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Multi-Granulation Double Fuzzy Rough Sets

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Abstract: In this article, we introduce two new rough set models based on the concept of double fuzzy relations. These models are called optimistic and pessimistic multi-granulation double fuzzy rough sets. We discuss their properties and explore the relationship between these new models and double fuzzy rough sets. Our study focuses on the lower and upper approximations of these models, which generalize the conventional rough set model. In addition, we suggest that the development of the multi-granulation double fuzzy rough set model is significant for the generalization of the rough set model.

Keywords: double fuzzy rough set; optimistic multi-granulation double fuzzy rough set; pessimistic multi-granulation double fuzzy rough set

MSC: 47H10; 47H09; 47H04; 46S40; 54H25

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1. Introduction and Preliminaries

The theory of rough sets, introduced by Pawlak [1,2], has become a well-established mathematical tool for studying uncertainty in a variety of applications and intelligent systems that deal with incomplete or inadequate information. The equivalence classes defined by the equivalence relation are used to determine the lower and upper approximations to approximate undefinable sets. Rough sets have been widely applied in various fields, including granular computing, graph theory, algebraic systems, partially ordered sets, medical diagnosis, data mining, and conflict analysis, among others [3–8].

The study of set theory involves a significant exploration of the generalization and extension of the rough set model. Qian et al. [9,10] introduced the multi-granulation rough set model, which is defined by a family of equivalence relations, as opposed to Pawlak's rough set model, which is defined by only one equivalence relation. The multi-granulation rough set model includes two types: the optimistic and pessimistic multi-granulation rough sets. The term "optimistic" is used to refer to the idea that in multi-independent granular structures, at least one granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set. Meanwhile, "pessimistic" denotes the idea that each granular structure must satisfy the inclusion relation between the equivalence class and the undefinable set. There have been several studies exploring multi-granulation rough set models based on various types of relations, leading to a number of intriguing ideas, such as those presented in [11–18].

On the other hand, one of these trends is to combine other theories dealing with uncertain knowledge, such as fuzzy set and rough set theory. The fuzzy set theory addresses potential uncertainties associated with erroneous cases, perceptions, and preferences, whereas approximate sets identify uncertainty caused by the ambiguity of information. As both types of uncertainty can arise in real-world problems, there have been numerous proposed approaches to combining fuzzy set theory with approximation set theory. Dubois and Prade introduced rough fuzzy sets and fuzzy rough sets based on approximations of fuzzy sets by crisp approximation spaces, as seen in [19,20]. Using the same framework, researchers have developed an approach to enhance coarse fuzzy rough sets and rough fuzzy sets, as demonstrated in [21–26].

Atanassov [27] proposed the concept of intuitionistic fuzzy sets, which provide membership and non-membership degrees for an element. This allows for more flexibility and efficiency when dealing with incomplete or inaccurate information compared with Zadeh's fuzzy sets [28].

The use of the term "intuitionistic" in relation to complete lattice L has generated some debate regarding its applicability. However, Garcia and Rodabaugh [29] definitively settled these doubts by demonstrating that this term is not appropriate for mathematics and its applications. As a result, they have adopted the name "double" for their work in this area.

Inspired and motivated by the recent works [1–30], our aim in this paper is to investigate and enhance the study of multi-granulation double fuzzy approximation spaces by exploring the double fuzzy upper and lower approximation operators. The framework focuses on introducing two types of double fuzzy sets using multiple pairs of double fuzzy relations on U and analyzing their relationship.

Throughout this paper, let $U = \{x_1, x_2, ...x_n\}$ be a nonempty and finite set of objects and $I = [0,1]$. A fuzzy set is a map from U to I. The set of all fuzzy sets on U is denoted by I^U. R is a fuzzy binary relation on U, i.e., $R(x,y) \in [0,1]$ for any $x, y \in U$. The set of all fuzzy binary relations on U is denoted by $I^{U\times U}$.

Definition 1 ([30]). *Let U and V be two arbitrary sets. A double fuzzy relation on $U \times V$ is a pair (R, R^*) of maps $R, R^* : U \times V \to I$ such that $R(x,y) \leq 1 - R^*(x,y)$ for all $(x,y) \in U \times V$. If $R, R^* : U \times U \to I$, then (R, R^*) is called a double fuzzy relation on U. $R(x,y)$ (resp. $R^*(x,y)$), referred to as the degree of relation (resp. non-relation) between x and y.*

Definition 2 ([30]). *Let U be an arbitrary universal set and (R, R^*) a double fuzzy relation on U. Then, for each fuzzy set λ on U, the pairs $(\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda)$, $(\overline{\mathcal{R}}_R\lambda, \overline{\mathcal{R}}^*_{R^*}\lambda)$ of maps $\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda, \overline{\mathcal{R}}_R\lambda$, $\overline{\mathcal{R}}^*_{R^*}\lambda : U \to I$ are called double fuzzy lower approximation and double fuzzy upper approximation of a fuzzy set λ, respectively, and are defined as follows: For all $x \in U$,*

$$(\underline{\mathcal{R}}_R\lambda)(x) = \bigwedge_{y\in U}((1 - R(x,y)) \vee \lambda(y)),\ (\underline{\mathcal{R}}^*_{R^*}\lambda)(x) = \bigvee_{y\in U}((1 - R^*(x,y)) \wedge (1 - \lambda(y)))$$

$$(\overline{\mathcal{R}}_R\lambda)(x) = \bigvee_{y\in U}(R(x,y) \wedge \lambda(y)),\ (\overline{\mathcal{R}}^*_{R^*}\lambda)(x) = \bigwedge_{y\in U}(R^*(x,y) \vee (1 - \lambda(y))).$$

The quaternary $(\underline{\mathcal{R}}_R\lambda, \underline{\mathcal{R}}^*_{R^*}\lambda, \overline{\mathcal{R}}_R\lambda, \overline{\mathcal{R}}^*_{R^*}\lambda)$ is called double fuzzy rough set of λ. The pairs $(\underline{\mathcal{R}}_R, \underline{\mathcal{R}}^*_{R^*})$, $(\overline{\mathcal{R}}_R, \overline{\mathcal{R}}^*_{R^*})$ of operators $\underline{\mathcal{R}}_R, \underline{\mathcal{R}}^*_{R^*}, \overline{\mathcal{R}}_R, \overline{\mathcal{R}}^*_{R^*} : I^U \to I^U$ are called double fuzzy lower approximation and double fuzzy upper approximation operators, respectively.

Definition 3 ([30]). *For all $x, y \in U$, a double fuzzy relation (R, R^*) on U is called as follows:*

(1) Double fuzzy reflexive if $R(x,x) = 1$ and $R^*(x,x) = 0$.

(2) Double fuzzy transitive if $R(x,z) \geq \bigvee_{y\in U}(R(x,y) \wedge R(y,z))$ and $R^*(x,z) \leq \bigwedge_{y\in U}(R^*(x,y) \vee R^*(y,z))$ $\forall z \in U$.

(3) Double fuzzy symmetric if $R(x,y) = R(y,x)$ and $R^*(x,y) = R^*(y,x)$.

2. Optimistic Multi-Granulation Double Fuzzy Rough Sets

In this section, we provide some concepts along with an example and discuss the optimistic multi-granulation double fuzzy rough sets based on multiple double fuzzy relations.

Definition 4. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. Then, for each fuzzy set λ on U, the pairs $(\underline{\mathcal{OR}_{R_1+R_2}}\lambda, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ and $(\overline{\mathcal{OR}_{R_1+R_2}}\lambda, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ of maps $\underline{\mathcal{OR}_{R_1+R_2}}\lambda$, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$, $\overline{\mathcal{OR}_{R_1+R_2}}\lambda$, $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda : U \to I$ are called optimistic two-granulation double fuzzy lower approximation and optimistic two-granulation double fuzzy upper approximation of a fuzzy set λ, respectively, and are defined as follows: For all $x \in U$,*

$$(\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee \lambda(y)) \right\};$$

$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge 1 - \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge 1 - \lambda(y)) \right\};$$

$$(\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\};$$

$$(\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\}.$$

The quaternary $(\underline{\mathcal{OR}_{R_1+R_2}}\lambda, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda, \overline{\mathcal{OR}_{R_1+R_2}}\lambda, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)$ is called optimistic two-granulation double fuzzy rough set of λ (in short, OTGDFRS). The pairs $(\underline{\mathcal{OR}_{R_1+R_2}}, \underline{\mathcal{OR}^*_{R_1^*+R_2^*}})$ and $(\overline{\mathcal{OR}_{R_1+R_2}}, \overline{\mathcal{OR}^*_{R_1^*+R_2^*}})$ of operators $\underline{\mathcal{OR}_{R_1+R_2}}$, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}$, $\overline{\mathcal{OR}_{R_1+R_2}}$, $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}} : U \to I$ are called optimistic two-granulation double fuzzy lower approximation and optimistic two-granulation double fuzzy upper approximation operators, respectively.

The OTGDFRS approximations are defined by many separate pairs of double fuzzy relations, whereas the normal double fuzzy rough approximations are represented by those produced by only one pair of double fuzzy relations, as can be seen from the preceding definition. In fact, when $(R_1, R_1^*) = (R_2, R_2^*)$, the OTGDFRS degenerates into a double fuzzy rough set. Put another way, a double fuzzy rough set model is a subset of the OTGDFRS.

Proposition 1. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. For each $\lambda \in I^U$, the following apply:*

(1) $\underline{\mathcal{OR}_{R_1+R_2}}\lambda = \underline{\mathcal{R}_{R_1}}\lambda \vee \underline{\mathcal{R}_{R_2}}\lambda$, and $\underline{\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}}\lambda = \underline{\underline{\mathcal{R}^*_{R_1^*}}}\lambda \wedge \underline{\underline{\mathcal{R}^*_{R_2^*}}}\lambda$.

(2) $\overline{\mathcal{OR}_{R_1+R_2}}\lambda = \overline{\mathcal{R}_{R_1}}\lambda \wedge \overline{\mathcal{R}_{R_2}}\lambda$, and $\overline{\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}}\lambda = \overline{\overline{\mathcal{R}^*_{R_1^*}}}\lambda \vee \overline{\overline{\mathcal{R}^*_{R_2^*}}}\lambda$.

Proof. The proofs follow directly from Definitions 2 and 4. □

From Definition 4, it is possible to determine the properties of the optimistic multi-granulation double fuzzy rough sets, as in the following.

Theorem 1. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. For each $\lambda \in I^U$, the following apply:*

(1) $\overline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$, and $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) $\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1} = \tilde{1}$, and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{OR}_{R_1+R_2}}\tilde{0} = \tilde{0}$, and $\overline{\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{OR}_{R_1+R_2}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}_{R_1+R_2}}\lambda$, and $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}}\lambda$.

(5) $\underline{\mathcal{OR}_{R_1+R_2}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}_{R_1+R_2}}\lambda$, and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}}\lambda$.

Proof. (1) For each $x \in U,\ \lambda \in I^U$, we have

$$\begin{aligned}
&\big(\tilde{1} - (\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)\big)(x) \\
= \quad & 1 - \left\{ \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \quad & \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \wedge \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \quad & \left\{ \bigvee_{y\in U} 1 - \{R_1^*(x,y) \vee 1 - \lambda(y)\} \right\} \wedge \left\{ \bigvee_{y\in U} 1 - \{R_2^*(x,y) \vee 1 - \lambda(y)\} \right\} \\
= \quad & \left\{ \bigvee_{y\in U} \{1 - R_1^*(x,y) \wedge \lambda(y)\} \right\} \wedge \left\{ \bigvee_{y\in U} \{1 - R_2^*(x,y) \wedge \lambda(y)\} \right\} \\
\geq \quad & \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\} \\
= \quad & (\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) \text{ for all } x \in U.
\end{aligned}$$

Hence, $\overline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) Since, for each $x \in U$, $\tilde{1}(x) = 1$, we obtain

$$\begin{aligned}
(\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1})(x) \quad &= \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee \tilde{1}(y)) \right\} \vee \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee \tilde{1}(y)) \right\} \\
&= 1 = \tilde{1}(x),
\end{aligned}$$

and

$$\begin{aligned}
(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1})(x) \quad &= \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge 1 - \tilde{1}(y)) \right\} \\
&\wedge \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge 1 - \tilde{1}(y)) \right\} \\
&= 0 = \tilde{0}(x).
\end{aligned}$$

Therefore, we obtain $\underline{\mathcal{OR}_{R_1+R_2}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) The proof is similar to the proof of (2).

(4) For each $x \in U$, we have

$$
\begin{aligned}
& \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda)(x) \\
= & \left\{\bigwedge_{y\in U}(R_1^*(x,y)\vee 1-(1-\lambda(y)))\right\}\vee\left\{\bigwedge_{y\in U}(R_2^*(x,y)\vee 1-(1-\lambda(y)))\right\} \\
= & \left\{1-\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\right\}\vee\left\{1-\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= & 1-\left\{\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\wedge\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= & 1-\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda(x).
\end{aligned}
$$

Thus, we obtain $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda)=\tilde{1}-\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, we can prove that $\overline{\mathcal{OR}_{R_1+R_2}}(\tilde{1}-\lambda)=\tilde{1}-\underline{\mathcal{OR}_{R_1+R_2}}\lambda$

(5) The proof is similar to the proof of (4). □

Theorem 2. *Let U be an arbitrary universal set, and (R_1,R_1^*) and (R_2,R_2^*) be double fuzzy relations on U. For each $\lambda,\mu\in I^U$, the following apply:*

(1) $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\wedge\mu)\le\underline{\mathcal{OR}_{R_1+R_2}}\lambda\wedge\underline{\mathcal{OR}_{R_1+R_2}}\mu$, *and* $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu)\ge\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\vee\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(2) $\overline{\mathcal{OR}_{R_1+R_2}}(\lambda\vee\mu)\ge\overline{\mathcal{OR}_{R_1+R_2}}\lambda\vee\overline{\mathcal{OR}_{R_1+R_2}}\mu$, *and*

$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\le\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\wedge\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(3) *If $\lambda\le\mu$, then* $\underline{\mathcal{OR}_{R_1+R_2}}\lambda\le\underline{\mathcal{OR}_{R_1+R_2}}\mu$, *and* $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\ge\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(4) *If $\lambda\le\mu$, then* $\overline{\mathcal{OR}_{R_1+R_2}}\lambda\le\overline{\mathcal{OR}_{R_1+R_2}}\mu$, *and* $\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\ge\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(5) $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\vee\mu)\ge\underline{\mathcal{OR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{OR}_{R_1+R_2}}\mu$, *and*

$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\le\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\wedge\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(6) $\overline{\mathcal{OR}_{R_1+R_2}}(\lambda\wedge\mu)\le\overline{\mathcal{OR}_{R_1+R_2}}\lambda\wedge\overline{\mathcal{OR}_{R_1+R_2}}\mu$, *and*

$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu)\ge\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda\vee\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

Proof. (1) For each $x\in U$ and $\lambda,\mu\in I^U$, we have

$$
\begin{aligned}
& (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda\wedge\mu))(x) \\
= & \left\{\bigwedge_{y\in U}((1-R_1(x,y))\vee(\lambda\wedge\mu)(y))\right\}\vee\left\{\bigwedge_{y\in U}((1-R_2(x,y))\vee(\lambda\wedge\mu)(y))\right\} \\
= & \left\{\left\{\bigwedge_{y\in U}((1-R_1(x,y))\vee(\lambda)(y)\right\}\wedge\left\{\bigwedge_{y\in U}((1-R_1(x,y))\vee(\mu)(y)\right\}\right\} \\
\vee & \left\{\left\{\bigwedge_{y\in U}((1-R_2(x,y))\vee(\lambda)(y)\right\}\wedge\left\{\bigwedge_{y\in U}((1-R_2(x,y))\vee(\mu)(y)\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{(\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_1}}\mu)(x)\right\} \vee \left\{(\underline{\mathcal{R}_{R_2}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x)\right\} \\
&\leq \left\{(\underline{\mathcal{R}_{R_1}}\lambda)(x) \bigvee (\underline{\mathcal{R}_{R_2}}\lambda)(x)\right\} \wedge \left\{(\underline{\mathcal{R}_{R_1}}\mu)(x) \bigvee (\underline{\mathcal{R}_{R_2}}\mu)(x)\right\} \\
&= (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) \wedge (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(x).
\end{aligned}
$$

Also, for each $x \in U$, we have

$$
\begin{aligned}
&(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) \\
&= \left\{\bigvee_{y\in U}((1-R_1^*(x,y)) \wedge 1-(\lambda\wedge\mu)(y))\right\} \wedge \left\{\bigvee_{y\in U}((1-R_2^*(x,y)) \wedge 1-(\lambda\wedge\mu)(y))\right\} \\
&= \left\{\bigvee_{y\in U}((1-R_1^*(x,y)) \wedge (1-\lambda(y) \vee 1-\mu(y))\right\} \wedge \left\{\bigvee_{y\in U}((1-R_2^*(x,y)) \wedge (1-\lambda(y) \vee 1-\mu(y))\right\} \\
&= \left\{\left\{\bigvee_{y\in U}((1-R_1^*(x,y)) \wedge (1-\lambda(y)\right\} \vee \left\{\bigvee_{y\in U}((1-R_1^*(x,y)) \wedge (1-\mu(y)\right\}\right\} \\
&\quad \wedge \left\{\left\{\bigvee_{y\in U}((1-R_2^*(x,y)) \wedge (1-\lambda(y)\right\} \vee \left\{\bigvee_{y\in U}((1-R_2^*(x,y)) \wedge (1-\mu(y)\right\}\right\} \\
&= \left\{(\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x)\right\} \wedge \left\{(\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x)\right\} \\
&\geq \left\{(\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \wedge (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x)\right\} \vee \left\{(\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \wedge (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x)\right\} \\
&= (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) \vee (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(x)
\end{aligned}
$$

(2) The proof is similar to the proof of (1).
(3) If $\lambda \leq \mu$, then for all $y \in U,\ \lambda(y) \leq \mu(y)$, we get

$$\bigwedge_{y\in U}(1-R_1(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U}(1-R_1(x,y) \vee \mu(y)) \tag{1}$$

and

$$\bigwedge_{y\in U}(1-R_2(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U}(1-R_2(x,y) \vee \mu(y)). \tag{2}$$

From Equations (1) and (2), we have

$$
\begin{aligned}
&\left\{\bigwedge_{y\in U}(1-R_1(x,y) \vee \lambda(y))\right\} \vee \left\{\bigwedge_{y\in U}(1-R_2(x,y) \vee \lambda(y))\right\} \\
&\leq \left\{\bigwedge_{y\in U}(1-R_1(x,y) \vee \mu(y))\right\} \vee \left\{\bigwedge_{y\in U}(1-R_2(x,y) \vee \mu(y))\right\}.
\end{aligned}
$$

Therefore, $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \underline{\mathcal{OR}_{R_1+R_2}}\mu$. Also,

$$\bigvee_{y\in U}(1-R_1^*(x,y) \wedge 1-\lambda(y)) \geq \bigvee_{y\in U}(1-R_1^*(x,y) \wedge 1-\mu(y)) \tag{3}$$

and

$$\bigvee_{y \in U}(1 - R_2^*(x,y) \wedge 1 - \lambda(y)) \geq \bigvee_{y \in U}(1 - R_2^*(x,y) \wedge 1 - \mu(y)). \quad (4)$$

From Equations (3) and (4), we have

$$\left\{\bigvee_{y \in U}(1 - R_1^*(x,y) \wedge 1 - \lambda(y))\right\} \wedge \left\{\bigvee_{y \in U}(1 - R_2^*(x,y) \wedge 1 - \lambda(y))\right\}$$
$$\geq \left\{\bigvee_{y \in U}(1 - R_1^*(x,y) \wedge 1 - \mu(y))\right\} \wedge \left\{\bigvee_{y \in U}(1 - R_2^*(x,y) \wedge 1 - \mu(y))\right\}$$

Hence, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

(4) The proof is similar to the proof of (3).

(5) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, from (3), we have

$$\underline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \underline{\mathcal{OR}_{R_1+R_2}}(\lambda \vee \mu) \text{ and } \underline{\mathcal{OR}_{R_1+R_2}}\mu \leq \underline{\mathcal{OR}_{R_1+R_2}}(\lambda \vee \mu).$$

Therefore, $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \vee \underline{\mathcal{OR}_{R_1+R_2}}\mu \leq \underline{\mathcal{OR}_{R_1+R_2}}(\lambda \vee \mu)$. Also, we have

$$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu) \text{ and } \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu).$$

This implies that $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \wedge \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu \geq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu)$.

(6) The proof is similar to the proof of (5). □

In the following example, we demonstrate that the converse of Theorem 2 (1) is not true.

Example 1. *Let $U = \{x, y, z\}$. Define $R_1, R_1^*, R_2, R_2^* : U \times U \to I$ as follows:*

$$R_1 = \begin{pmatrix} 0.2 & 0.4 & 0.7 \\ 0.9 & 0.3 & 0.7 \\ 0.6 & 0.3 & 0.2 \end{pmatrix} \qquad R_1^* = \begin{pmatrix} 0.1 & 0.6 & 0.2 \\ 0.0 & 0.6 & 0.2 \\ 0.3 & 0.0 & 0.7 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0.1 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.8 \\ 0.7 & 0.2 & 0.3 \end{pmatrix} \qquad R_2^* = \begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0.1 & 0.6 & 0.1 \\ 0.3 & 0.6 & 0.6 \end{pmatrix}$$

Define $\lambda, \mu \in I^U$ as follows:

$$\lambda = \{(x, 0.5), (y, 0.7), (z, 0.1)\},$$
$$\mu = \{(x, 0.4), (y, 0.2), (z, 0.8)\},$$
$$\lambda \wedge \mu = \{(x, 0.4), (y, 0.2), (z, 0.1)\}.$$

Then,

$$(\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(x) = 0.4, (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(y) = 0.3, (\underline{\mathcal{OR}_{R_1+R_2}}\lambda)(z) = 0.5$$
$$(\underline{\mathcal{OR}_{R_1+R_2}}\mu)(x) = 0.6, (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(y) = 0.6, (\underline{\mathcal{OR}_{R_1+R_2}}\mu)(z) = 0.4$$
$$(\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(x) = 0.4, (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(y) = 0.2, (\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu))(z) = 0.4$$

Therefore, $\underline{\mathcal{OR}_{R_1+R_2}}(\lambda \wedge \mu) \not\geq \underline{\mathcal{OR}_{R_1+R_2}}\lambda \wedge \underline{\mathcal{OR}_{R_1+R_2}}\mu$.

$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(x) = 0.6, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(y) = 0.8, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)(z) = 0.5$$
$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(x) = 0.6, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(y) = 0.6, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu)(z) = 0.6$$
$$(\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) = 0.7, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(y) = 0.8, (\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(z) = 0.6.$$

Therefore, $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu) \nleq \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \vee \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\mu$.

Theorem 3. *Let* (R_1, R_1^*) *and* (R_2, R_2^*) *be double fuzzy relations on an universal set* U. *Then, the following statements are equivalent:*

(1) (R_1, R_1^*) *and* (R_2, R_2^*) *are double fuzzy reflexive relations.*

(2) $\lambda \leq \overline{\mathcal{OR}_{R_1+R_2}}\lambda$, *and* $\tilde{1} - \lambda \geq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(3) $\underline{\mathcal{OR}_{R_1+R_2}}\lambda \leq \lambda$, *and* $\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda \geq \tilde{1} - \lambda$.

Proof. (1) $\Rightarrow$ (2) Let (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy reflexive relations. Then, $R_i(x, x) = 1$, and $R_i^*(x, x) = 0$ for all $i \in \{1, 2\}$ and $x \in U$. Therefore,

$$\begin{aligned}
\lambda(x) &= 1 \wedge \lambda(x) \\
&= \{R_1(x,x) \wedge \lambda(x)\} \wedge \{R_2(x,x) \wedge \lambda(x)\} \\
&\leq \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\} \\
&= \overline{\mathcal{OR}_{R_1+R_2}}\lambda
\end{aligned}$$

and

$$\begin{aligned}
\tilde{1} - \lambda(x) &= 0 \vee \tilde{1} - \lambda(x) \\
&= \left\{R_1^*(x,x) \vee \tilde{1} - \lambda(x)\right\} \vee \left\{R_2^*(x,x) \vee \tilde{1} - \lambda(x)\right\} \\
&\geq \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee \tilde{1} - \lambda(y)) \right\} \vee \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee \tilde{1} - \lambda(y)) \right\} \\
&= \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda.
\end{aligned}$$

(2) $\Rightarrow$ (1) Suppose that there exist some $x \in U$ such that $R_i(x, x) = a_i \neq 1$ and $R_i^*(x, x) = b_i \neq 0$ for all $i \in \{1, 2\}$; then, we can define fuzzy set $\delta_x : U \longrightarrow I$ as

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases}$$

Then,

$$\begin{aligned}
\overline{\mathcal{OR}_{R_1+R_2}}\delta_x(x) &= \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \delta_x(y)) \right\} \wedge \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \delta_x(y)) \right\} \\
&= R_1(x,x) \wedge R_2(x,x) \\
&= a_1 \wedge a_2 \neq 1 = \delta_x(x)
\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\delta_x(x) &= \left\{\bigwedge_{y\in U}(R_1^*(x,y)\vee 1-\delta_x(y))\right\}\vee\left\{\bigwedge_{y\in U}(R_2^*(x,y)\vee 1-\delta_x(y))\right\}\\ &= R_1^*(x,x)\vee R_2^*(x,x)\\ &= b_1\vee b_2\neq 0 = 1-\delta_x(x).\end{aligned}$$

Therefore $\delta_x \nleq \overline{\mathcal{OR}_{R_1+R_2}}\delta_x$ and $\tilde{1}-\delta_x \ngeq \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\delta_x$. This is a contradiction. Hence, $R_i(x,x)=1$, and $R_i^*(x,x)=0$ for all $i\in\{1,2\}$ and $x\in U$.
(2) $\Leftrightarrow$ (3) It is easy to show this from Theorem 1 ((4) and (5)). □

Theorem 4. *Let (R_1,R_1^*) and (R_2,R_2^*) be double fuzzy relations on an universal set U. Then, the following statements are equivalent:*

(1) *(R_1,R_1^*) and (R_2,R_2^*) are double fuzzy transitive relations.*
(2) $\overline{\mathcal{OR}_{R_1+R_2}}(\overline{\mathcal{OR}_{R_1+R_2}}\lambda)\leq\overline{\mathcal{OR}_{R_1+R_2}}\lambda$, *and*
$\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda)\geq\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.
(3) $\underline{\mathcal{OR}_{R_1+R_2}}(\underline{\mathcal{OR}_{R_1+R_2}}\lambda)\geq\underline{\mathcal{OR}_{R_1+R_2}}\lambda$, *and*
$\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\underline{\mathcal{OR}^*_{R_1^*+R_2^*}})\leq\underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

Proof. (1) $\Leftrightarrow$ (2) For each $\lambda\in I^U$,

$$\begin{aligned}&\overline{\mathcal{OR}_{R_1+R_2}}(\overline{\mathcal{OR}_{R_1+R_2}}\lambda)(x)\\ =\ &\left\{\bigvee_{y\in U}(R_1(x,y)\wedge(\overline{\mathcal{OR}_{R_1+R_2}})(y)\right\}\wedge\left\{\bigvee_{y\in U}(R_2(x,y)\wedge(\overline{\mathcal{OR}_{R_1+R_2}})(y)\right\}\\ &= b_1\vee b_2\neq 0 = 1-\delta_x(x).\end{aligned}$$

□

As part of the extension of the optimistic two-granulation double fuzzy rough set, we will introduce the optimistic multi-granulation double fuzzy rough set (in short, OMGDFRS) and its associated properties.

Definition 5. *Let U be an arbitrary set and the pairs (R_i,R_i^*), such that $1\leq i\leq m$, double fuzzy relations on U. Then, $(U,\mathcal{R},\mathcal{R}^*)$ is called the multi-granulation double fuzzy approximation space (in short, MGDFAS), where $\mathcal{R}=\{R_1,R_2,...R_i\}$ and $\mathcal{R}^*=\{R_1^*,R_2^*,...R_i^*\}$.*

Definition 6. *Let $(U,\mathcal{R},\mathcal{R}^*)$ be an MGDFAS. Then, for each fuzzy set λ on U, the pairs $(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda,\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ and $(\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda,\overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ of maps $\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda,\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda$, $\overline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda,\overline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda : U\rightarrow I$ are called optimistic multi-granulation double fuzzy lower approximation and optimistic multi-granulation double fuzzy upper approximation of a fuzzy set λ, respectively, and are defined as follows: For all $x\in U$,*

$$(\underline{\mathcal{OR}_{\sum_{i=1}^m R_i}}\lambda)(x)=\bigvee_{i=1}^m\bigwedge_{y\in U}((1-R_i(x,y))\vee\lambda(y)),$$

$$(\underline{\mathcal{OR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x)=\bigwedge_{i=1}^m\bigvee_{y\in U}((1-R_i^*(x,y))\wedge 1-\lambda(y)),$$

$$(\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda)(x) = \bigwedge_{i=1}^{m} \bigvee_{y\in U} (R_i(x,y) \wedge \lambda(y)),$$

$$(\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda)(x) = \bigvee_{i=1}^{m} \bigwedge_{y\in U} (R_i^*(x,y) \vee 1 - \lambda(y)).$$

The quaternary $(\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda, \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda, \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda, \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda)$ *is called optimistic multi-granulation double fuzzy rough set of* λ *(in short, OMGDFRS).*

The pairs $(\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}, \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*})$ *and* $(\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}, \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*})$ *of operators* $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}, \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}, \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}, \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*} : I^U \to I^U$ *are called optimistic multi-granulation double fuzzy lower approximation and optimistic multi-granulation double fuzzy upper approximation operators, respectively.*

Proposition 2. *Let* $(U, \mathcal{R}, \mathcal{R}^*)$ *be an MGDFAS. For each* $\lambda \in I^U$*, the following apply:*

(1) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda = \bigvee_{i=1}^{m} \underline{\mathcal{R}_{R_i}}\lambda$*, and* $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda = \bigwedge_{i=1}^{m} \underline{\mathcal{R}^*_{R_i^*}}\lambda$.

(2) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda = \bigwedge_{i=1}^{m} \overline{\mathcal{R}_{R_i}}\lambda$*, and* $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda = \bigvee_{i=1}^{m} \overline{\mathcal{R}^*_{R_i^*}}\lambda$.

Proof. The proof is similar to the proof of Proposition 1. □

Theorem 5. *Let* $(U, \mathcal{R}, \mathcal{R}^*)$ *be an MGDFAS. For each* $\lambda \in I^U$*, the following apply:*

(1) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \leq \tilde{1} - \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$*, and* $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \geq \tilde{1} - \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$.

(2) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\tilde{1} = \tilde{1}$*, and* $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\tilde{0} = \tilde{0}$*, and* $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda$*, and* $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$.

(5) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda$*, and* $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda$.

Proof. The proof is similar to the proof of Theorem 1. □

Theorem 6. *Let* $(U, \mathcal{R}, \mathcal{R}^*)$ *be an MGDFAS. For each* $\lambda, \mu \in I^U$*, the following apply:*

(1) $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \wedge \mu) \leq \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \wedge \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$*, and*

$\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \wedge \mu) \geq \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \vee \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(2) $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}(\lambda \vee \mu) \geq \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \vee \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$*, and*

$\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}(\lambda \vee \mu) \leq \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \wedge \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(3) *If* $\lambda \leq \mu$*, then* $\underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \leq \underline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$*, and* $\underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \geq \underline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(4) *If* $\lambda \leq \mu$*, then* $\overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\lambda \leq \overline{\mathcal{OR}}_{\sum_{i=1}^{m} R_i}\mu$*, and* $\overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\lambda \geq \overline{\mathcal{OR}^*}_{\sum_{i=1}^{m} R_i^*}\mu$.

(5) $\underline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}(\lambda \vee \mu) \geq \underline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\lambda \vee \underline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\mu$, and
$\underline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \vee \mu) \leq \underline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \wedge \underline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(6) $\overline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}(\lambda \wedge \mu) \leq \overline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\lambda \wedge \overline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\mu$, and
$\overline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \wedge \mu) \geq \overline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \vee \overline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

Proof. The proof is similar to the proof of Theorem 2. □

3. Pessimistic Multi-Granulation Double Fuzzy Rough Sets

In this section, we provide the pessimistic multi-granulation double fuzzy rough sets based on multiple double fuzzy relations and discuss the relationship between optimistic multi-granulation double fuzzy rough sets and pessimistic multi-granulation double fuzzy rough sets.

Definition 7. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. Then, for each fuzzy set λ on U, the pairs $(\underline{\mathcal{PR}_{R_1+R_2}}\lambda, \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)$ and $(\overline{\mathcal{PR}_{R_1+R_2}}\lambda, \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)$ of maps $\underline{\mathcal{PR}_{R_1+R_2}}\lambda, \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda, \overline{\mathcal{PR}_{R_1+R_2}}\lambda, \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda : U \to I$ are called pessimistic two-granulation double fuzzy lower approximation and pessimistic two-granulation double fuzzy upper approximation of a fuzzy set λ, respectively, and are defined as follows: For all $x \in U$,*

$$(\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigwedge_{y \in U} ((1 - R_1(x,y)) \vee \lambda(y)) \right\} \wedge \left\{ \bigwedge_{y \in U} ((1 - R_2(x,y)) \vee \lambda(y)) \right\};$$

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigvee_{y \in U} ((1 - R_1^*(x,y)) \wedge 1 - \lambda(y)) \right\} \vee \left\{ \bigvee_{y \in U} ((1 - R_2^*(x,y)) \wedge 1 - \lambda(y)) \right\};$$

$$(\overline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) = \left\{ \bigvee_{y \in U} (R_1(x,y) \wedge \lambda(y)) \right\} \vee \left\{ \bigvee_{y \in U} (R_2(x,y) \wedge \lambda(y)) \right\};$$

$$(\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(x) = \left\{ \bigwedge_{y \in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \wedge \left\{ \bigwedge_{y \in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\}.$$

The quaternary $(\underline{\mathcal{PR}_{R_1+R_2}}\lambda, \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda, \overline{\mathcal{PR}_{R_1+R_2}}\lambda, \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)$ is called pessimistic two-granulation double fuzzy rough set of λ (in short, PTGDFRS). The pairs $(\underline{\mathcal{PR}_{R_1+R_2}}, \underline{\mathcal{PR}^*_{R_1^*+R_2^*}})$ and $(\overline{\mathcal{PR}_{R_1+R_2}}, \overline{\mathcal{PR}^*_{R_1^*+R_2^*}})$ of operators $\underline{\mathcal{PR}_{R_1+R_2}}, \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}, \overline{\mathcal{PR}_{R_1+R_2}}, \overline{\mathcal{PR}^*_{R_1^*+R_2^*}} : U \to I$ are called pessimistic two-granulation double fuzzy lower approximation and pessimistic two-granulation double fuzzy upper approximation operators, respectively.

The PTGDFRS approximations are defined by many separate pairs of double fuzzy relations, whereas the normal double fuzzy rough approximations are represented by only one pair of double fuzzy relations. This can be observed from the above definition. In fact, when $(R_1, R_1^*) = (R_2, R_2^*)$, the PTGDFRS degenerates into a double fuzzy rough set. This means that a double fuzzy rough set is a subset of the PTGDFRS.

Proposition 3. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. For each $\lambda \in I^U$, the following apply:*

(1) $\underline{\mathcal{PR}_{R_1+R_2}}\lambda = \underline{\mathcal{R}_{R_1}}\lambda \wedge \underline{\mathcal{R}_{R_2}}\lambda$, *and* $\underline{\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\lambda = \underline{\underline{\mathcal{R}^*_{R_1^*}}}\lambda \vee \underline{\underline{\mathcal{R}^*_{R_2^*}}}\lambda$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda = \overline{\mathcal{R}_{R_1}}\lambda \vee \overline{\mathcal{R}_{R_2}}\lambda$, *and* $\overline{\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\lambda = \overline{\overline{\mathcal{R}^*_{R_1^*}}}\lambda \wedge \overline{\overline{\mathcal{R}^*_{R_2^*}}}\lambda$.

Proof. They can be proved using Definition 2 and Definition 7. □

From Definition 7, we can obtain the following result for the pessimistic multi-granulation double fuzzy rough sets.

Theorem 7. *Let U be an arbitrary universal set, and* (R_1, R_1^*) *and* (R_2, R_2^*) *be double fuzzy relations on U. For each* $\lambda \in I^U$, *the following apply:*

(1) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$, *and* $\underline{\mathcal{PR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$.

(2) $\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1} = \tilde{1}$, *and* $\underline{\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{PR}_{R_1+R_2}}\tilde{0} = \tilde{0}$, *and* $\overline{\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda) = \tilde{1} - \underline{\mathcal{PR}_{R_1+R_2}}\lambda$, *and* $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda) = \tilde{1} - \underline{\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\lambda$.

(5) $\underline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda) = \tilde{1} - \overline{\mathcal{PR}_{R_1+R_2}}\lambda$, *and* $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda) = \tilde{1} - \overline{\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}}\lambda$.

Proof. (1) For each $x \in U,\ \lambda \in I^U$, we have

$$\begin{aligned}
&\big(\tilde{1} - (\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)\big)(x) \\
= \ & 1 - \left\{ \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \wedge \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \ & \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_1^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \vee \left\{ 1 - \left\{ \bigwedge_{y\in U} (R_2^*(x,y) \vee 1 - \lambda(y)) \right\} \right\} \\
= \ & \left\{ \bigvee_{y\in U} 1 - \{R_1^*(x,y) \vee 1 - \lambda(y)\} \right\} \vee \left\{ \bigvee_{y\in U} 1 - \{R_2^*(x,y) \vee 1 - \lambda(y)\} \right\}
\end{aligned}$$

$$\begin{aligned}
= \ & \left\{ \bigvee_{y\in U} \{1 - R_1^*(x,y) \wedge \lambda(y)\} \right\} \vee \left\{ \bigvee_{y\in U} \{1 - R_2^*(x,y) \wedge \lambda(y)\} \right\} \\
\geq \ & \left\{ \bigvee_{y\in U} (R_1(x,y) \wedge \lambda(y)) \right\} \vee \left\{ \bigvee_{y\in U} (R_2(x,y) \wedge \lambda(y)) \right\} \\
= \ & (\overline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) \text{ for all } x \in U.
\end{aligned}$$

Hence, $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \tilde{1} - \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, we have

$$\underline{\mathcal{PR}_{R_1+R_2}}\lambda \geq \tilde{1} - \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda.$$

(2) Since, for each $x \in U$, $\tilde{1}(x) = 1$, we obtain

$$\begin{aligned}
(\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1})(x) &= \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee \tilde{1}(y)) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee \tilde{1}(y)) \right\} \\
&= 1 = \tilde{1}(x),
\end{aligned}$$

and

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\tilde{1})(x) = \left\{\bigvee_{y\in U}((1-R_i(x,y))\wedge 1-\tilde{1}(y))\right\} \vee \left\{\bigvee_{y\in U}((1-R_i(x,y))\wedge 1-\tilde{1}(y))\right\}$$
$$= 0 = \tilde{0}(x).$$

Therefore, we obtain $\underline{\mathcal{PR}_{R_1+R_2}}\tilde{1} = \tilde{1}$ and $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\tilde{1} = \tilde{0}$.

(3) The proof follows steps similar to those of the proof of (2).

(4) For each $x \in U$, we have

$$\begin{aligned}
&\overline{\mathcal{OR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda)(x) \\
= \;& \left\{\bigwedge_{y\in U}(R_1^*(x,y)\vee 1-(1-\lambda(y)))\right\} \wedge \left\{\bigwedge_{y\in U}(R_2^*(x,y)\vee 1-(1-\lambda(y)))\right\} \\
= \;& \left\{1-\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \wedge \left\{1-\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= \;& 1-\left\{\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\} \vee \left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}\right\} \\
= \;& 1-\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda(x).
\end{aligned}$$

Thus, we obtain $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda$. Similarly, we can prove $\overline{\mathcal{PR}_{R_1+R_2}}(\tilde{1}-\lambda) = \tilde{1}-\underline{\mathcal{PR}_{R_1+R_2}}\lambda$.

(5) The proof follows steps similar to those of the proof of (4). □

Theorem 8. *Let U be an arbitrary universal set, and (R_1, R_1^*) and (R_2, R_2^*) be double fuzzy relations on U. For each $\lambda, \mu \in I^U$, the following hold:*

(1) $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\wedge\mu) = \underline{\mathcal{PR}_{R_1+R_2}}\lambda \wedge \underline{\mathcal{PR}_{R_1+R_2}}\mu$, *and*
$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu) = \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \vee \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu) = \overline{\mathcal{PR}_{R_1+R_2}}\lambda \vee \overline{\mathcal{PR}_{R_1+R_2}}\mu$, *and*
$\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu) = \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \wedge \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(3) *If $\lambda \leq \mu$, then* $\underline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \underline{\mathcal{PR}_{R_1+R_2}}\mu$, *and* $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \geq \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(4) *If $\lambda \leq \mu$, then* $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \leq \overline{\mathcal{PR}_{R_1+R_2}}\mu$, *and* $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \geq \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(5) $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu) \geq \underline{\mathcal{PR}_{R_1+R_2}}\lambda \vee \underline{\mathcal{PR}_{R_1+R_2}}\mu$, *and*
$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu) \leq \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \wedge \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

(6) $\overline{\mathcal{PR}_{R_1+R_2}}(\lambda\wedge\mu) \leq \overline{\mathcal{PR}_{R_1+R_2}}\lambda \wedge \overline{\mathcal{PR}_{R_1+R_2}}\mu$, *and*
$\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\wedge\mu) \geq \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \vee \overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

Proof. (1) For each $x \in U,\ \lambda, \mu \in I^U$,

$$
\begin{aligned}
& (\underline{\mathcal{PR}_{R_1+R_2}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_1(x,y)) \vee (\mu)(y) \right\} \right\} \\
\wedge & \left\{ \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\lambda)(y) \right\} \wedge \left\{ \bigwedge_{y\in U} ((1 - R_2(x,y)) \vee (\mu)(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_1}}\mu)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}_{R_2}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
= & \left\{ (\underline{\mathcal{R}_{R_1}}\lambda)(x) \wedge (\underline{\mathcal{R}_{R_2}}\lambda)(x) \right\} \wedge \left\{ (\underline{\mathcal{R}_{R_1}}\mu)(x) \wedge (\underline{\mathcal{R}_{R_2}}\mu)(x) \right\} \\
= & (\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(x) \wedge (\underline{\mathcal{PR}_{R_1+R_2}}\mu)(x).
\end{aligned}
$$

Also, for each $x \in U$,

$$
\begin{aligned}
& (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \wedge \mu))(x) \\
= & \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge 1 - (\lambda \wedge \mu)(y)) \right\} \\
= & \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \\
\vee & \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \vee 1 - \mu(y)) \right\} \\
= & \left\{ \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_1^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
\vee & \left\{ \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \lambda(y) \right\} \vee \left\{ \bigvee_{y\in U} ((1 - R_2^*(x,y)) \wedge (1 - \mu(y) \right\} \right\} \\
= & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \right\} \vee \left\{ (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
= & \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\lambda)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\lambda)(x) \right\} \vee \left\{ (\underline{\mathcal{R}^*_{R_1^*}}\mu)(x) \vee (\underline{\mathcal{R}^*_{R_2^*}}\mu)(x) \right\} \\
= & (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(x) \vee (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(x)
\end{aligned}
$$

(2) The proof follows steps similar to those of the proof of (1).
(3) If $\lambda \leq \mu$, then for all $y \in U,\ \lambda(y) \leq \mu(y)$. Therefore, we have

$$
\bigwedge_{y\in U} (1 - R_1(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U} (1 - R_1(x,y) \vee \mu(y)) \tag{5}
$$

and

$$
\bigwedge_{y\in U} (1 - R_2(x,y) \vee \lambda(y)) \leq \bigwedge_{y\in U} (1 - R_2(x,y) \vee \mu(y)). \tag{6}
$$

From Equations (5) and (6), we have

$$\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\lambda(y))\right\}\wedge\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\lambda(y))\right\}$$
$$\leq\left\{\bigwedge_{y\in U}(1-R_1(x,y)\vee\mu(y))\right\}\wedge\left\{\bigwedge_{y\in U}(1-R_2(x,y)\vee\mu(y))\right\}$$

Thus, $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{PR}_{R_1+R_2}}\mu$, also,

$$\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y)) \tag{7}$$

and

$$\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\geq\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y)). \tag{8}$$

From Equations (7) and (8), we have

$$\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\lambda(y))\right\}\vee\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\lambda(y))\right\}$$
$$\geq\left\{\bigvee_{y\in U}(1-R_1^*(x,y)\wedge 1-\mu(y))\right\}\vee\left\{\bigvee_{y\in U}(1-R_2^*(x,y)\wedge 1-\mu(y))\right\}$$

Thus, $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.
(4) The proof follows steps similar to those of the proof of (3).
(5) Since $\lambda\leq\lambda\vee\mu$ and $\mu\leq\lambda\vee\mu$, by (3), we have

$$\underline{\mathcal{PR}_{R_1+R_2}}\lambda\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{PR}_{R_1+R_2}}\mu\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu).$$

Therefore, $\underline{\mathcal{PR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{PR}_{R_1+R_2}}\mu\leq\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)$. Also, we have

$$\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)\text{ and }\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu).$$

This implies that $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda\wedge\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu\geq\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda\vee\mu)$.
(6) The proof follows steps similar to those of the proof of (5). □

In the following example, we show that the converse of Theorem 8 (5) does not hold true.

Example 2. *Let $U=\{x,y,z\}$. Define $R_1,R_1^*,R_2,R_2^*:U\times U\to I$ as in Example 1 and $\lambda,\mu\in I^U$ as in Example 1. Then,*

$$(\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(x)=0.3,(\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(y)=0.2,(\underline{\mathcal{PR}_{R_1+R_2}}\lambda)(z)=0.5,$$

$$(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(x)=0.5,(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(y)=0.4,(\underline{\mathcal{PR}_{R_1+R_2}}\mu)(z)=0.4,$$

$$(\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu))(x)=0.7,(\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu))(y)=0.5,(\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu))(z)=0.5,$$

Therefore, $\underline{\mathcal{PR}_{R_1+R_2}}(\lambda\vee\mu)\nleq\underline{\mathcal{PR}_{R_1+R_2}}\lambda\vee\underline{\mathcal{PR}_{R_1+R_2}}\mu$.

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(x)=0.8,(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(y)=0.9,(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda)(z)=0.5,$$

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(x) = 0.7, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(y) = 0.6, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu)(z) = 0.8,$$

$$(\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu))(x) = 0.5, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu))(y) = 0.5, (\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu))(z) = 0.5.$$

Therefore, $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}(\lambda \vee \mu) \ngeq \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \wedge \underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\mu$.

We are now extending the pessimistic two-granulation double fuzzy rough set. We present the pessimistic multi-granulation double fuzzy rough set (in short, PMGDFRS) and its properties.

Definition 8. *Let $(U, \mathcal{R}, \mathcal{R}^*)$ be an MGDFAS such that $1 \le i \le m$. Then, for each fuzzy set λ on U, the pairs $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ and $(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ of maps $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda$, $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda : U \to I$ are called pessimistic multi-granulation double fuzzy lower approximation and pessimistic multi-granulation double fuzzy upper approximation of a fuzzy set λ, respectively, and are defined as follows: For all $x \in U$,*

$$(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigwedge_{i=1}^{m} \bigwedge_{y \in U} ((1 - R_i(x,y)) \vee \lambda(y))$$

$$(\underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigvee_{i=1}^{m} \bigvee_{y \in U} ((1 - R_i^*(x,y)) \wedge 1 - \lambda(y))$$

$$(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda)(x) = \bigvee_{i=1}^{m} \bigvee_{y \in U} (R_i(x,y) \wedge \lambda(y))$$

$$(\overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)(x) = \bigwedge_{i=1}^{m} \bigwedge_{y \in U} (R_i^*(x,y) \vee 1 - \lambda(y)).$$

The quaternary $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda, \overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda)$ is called pessimistic multi-granulation double fuzzy rough set of λ (in short, PMGDFRS).

The pairs $(\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}})$ and $(\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}})$ of operators $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}$, $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}, \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}} : I^U \to I^U$ are called pessimistic multi-granulation double fuzzy lower approximation and pessimistic multi-granulation double fuzzy upper approximation operators, respectively.

Proposition 4. *Let $(U, \mathcal{R}, \mathcal{R}^*)$ be an MGDFAS. For each $\lambda \in I^U$, the following hold:*

(1) $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda = \bigwedge_{i=1}^{m} \underline{\mathcal{R}_{R_i}}\lambda$*, and* $\underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda = \bigvee_{i=1}^{m} \underline{\mathcal{R}^*_{R_i^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda = \bigvee_{i=1}^{m} \overline{\mathcal{R}_{R_i}}\lambda$*, and* $\overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda = \bigwedge_{i=1}^{m} \overline{\mathcal{R}^*_{R_i^*}}\lambda$.

Proof. The proof follows steps similar to those of the proof of Proposition 3. □

Theorem 9. *Let $(U, \mathcal{R}, \mathcal{R}^*)$ be an MGDFAS. For each $\lambda \in I^U$, the following apply:*

(1) $\overline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda \le \tilde{1} - \overline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda$*, and* $\underline{\mathcal{PR}_{\sum_{i=1}^m R_i}}\lambda \ge \tilde{1} - \underline{\mathcal{PR}^*_{\sum_{i=1}^m R_i^*}}\lambda$.

(2) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\tilde{1} = \tilde{1}$, *and* $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\tilde{1} = \tilde{0}$.

(3) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\tilde{0} = \tilde{0}$, *and* $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\tilde{0} = \tilde{1}$.

(4) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$, *and* $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\tilde{1} - \lambda) = \tilde{1} - \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

(5) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$, *and* $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\tilde{1} - \lambda) = \tilde{1} - \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

Proof. The proof follows steps similar to those of the proof of Theorem 7. □

Theorem 10. *Let* $(U, \mathcal{R}, \mathcal{R}^*)$ *be an MGDFAS. For each* $\lambda, \mu \in I^U$, *the following apply:*

(1) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \wedge \mu) = \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \wedge \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and*

$\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \wedge \mu) = \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \vee \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \vee \mu) = \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \vee \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and*

$\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \vee \mu) = \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \wedge \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(3) *If* $\lambda \le \mu$, *then* $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \le \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and* $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \ge \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(4) *If* $\lambda \le \mu$, *then* $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \le \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and* $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \ge \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(5) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \vee \mu) \ge \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \vee \underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and*

$\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \vee \mu) \le \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \wedge \underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

(6) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}(\lambda \wedge \mu) \le \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \wedge \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\mu$, *and*

$\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}(\lambda \wedge \mu) \ge \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \vee \overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\mu$.

Proof. The proof follows steps similar to those of the proof of Theorem 8. □

The following propositions show the relationship between optimistic multi-granulation double fuzzy rough sets and pessimistic multi-granulation double fuzzy rough sets.

Proposition 5. *Let* U *be an arbitrary universal set, and* (R_1, R_1^*) *and* (R_2, R_2^*) *be double fuzzy relations on* U. *For each* $i \in \{1, 2\}$ *and* $\lambda \in I^U$, *the following hold:*

(1) $\underline{\mathcal{PR}_{R_1+R_2}}\lambda \le \underline{\mathcal{R}_{R_i}}\lambda \le \underline{\mathcal{OR}_{R_1+R_2}}\lambda$, *and* $\underline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \ge \underline{\mathcal{R}^*_{R_i^*}}\lambda \ge \underline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{R_1+R_2}}\lambda \ge \overline{\mathcal{R}_{R_i}}\lambda \ge \overline{\mathcal{OR}_{R_1+R_2}}\lambda$, *and* $\overline{\mathcal{PR}^*_{R_1^*+R_2^*}}\lambda \le \overline{\mathcal{R}^*_{R_i^*}}\lambda \le \overline{\mathcal{OR}^*_{R_1^*+R_2^*}}\lambda$.

Proof. Based on Proposition 1 and Proposition 3, we can prove this. □

Proposition 6. *Let* $(U, \mathcal{R}, \mathcal{R}^*)$ *be an MGDFAS. For each* $\lambda \in I^U$ *and* $1 \le i \le m$, *the following apply:*

(1) $\underline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \le \underline{\mathcal{R}_{R_i}}\lambda \le \underline{\mathcal{OR}_{\sum_{i=1}^{m} R_i}}\lambda$, *and* $\underline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \ge \underline{\mathcal{R}^*_{R_i^*}}\lambda \ge \underline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

(2) $\overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda \ge \overline{\mathcal{R}_{R_i}}\lambda \ge \overline{\mathcal{PR}_{\sum_{i=1}^{m} R_i}}\lambda$, *and* $\overline{\mathcal{PR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda \le \overline{\mathcal{R}^*_{R_i^*}}\lambda \le \overline{\mathcal{OR}^*_{\sum_{i=1}^{m} R_i^*}}\lambda$.

Proof. Based on Proposition 2 and Proposition 4, we can prove this. □

4. Conclusions

In this article, we discover that rough set theory is a potent theory with numerous applications in the artificial intelligence fields of pattern recognition, machine learning, and automated knowledge acquisition. In this study, the idea of double fuzzy rough sets, which are seen as a generalization of fuzzy rough sets, is introduced. The contribution of this paper is that it has constructed two different types of multi-granulation double fuzzy rough sets associated with granular computing, in which double approximation operators are based on multiple double fuzzy relations.

Additionally, we draw the conclusion that rough sets, multi-granulation fuzzy rough sets models, double fuzzy rough sets models, and multi-granulation rough set models are special cases of the two types of multi-granulation double fuzzy rough sets by analyzing their definitions.

The conclusion of the construction of the new types of multi-granulation double fuzzy rough set models is an extension of granular computing and is meaningful compared with the generalization of rough set theory.

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