{V}) \Leftrightarrow z \in ((f \times f)^{-1}(\mathcal{V}))[x]$. Therefore, if $\mathcal{V}[f(x)] \subseteq \mathcal{B}$, then $f^{-1}(\mathcal{V}[f(x)]) = ((f \times f)^{-1}\mathcal{V})[x] \subseteq f^{-1}(\mathcal{B})$.
(2) Since $(f \times f)^{-1}(\mathcal{V}[f^{-1}(\mathcal{B})]) = f^{-1}(\mathcal{V}[f(f^{-1}(\mathcal{B}))]) \subseteq f^{-1}(\mathcal{V}[\mathcal{B}])$, then by putting $\mathcal{U} = (f \times f)^{-1}(\mathcal{V})$ we obtain
$$\begin{aligned}
\tau^{(1)}_{\mathfrak{V}}(\mathcal{B}) &= \bigvee_{\mathcal{V}[\mathcal{B}^c] \cap \mathcal{V}[\mathcal{B}] = \phi} \mathfrak{V}(\mathcal{V}) \\
&= \bigvee_{f^{-1}(\mathcal{V}[\mathcal{B}^c] \cap \mathcal{V}[\mathcal{B}]) = \phi} \mathfrak{V}(\mathcal{V}) \\
&\ll \bigvee_{f^{-1}(\mathcal{V}[\mathcal{B}^c] \cap \mathcal{V}[\mathcal{B}]) = \phi} \mathfrak{U}((f \times f)^{-1}(\mathcal{V})) \\
&\leq \bigvee_{\mathcal{U}[f^{-1}(\mathcal{B}^c)] \cap \mathcal{U}[f^{-1}(\mathcal{B})] = \phi} \mathfrak{U}(\mathcal{U}) \\
&= \tau^{(1)}_{\mathfrak{U}}(f^{-1}(\mathcal{B})).
\end{aligned}$$

Therefore $f : (\mathcal{X}, \tau^{(1)}_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau^{(1)}_{\mathfrak{V}})$ is an L-fuzzifying continuous function.
(3) If $\mathcal{B} = \phi$, then the result holds. Now, suppose $\mathcal{B} \neq \phi$. Since f is injective, we have $(f \times f)^{-1}(\mathcal{U}(\mathcal{B})) = \mathcal{U}(f^{-1}(\mathcal{B}))$. Indeed, if $x \neq y$ we have $(x, y) \in (f \times f)^{-1}(\mathcal{U}(\mathcal{B})) \Leftrightarrow (f(x), f(y)) \in \mathcal{U}(\mathcal{B}) \Leftrightarrow f(x), f(y) \in \mathcal{B} \Leftrightarrow x, y \in f^{-1}(\mathcal{B}) \Leftrightarrow (x, y) \in \mathcal{U}(f^{-1}(\mathcal{B}))$. Therefore $\tau^{(2)}_{\mathfrak{V}}(\mathcal{B}) = \mathfrak{V}(\mathcal{V}(\mathcal{B})) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}(\mathcal{B}))) = \mathfrak{U}(\mathcal{V}(f^{-1}(\mathcal{B}))) = \tau^{2}_{\mathfrak{U}}(f^{-1}(\mathcal{B}))$. Hence $f : (\mathcal{X}, \tau^{(2)}_{\mathfrak{U}}) \longrightarrow (\mathcal{Y}, \tau^{(2)}_{\mathfrak{V}})$ is L-fuzzifying continuous.
(4) Suppose that $\mathcal{B} \in \tau_{\mathfrak{V}_\alpha}$. Then for every $y \in \mathcal{B}$, there exists $\mathcal{V} \in \mathfrak{V}_\alpha$ such that $\mathcal{V}[y] \subseteq \mathcal{B}$. Hence $\mathfrak{V}(\mathcal{V}) \geq \alpha$ and $\mathcal{V}[y] \subseteq \mathcal{B}$. Since f is a surjective LB-fuzzifying uniformly continuous function, then $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f \times f)^{-1}(\mathcal{V}))$ and $((f \times f)^{-1}(\mathcal{V}))[f^{-1}(y)] \subseteq f^{-1}(\mathcal{B})$, where $x \in f^{-1}(\mathcal{B})$ and $f(x) = y$. Hence $(f \times f)^{-1}(\mathcal{V}) \in \mathfrak{U}_\alpha$ and $((f \times f)^{-1}(\mathcal{V}))[f^{-1}(y)] \subseteq f^{-1}(\mathcal{B})$. So, $f^{-1}(\mathcal{B}) \in \tau_{\mathfrak{U}_\alpha}$. Therefore

$$\tau^{(3)}_{\mathfrak{V}}(\mathcal{B}) = \bigvee_{\alpha \in \mathcal{L}-\{\perp\},\ \mathcal{B} \in \tau_{\mathfrak{V}_\alpha}} \alpha \leq \bigvee_{\alpha \in \mathcal{L}-\{\perp\},\ f^{-1}(\mathcal{B}) \in \tau_{\mathfrak{U}_\alpha}} \alpha = \tau^{(3)}_{\mathfrak{U}}(f^{-1}(\mathcal{B}))$$

Hence $f : (\mathcal{X}, \tau^{(3)}_{\mathfrak{U}}) \rightarrow (\mathcal{Y}, \tau^{(3)}_{\mathfrak{V}})$ is L-fuzzifying continuous. □

Theorem 6.2 *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a function, and let $\mathfrak{V}$ be an LB-fuzzifying uniformity on $\mathcal{Y}$. Then the function $\mathfrak{U} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ defined by $\mathfrak{U}(\mathcal{U}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c))$ is an LB-fuzzifying uniformity on $\mathcal{X}$.*

Proof $\mathfrak{U}(\mathcal{X} \times \mathcal{X}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{X} \times \mathcal{X})^c) = \mathfrak{V}(\mathcal{Y}^2) = \top$. So, $\mathfrak{U}$ is a normal function. Now, we need to prove (LBFU1)-(LBFU5).

(LBFU1): If $\mathfrak{U}(\mathcal{U}) \neq \perp$, then $\mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \neq \perp$. So, $\Delta_{\mathcal{Y}} \subseteq \mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)$. Hence $\Delta_{\mathcal{X}} \subseteq \mathcal{U}$.

(LBFU2): $\mathfrak{U}(\mathcal{U}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \ll \mathfrak{V}((\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c))^{-1}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)((\mathcal{U}^{-1})^c)) = \mathfrak{U}(\mathcal{U}^{-1})$. Hence $\mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{U}^{-1})$

(LBFU3): Suppose $\mathfrak{U}(\mathcal{U}) \neq \perp$. Then $\mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \neq \perp$. Hence, we have

$$\mathfrak{U}(\mathcal{U}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \ll \bigvee_{\mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c) \circ \mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c) \subseteq \mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)} \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c))$$

$$= \bigvee_{(f \times f)(\mathcal{U}^c) \subseteq (f \times f)((\mathcal{V} \circ \mathcal{V})^c)} \mathfrak{U}(\mathcal{V}) \leq \bigvee_{\mathcal{U}^c \subseteq (\mathcal{V} \circ \mathcal{V})^c} \mathfrak{U}(\mathcal{V}) = \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{U}(\mathcal{V}).$$

Therefore $\mathfrak{U}(\mathcal{U}) \ll \bigvee_{\mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}} \mathfrak{U}(\mathcal{V})$.

(LBFU4):

$$\begin{aligned} \mathfrak{U}(\mathcal{U}) \wedge \mathfrak{U}(\mathcal{V}) &= \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \wedge \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c)) \\ &\ll \mathfrak{V}((\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \cap (\mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c))) \\ &= \mathfrak{V}(\mathcal{Y}^2 \backslash ((f \times f)(\mathcal{U}^c) \cup (f \times f)(\mathcal{V}^c))) \\ &= \mathfrak{V}(\mathcal{Y}^2 \backslash ((f \times f)(\mathcal{U}^c \cup \mathcal{V}^c))) \\ &= \mathfrak{V}(\mathcal{Y}^2 \backslash ((f \times f)(\mathcal{U} \cap \mathcal{V})^c)) = \mathfrak{U}(\mathcal{U} \cap \mathcal{V}). \end{aligned}$$

(LBFU5): Suppose that $\mathfrak{U}(\mathcal{U}) \neq \perp$ and $\mathcal{U} \subseteq \mathcal{V}$. Then $\mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \neq \perp$ and we obtain $\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c) \subseteq Y^2 \backslash (f \times f)(\mathcal{V}^c)$. Therefore, $\mathfrak{U}(\mathcal{U}) = \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)) \ll \mathfrak{V}(\mathcal{Y}^2 \backslash (f \times f)(\mathcal{V}^c)) = \mathfrak{U}(\mathcal{V})$. Hence $\mathfrak{U}(\mathcal{U}) \ll \mathfrak{U}(\mathcal{V})$. Thus $\mathfrak{U}$ is an LB-fuzzifying uniformity on $\mathcal{X}$ □

Theorem 6.3 *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a function, and let $\mathfrak{V}$ be an LB-fuzzifying uniformity on $\mathcal{Y}$. Define $\mathfrak{U} : \mathcal{P}(\mathcal{X} \times \mathcal{X}) \longrightarrow \mathcal{L}$ by $\mathfrak{U}(\mathcal{U}) = \bigvee_{(f \times f)^{-1}(\mathcal{V}) \subseteq \mathcal{U}} \mathfrak{V}(\mathcal{V})$. Then $\mathfrak{U}$ is an LB-fuzzifying uniformity on $\mathcal{X}$. Also, if $\mathcal{L}$ is a complete chain, then $\mathfrak{U}$ is the smallest LB-fuzzifying uniformity such that f is LB-fuzzifying uniformly continuous.*

Proof If $(f \times f)^{-1}(\mathcal{V}) \subseteq \mathcal{U}$, then $\mathcal{V} \subseteq \mathcal{Y}^2 \backslash (f \times f)(\mathcal{U}^c)$. So,

$\bigvee_{(f\times f)^{-1}(\mathcal{V})\subseteq\mathcal{U}} \mathfrak{V}(\mathcal{V}) \leq \mathfrak{V}(\mathcal{Y}^2\backslash(f\times f)(\mathcal{U}^c))$. On the other hand, if $\mathcal{M} = \mathcal{Y}^2 \setminus (f \times f)(\mathcal{U}^c)$, then $(f \times f)^{-1}(\mathcal{M}) \subseteq \mathcal{U}$ and $\mathfrak{V}(\mathcal{M}) \leq \bigvee_{(f\times f)^{-1}(\mathcal{M})\subseteq\mathcal{U}} \mathfrak{V}(\mathcal{M})$. Hence $\bigvee_{(f\times f)^{-1}(\mathcal{V})\subseteq\mathcal{U}} \mathfrak{V}(\mathcal{V}) = \mathfrak{V}(\mathcal{Y}^2\backslash(f\times f)(\mathcal{U}^c))$. Therefore, from Theorem 6.2 we obtain that $\mathfrak{U}$ is an *LB*-fuzzifying uniformity on $\mathcal{X}$. To prove that $\mathfrak{U}$ is the smallest *LB*-fuzzifying uniformity, we know from the definition of $\mathfrak{U}$ that $\mathfrak{V}(\mathcal{V}) \leq \mathfrak{U}((f\times f)^{-1}(\mathcal{V}))$, $\mathcal{V} \in \mathcal{P}(\mathcal{Y}\times\mathcal{Y})$. Since $\mathcal{L}$ is a complete chain, we deduce that $\mathfrak{V}(\mathcal{V}) \ll \mathfrak{U}((f\times f)^{-1}(\mathcal{V}))$. So, $f : (\mathcal{X}, \mathfrak{U}) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ is *LB*-fuzzifying uniformly continuous. Now, suppose that $f : (\mathcal{X}, \mathfrak{U}^*) \longrightarrow (\mathcal{Y}, \mathfrak{V})$ is *LB*-fuzzifying uniformly continuous. Then $\mathfrak{U}(\mathcal{U}) = \bigvee_{(f\times f)^{-1}(\mathcal{V})\subseteq\mathcal{U}} \mathfrak{V}(\mathcal{V}) \ll \bigvee_{(f\times f)^{-1}(\mathcal{V})\subseteq\mathcal{U}} \mathfrak{U}^*((f\times f)^{-1}(\mathcal{V})) \leq \mathfrak{U}^*(\mathcal{U})$, $\forall\mathcal{U} \in \mathcal{P}(\mathcal{X}\times\mathcal{X})$. □

Proposition 6.1 *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a function, and let $\mathfrak{V}$ be an LB-fuzzifying uniformity on $\mathcal{Y}$. Then $\mathfrak{S}((f\times f)^{-1}(\mathcal{U})) = \mathfrak{V}(\mathcal{V})$ is a subbase of some LB-fuzzifying uniformity $\mathfrak{U}$ on $\mathcal{X}$.*

Proof It suffices to check (LBS1)-(LBS3) for $\mathfrak{S}$. Since $\mathfrak{V}$ is an *LB*-fuzzifying uniformity, then $\mathfrak{V}(\mathcal{Y}\times\mathcal{Y}) = \top$. Hence $\mathfrak{S}((f\times f)^{-1}(\mathcal{Y}\times\mathcal{Y})) = \mathfrak{V}(\mathcal{Y}\times\mathcal{Y}) = \mathfrak{S}(\mathcal{X}\times\mathcal{X}) = \top$. Therefore $\mathfrak{S}$ is normal function.
(LBS1): If $\mathfrak{S}((f\times f)^{-1}(\mathcal{V})) \neq \bot$, then $\mathfrak{V}(\mathcal{V}) \neq \bot$. So, $\Delta_{\mathcal{Y}} \subseteq \mathcal{V}$, where $\Delta_{\mathcal{Y}} = \{(f(x), f(x)) \mid x \in \mathcal{X}\}$. Hence $\Delta \subseteq (f\times f)^{-1}(\mathcal{V})$ and $\Delta = \{(x, x) \mid x \in \mathcal{X}\}$.
(LBS2): $\mathfrak{S}((f\times f)^{-1}(\mathcal{U})) = \mathfrak{V}(\mathcal{U}) \ll \mathfrak{V}(\mathcal{U}^{-1}) = \mathfrak{S}((f\times f)^{-1}(\mathcal{U}^{-1})) \leq \bigvee_{\mathcal{V}\subseteq(f\times f)^{-1}(\mathcal{U}^{-1})} \mathfrak{S}(\mathcal{V}) = \bigvee_{\mathcal{V}\subseteq((f\times f)^{-1}(\mathcal{U}))^{-1}} \mathfrak{S}(\mathcal{V})$. Therefore $\mathfrak{S}((f\times f)^{-1}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\subseteq((f\times f)^{-1}(\mathcal{U}))^{-1}} \mathfrak{S}(\mathcal{V})$.
(LBS3): $\mathfrak{S}((f\times f)^{-1}(\mathcal{U})) = \mathfrak{V}(\mathcal{U}) \ll \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{V}(\mathcal{V}) = \bigvee_{\mathcal{V}\circ\mathcal{V}\subseteq\mathcal{U}} \mathfrak{S}((f\times f)^{-1}(\mathcal{V}))$

$$\leq \bigvee_{(f\times f)^{-1}(\mathcal{V}\circ\mathcal{V})\subseteq(f\times f)^{-1}(\mathcal{U})} \mathfrak{S}((f\times f)^{-1}(\mathcal{V})) = \bigvee_{(f\times f)^{-1}(\mathcal{V})\circ(f\times f)^{-1}(\mathcal{V})\subseteq(f\times f)^{-1}(\mathcal{U})} \mathfrak{S}((f\times f)^{-1}(\mathcal{V})).$$

Put $(f\times f)^{-1}(\mathcal{U}) = \mathcal{M}$ and $(f\times f)^{-1}(\mathcal{V}) = \mathcal{N}$, we obtain $\mathfrak{S}(\mathcal{M}) \ll \bigvee_{\mathcal{N}\circ\mathcal{N}\subseteq\mathcal{M}} \mathfrak{S}(\mathcal{N})$. Hence $\mathfrak{S}$ is a subbase of some *LB*-fuzzifying uniformity $\mathfrak{U}$ on $\mathcal{X}$. □

7 Conclusion

Uniformity is an important concept in point-set topology. It is close to the standard notion of topology thus it constitutes a convenient tool for the investigation of this structure. The concept of way below relation had never been connected with the structure of uniform spaces, although both have appeared separately in the literature. Our research has shown that these two concepts can be linked in an efficient manner. This connection produces noteworthy fundamental results, therefore we can safely claim that the new model deserves further consideration.

Technically speaking, the present work defines LBFU spaces. Some of their properties are investigated in terms of bases and subbases in Theorems 4.1 and 4.2. Furthermore, explicit relations between the notions of LBFU space and *L*-fuzzifying topological space are shown in Theorems 5.1, 5.2, 5.3 and 5.4. Also, some properties of closure and interior are given in Theorems 5.5, 5.6 and 5.7.

Additionally, we believe that it would be interesting to extend this approach to other structures such as proximity, pre-uniformity, topogenous, syntopogenous, homotopy, *et cetera*. We intend to investigate all these issues in future research works.

Another avenue of discussion concerns evolutionary biology, as central notions in this field are intrinsically topological (Stadler and Stadler 2004). Classical models of population genetics and quantitative genetics presuppose a natural framework for studying the evolution of phenotypic adaptation and the process of speciation which is a Euclidean vector space. In this regard, it is interesting to develop a mathematical framework that contains graphs, recombination sets, and Euclidean vector spaces as special cases. If phenotypes are organized according to genetic accessibility, the resulting space lacks a metric and is formalized by an unfamiliar structure. In future studies we shall investigate if and to what extent, LB-fuzzifying uniformities serve this purpose. We expect that patterns of phenotypic evolution—such as punctuation, irreversibility, or modularity—result from the properties of this space. Also by inspiration from Stadler and Stadler (2004), future studies might focus on the applicability of LB-fuzzifying uniformity to combinatorial search spaces, fitness landscapes, evolutionary trajectories, and artificial chemistry.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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