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FUZZY γ-OPEN SETS AND FUZZY γ-CONTINUITY IN FUZZIFYING TOPOLOGY

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Abstract. The concepts of fuzzifying γ-open sets and fuzzifying γ-closed sets are studied and some interesting results (Theorems 5.4 and 5.5) are obtained. Also, the concept of fuzzifying γ-continuity are introduced and some important characterisations (Theorem 6.1) are obtained. Furthermore, some compositions of fuzzifying γ-continuity and fuzzifying continuity are presented (Theorem 6.2).

1. Introduction. In 1991, Ying [7] used the semantic method of continuous valued logic to propose the so-called fuzzifying topology as a preliminary of the research on bifuzzy topology and elementally develop topology in the theory of fuzzy sets from a completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns each crisp subset of X to a certain degree of being open, other than being definitely open or not. Andrijević [3] introduced the concepts of b-open sets and b-closed sets in general topology. In [4] Hanafy used the term γ-open sets instead of b-open sets and studied the concepts of γ-open sets and γ-continuity in fuzzy topology. Follow to Hanafy, we use the terms γ-open sets and γ-continuity. In the present paper the concepts of fuzzifying γ-open sets, fuzzifying γ-closed sets and fuzzifying γ-neighbourhoods are introduced and some of their properties are examined. Also, in the framework of fuzzifying topology, the concepts of γ-derived sets, γ-closure operation and γ-interior operation are established and some of their properties are discussed. In the last section, we introduce the concept of fuzzifying γ-continuity as a unary fuzzy predicate and the characterisations of γ-continuity in fuzzifying topology are presented.

2. Preliminaries. We present the fuzzy logical and corresponding set theoretical notations [7, 8] since we need them in this paper.

For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values in the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) (a) $[\alpha] = \alpha(\alpha \in [0,1])$;

(b) $[\varphi \wedge \psi] = \min([\varphi], [\psi])$;

(c) $[\varphi \rightarrow \psi] = \min(1, 1 - [\varphi] + [\psi])$.

(2) If $\tilde{A} \in \mathcal{F}(X), [x \in \tilde{A}] := \tilde{A}(x)$.

(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition the following derived formulae are given,

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(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$;

(2) $[\varphi \vee \psi] = [\neg(\neg\varphi \wedge \neg\psi)] = \max([\varphi], [\psi])$;

(3) $[\varphi \mathbin{\dot\wedge} \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;

(4) $[\varphi \mathbin{\dot\wedge} \psi] = [\neg(\varphi \to \neg\psi)] = \max(0, [\varphi] + [\psi] - 1)$;

(5) $[\varphi \dot\vee \psi] := [\neg\varphi \to \psi] = \min(1, [\varphi] + [\psi])$;

(6) $[\exists x\varphi(x)] := [\neg\forall x \neg\varphi(x)] = \sup\limits_{x \in X} [\varphi(x)]$;

(7) If $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$, then

(a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \to x \in \tilde{B})] = \inf\limits_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$;

(b) $[\tilde{A} \equiv \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A})]$;

(c) $[\tilde{A} \doteq \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \mathbin{\dot\wedge} (\tilde{B} \subseteq \tilde{A})]$,

where $\mathcal{F}(X)$ is the family of all fuzzy sets in X.

We do often not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying did [7 – 9]. For the definitions and results in fuzzifying topology which are used in the sequel we refer to [7 – 9].

We now give some definitions and results which are useful in the rest of the present paper.

Definition 2.1 [7]. Let X be a universe of discourse, $\tau \in \mathcal{F}(P(X))$ satisfy the following conditions:

(1) $\tau(X) = 1, \tau(\phi) = 1$;

(2) for any $A, B, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(3) for any $\{A_\lambda : \lambda \in \Lambda\}, \tau(\bigcup\limits_{\lambda \in \Lambda} A_\lambda) \geq \bigwedge\limits_{\lambda \in \Lambda} \tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 2.2 [7]. The family of fuzzifying closed sets, denoted by $F \in \mathcal{F}(P(X))$, is defined as follows: $A \in F := X \sim A \in \tau$, where $X \sim A$ is the complement of A.

Definiton 2.3 [7]. Let $x \in X$. The neighbourhood system of x, denoted by $N_x \in \mathcal{F}(P(X))$, is defined as follows: $N_x(A) = \sup\limits_{x \in B \subseteq A} \tau(B)$.

Definiton 2.4 (Lemma 5.2 [7]). The closure $\bar{A}$ of A is defined as $\bar{A}(x) = 1 - N_x(X \sim A)$.

In Theorem 5.3 [7], Ying proved that the closure$^-$: $P(X) \to \mathcal{F}(X)$ is a fuzzifying closure operator (see Definition 5.3 [7]) because its extension$^-$: $\mathcal{F}(X) \to \mathcal{F}(X), \bar{\tilde{A}} = \bigcup\limits_{\alpha \in [0,1]} \alpha \overline{\tilde{A}_\alpha}, \tilde{A} \in \mathcal{F}(X)$, where $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}$ is the α-cut of A and $\alpha\tilde{A}(x) = \alpha \wedge \tilde{A}(x)$ satisfies the following Kuratowski closure axioms:

(1) $\models \bar{\phi} \equiv \phi$;

(2) for any $\tilde{A} \in \mathcal{F}(X), \models \tilde{A} \subseteq \bar{\tilde{A}}$;

(3) for any $\tilde{A}, \tilde{B} \in \mathcal{F}(X), \models \overline{\tilde{A} \cup \tilde{B}} \equiv \bar{\tilde{A}} \cup \bar{\tilde{B}}$;

(4) for any $\tilde{A} \in \mathcal{F}(X), \models \overline{(\bar{\tilde{A}})} \subseteq \bar{\tilde{A}}$.

Definition 2.5 [8]. For any $A \subseteq X$, the fuzzy set of interior points of A is called the interior of A, and given as follows: $A^\circ(x) = N_x(A)$.

From Lemma 3.1 [7] and the definitions of $N_x(A)$ and A° we have $\tau(A) = \inf\limits_{x \in A} A^\circ(x)$.

Definition 2.6 [5]. For any $\tilde{A} \in \mathcal{F}(X), \models (\tilde{A})^\circ \equiv X \sim \overline{(X \sim \tilde{A})}$.

Lemma 2.1 [5]. If $[\tilde{A} \subseteq \tilde{B}] = 1$, then (1) $\models \bar{\tilde{A}} \subseteq \bar{\tilde{B}}$; (2) $\models (\tilde{A})^\circ \subseteq (\tilde{B})^\circ$.

Lemma 2.2 [5]. Let (X,τ) be a fuzzifying topological space. For any $\tilde{A}, \tilde{B}$,
(1) $\models X^\circ \equiv X$; (2) $\models (\tilde{A})^\circ \subseteq \tilde{A}$; (3) $\models (\tilde{A}\cap\tilde{B})^\circ \equiv (\tilde{A})^\circ \cap (\tilde{B})^\circ$; (4) $\models (\tilde{A})^{\circ\circ} \supseteq (\tilde{A})^\circ$.

Lemma 2.3 [5]. Let (X,τ) be a fuzzifying topological space. For any $\tilde{A} \in \mathcal{F}(X)$,
(1) $\models X \sim (\tilde{A})^{\circ-} \equiv (X \sim \tilde{A})^{-\circ}$; (2) $\models X \sim (\tilde{A})^{-\circ} \equiv (X\sim\tilde{A})^{\circ-}$.

Lemma 2.4 [2,5]. If $[\tilde{A}\subseteq\tilde{B}] = 1$, then (1) $\models (\tilde{A})^{\circ-} \subseteq (\tilde{B})^{\circ-}$; (2) $\models (\tilde{A})^{-\circ} \subseteq (\tilde{B})^{-\circ}$.

Definition 2.7. Let (X,τ) be a fuzzifying topological space.
(1) The family of fuzzifying α-open [6] (resp. semi-open [5], pre-open [2], β-open [1]) sets, denoted by $\alpha\tau$ (resp. $S\tau, P\tau, \beta\tau) \in \mathcal{F}(P(X))$, is defined as follows:
$A \in \alpha\tau$ (resp. $S\tau, P\tau, \beta\tau) := \forall x(x \in A \to x \in A^{\circ-\circ}$ (resp. $A^{\circ-}, A^{-\circ}, A^{-\circ-}$)).
(2) The family of fuzzifying α-closed [6] (resp. semi-closed [5], pre-closed [2], β-closed [1]) sets, denoted by αF (resp. $SF, PF, \beta F) \in \mathcal{F}(P(X))$, is defined as follows:
$A \in \alpha F$ (resp. $SF, PF, \beta F) := X \sim A \in \alpha\tau$ (resp. $S\tau, P\tau, \beta\tau$).

Definition 2.8 [9]. Let $(X,\tau), (Y,U)$ be two fuzzifying topological spaces. A unary fuzzy predicate $C \in \mathcal{F}(Y^X)$, called fuzzifying continuity, is given as follows:

$$C(f) := \forall u(u \in U \to f^{-1}(u) \in \tau).$$

3. Fuzzifying γ-open sets

Definition 3.1. Let (X,τ) be a fuzzifying topological space.
(1) The family of fuzzifying γ-open sets, denoted by $\gamma\tau \in \mathcal{F}(P(X))$, is defined as follows:

$$A \in \gamma\tau := \forall x(x \in A \to x \in A^{\circ-} \cup A^{-\circ}).$$

(2) The family of fuzzifying γ-closed sets, denoted by $\gamma F \in \mathcal{F}(P(X))$, is defined as follows:

$$A \in \gamma F := X \sim A \in \gamma\tau.$$

Theorem 3.1. Let (X,τ) be a fuzzifying topological space. Then
(1) $\gamma\tau(X) = 1, \gamma\tau(\phi) = 1$;
(2) for any $\{A_\lambda : \lambda \in \Lambda\}, \gamma\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda} \gamma\tau(A_\lambda)$.

Proof. The proof of (1) is straightforward.
(2) From Lemma 2.4,

$$\models A_\lambda^{\circ-} \subseteq \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{\circ-} \text{ and } \models A_\lambda^{-\circ} \subseteq \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ}.$$

So,

$$\begin{aligned}\gamma\tau\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) &= \inf_{x\in\bigcup_{\lambda\in\Lambda} A_\lambda} \max\left(\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{\circ-}(x), \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ}(x)\right)\\ &= \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda} \max\left(\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{\circ-}(x), \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^{-\circ}(x)\right)\\ &\geq \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda} \max(A_\lambda^{\circ-}(x), A_\lambda^{-\circ}(x)) = \bigwedge_{\lambda\in\Lambda}\gamma\tau(A_\lambda).\end{aligned}$$

Theorem 3.2. Let (X,τ) be a fuzzifying topological space. Then
(1) $\gamma F(X)=1, \gamma F(\phi)=1$; (2) $\gamma F(\bigcap_{\lambda\in\Lambda} A_\lambda)\geq \bigwedge_{\lambda\in\Lambda}\gamma F(A_\lambda)$.

Proof. From Theorem 3.1 the proof is obtained.

Theorem 3.3. Let (X,τ) be a fuzzifying topological space. Then
(1) (a) $\models \tau\subseteq\alpha\tau$; (b) $\models \alpha\tau\subseteq P\tau$; (c) $\models \alpha\tau\subseteq S\tau$; (d) $\models S\tau\subseteq\gamma\tau$; (e) $\models P\tau\subseteq\gamma\tau$; (f) $\models \gamma\tau\subseteq\beta\tau$.

(2) (a) $\models F\subseteq\alpha F$; (b) $\models \alpha F\subseteq PF$; (c) $\models \alpha F\subseteq SF$; (d) $\models SF\subseteq\gamma F$; (e) $\models PF\subseteq\gamma F$; (f) $\models \gamma F\subseteq\beta F$.

Proof. From the properties of the fuzzifying interior and the fuzzifying closure operations and from Theorem 2.2 (3) [8] we have
(1) (a) $[A\in\tau]=[A\subseteq A^{\circ}]\leq[A\subseteq A^{\circ-\circ}]=[A\in\alpha\tau]$.
(b) $[A\in\alpha\tau]=[A\subseteq A^{\circ-\circ}]\leq[A\subseteq A^{-\circ}]=[A\in P\tau]$.
(c) $[A\in\alpha\tau]=[A\subseteq A^{\circ-\circ}]\leq[A\subseteq A^{\circ-}]=[A\in S\tau]$.
(d) $\gamma\tau(A)=\inf_{x\in A}\max(A^{\circ-}(x),A^{-\circ}(x))\geq\inf_{x\in A}A^{-\circ}(x)=P\tau(A)$.
(e) $\gamma\tau(A)=\inf_{x\in A}\max(A^{\circ-}(x),A^{-\circ}(x))\geq\inf_{x\in A}A^{\circ-}(x)=S\tau(A)$.
(f) $\gamma\tau(A)=\inf_{x\in A}\max(A^{\circ-}(x),A^{-\circ}(x))\leq\inf_{x\in A}A^{-\circ-}(x)=\beta\tau(A)$.
(2) The proof is obtained from (1).

Remark 3.1. In crisp setting, i.e., in case that the underlying fuzzifying topology is the ordinary topology, one can have
(1) $\models A\in\tau\wedge B\in\gamma\tau\to A\cap B\in\gamma\tau$; (2) $\models A\in\alpha\tau\wedge B\in\gamma\tau\to A\cap B\in\gamma\tau$.
But these statements may not be true in fuzzifying topology as illustrated by the following counterexample:

Counterexample 3.1. Let $X=\{a,b,c\}$ and let τ be a fuzzifying topology on X defined as follows: $\tau(X)=\tau(\phi)=\tau(\{a\})=\tau(\{a,c\})=1, \tau(\{b\})=\tau(\{a,b\})=0$ and $\tau(\{c\})=\tau(\{b,c\})=\frac{1}{8}$. From the definitions of the interior and the closure of a subset of X and the interior and the closure of a fuzzy subset of X we have $\tau(\{b,c\})=\frac{1}{8}, \alpha\tau(\{b,c\})=\frac{1}{8}, \gamma\tau(\{a,b\})=\frac{7}{8}$ and $\gamma\tau(\{b\})=0$.

Theorem 3.4. $\models A\in\gamma F\leftrightarrow\forall x(x\in A^{-\circ}\cap A^{\circ-}\to x\in A)$.

Proof.

$$\begin{aligned}[\forall x(x\in A^{-\circ}\cap A^{\circ-}\to x\in A)]&=[\forall x(x\in X\sim A\to x\in(X\sim(A^{-\circ}\cap A^{\circ-}))]\\&=[\forall x(x\in X\sim A\to x\in((X\sim A^{-\circ})\cup(X\sim A^{\circ-}))]\\&=[\forall x(x\in X\sim A\to x\in((X\sim A)^{\circ-}\cup(X\sim A)^{-\circ})]\\&=[X\sim A\in\gamma\tau]=[A\in\gamma F].\end{aligned}$$

Theorem 3.5. $\models A\in\gamma\tau\leftrightarrow\forall x(x\in A\to\exists B(B\in\gamma\tau\wedge x\in B\subseteq A))$.

Proof. $[\forall x(x\in A\to\exists B(B\in\gamma\tau\wedge x\in B\subseteq A))]=\inf_{x\in A}\sup_{x\in B\subseteq A}\gamma\tau(B)$. First, we have $\inf_{x\in A}\sup_{x\in B\subseteq A}\gamma\tau(B)\geq\gamma\tau(A)$. On the other hand, let $\beta_x=\{B: x\in B\subseteq A\}$. Then, for any

$f \in \prod_{x\in A} \beta_x$, we have $\bigcup_{x\in A} f(x) = A$ and furthermore $\gamma\tau(A) = \gamma\tau(\bigcup_{x\in A} f(x)) \geq \inf_{x\in A} \gamma\tau(f(x))$. Hence, $\gamma\tau(A) \geq \sup_{f\in \prod_{x\in A} \beta_x} \inf_{x\in A} \gamma\tau(f(x)) = \inf_{x\in A} \sup_{x\in B\subseteq A} \gamma\tau(B)$.

4. Fuzzifying γ-neighbourhood structure.

Definition 4.1. Let $x \in X$. The γ-neighbourhood system of x, denoted by $\gamma N_x \in \mathcal{F}(P(X))$, is defined as $\gamma N_x(A) = \sup_{x\in B\subseteq A} \gamma\tau(B)$.

Theorem 4.1. $\models A \in \gamma\tau \leftrightarrow \forall x(x \in A \to \exists B(B \in \gamma N_x \wedge B \subseteq A))$.

Proof. By Theorem 3.5 we have

$$[\forall x(x \in A \to \exists B(B \in \gamma N_x \wedge B \subseteq A))] = \inf_{x\in A} \sup_{B\subseteq A} \gamma N_x(B) = \inf_{x\in A} \sup_{B\subseteq A} \sup_{x\in C\subseteq B} \gamma\tau(C)$$
$$= \inf_{x\in A} \sup_{x\in C\subseteq A} \gamma\tau(C) = [A \in \gamma\tau].$$

Corollary 4.1. $\inf_{x\in A} \gamma N_x(A) = \gamma\tau(A)$.

Theorem 4.2. The mapping $\gamma N : X \to \mathcal{F}^N(P(X)), x \mapsto \gamma N_x$, where $\mathcal{F}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$, has the following properties:

(1) $\models A \in \gamma N_x \to x \in A$;
(2) $\models A \subseteq B \to (A \in \gamma N_x \to B \in \gamma N_x)$;
(3) $\models A \in \gamma N_x \to \forall H(H \in \gamma N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \gamma N_y))$.

Proof. (1) If $[A \in \gamma N_x] = \sup_{x\in H\subseteq A} \gamma\tau(H) > 0$, then there exists H_o such that $x \in H_o \subseteq A$. Now, we have $[x \in A] = 1$. Therefore, $[A \in \gamma N_x] \leq [x \in A]$ holds always.

(2) Immediate.

(3) $[\exists H(H \in \gamma N_x \wedge H \subseteq A \wedge \forall y(y \in H \to H \in \gamma N_y))] = \sup_{H\subseteq A} (\gamma N_x(H) \wedge \inf_{y\in H} \gamma N_y(H)) = \sup_{H\subseteq A} (\gamma N_x(H) \wedge \gamma\tau(H)) = \sup_{H\subseteq A} \gamma\tau(H) \geq \sup_{x\in H\subseteq A} \gamma\tau(H) = [A \in \gamma N_x]$.

5. Fuzzifying γ-derived sets, fuzzifying γ-closure, fuzzifying γ-interior.

Definition 5.1. Let (X, τ) be a fuzzifying topological space. The fuzzifying γ-derived set $\gamma - d(A)$ of A is defined as follows:

$$\gamma - d(A)(x) = \inf_{B\cap(A-\{x\})=\emptyset} (1 - \gamma N_x(B)).$$

Lemma 5.1. $\gamma - d(A)(x) = 1 - \gamma N_x((X \sim A) \cup \{x\})$.

Proof.

$$\gamma - d(A)(x) = 1 - \sup_{B\cap(A-\{x\})=\emptyset} \gamma N_x(B) = 1 - \gamma N_x((X \sim A) \cup \{x\}).$$

Theorem 5.1. For any $A, \models A \in \gamma F \leftrightarrow \gamma - d(A) \subseteq A$.

Proof.

$$[\gamma - d(A) \subseteq A] = \inf_{x\in X\sim A} (1 - \gamma - d(A)(x)) = \inf_{x\in X\sim A} \gamma N_x((X \sim A) \cup \{x\})$$
$$= \inf_{x\in X\sim A} \sup_{x\in H\subseteq X\sim A} \gamma\tau(H) = \gamma\tau(X \sim A) = \gamma F(A).$$

Definition 5.2. Let (X,τ) be a fuzzifying topological space. The γ-closure of A is defined as follows: $\gamma - cl(A)(x) = \inf_{x\notin B\supseteq A}(1-\gamma F(B))$.

Theorem 5.2. For any x, A,

(1) $\gamma - cl(A)(x) = 1 - \gamma N_x(X\sim A)$;

(2) $\models \gamma - cl(\phi) \equiv \phi$;

(3) $\models A \subseteq \gamma - cl(A)$.

Proof. (1) $\gamma - cl(A)(x) = \inf_{x\notin B\supseteq A}(1-\gamma F(B)) = \inf_{x\in X\sim B\subseteq X\sim A}(1-\gamma\tau(X\sim B)) = 1 - \sup_{x\in X\sim B\subseteq X\sim A}\gamma\tau(X\sim B) = 1-\gamma N_x(X\sim A)$.

(2) $\gamma - cl(\phi)(x) = 1-\gamma N_x(X\sim\phi) = 0$.

(3) It is clear that if $x\notin A$, then $\gamma N_x(A) = 0$. If $x\in A$, then $\gamma - cl(A)(x) = 1-\gamma N_x(X\sim A) = 1$. Then $[A\subseteq \gamma - cl(A)] = 1$.

Lemma 5.2. For any $A\in P(X)$ and $\tilde{B}\in \mathcal{F}(X), [\tilde{B}\subseteq A] = [\tilde{B}\cup A\subseteq A]$.

Theorem 5.3. For any x, A,

(1) $\models \gamma - cl(A) \equiv \gamma - d(A)\cup A$;

(2) $\models x\in\gamma - cl(A) \leftrightarrow \forall B(B\in\gamma N_x \to A\cap B\neq\phi)$;

(3) $\models A\equiv \gamma - cl(A) \leftrightarrow A\in\gamma F$.

Proof. (1) Applying Lemma 5.1 and Theorem 5.2 (3) we have $x\in\gamma - d(A)\cup A = \max(1-\gamma N_x((X\sim A)\cup\{x\}), A(x)) = \gamma - cl(A)(x)$.

(2) $[\forall B(B\in\gamma N_x\to A\cap B\neq\phi)] = \inf_{B\subseteq X\sim A}(1-\gamma N_x(B)) = 1-\gamma N_x(X\sim A) = [x\in \gamma - cl(A)]$.

(3) Since $[A\subseteq A\cup\gamma - d(A)] = 1$, from Theorem 5.1, Lemma 5.2 and (1) above we have

$$\begin{aligned} SF(A) &= [\gamma - d(A)\subseteq A] = [\gamma - d(A)\cup A\subseteq A] = [\gamma - d(A)\cup A\subseteq A]\wedge[A\subseteq\gamma - d(A)\cup A] \\ &= [\gamma - d(A)\cup A\equiv A] = [\gamma - cl(A)\equiv A]. \end{aligned}$$

Theorem 5.4. For any $A, B, \models B\dot{\equiv}\gamma - cl(A)\to B\in\gamma F$.

Proof. If $[A\subseteq B] = 0$, then $[B\dot{\equiv}\gamma - cl(A)] = 0$. Now, we suppose that $[A\subseteq B] = 1$, then we have $[B\subseteq\gamma - cl(A)] = 1 - \sup_{x\in B\sim A}\gamma N_x(X\sim A), [\gamma - cl(A)\subseteq B] = \inf_{x\in X\sim B}\gamma N_x(X\sim A)$. So, $[B\dot{\equiv}\gamma - cl(A)] = \max(0, \inf_{x\in X\sim B}\gamma N_x(X\sim A) - \sup_{x\in B\sim A}\gamma N_x(X\sim A))$.

If $[B\dot{\equiv}\gamma - cl(A)] > t$, then $\inf_{x\in X\sim B}\gamma N_x(X\sim A) > t + \sup_{x\in B\sim A}\gamma N_x(X\sim A)$. For any $x\in X\sim B$, $\sup_{x\in C\subseteq X\sim A}\gamma\tau(C) > t + \sup_{x\in B\sim A}\gamma N_x(X\sim A)$, i.e., there exists C_x such that $x\in C_x\subseteq X\sim A$ and $\gamma\tau(C_x) > t + \sup_{x\in B\sim A}\gamma N_x(X\sim A)$. Now we want to prove $C_x\subseteq X\sim B$. If not, then there exists $x'\in B\sim A$ with $x'\in C_x$. Hence, $\sup_{x\in B\sim A}\gamma N_x(X\sim A)\geq\gamma N_{x'}(X\sim A)\geq\gamma\tau(C_x) > t + \sup_{x\in B\sim A}\gamma N_x(X\sim A)$. This is a contradiction. Therefore, $\gamma F(B) = \gamma\tau(X\sim B) = \inf_{x\in X\sim B}\gamma N_x(X\sim B)\geq\inf_{x\in X\sim B}\gamma\tau(C_x) > t + \sup_{x\in B\sim A}\gamma N_x(X\sim A) > t$. Since t is arbitrary, it holds that $[B\dot{\equiv}\gamma - cl(A)]\leq[B\in\gamma F]$.

Definition 5.3. Let (X,τ) be a fuzzifying topological space. For any $A\subseteq X$, the γ-interior of A is given as follows: $\gamma - int(A)(x) = \gamma N_x(A)$.

Theorem 5.5. For any x, A and B,

(1) $\models B \in \gamma\tau \wedge B \subseteq A \to B \subseteq \gamma - int(A)$;
(2) $\models A \equiv \gamma - int(A) \leftrightarrow A \in \gamma\tau$;
(3) $\models x \in \gamma - int(A) \leftrightarrow x \in A \wedge x \in (X \sim \gamma - d(X \sim A))$;
(4) $\models \gamma - int(A) \equiv X \sim \gamma - cl(X \sim A)$;
(5) $\models B \dot{\equiv} \gamma - int(A) \to B \in \gamma\tau$.

Proof. (1) If $B \not\subseteq A$, then $[B \in \gamma\tau \wedge B \subseteq A] = 0$. If $B \subseteq A$, then $[B \subseteq \gamma - int(A)] = \inf_{x \in B} \gamma - int(A)(x) = \inf_{x \in B} \gamma N_x(A) \geq \inf_{x \in B} \gamma N_x(B) = \gamma\tau(B) = [B \in \gamma\tau \wedge B \subseteq A]$.

(2) $[A \equiv \gamma - int(A)] = \min(\inf_{x \in A} \gamma - int(A)(x), \inf_{x \in X \sim A}(1 - \gamma - int(A)(x)) = \inf_{x \in A} \gamma - int(A)(x) = \inf_{x \in A} \gamma N_x(A) = \gamma\tau(A) = [A \in \gamma\tau]$.

(3) If $x \notin A$, then $[x \in \gamma - int(A)] = 0 = [x \in A \wedge x \in (X \sim \gamma - d(X \sim A))]$. If $x \in A$, then $[x \in \gamma - d(X \sim A)] = 1 - \gamma N_x(A \cup \{x\}) = 1 - \gamma N_x(A) = 1 - \gamma - int(A)(x)$, so that $[x \in A \wedge x \in (X \sim \gamma - d(X \sim A))] = [x \in \gamma - int(A)]$.

(4) It follows from Theorem 5.2(1).

(5) From (4) and Theorem 5.4, we have $[B \dot{\equiv} \gamma - int(A)] = [X \sim B \dot{\equiv} \gamma - cl(X \sim A)] \leq [X \sim B \in \gamma F] = [B \in \gamma\tau]$.

6. Fuzzifying γ-continuous functions.

Definition 6.1. Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces. A unary fuzzy predicate $\gamma C \in \mathcal{F}(Y^X)$, called fuzzy γ-continuity, is given as

$$\gamma C(f) := \forall u(u \in U \to f^{-1}(u) \in \gamma\tau).$$

Definition 6.2. Let $(X, \tau), (Y, U)$ be two fuzzifying topological spaces. For any $f \in Y^X$, we define the unary fuzzy predicates $\gamma_j \in \mathcal{F}(Y^X)$ where $j = 1, 2, \cdots, 5$ as follows:

(1) $\gamma_1(f) := \forall B(B \in F_Y \to f^{-1}(B) \in \gamma F_X)$, where F_Y is the family of closed subsets of Y and γF_X is the family of γ-closed subsets of X;
(2) $\gamma_2(f) := \forall x \forall u(u \in N_{f(x)} \to f^{-1}(u) \in \gamma N_x)$, where N is the neighbourhood system of Y and γN is the γ-neighbourhood system of X;
(3) $\gamma_3(f) := \forall x \forall u(u \in N_{f(x)} \to \exists v(f(v) \subseteq u \to v \in \gamma N_x))$;
(4) $\gamma_4(f) := \forall A(f(\gamma - cl_X(A)) \subseteq cl_Y(f(A)))$;
(5) $\gamma_5(f) := \forall B(\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))$.

Theorem 6.1. $\models f \in \gamma C \leftrightarrow f \in \gamma_j, j = 1, 2, \cdots, 5$.

Proof. (1) We prove that $\models f \in \gamma C \leftrightarrow f \in \gamma_1$.

$$\begin{aligned}
[f \in \gamma_1] &= \inf_{F \in P(Y)} \min(1, 1 - F_Y(F) + \gamma F_X(f^{-1}(F))) \\
&= \inf_{F \in P(Y)} \min(1, 1 - U(Y - F) + \gamma\tau(X \sim f^{-1}(F))) \\
&= \inf_{F \in P(Y)} \min(1, 1 - U(Y - F) + \gamma\tau(Y - F))) \\
&= \inf_{u \in P(Y)} \min(1, 1 - U(u) + \gamma\tau(f^{-1}(u))) \\
&= [f \in \gamma C].
\end{aligned}$$

(2) We prove that $\models f \in \gamma C \leftrightarrow f \in \gamma_2$. First, we prove that $\gamma_2(f) \geq \gamma C(f)$. If $N_{f(x)}(u) \leq \gamma N_x(f^{-1}(u))$, then the result holds. Suppose $N_{f(x)}(u) > \gamma N_x(f^{-1}(u))$. It is

clear that if $f(x) \in A \subseteq u$ then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then, we have

$$\begin{aligned} N_{f(x)}(u) - \gamma N_x(f^{-1}(u)) &= \sup_{f(x)\in A\subseteq u} U(A) - \sup_{x\in B\subseteq f^{-1}(u)} \gamma\tau(B) \\ &\leq \sup_{f(x)\in A\subseteq u} U(A) - \sup_{f(x)\in A\subseteq u} \gamma\tau(f^{-1}(A)) \\ &\leq \sup_{f(x)\in A\subseteq u} (U(A) - \gamma\tau(f^{-1}(A))). \end{aligned}$$

So, $1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u)) \geq \inf_{f(x)\in A\subseteq u} (1 - U(A) + \gamma\tau(f^{-1}(A)))$ and

$$\begin{aligned} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) &\geq \inf_{f(x)\in A\subseteq u} \min(1, 1 - U(A) + \gamma\tau(f^{-1}(A))) \\ &\geq \inf_{v\in P(Y)} \min(1, 1 - U(v) + \gamma\tau(f^{-1}(v))) = \gamma C(f). \end{aligned}$$

Hence, $\inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) \geq [f \in \gamma C]$.

Secondly, we prove that $\gamma C(f) \geq \gamma_2(f)$. From Corollary 4.1, we have

$$\begin{aligned} \gamma C(f) &= \inf_{u\in P(Y)} \min(1, 1 - U(u) + \gamma\tau(f^{-1}(u))) \\ &= \inf_{u\in P(Y)} \min(1, 1 - \inf_{f(x)\in u} N_{f(x)}(u) + \inf_{x\in f^{-1}(u)} \gamma N_x(f^{-1}(u))) \\ &= \inf_{u\in P(Y)} \min(1, 1 - \inf_{x\in f^{-1}(u)} N_{f(x)}(u) + \inf_{x\in f^{-1}(u)} \gamma N_x(f^{-1}(u))) \\ &\geq \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) = \gamma_2(f). \end{aligned}$$

(3) We prove that $\models f \in \gamma_2 \leftrightarrow f \in \gamma_3$. Since γN_x is monotonous (Theorem 4.2 (2)), it is clear that $\sup_{v\in P(X), f(v)\subseteq u} \gamma N_x(v) = \sup_{v\in P(X), v\subseteq f^{-1}(u)} \gamma N_x(v) = \gamma N_x(f^{-1}(u))$.

Then,

$$\begin{aligned} \gamma_3(f) &= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N_{f(x)}(u) + \sup_{v\in P(X), f(v)\subseteq u} \gamma N_x(v)) \\ &= \inf_{x\in X} \inf_{u\in P(Y)} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) = \gamma_2(f). \end{aligned}$$

(4) We prove that $\models f \in \gamma_4 \leftrightarrow f \in \gamma_5$.

Firstly, for each $B \in P(Y)$, there exists $A \in P(X)$ such that $f^{-1}(B) = A$ and $f(A) \subseteq B$. So, $[\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq [\gamma - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))]$.

Hence,

$\gamma_5(f) = \inf_{B\in P(Y)} [\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \geq \inf_{A\in P(X)} [\gamma - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] = \gamma_4(f)$.

Secondly, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, $[\gamma - cl(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \leq [\gamma - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))]$. Thus,

$$\begin{aligned} \gamma_4(f) &= \inf_{A\in P(X)} [\gamma - cl_X(A) \subseteq f^{-1}(cl_Y(f(A)))] \\ &\geq \inf_{B\in P(Y), B=f(A)} [\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] \\ &\geq \inf_{B\in P(Y)} [\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))] = \gamma_5(f). \end{aligned}$$

(5) We prove that $\models f \in \gamma_5 \leftrightarrow f \in \gamma_2$. From Theorem 5.2 (1) we have

$$\begin{aligned}\gamma_5(f) &= \forall B(\gamma - cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B)))\\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-\gamma N_x(X \sim f^{-1}(B))) + 1 - N_{f(x)}(Y \sim B))\\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1 - N_{f(x)}(Y \sim B) + \gamma N_x(X \sim f^{-1}(B)))\\ &= \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1 - N_{f(x)}(u) + \gamma N_x(f^{-1}(u))) = \gamma_2(f).\end{aligned}$$

Theorem 6.2. Let $(X,\tau), (Y,U), (Z,V)$ be three fuzzifying topological spaces. For any $f \in Y^X, g \in Z^Y$, (1) $\models \gamma C(f) \to (C(g) \to \gamma C(g\circ f))$; (2) $\models C(g) \to (\gamma C(f) \to \gamma C(g \circ f))$.

Proof. (1) We need to prove that $[\gamma C(f)] \leq [C(g) \to \gamma C(g\circ f)]$. If $[C(g)] \leq [\gamma C(g\circ f)]$, then the result holds. If $[C(g)] > [\gamma C(g \circ f)]$, then

$$\begin{aligned}&[C(g)] - [\gamma C(g\circ f)]\\ &= \inf_{v\in P(Z)} \min(1, 1 - V(v) + U(g^{-1}(v))) - \inf_{v\in P(z)} \min(1, 1 - V(v) + \gamma\tau(g\circ f)^{-1}(v))\\ &\leq \sup_{v\in P(Z)} (U(g^{-1}(v)) - \gamma\tau(g\circ f)^{-1}(v)) \leq \sup_{u\in P(Y)} (U(u) - \gamma\tau(f^{-1}(u)).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}[C(g)] \to [\gamma C(g\circ f)] &= \min(1, 1 - [C(g)] + [\gamma C(g \circ f)])\\ &\geq \inf_{u\in P(Y)} \min(1, 1 - U(u) + \gamma\tau(f^{-1}(u))) = \gamma C(f).\end{aligned}$$

(2) Since the conjunction $\wedge$ is commutative, from (1) above one can deduce that $[C(g) \to (\gamma C(f) \to \gamma C(g\circ f))] = [\neg(C(g) \wedge \gamma C(f) \wedge \neg\gamma C(g\circ f))] =$ $[\neg(\gamma C(f) \wedge C(g) \wedge \neg\gamma C(g\circ f))] = [\gamma C(f) \to (C(g) \to \gamma C(g \circ f))] = 1$.

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