sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2023-0027
Tatra Mt. Math. Publ. **85** (2023), 101–120

OPERATION PC-OPEN SETS AND OPERATION PC-SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

Fathi Hesham Khedr — Osama Rashed Sayed*

Department of Mathematics, Faculty of Science, Assiut University, EGYPT

ABSTRACT. In the present paper, we introduce new types of generalized closed sets called ij-pre-generalized closed sets and study some of their properties in bitopological spaces. Also, we use them to construct new types of separation axioms. Further, we introduce and study the concepts of pairwise operation pc-open sets and pairwise operation pc-separation axioms in bitopological spaces. Several interesting characterizations of different spaces are discussed. The relationships between these spaces are given.

1. Introduction and preliminaries

The study of bitopological spaces $(\mathcal{X}, \tau_1, \tau_2)$ was first initiated in [11] by J. C. Kelly. This generalization of topological space was noticed, and many articles were written in this field. The study of pre-open sets and pre-continuity in topological spaces was initiated by Mashhour et al. [17]. Analogous to generalized closed sets which were introduced by Levine [16], Maki et al. [18] introduced the concept of pre-generalized closed sets in topological spaces. In [7, 9, 14], these concepts were extended to bitopological spaces. In [12], a new class of pre-open sets called λ_{pc}-open sets in topological spaces were introduced and studied. Also, by using the notions of λ_{pc}-closed and λ_{pc}-open sets, the concepts of $\lambda_{pc} - T_i (i = 0, \frac{1}{2}, 1, 2)$ and $\lambda_{pc} - R_j (j = 0, 1)$ spaces were investigated.

2020 Mathematics Subject Classification: 54C05, 54A10, 54C55, 54D10, 54D15.
Keywords: bitopological spaces, bispaces, operations, preopen sets, separation axioms.

*The corresponding author.

Further, several properties and characterizations of these spaces were obtained. In 1979, Kasahara [10] introduced the concept of operations on topological spaces and studied the concept of closed graphs of a function. Ahmad and Hussain [2] continued studying the properties of operations on topological spaces introduced by Kasahara. Further investigations of these concept were given in [6,21]. Chattopadhyay [4] defined other new types of separation axioms (pre-$T_i, i = 0,1,2$), and Caldas et al. [3] defined pre-R_1, and pre-R_0 spaces. Also, in [18], the space pre-$T_{\frac{1}{2}}$ was studied. The concept of operations on bitopological spaces was introduced and studied in [1,13]. In [13], the authors used a different technique of that in [1] to study the concepts of operations on bitopological spaces by generalizing the results obtained in [10,21] to such spaces. In [22], the authors introduced and studied the notion of (i, j)-ω-preopen sets as a generalization of (i, j)--preopen sets in bitopological space. The rest of this paper is organized as follows. This section contains some necessary concepts and properties. In Section 2, we introduce a new type of closed sets in bitopological space called ij-pre-generalized closed set and study some of its properties. In Section 3, we define and characterize a pairwise pre-T_k, $k = 0, \frac{1}{2}, 1, 2$ and a pairwise pre-R_k, $k = 0, 1$ spaces. Also, for a subset $\mathcal{A}$ of bitopological space $\mathcal{X}$, we introduce a new class of sets denoted by $\mathcal{A}^{\mathcal{P}\bigwedge_{ij}}$ and investigate some of its properties. The purpose of Section 4 is to introduce and study the concepts of pairwise operation pc-open sets. In Section 5, we introduce the concept of $ij - g\mu_{pc}$-closed set and study some of its properties. In Section 6, we generalize the results in [12] to the setting of bitopological space and introduce a new type of separation axioms called pairwise $\mu_{pc} - T_k, k \in \{0, \frac{1}{2}, 1, 2\}$ and pairwise $\mu_{pc} - R_k, k \in \{0, 1\}$ spaces. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research.

Throughout this paper, $(\mathcal{X}, \tau_1, \tau_2)$ (or $\mathcal{X}$, for short) denotes a bitopological space (or bispace, for short) on which no separation axioms are assumed unless otherwise mentioned. For a subset $\mathcal{A}$ of $\mathcal{X}$, we shall denote the closure of $\mathcal{A}$ and the interior of $\mathcal{A}$ with respect to τ_i by $i - \mathfrak{Cl}(\mathcal{A})$ and $i - \mathfrak{Int}(\mathcal{A})$, respectively, for $i = 1, 2$. Also, $i, j = 1, 2$ and $i \neq j$.

A subset $\mathcal{A}$ of a space $\mathcal{X}$ is said to be ij-preopen [7,14] if $\mathcal{A} \subseteq i - \mathfrak{Int}\big(j - \mathfrak{Cl}(\mathcal{A})\big)$. The complement of an ij-preopen set is called ij-preclosed. Equivalently, a set $\mathcal{F}$ is ij-preclosed if $i - \mathfrak{Cl}\big(j - \mathfrak{Int}(\mathcal{F})\big) \subseteq \mathcal{F}$. The family of all ij-preopen (resp. ij-preclosed) sets of $\mathcal{X}$ is denoted by $ij - \mathfrak{PO}(\mathcal{X})$ (resp. $ij - \mathfrak{PC}(\mathcal{X})$). The ij-pre interior set of $\mathcal{A}$ [20] denoted by $ij - \mathfrak{pInt}(\mathcal{A})$ is defined as the union of all ij-preopen sets contained in $\mathcal{A}$. The intersection of all ij-preclosed sets containing $\mathcal{A}$ is called the ij-pre closure of $\mathcal{A}$ [14] and is denoted by $ij - \mathfrak{pCl}(\mathcal{A})$. A subset $\mathcal{A}$ of a space $\mathcal{X}$ is called an ij-generalized closed (ij-g-closed, for short) [5] if $j - \mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$ whenever $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is τ_i -open in $\mathcal{X}$.

2. ij-pre-generalized closed sets

The purpose of this section is to introduce a new type of closed sets in bispace called ij-pre-generalized closed set and to investigate some of its properties.

Definition 2.1. A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is called ij-pre-generalized closed (ij-pg-closed, for short) if $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{U}$ whenever $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is an ij-preopen set. If $\mathcal{A} \subseteq \mathcal{X}$ is ij-pg-closed, then it is said to be pairwise pre-generalized closed (for short, pairwise pg-closed).

The notions of ij-g-closed sets and ij-pg-closed sets are independent. See the following example.

Example 2.1. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \mathcal{X}\}$ and $\tau_2 = \{\phi, \{\beta\}, \{\gamma, \delta\}, \{\beta, \gamma, \delta\}, \mathcal{X}\}$. If $\mathcal{A} = \{\beta, \gamma, \delta\}$, then $\mathcal{A}$ is an 12-g-closed set but it is not 12-pg-closed. Also, if $\mathcal{A} = \{\alpha, \gamma\}$, then $\mathcal{A}$ is 12-pg-closed set but it is not 12-g-closed.

Proposition 2.1. Every ji-preclosed set is ij-pg-closed.

Proof. A set $\mathcal{A} \subseteq \mathcal{X}$ is ji-preclosed if and only if $ji - \mathfrak{pCl}(\mathcal{A}) = \mathcal{A}$. Thus, $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{U}$ for every $\mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X})$ and $\mathcal{A} \subseteq \mathcal{U}$. □

The following example shows that the union of two ij-pg-closed sets need not be ij-pg-closed.

Example 2.2. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \mathcal{X}\}$ and $\tau_2 = \{\phi, \{\alpha, \gamma\}, \{\beta, \delta\}, \mathcal{X}\}$. If $\mathcal{A} = \{\alpha\}$ and $\mathcal{B} = \{\gamma\}$, then $\mathcal{A}$ and $\mathcal{B}$ are 21-pg-closed sets but $\mathcal{A} \cup \mathcal{B}$ is not 21-pg-closed set.

Proposition 2.2. If $\mathcal{A}$ is an ij-pg-closed set and $\mathcal{A} \subseteq \mathcal{B} \subseteq ji - \mathfrak{pCl}(\mathcal{A})$, then so is $\mathcal{B}$.

Proof. Suppose that $\mathcal{B} \subseteq \mathcal{U}$, where $\mathcal{U}$ is an ij-preopen set. Since $\mathcal{A}$ is ij-pg--closed and $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{U}$. By hypothesis, $\mathcal{B} \subseteq ji - \mathfrak{pCl}(\mathcal{A})$, and so, $ji - \mathfrak{pCl}(\mathcal{B}) \subseteq ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{U}$. Therefore, $\mathcal{B}$ is an ij-pg-closed set. □

Proposition 2.3. Let $\mathcal{X}$ be a bispace. Then for each $\mathrm{x} \in \mathcal{X}$, either $\{\mathrm{x}\}$ is ij-preclosed or $\mathcal{X} \backslash \{\mathrm{x}\}$ is ij-pg-closed.

Proof. Without loss of all generality, we can assume that $\{\mathrm{x}\}$ is not ij-preclosed set. Since $\mathcal{X} \backslash \{\mathrm{x}\}$ is not ij-preopen set, the only ij-preopen set containing $\mathcal{X} \backslash \{\mathrm{x}\}$ is $\mathcal{X}$. Hence, $ji - \mathfrak{pCl}(\mathcal{X} \backslash \{\mathrm{x}\}) \subseteq \mathcal{X}$ and $\mathcal{X} \backslash \{\mathrm{x}\}$ is ij-pg-closed. □

Proposition 2.4. If $\mathcal{A}$ is ij-pg-closed, then $ji - \mathfrak{pCl}(\mathcal{A}) \backslash \mathcal{A}$ contains no nonempty ij-preclosed set.

P r o o f. Suppose that $\mathcal{F}$ is an ij-preclosed set such that $\mathcal{F} \subseteq ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}$. Then, $\mathcal{F} \subseteq \mathcal{X}\backslash\mathcal{A}$, and so $\mathcal{A} \subseteq \mathcal{X}\backslash\mathcal{F}$. Since $\mathcal{A}$ is ij-pg-closed, $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{X}\backslash\mathcal{F}$, and so $\mathcal{F} \subseteq \mathcal{X}\backslash ji - \mathfrak{pCl}(\mathcal{A})$. Thus, $\mathcal{F} \subseteq \big(\mathcal{X}\backslash ji - \mathfrak{pCl}(\mathcal{A})\big) \cap (ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}) = \phi$. As a result, $\mathcal{F}$ is empty. □

Corollary 2.1. *Let $\mathcal{A}$ be an ij-pg-closed set. Then, $\mathcal{A}$ is ji-preclosed if and only if $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}$ is ij-preclosed.*

P r o o f. If $\mathcal{A}$ is ji-preclosed, then $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A} = \phi$ which is ij-preclosed. Conversely, suppose that $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}$ is an ij-preclosed set. Then, $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}$ does not contain a non-empty ij-preclosed subset. Since $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A}$ is ij-preclosed, $ji - \mathfrak{pCl}(\mathcal{A})\backslash\mathcal{A} = \phi$. Therefore, $\mathcal{A}$ is a ji-preclosed set. □

3. Pairwise pre-T_k and pre-R_k spaces

In this section, we define a pairwise pre-T_k, $k = 0, \frac{1}{2}, 1, 2$ and pairwise pre-R_k, $k = 0, 1$-spaces and give some of their properties. Also, for a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we introduce a new class of sets denoted by $\mathcal{A}^{\mathcal{P}\wedge_{ij}}$ and investigate some of its properties. Also, we use it to give a characterization of a pairwise pre-R_0-space.

Definition 3.1. A bispace $\mathcal{X}$ is said to be:

(1) Pairwise pre-T_0-space (pairwise pT_0-space, for short) if for each two distinct points of $\mathcal{X}$ there exists an ij-preopen set or a ji-preopen set containing one of them but not the other;

(2) Pairwise pre-$T_{\frac{1}{2}}$-space (pairwise $pT_{\frac{1}{2}}$-space, for short) if every an ij-gp--closed set is a ji-preclosed;

(3) Pairwise pre-T_1-space (pairwise pT_1-space, for short) if for each two distinct points $\mathrm{x}, \mathrm{y} \in \mathcal{X}$, there exist an ij-preopen set $\mathcal{U}$ containing x but not y and a ji-preopen set $\mathcal{V}$ containing y but not x;

(4) Pairwise pre-T_2-space (pairwise pT_2-space, for short) if for each two distinct points $\mathrm{x}, \mathrm{y} \in \mathcal{X}$, there exist an ij-preopen set $\mathcal{U}$ and a ji-preopen set $\mathcal{V}$ such $\mathrm{x} \in \mathcal{U}$, $\mathrm{y} \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \phi$;

(5) Pairwise pre-R_0-space (pairwise pR_0-space, for short) if for each an ij--preopen set $\mathcal{U}$, $\mathrm{x} \in \mathcal{U}$ implies that $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \subseteq \mathcal{U}$;

(6) Pairwise pre-R_1-space (pairwise pR_1-space, for short) if for each two distinct points $\mathrm{x}, \mathrm{y} \in \mathcal{X}$ such that $ij - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ji - \mathfrak{pCl}(\{\mathrm{y}\})$, there exist an ij-preopen set $\mathcal{U}$ and a ji-preopen set $\mathcal{V}$ such that $\mathrm{y} \in \mathcal{U}$, $\mathrm{x} \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \phi$.

Proposition 3.1. A bispace $\mathcal{X}$ is a pairwise pT_0-space if and only if for any $\mathrm{x}, \mathrm{y} \in \mathcal{X}$, such that $\mathrm{x} \neq \mathrm{y}$, $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ij - \mathfrak{pCl}(\{\mathrm{y}\})$.

P r o o f. Suppose that $\mathcal{X}$ is a pairwise pT_0-space and $\mathrm{x}, \mathrm{y} \in \mathcal{X}$, such that $\mathrm{x} \neq \mathrm{y}$. Then, without loss generality, we can assume that there exists an ij-preopen set $\mathcal{U}$ such that $\mathrm{x} \in \mathcal{U}$ and $\mathrm{y} \notin \mathcal{U}$. Thus, $\{\mathrm{y}\} \cap \mathcal{U} = \phi$. This means that $\mathrm{x} \notin ij - \mathfrak{pCl}(\{\mathrm{y}\})$. Since, $\mathrm{x} \in ji - \mathfrak{pCl}(\{\mathrm{x}\})$, $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ij - \mathfrak{pCl}(\{\mathrm{y}\})$. Conversely, suppose that for any $\mathrm{x}, \mathrm{y} \in \mathcal{X}$, such that $\mathrm{x} \neq \mathrm{y}$, $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ij - \mathfrak{pCl}(\{\mathrm{y}\})$. Thus, either $\mathrm{y} \notin ji - \mathfrak{pCl}(\{\mathrm{x}\})$ or $\mathrm{x} \notin ij - \mathfrak{pCl}(\{\mathrm{y}\})$. Without loss generality, we can assume that $\mathrm{y} \notin ji - \mathfrak{pCl}(\{\mathrm{x}\})$. Hence, there exists a ji-preopen $\mathcal{U}$ such that $\mathrm{y} \in \mathcal{U}$ and $\{\mathrm{x}\} \cap \mathcal{U} = \phi$, i.e., $\mathrm{x} \notin \mathcal{U}$. Therefore, $\mathcal{X}$ is a pairwise pT_0-space. □

Proposition 3.2. A bispace $\mathcal{X}$ is a pairwise $pT_{\frac{1}{2}}$-space if and only if every singleton set is either ji-preopen or ij-preclosed.

P r o o f. Suppose that $\{\mathrm{x}\}$ is not an ij-preclosed set. Then, by Proposition 2.3, $\mathcal{X} \backslash \{\mathrm{x}\}$ is an ij-pg-closed set. Since $\mathcal{X}$ is a pairwise $pT_{\frac{1}{2}}$-space, $\mathcal{X} \backslash \{\mathrm{x}\}$ is ji-preclosed and $\{\mathrm{x}\}$ is ji-preopen. Conversely, suppose that $\mathcal{F}$ is an ij-pg-closed set. For any $x \in ji - \mathfrak{pCl}(\mathcal{F})$, $\{\mathrm{x}\}$ is either ji-preopen or ij-preclosed by the assumption.

Case 1. Suppose $\{\mathrm{x}\}$ is a ji-preopen set. Since, $\{\mathrm{x}\} \cap \mathcal{F} \neq \phi$, $\mathrm{x} \in \mathcal{F}$.

Case 2. Suppose that $\{\mathrm{x}\}$ is an ij-preclosed set. If $\mathrm{x} \notin \mathcal{F}$, then $\{\mathrm{x}\} \subseteq ji - \mathfrak{pCl}(\mathcal{F}) \backslash \mathcal{F}$ which contradicts Proposition 2.4. Thus, $\mathrm{x} \in \mathcal{F}$. From the above two cases we conclude that $\mathcal{F}$ is a ji-preclosed set. Hence, $\mathcal{X}$ is a pairwise $pT_{\frac{1}{2}}$-space. □

Proposition 3.3. A bispace $\mathcal{X}$ is a pairwise pT_1-space if and only if every singleton is pairwise preclosed.

P r o o f. Suppose that $\mathcal{X}$ is a pairwise pT_1-space. For every singleton $\{\mathrm{x}\}$ we have $\{\mathrm{x}\} \subseteq ij - \mathfrak{pCl}(\{\mathrm{x}\})$. For every point $\mathrm{y} \in \mathcal{X}$ different from x, there exists an ij-preopen set $\mathcal{U}$ such that $\mathrm{y} \in \mathcal{U}$ and $\mathrm{x} \notin \mathcal{U}$. Thus, $\{\mathrm{x}\} \cap \mathcal{U} = \phi$ and $\mathrm{y} \notin ij - \mathfrak{pCl}(\{\mathrm{x}\})$. Then, $\{\mathrm{x}\} = ij - \mathfrak{pCl}(\{\mathrm{x}\})$, and hence $\{\mathrm{x}\}$ is ij-preclosed. Now, for every $\mathrm{x} \neq \mathrm{y}$, we have $\mathrm{y} \in \mathcal{X} \backslash \{\mathrm{x}\}$. So, there exists a ji-preopen set $\mathcal{V}_{\mathrm{y}}$ such that $\mathrm{y} \in \mathcal{V}_{\mathrm{y}}$ but $\mathrm{x} \notin \mathcal{V}_{\mathrm{y}}$. Therefore, $\mathrm{y} \in \mathcal{V}_{\mathrm{y}} \subseteq \mathcal{X} \backslash \{\mathrm{x}\}$. Hence, $X \backslash \{\mathrm{x}\}$ is ji-preopen, and thus $\{\mathrm{x}\}$ is ji-preclosed. Conversely, suppose that every singleton is pairwise preclosed. Then, for any $\mathrm{x} \in \mathcal{X}$ we have $\{\mathrm{x}\} = ji - \mathfrak{pCl}(\{\mathrm{x}\})$ and $\{\mathrm{x}\} = ij - \mathfrak{pCl}(\{\mathrm{x}\})$. Hence for every $\mathrm{x}, \mathrm{y} \in \mathcal{X}$ such that $\mathrm{x} \neq \mathrm{y}$, we have $X \backslash ji - \mathfrak{pCl}(\{\mathrm{x}\})$, a ji-preopen set containing y but not x. Similarly, $X \backslash ji - \mathfrak{pCl}(\{\mathrm{y}\})$ is an ij-preopen set containing x but not y. Thus, $\mathcal{X}$ is a pairwise pT_1-space. □

Corollary 3.1. *Every pairwise pT_1-space is pairwise $pT_{\frac{1}{2}}$.*

P r o o f. The proof is an immediate consequence of Propositions 3.2 and 3.3. □

Definition 3.2. A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is said to be an ij-pre-neighborhood (ij-pre-nbd, for short) of a point x in $\mathcal{X}$ if there exists an ij-preopen set $\mathcal{U}$ containing x and contained in $\mathcal{A}$.

Theorem 3.1. *A bispace $\mathcal{X}$ is pairwise pT_2 if and only if the intersection of all ij-preclosed ji-pre-npds of a point* $\mathrm{x} \in \mathcal{X}$ *is reduced to* $\{\mathrm{x}\}$.

Proof. Let $\mathcal{X}$ be a pairwise pT_2-space and $\mathrm{x} \in \mathcal{X}$. For each $\mathrm{y} \in \mathcal{X}$ such that $\mathrm{y} \neq \mathrm{x}$, there exist an ij-preopen set $\mathcal{G}$ and a ji-preopen set $\mathcal{H}$ such that $\mathrm{x} \in \mathcal{H}$, $\mathrm{y} \in \mathcal{G}$ and $\mathcal{G} \cap \mathcal{H} = \phi$. Hence, $\mathrm{x} \in \mathcal{H} \subseteq \mathcal{X} \backslash \mathcal{G}$ and $\mathrm{y} \notin \mathcal{X} \backslash \mathcal{G}$. Therefore, $\mathcal{X} \backslash \mathcal{G}$ is ij--preclosed ji-pre-nbd of x which y does not belong. Consequently, the intersection of all ij-preclosed ji-pre-nbds of x is reduced to $\{\mathrm{x}\}$.

Conversely, suppose that $\mathrm{x}, \mathrm{y} \in X$ such that $\mathrm{x} \neq \mathrm{y}$. Then, there exists an ij-preclosed ji-pre-nbd $\mathcal{U}$ of x which x does not belong. Now, there exists a ji--pre open set $\mathcal{G}$ such that $\mathrm{x} \in \mathcal{G} \subseteq \mathcal{U}$. Thus, $\mathcal{G}$ is a ji-preopen set and $\mathcal{X} \backslash \mathcal{U}$ is an ij-preopen set, $\mathrm{x} \in \mathcal{G}$, $\mathrm{y} \in \mathcal{X} \backslash \mathcal{U}$ and $\mathcal{G} \cap \mathcal{X} \backslash \mathcal{U} = \phi$. Hence, $\mathcal{X}$ is a pairwise pT_2-space. □

Proposition 3.4. A bispace $\mathcal{X}$ is a pairwise pre-R_1-space if and only if for each two point of $\mathrm{x}, \mathrm{y} \in \mathcal{X}$ such that $ij - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ji - \mathfrak{pCl}(\{\mathrm{y}\})$, there exist an ij--preopen set $\mathcal{U}$ and a ji-preopen set $\mathcal{V}$ such that $ij - \mathfrak{pCl}(\{\mathrm{x}\}) \subseteq \mathcal{V}$, $ji - \mathfrak{pCl}(\{\mathrm{y}\}) \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} = \phi$.

Proof. Suppose that $\mathcal{X}$ is a pairwise pre-R_1-space and $\mathrm{x}, \mathrm{y} \in \mathcal{X}$ such that $ij - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ji - \mathfrak{pCl}(\{\mathrm{y}\})$. Then, there exist an ij-preopen set $\mathcal{U}$ and a ji--preopen set $\mathcal{V}$ such that $\mathrm{x} \in \mathcal{V}, \mathrm{y} \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} = \phi$. Since every pairwise pre-R_1-space is pairwise pre-R_0-space, $\mathrm{x} \in \mathcal{V}$ which implies $ij - \mathfrak{pCl}(\{\mathrm{x}\}) \subseteq \mathcal{V}$ and $ji - \mathfrak{pCl}(\{\mathrm{y}\}) \subseteq \mathcal{U}$. Hence, the result is obtained. The converse is obvious. □

Corollary 3.2. *Every pairwise pre-R_1-space is a pairwise pre-R_0.*

Proof. Suppose $\mathcal{X}$ is a pairwise pre-R_1-space, $\mathcal{G}$ any ij-preopen set and $\mathrm{x} \in \mathcal{G}$. For each point $\mathrm{y} \in \mathcal{X} \backslash \mathcal{G}$, we have $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \neq ij - \mathfrak{pCl}(\{\mathrm{y}\})$. So, by Proposition 3.4, there exist an ij-preopen set $\mathcal{U}_{\mathrm{y}}$ and a ji-preopen set $\mathcal{V}_{\mathrm{y}}$ such that $\mathrm{x} \in \mathcal{U}_{\mathrm{y}}, \mathrm{y} \in \mathcal{V}_{\mathrm{y}}$ and $\mathcal{U}_{\mathrm{y}} \cap \mathcal{V}_{\mathrm{y}} = \phi$. If $\mathcal{A} = \bigcup\{\mathcal{V}_{\mathrm{y}} : \mathrm{y} \in \mathcal{X} \backslash \mathcal{G}\}$, then $\mathcal{X} \backslash \mathcal{G} \subseteq \mathcal{A}$ and $\mathrm{x} \notin \mathcal{A}$. Since $\mathcal{A}$ is a ji-preopen set, $ji - \mathfrak{pCl}(\{\mathrm{x}\}) \subseteq \mathcal{X} \backslash \mathcal{A} \subseteq \mathcal{G}$. Hence, $\mathcal{X}$ is a pairwise pre-R_0-space. □

Lemma 3.1. *In every bispace $\mathcal{X}$, each singleton is an ij-preopen or a ji-preclosed.*

Proof. Suppose that $\mathcal{X}$ is a bispace, $\mathrm{x} \in \mathcal{X}$ and $\{\mathrm{x}\}$ is not a ji-preclosed set. Then, $j - \mathfrak{Cl}(i - \mathfrak{Int}(\{\mathrm{x}\}) \nsubseteq \{\mathrm{x}\}$. Thus, $i - \mathfrak{Int}(\{\mathrm{x}\}) = \{\mathrm{x}\}$. Therefore, $\{\mathrm{x}\}$ is an ij-preopen set. □

Theorem 3.2. *A bispace $\mathcal{X}$ is a pairwise pre-R_0-space if and only if it is a pairwise pre-T_1-space.*

P r o o f. Suppose that $\mathcal{X}$ is a pairwise pre-R_0-space. For each point $x \in \mathcal{X}$, by Lemma 3.1, $\{x\}$ is ij-preopen or ji-preclosed in $\mathcal{X}$. If $\{x\}$ is ij-preopen, then we have $ji - \mathfrak{pCl}(\{x\}) \subseteq \{x\}$ and hence, $\{x\}$ is ji-preclosed. Thus, pairwise preclosed. Hence, $\mathcal{X}$ is a pairwise pre-T_1-space.

Conversely, assume that $\mathcal{U}$ is any ij-preopen subset of $\mathcal{X}$ and $x \in \mathcal{U}$. Since $\{x\}$ is a pairwise preclosed set, $ji - \mathfrak{pCl}(\{x\}) = \{x\} \subseteq \mathcal{U}$. Therefore, $\mathcal{X}$ is a pairwise pre-R_0-space. □

Theorem 3.3. *A bispace $\mathcal{X}$ is pairwise pre-R_1 if and only if it is pairwise pre-T_2.*

P r o o f. Let $\mathcal{X}$ be a pairwise pre-T_2-space. Then, $\mathcal{X}$ is pairwise pre-T_1. If $x, y \in \mathcal{X}$ such that $ij - \mathfrak{pCl}(\{x\}) \neq ji - \mathfrak{pCl}(\{y\})$, then $x \neq y$. So, there exist an ij-preopen set $\mathcal{U}$ containing x but not y and a ji-preopen set $\mathcal{V}$ containing y but not x. Hence, $ji - \mathfrak{pCl}(\{x\}) = \{x\} \subseteq \mathcal{U}$ and $ij - \mathfrak{pCl}(\{y\}) = \{y\} \subseteq V$. Therefore, $\mathcal{X}$ is a pairwise pre-R_1-space. Conversely, suppose that $\mathcal{X}$ is pairwise pre-R_1. By Corollary 3.2, $\mathcal{X}$ is pairwise pre-R_0 and hence, by Theorem 3.2, it is pairwise pre-T_1. Assume that $x, y \in \mathcal{X}$ such that $x \neq y$. Since $ij - \mathfrak{pCl}(\{x\}) = \{x\} \neq \{y\} = ji - \mathfrak{pCl}(\{y\})$, there exist disjoint ij-preopen set $\mathcal{U}$ and ji-preopen set $\mathcal{V}$ such that $x \in \mathcal{V}$ and $y \in \mathcal{U}$. Hence, $\mathcal{X}$ is a pairwise pre-T_2-space. □

Remark 3.1. From the definitions and previous results, we can get the following diagram of implications: Pairwise pT_2 ⇒ Pairwise pT_1 ⇒ Pairwise $pT_{\frac{1}{2}}$ ⇒ Pairwise pT_0. The converses of this implications are not true in general. See the following examples.

Example 3.1. Let $(\mathcal{X}, \tau_1, \tau_2)$ be a bispace, where $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}, \mathcal{X}\}$, and $\tau_2 = \{\phi, \{\beta\}, \{\gamma, \delta\}, \{\beta, \gamma, \delta\}, \mathcal{X}\}$. Then, $(\mathcal{X}, \tau_1, \tau_2)$ is a pairwise $pT_{\frac{1}{2}}$ but it is not a pairwise pR_0-space.

Example 3.2. Let $(\mathcal{X}, \tau_1, \tau_2)$ be a bispace, where $\mathcal{X} = \{\alpha, \beta, \gamma, \delta, \epsilon\}, \tau_1 = \{\phi, \{\epsilon\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \beta, \epsilon\}, \{\gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma, \delta\}, \mathcal{X}\}$, and $\tau_2 = \{\phi, \{\alpha\}, \{\gamma\}, \{\epsilon\}, \{\alpha, \gamma\}, \{\alpha, \epsilon\}, \{\alpha, \epsilon\}, \{\alpha, \gamma, \epsilon\}, \{\alpha, \gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma, \epsilon\}, \mathcal{X}\}$. Then, $\mathcal{X}$ is a pairwise pT_0-space but it is not a pairwise pT_1-space.

Definition 3.3. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we define $\mathcal{A}^{\mathcal{P}\wedge_{ij}}$ as follows

$$\mathcal{A}^{\mathcal{P}\wedge_{ij}} = \bigcap\{\mathcal{U} : \mathcal{A} \subseteq \mathcal{U},\ \mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X})\}.$$

Theorem 3.4. *Let $\mathcal{A}$ be a subset of a bispace $(\mathcal{X}, \tau_1, \tau_2)$. Then,*

$$\mathcal{A}^{\mathcal{P}\wedge_{ij}} = \left(\mathcal{A}^{\mathcal{P}\wedge_{ij}}\right)^{\mathcal{P}\wedge_{ij}}.$$

P r o o f.

$$\begin{aligned}\left(\mathcal{A}^{\mathcal{P}\wedge_{ij}}\right)^{\mathcal{P}\wedge_{ij}} &= \bigcap\{\mathcal{U} : \mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X}), \mathcal{A}^{\mathcal{P}\wedge_{ij}} \subseteq \mathcal{U}\} \\ &= \bigcap\{\mathcal{U} : \mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X}), \bigcap\{\mathcal{V} : \mathcal{V} \in ij - \mathfrak{PO}(\mathcal{X}), \mathcal{A} \subseteq \mathcal{V}\} \subseteq \mathcal{U}\} \\ &\subseteq \bigcap\{\mathcal{U} : \mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X}), \mathcal{A} \subseteq \mathcal{U}\} = \mathcal{A}^{\mathcal{P}\wedge_{ij}}.\end{aligned}$$

This shows that $\left(\mathcal{A}^{\mathcal{P}\wedge_{ij}}\right)^{\mathcal{P}\wedge_{ij}} \subseteq \mathcal{A}^{\mathcal{P}\wedge_{ij}}$.

On the other hand, we have $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{P}\wedge_{ij}}$ for each $\mathcal{A} \subseteq \mathcal{X}$. Thus,

$$\mathcal{A}^{\mathcal{P}\wedge_{ij}} \subseteq \left(\mathcal{A}^{\mathcal{P}\wedge_{ij}}\right)^{\mathcal{P}\wedge_{ij}}. \qquad \square$$

Lemma 3.2. *A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is ij-pg-closed if and only if*

$$ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{A}^{\mathcal{P}\wedge_{ij}}.$$

P r o o f. Let $\mathcal{A}$ be an ij-pg-closed set. Then, for each ij-preopen set $\mathcal{U}$ containing $\mathcal{A}$ we have $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{U}$. Hence,

$$ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \bigcap\{\mathcal{U} : \mathcal{U} \in ij - \mathfrak{PO}(\mathcal{X}), \quad \mathcal{A} \subseteq \mathcal{U}\} = \mathcal{A}^{\mathcal{P}\wedge_{ij}}.$$

Conversely, assume that $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{A}^{\mathcal{P}\wedge_{ij}}$. Then, $ji - \mathfrak{pCl}(\mathcal{A}) \subseteq \mathcal{U}$ for each ij-preopen set containing $\mathcal{A}$. This shows that $\mathcal{A}$ is an ij-pg-closed set. $\square$

Theorem 3.5. *For a bispace $\mathcal{X}$, the following statements are equivalent:*

(i) *$\mathcal{X}$ is a pairwise pR_0-space;*

(ii) *For any $x \in \mathcal{X}, ij - \mathfrak{pCl}(\{x\}) \subseteq \{x\}^{\mathcal{P}\wedge_{ji}}$;*

(iii) *For any $x, y \in \mathcal{X}, y \in \{x\}^{\mathcal{P}\wedge_{ij}}$ if and only if $x \in \{y\}^{\mathcal{P}\wedge_{ji}}$;*

(iv) *For any $x, y \in \mathcal{X}, y \in ij - \mathfrak{pCl}(\{x\})$ if and only if $x \in ji - \mathfrak{pCl}(\{y\})$;*

(v) *For any ij-preclosed set $\mathcal{F}$ and a point $x \notin \mathcal{F}$, there exists a ji-preopen set $\mathcal{U}$ such that $x \notin \mathcal{U}, \mathcal{F} \subseteq \mathcal{U}$;*

(vi) *Each ij-preclosed $\mathcal{F}$ can be expressed as the intersection of all ji-preopen sets containing $\mathcal{F}$;*

(vii) *Each ij-preopen set $\mathcal{U}$ can be expressed as the union of a ji-preclosed sets contained in $\mathcal{U}$;*

(viii) *For each ij-preclosed set $\mathcal{F}$, $x \notin \mathcal{F}$ implies $ji - \mathfrak{pCl}(\{x\}) \cap \mathcal{F} = \phi$.*

P r o o f.

(i)⇒(ii): By Definition 3.3, for any $x \in \mathcal{X}$ we have $\{x\}^{\mathcal{P}\wedge_{ji}} = \bigcap \mathcal{U} : x \in \mathcal{U}$, $\mathcal{U} \in ji - \mathfrak{PO}(\mathcal{X})$. Since $\mathcal{X}$ is a pairwise pR_0-space, each ji-preopen set $\mathcal{U}$ containing x contains $ij - \mathfrak{pCl}(\{x\})$. Hence, $ij - \mathfrak{pCl}(\{x\}) \subseteq \{x\}^{\mathcal{P}\wedge_{ji}}$.

(ii)⇒(iii): For any $x, y \in \mathcal{X}$, if $y \in \{x\}^{\mathcal{P}\wedge_{ji}}$, then $x \in ij - \mathfrak{pCl}(\{y\})$. Since $ij - \mathfrak{pCl}(\{y\}) \subseteq \{y\}^{\mathcal{P}\wedge_{ji}}$, by (ii), $x \in \{y\}^{\mathcal{P}\wedge_{ji}}$.

(iii)⇒(iv): For any $x, y \in \mathcal{X}$ if $y \in ij - \mathfrak{pCl}(\{x\})$, then $x \in \{y\}^{\mathcal{P}\wedge_{ji}}$. Thus, by (iii), $y \in \{x\}^{\mathcal{P}\wedge_{ji}}$, and so $x \in ji - \mathfrak{pCl}(\{y\})$.

(iv)⇒(v): Let $\mathcal{F}$ be an ij-preclosed set and $x \notin \mathcal{F}$. Then for any $y \in \mathcal{F}$, $ij - \mathfrak{pCl}(\{y\}) \subseteq \mathcal{F}$ and $x \notin ij - \mathfrak{pCl}(\{y\})$. By (iv), $x \notin ij - \mathfrak{pCl}(\{y\})$ and $y \notin ji - \mathfrak{pCl}(\{x\})$. Hence, there exists a ji-preopen set $\mathcal{U}_y$ such that $y \in \mathcal{U}_y$ and $x \notin \mathcal{U}_y$. Suppose that $\mathcal{U} = \bigcup_{y \in \mathcal{F}} \{\mathcal{U}_y : y \in \mathcal{U}_y$ and $x \notin \mathcal{U}_y, \mathcal{U}_y$ is ij-preopen$\}$. Then, $\mathcal{U}$ is a ji-preopen set such that $x \notin \mathcal{U}$ and $\mathcal{F} \subseteq \mathcal{U}$.

(v)⇒(vi): Suppose that $\mathcal{F}$ is an ij-preclosed set and $\mathcal{H} = \bigcap \{\mathcal{U} : \mathcal{F} \subseteq \mathcal{U}$, $\mathcal{U}$ is ji-preopen$\}$. Then $\mathcal{F} \subseteq \mathcal{H}$, and we show that $\mathcal{H} \subseteq \mathcal{F}$. Let $x \notin \mathcal{F}$. Then, by (v), there exists a ji-preopen set $\mathcal{U}$ such that $x \notin \mathcal{U}$ and $\mathcal{F} \subseteq \mathcal{U}$. Hence, $x \notin \mathcal{H}$, and so, $\mathcal{H} \subseteq \mathcal{F}$. Thus, $\mathcal{F} = \mathcal{H}$.

(vi)⇒(vii): Obvious.

(vii)⇒(viii): Suppose that $\mathcal{F}$ is an ij-preclosed set and $x \notin \mathcal{F}$. Then, $\mathcal{X} \backslash \mathcal{F} = \mathcal{U}$ is an ij-preopen set containing x. Then, by (vii), there exists a ji-preclosed set $\mathcal{H}$ such that $x \in \mathcal{H} \subseteq \mathcal{U}$ and so, $ji - \mathfrak{pCl}(\{x\}) \subseteq \mathcal{U}$. Thus, $ji - \mathfrak{pCl}(\{x\}) \cap \mathcal{F} = \phi$.

(viii)⇒(i): Suppose that $\mathcal{U}$ is an ij-preopen set and $x \in \mathcal{U}$. Then, $x \notin \mathcal{X} \backslash \mathcal{U}$ which is ij-preclosed set and, by (viii), $ji - \mathfrak{pCl}(\{x\}) \cap \mathcal{X} \backslash \mathcal{U} = \phi$. Thus, $ji - \mathfrak{pCl}(\{x\}) \subseteq \mathcal{U}$. Therefore, $\mathcal{X}$ is pairwise pR_0-space. □

4. A pairwise operation pc-open sets

The purpose of this section is to introduce and study the concepts of the pairwise operation pc-open sets.

Definition 4.1. Let $(\mathcal{X}, \tau_1, \tau_2)$ be a bispace. A mapping $\mu : ij - \mathfrak{PO}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$ is called an ij-p-operation on $ij - \mathfrak{PO}(\mathcal{X})$ if $\mathcal{V} \subseteq \mathcal{V}^{\mu}$, for each nonempty set $\mathcal{V} \in ij - \mathfrak{PO}(\mathcal{X})$ and $\phi^{\mu} = \phi$, where $\mathcal{V}^{\mu}$ (or $\mu(\mathcal{V})$) is the image of $\mathcal{V}$ under μ.

Example 4.1. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \big\{\mathcal{X}, \phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}\big\}$, and $\tau_2 = \big\{\mathcal{X}, \phi, \{\beta\}, \{\gamma, \delta\}, \{\beta, \gamma, \delta\}\big\}$. Then, $12 - \mathfrak{PO}(\mathcal{X}) = \big\{\mathcal{X}, \phi, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}, \{\beta, \gamma, \delta\}\big\}$. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{U}) = 1 - \mathfrak{Cl}(\mathcal{U})$ for all $\mathcal{U} \in 12 - \mathfrak{PO}(\mathcal{X})$. Then, μ is 12-p-operation.

Definition 4.2. A mapping $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$ is a pairwise p-operation if $\mathcal{A} \subseteq \mathcal{A}^{\mu}$ for each $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X})$ and $\phi^{\mu} = \phi$.

Example 4.2. Let $(\mathcal{X}, \tau_1, \tau_2)$ be as in Example 4.1. Then, $21 - \mathfrak{PO}(\mathcal{X}) = \{\phi, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\beta, \delta\}, \{\gamma, \delta\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}, \{\beta, \gamma, \delta\}, \mathcal{X}\}$. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{U}) = 2 - \mathfrak{Cl}(\mathcal{U})$ for all $\mathcal{U} \in 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X})$. Then, μ is a pairwise p-operation.

Definition 4.3. Let $\mathcal{X}$ be a bispace and $\mu : ij - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be an ij--p-operation on $ij - \mathfrak{PO}(\mathcal{X})$. A subset $\mathcal{A} \subseteq \mathcal{X}$ is called $ij - \mu_p$-open if for each $x \in \mathcal{A}$ there exists an ij-preopen set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $\mathcal{U}^{\mu} \subseteq \mathcal{A}$.

Definition 4.4. An $ij - \mu_p$-open subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is called $ij - \mu_{pc}$-open if for each $x \in \mathcal{A}$, there exists a j-closed subset $\mathcal{F}$ of $\mathcal{X}$ such that $x \in \mathcal{F} \subseteq \mathcal{A}$. The complement of an $ij - \mu_{pc}$-open set is $ij - \mu_{pc}$-closed. The family of all $ij - \mu_{pc}$-open (resp. $ij - \mu_{pc}$-closed) subsets of a bispace $\mathcal{X}$ is denoted by $ij - \mu_{pc}\mathfrak{O}(\mathcal{X})$ (resp. $ij - \mu_{pc}\mathfrak{C}(\mathcal{X})$). If a subset $\mathcal{A}$ of a space $\mathcal{X}$ is $ij - \mu_{pc}$-closed and $ji - \mu_{pc}$-closed, then it is pairwise μ_{pc}-closed.

Example 4.3. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta, \epsilon\}$, $\tau_1 = \{\mathcal{X}, \phi, \{\epsilon\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \beta, \epsilon\}, \{\gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma, \delta\}\}$, and $\tau_2 = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\epsilon\}, \{\alpha, \gamma\}, \{\alpha, \epsilon\}, \{\gamma, \epsilon\}, \{\alpha, \gamma, \epsilon\}, \{\alpha, \gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma, \epsilon\}\}$. Then, $12 - \mathfrak{PO}(\mathcal{X}) = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\epsilon\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \gamma\}, \{\alpha, \epsilon\}, \{\gamma, \epsilon\}, \{\alpha, \beta, \epsilon\}, \{\gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma\}, \{\alpha, \gamma, \delta\}, \{\alpha, \beta, \gamma, \delta\}\}$. Let

$$\mu : 12 - \mathfrak{PO}(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X}) \quad \text{be defined by} \quad \mu(\mathcal{U}) = 1 - \mathfrak{Cl}(\mathcal{U}).$$

Then, $12 - \mu_p\mathfrak{O}(\mathcal{X}) = \{\mathcal{X}, \phi, \{\epsilon\}, \{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \beta, \epsilon\}, \{\gamma, \delta, \epsilon\}, \{\alpha, \beta, \gamma, \delta\}\}$.

Proposition 4.1. For any bispace $\mathcal{X}$, $ij - \mu_{pc}\mathfrak{O}(\mathcal{X}) \subseteq ij - \mu_p\mathfrak{O}(\mathcal{X}) \subseteq ij - \mathfrak{PO}(\mathcal{X})$.

P r o o f. The proof is easy and hence omitted. □

The following example shows that the equality in the above proposition may not be true in general.

Example 4.4. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\mathcal{X}, \phi, \{\alpha, \beta\}, \{\gamma, \delta\}\}$ and $\tau_2 = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}, \{\alpha, \beta, \delta\}\}$. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 1 - \mathfrak{Cl}(\mathcal{A})$ for each $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X})$. Then, $\{\alpha, \beta\}$ is 12-μ_p-open but it is not 12-μ_{pc}-open. Also, $\{\alpha, \gamma\}$ is a 12-preopen set but it is not 12-μ_p-open.

The following example shows that $ij - \mu_{pc}\mathfrak{O}(\mathcal{X})$ is not comparable with

$$\tau_i, i \in \{1, 2\}.$$

Example 4.5. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}, \{\beta, \delta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta, \delta\}\}$, $\tau_2 = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\delta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\alpha, \gamma\}, \{\gamma, \delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \gamma, \delta\}, \{\beta, \gamma, \delta\}\}$ and let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 2 - \mathfrak{Cl}(\mathcal{A})$ for all $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X})$. Then,

(a) $\{\gamma\}$ is both τ_1- and τ_2-open set but it is not 12-μ_{pc}-open.

(b) $\{\alpha, \beta, \delta\}$ is a 12-μ_{pc}-open set but it is not τ_2-open.

Proposition 4.2. In a bispace $\mathcal{X}$, the union of any collection of $ij-\mu_{pc}$-open sets is an $ij-\mu_{pc}$-open set.

Proof. Let $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ be any collection of $ij-\mu_{pc}$-open sets in $\mathcal{X}$. Since $\mathcal{A}_\alpha$ is $ij-\mu_{pc}$-open for all $\alpha\in I$, there exists an ij-preopen set $\mathcal{U}$ such that $\mu(\mathcal{U})\subseteq\mathcal{A}_\alpha\subseteq\bigcup_{\alpha\in I}\mathcal{A}_\alpha$. Therefore, $\bigcup_{\alpha\in I}\mathcal{A}_\alpha$ is an $ij-\mu_{pc}$-open. Suppose that $x\in\bigcup_{\alpha\in I}\mathcal{A}_\alpha$. Then, there exists $\alpha_0\in I$ such that $x\in\mathcal{A}_{\alpha_0}$ and so, there exists a j-closed set $\mathcal{F}$ such that $x\in\mathcal{F}\subseteq\mathcal{A}_{\alpha_0}\subset\bigcup_{\alpha\in I}\mathcal{A}_\alpha$. Therefore, $\bigcup_{\alpha\in I}\mathcal{A}_\alpha$ is an $ij-\mu_{pc}$-open set in $\mathcal{X}$. □

The following example shows that the intersection of two $ij-\mu_{pc}$-open sets need not be $ij-\mu_{pc}$-open.

Example 4.6. Consider the set of integers $\mathbb{Z}$, τ_1 to be the excluding point topology on $\mathbb{Z}$, where $e=0$, and τ_2 the cofinite topology on $\mathbb{Z}$, i.e., $\tau_1=\{\mathcal{U}\subseteq\mathbb{Z}:0\notin\mathcal{U}\}\cup\{\mathbb{Z}\}$ and $\tau_2=\{\phi\}\cup\{\mathcal{U}\subseteq\mathbb{Z}:\mathcal{U}^c \text{ is finite}\}$. Then, $12-\mathfrak{PO}(\mathcal{X})$ is both the set of all infinite subsets of $\mathbb{Z}$ and the set of all finite subsets of $\mathbb{Z}$ which is not containing 0. Let $\mu:12-\mathfrak{PO}(\mathcal{X})\to\mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{U})=1-\mathfrak{Cl}(\mathcal{U})$. Then, $12-\mu_p-\mathfrak{O}(\mathcal{X})$ is the set of all infinite subsets of $\mathbb{Z}$ containing 0. Also, $12-\mu_{pc}-\mathfrak{O}(\mathcal{X})=12-\mu_p-\mathfrak{O}(\mathcal{X})$. The sets $A=\{\dots,-2,-1,0,1\}$ and $B=\{-1,0,1,2,\dots\}$ are both 12-μ_{pc}-open subsets of $\mathbb{Z}$ but their intersection is not.

Proposition 4.3. A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is $ij-\mu_{pc}$-open if and only if for each $x\in\mathcal{A}$ there exists an $ij-\mu_{pc}$-open set $\mathcal{B}$ such that $x\in\mathcal{B}\subseteq\mathcal{A}$.

Proof. If $\mathcal{A}$ is an $ij-\mu_{pc}$-open set, then for each $x\in\mathcal{A}$, $x\in\mathcal{A}\subseteq\mathcal{A}$. Conversely, suppose that there exists an $ij-\mu_{pc}$-open set $\mathcal{B}_x$ such that $x\in\mathcal{B}_x\subseteq\mathcal{A}$. Thus, $\mathcal{A}=\bigcup\mathcal{B}_x$ and, by Proposition 4.2, $\mathcal{A}$ is an $ij-\mu_{pc}$-open set. □

Definition 4.5. Let $\mathcal{X}$ be a bispace. An ij-p-operation μ is said to be ij-p-regular if for every two ij-preopen sets $\mathcal{U}$ and $\mathcal{V}$ containing $x\in\mathcal{X}$, there exists an ij-preopen set $\mathcal{W}$ containing x such that $\mu(\mathcal{W})\subseteq\mu(\mathcal{U})\cap\mu(\mathcal{V})$.

Theorem 4.1. *If μ is an ij-regular ij-p-operation on a bispace $\mathcal{X}$, then the intersection of two $ij-\mu_{pc}$-open subsets of $\mathcal{X}$ is $ij-\mu_{pc}$-open.*

Proof. Let μ be an ij-regular ij-p-operation on a bispace $\mathcal{X}$, and $\mathcal{A}$ and $\mathcal{B}$ be two $ij-\mu_{pc}$-open subsets of $\mathcal{X}$. Let $x\in\mathcal{A}\cap\mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are $ij-\mu_{pc}$-open sets, there exist ij-preopen sets $\mathcal{U}$ and $\mathcal{V}$ such that $x\in\mathcal{U}$, $\mu(\mathcal{U})\subseteq\mathcal{A}$, $x\in\mathcal{V}$, and $\mu(\mathcal{V})\subseteq\mathcal{B}$. Since μ is ij-regular, there exists an ij-preopen set $\mathcal{W}$ containing x such that $\mu(\mathcal{W})\subseteq\mu(\mathcal{U})\cap\mu(\mathcal{V})\subseteq\mathcal{A}\cap\mathcal{B}$. Therefore, $\mathcal{A}\cap\mathcal{B}$ is an $ij-\mu_p$-open set. Again, for each $x\in\mathcal{A}\cap\mathcal{B}$, and since $\mathcal{A}$ and $\mathcal{B}$ are $ij-\mu_{pc}$-open sets, there exist j-closed sets $\mathcal{E}$ and $\mathcal{F}$ such that $x\in\mathcal{E}\subseteq\mathcal{A}$ and $x\in\mathcal{F}\subseteq\mathcal{B}$. Therefore, $x\in\mathcal{E}\cap\mathcal{F}\subseteq\mathcal{A}\cap\mathcal{B}$. Since $\mathcal{E}\cap\mathcal{F}$ is j-closed, $\mathcal{A}\cap\mathcal{B}$ is $ij-\mu_{pc}$-open subsets of $\mathcal{X}$. □

Definition 4.6. Let $\mathcal{X}$ be a bispace and $\mathcal{A} \subseteq \mathcal{X}$. A point $x \in \mathcal{X}$ is called $ij-\mu_{pc}$-limit point of $\mathcal{A}$ if every $ij-\mu_{pc}$-open set containing x contains a point of $mathcalA$ different from x. The set of all $ij-\mu_{pc}$-limit points of $\mathcal{A}$ is called the $ij-\mu_{pc}$-derived set of $\mathcal{A}$ and is denoted by $ij-\mu_{pc}\mathfrak{d}(\mathcal{A})$. The $ij-\mu_{pc}$-closure of $\mathcal{A}$, denoted by $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$, is the intersection of all $ij-\mu_{pc}$-closed sets containing $\mathcal{A}$. The $ij-\mu_{pc}$-interior of $\mathcal{A}$, denoted by $ij-\mu_{pc}\mathfrak{Int}(\mathcal{A})$, is the union of all $ij-\mu_{pc}$-open sets contained in $\mathcal{A}$.

The following proposition gives the properties of $ij-\mu_{pc}$-closure of a set.

Proposition 4.4. For subsets $\mathcal{A}$ and $\mathcal{B}$ of a bispace $\mathcal{X}$, we have:

(1) $\mathcal{A} \subset ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$;

(2) $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$ is an $ij-\mu_{pc}$-closed set;

(3) $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$ is the smallest $ij-\mu_{pc}$-closed set containing $\mathcal{A}$;

(4) $\mathcal{A}$ is $ij-\mu_{pc}$-closed if and only if $A = ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$;

(5) $ij-\mu_{pc}\mathfrak{Cl}(\phi) = \phi$ and $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{X}) = \mathcal{X}$;

(6) If $\mathcal{A} \subseteq \mathcal{B}$, the $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq ij-\mu_{pc}\mathfrak{Cl}(\mathcal{B})$;

(7) $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) \cup ij-\mu_{pc}\mathfrak{Cl}(\mathcal{B}) \subseteq ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A} \cup \mathcal{B})$;

(8) $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A} \cap \mathcal{B}) \subseteq ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) \cap ij-\mu_{pc}\mathfrak{Cl}(\mathcal{B})$.

P r o o f. The proof, being easy, is omitted. □

In general, equalities in (7) and (8) in the above proposition need not be true as shown by the following example.

Example 4.7. Let $\mathcal{X}=\{\alpha,\beta,\gamma,\delta\}$, $\tau_1=\big\{\mathcal{X},\phi,\{\alpha\},\{\gamma\},\{\alpha,\gamma\},\{\beta,\delta\},\{\beta,\gamma,\delta\},\{\alpha,\beta,\delta\}\big\}$ and $\tau_2 = \big\{\mathcal{X},\phi,\{\beta\},\{\gamma\},\{\delta\},\{\beta,\gamma\},\{\beta,\delta\},\{\gamma,\delta\},\{\alpha,\gamma\},\{\alpha,\beta,\gamma\},\{\alpha,\gamma,\delta\},\{\beta,\gamma,\delta\}\big\}$. Let $\mu : 12-\mathfrak{PO}(\mathcal{X}) \cup 21-\mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 1-\mathfrak{Cl}(\mathcal{A})$ for each $\mathcal{A} \in 12-\mathfrak{PO}(\mathcal{X}) \cup 21-\mathfrak{PO}(\mathcal{X})$.

(a) Let $\mathcal{A}=\{\beta,\gamma\}$ and $\mathcal{B}=\{\gamma,\delta\}$. Then, $21-\mu_{pc}\mathfrak{Cl}(\mathcal{A})=\mathcal{X}$, $21-\mu_{pc}\mathfrak{Cl}(\mathcal{B})=\mathcal{X}$ and $21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}\cap\mathcal{B})=\{\gamma\}$. Hence,$21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}\cap\mathcal{B}) \neq 21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) \cap 21-\mu_{pc}\mathfrak{Cl}(\mathcal{B})$.

(b) Let $\mathcal{A}=\{\gamma\}$ and $\mathcal{B}=\{\beta,\delta\}$. Then, $21-\mu_{pc}\mathfrak{Cl}(\mathcal{A})=\{\gamma\}$, $21-\mu_{pc}\mathfrak{Cl}(\mathcal{B}) = \{\beta,\delta\}$ and $21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}\cup\mathcal{B})=\mathcal{X}$. This shows that $21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}\cup\mathcal{B}) \neq 21-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) \cup 21-\mu_{pc}\mathfrak{Cl}(\mathcal{B})$.

Proposition 4.5. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have $ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A}) = \mathcal{A} \cup ij-\mu_{pc}\mathfrak{d}(\mathcal{A})$.

P r o o f. The proof is straightforward. □

Theorem 4.2. *Let $\mathcal{A}$ be a subset of a bispace $\mathcal{X}$ and $x \in \mathcal{X}$. Then, $x \in ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$ if and only if $\mathcal{V} \cap \mathcal{A} \neq \phi$ for every $ij - \mu_{pc}$-open set $\mathcal{V}$ containing x.*

Proof. Let $x \in ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$ and $\mathcal{V} \cap \mathcal{A} = \phi$ for some $ij - \mu_{pc}$-open set $\mathcal{V}$ containing x. Then, $\mathcal{X}\backslash\mathcal{V}$ is an $ij - \mu_{pc}$-closed set and $\mathcal{A} \subseteq \mathcal{X}\backslash\mathcal{V}$. Therefore, $ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{X}\backslash\mathcal{V}$ which implies that $x \in \mathcal{X}\backslash\mathcal{V}$, a contradiction. Thus, $\mathcal{V}\cap\mathcal{A}\neq\phi$. Conversely, let $\mathcal{A} \subseteq \mathcal{X}$ and $x \in \mathcal{X}$ such that every $ij-\mu_{pc}$-open $\mathcal{V}$ containing x, $\mathcal{V}\cap\mathcal{A} \neq \phi$. If $x \notin ij-\mu_{pc}\mathfrak{Cl}(\mathcal{A})$, then there exist an $ij-\mu_{pc}$-closed set $\mathcal{F}$ such that $\mathcal{A} \subseteq \mathcal{F}$ and $x \notin \mathcal{F}$. Hence, $\mathcal{X}\backslash\mathcal{F}$ is an $ij - \mu_{pc}$-open set containing x with $(\mathcal{X}\backslash\mathcal{F}) \cap \mathcal{A} = \phi$, a contradiction. Therefore, $x \in ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$. □

Proposition 4.6. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have $ij - \mathfrak{pCl}(\mathcal{A}) \subseteq ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$.

Proof. The proof is straightforward. □

The following example shows that the equality in the above proposition need not be true.

Example 4.8. Let $(\mathcal{X}, \tau_1, \tau_2)$ be as in Example 4.7. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 2 - \mathfrak{Cl}(\mathcal{A})$ for each $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X})$. Let $\mathcal{A} = \{\alpha, \beta\}$. Then, $21 - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) = \mathcal{X}$ and $21 - \mathfrak{pCl}(\mathcal{A}) = \{\alpha, \beta\}$.

The following proposition gives the properties of $ij - \mu_{pc}$-interior of a set.

Proposition 4.7. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have:

(1) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A})$ is an $ij - \mu_{pc}$-open set;

(2) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \subseteq \mathcal{A}$;

(3) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A})$ is the largest $ij - \mu_{pc}$-open set contained in $\mathcal{A}$;

(4) $\mathcal{A}$ is $ij - \mu_{pc}$-open if and only if $\mathcal{A} = ij - \mu_{pc}\mathfrak{Int}(\mathcal{A})$;

(5) $ij - \mu_{pc}\mathfrak{Int}\big(ij - \mu_{pc}\mathfrak{Int}(\mathcal{A})\big) = ij - \mu_{pc}\mathfrak{Int}(\mathcal{A})$;

(6) If $\mathcal{A} \subseteq \mathcal{B}$, then $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \subseteq ij - \mu_{pc}\mathfrak{Int}(\mathcal{B})$;

(7) $ij - \mu_{pc}\mathfrak{Int}(\phi) = \phi, ij - \mu_{pc}\mathfrak{Int}(\mathcal{X}) = \mathcal{X}$;

(8) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \cup ij - \mu_{pc}\mathfrak{Int}(\mathcal{B}) \subseteq ij - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cup \mathcal{B})$;

(9) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cap \mathcal{B}) \subseteq ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \cap ij - \mu_{pc}\mathfrak{Int}(\mathcal{B})$.

Proof. The proof is obvious. □

In general, equalities in (8) and (9) in the above proposition need not be true as shown by the following example.

Example 4.9. Let $(\mathcal{X}, \tau_1, \tau_2)$ be as in Example 4.7. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 1 - \mathfrak{Cl}(\mathcal{A})$ for each $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X})$.

(a) Let $\mathcal{A} = \{\alpha\}$ and $\mathcal{B} = \{\gamma\}$. Then, $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = \phi$, $21 - \mu_{pc}\mathfrak{Int}(\mathcal{B}) = \{\gamma\}$ and $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cup \mathcal{B}) = \{\alpha, \gamma\}$. This shows that $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \cup 21 - \mu_{pc}\mathfrak{Int}(\mathcal{B}) \cup 21 - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cup \mathcal{B})$.

(b) Let $\mathcal{A} = \{\alpha, \beta, \gamma\}$ and $\mathcal{B} = \{\alpha, \beta, \delta\}$. Then, $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = \{\alpha, \gamma\}$, $21 - \mu_{pc}\mathfrak{Int}(\mathcal{B}) = \{\alpha, \beta, \delta\}$ and $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cap \mathcal{B}) = \phi$. This shows that $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \cap 21 - \mu_{pc}\mathfrak{Int}(\mathcal{B}) \neq 21 - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cap \mathcal{B})$.

Proposition 4.8. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \subseteq ij - \mathfrak{pInt}(\mathcal{A})$.

Proof. Obvious. □

The following example shows that the equality in the above proposition need not be true.

Example 4.10. Let $(\mathcal{X}, \tau_1, \tau_2)$ be as in Example 4.7. Let $\mu : 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be defined by $\mu(\mathcal{A}) = 2 - \mathfrak{Cl}(\mathcal{A})$ for each $\mathcal{A} \in 12 - \mathfrak{PO}(\mathcal{X}) \cup 21 - \mathfrak{PO}(\mathcal{X})$. Let $\mathcal{A} = \{\alpha, \beta, \gamma\}$. Then, $21 - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = \{\alpha, \gamma\}$ and $21 - \mathfrak{pInt}(\mathcal{A}) = \{\alpha, \beta, \gamma\}$.

Proposition 4.9. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = \mathcal{A} \backslash ij - \mu_{pc}\mathfrak{d}(\mathcal{X} \backslash \mathcal{A})$.

Proof. Obvious. □

The following proposition gives the relations between the $ij - \mu_{pc}$-closure and $ij - \mu_{pc}$-interior of a set.

Proposition 4.10. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we have:

(a) $\mathcal{X} \backslash ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = ij - \mu_{pc}\mathfrak{Cl}(\mathcal{X} \backslash \mathcal{A})$;

(b) $ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) = \mathcal{X} \backslash ij - \mu_{pc}\mathfrak{Int}(\mathcal{X} \backslash \mathcal{A})$;

(c) $\mathcal{X} \backslash ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) = ij - \mu_{pc}\mathfrak{Int}(\mathcal{X} \backslash \mathcal{A})$;

(d) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) = \mathcal{X} \backslash ij - \mu_{pc}\mathfrak{Cl}(\mathcal{X} \backslash \mathcal{A})$.

Proof. Obvious. □

Theorem 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of a bispace $\mathcal{X}$ and let $\mu : ij - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ be an ij-p-regular ij-p-operation, then:*

(a) $ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A} \cup \mathcal{B}) = ij - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \cup ij - \mu_{pc}\mathfrak{Cl}(\mathcal{B})$;

(b) $ij - \mu_{pc}\mathfrak{Int}(\mathcal{A} \cap \mathcal{B}) = ij - \mu_{pc}\mathfrak{Int}(\mathcal{A}) \cap ij - \mu_{pc}\mathfrak{Int}(\mathcal{B})$.

Proof. Obvious. □

5. $ij - g\mu_{pc}$-closed sets

In this section, we introduce the concept of $ij - g\mu_{pc}$-closed set and study some of its properties.

Definition 5.1. A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is called ij-generalized μ_{pc}-closed ($ij - g\mu_{pc}$-closed, for short) if $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$ whenever $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is an $ij - \mu_{pc}$-open set in $\mathcal{X}$. The complement of an ij-generalized μ_{pc}-closed set is ij-generalized μ_{pc}-open ($ij - g\mu_{pc}$-open, for short).

Theorem 5.1. *If $\mathcal{A}$ is an $ij - g\mu_{pc}$-closed subset of a bispace $\mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{X}$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$, then $\mathcal{B}$ is $ij - g\mu_{pc}$-closed.*

Proof. Let $\mathcal{A}$ be an $ij - g\mu_{pc}$-closed set and $\mathcal{A} \subseteq \mathcal{B} \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$. Let $\mathcal{U}$ be an $ij - \mu_{pc}$-open set such that $\mathcal{B} \subseteq \mathcal{U}$. Then, $\mathcal{A} \subseteq \mathcal{U}$ and so, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$. Now, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{B}) \subseteq ji - \mu_{pc}\mathfrak{Cl}\big(ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})\big) = ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$. This implies that $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{B}) \subseteq \mathcal{U}$ and $\mathcal{B}$ is $ij - g\mu_{pc}$-closed. □

Lemma 5.1. *Let μ be a pairwise p-operation on a bispace $\mathcal{X}$. Then for each $x \in \mathcal{X}$, $\{x\}$ is either an $ij - \mu_{pc}$-closed or $ij - g\mu_{pc}$-open set in $\mathcal{X}$.*

Proof. Suppose that $\{x\}$ is not an $ij - \mu_{pc}$-closed set. Then, $\mathcal{X} \backslash \{x\}$ is not $ij - \mu_{pc}$-open. So, the only $ij - \mu_{pc}$-open set containing $\mathcal{X} \backslash \{x\}$ is $\mathcal{X}$. Thus, $\mathcal{X} \backslash \{x\}$ is an $ij - g\mu_{pc}$-closed set and $\{x\}$ is $ij - g\mu_{pc}$-open. □

Proposition 5.1. A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is $ij - g\mu_{pc}$-closed if and only if $ij - \mu_{pc}\mathfrak{Cl}(\{x\}) \cap \mathcal{A} \neq \phi$, for every $x \in ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$.

Proof. Let $\mathcal{U}$ be an $ij - \mu_{pc}$-open set such that $\mathcal{A} \subseteq \mathcal{U}$ and $x \in ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$. Then, there exists $z \in ij - \mu_{pc}\mathfrak{Cl}(\{x\})$ and $z \in \mathcal{A} \subseteq \mathcal{U}$. By Theorem 4.2, we have $\mathcal{U} \cap \{x\} \neq \phi$. Hence, $x \in \mathcal{U}$ and $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$. Therefore, $\mathcal{A}$ is an $ij - g\mu_{pc}$-closed set. Conversely, suppose that $x \in ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$ such that $ij - \mu_{pc}\mathfrak{Cl}(\{x\}) \cap \mathcal{A} = \phi$. Since $\mathcal{A} \subseteq X \backslash ij - \mu_{pc}\mathfrak{Cl}(\{x\})$ and $\mathcal{A}$ is an $ij - g\mu_{pc}$-closed set, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{X} \backslash ij - \mu_{pc}\mathfrak{Cl}(\{x\})$. Hence, $x \notin ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$, a contradiction. Therefore, $ij - \mu_{pc}\mathfrak{Cl}(\{x\}) \cap \mathcal{A} \neq \phi$. □

Theorem 5.2. *Let $\mathcal{A}$ be an $ij - g\mu_{pc}$-closed subset of a bispace $\mathcal{X}$. Then, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \backslash \mathcal{A}$ does not contain any nonempty $ij - \mu_{pc}$-closed set in $\mathcal{X}$.*

Proof. Let $\mathcal{A}$ be $\mathcal{F}$ be two $ij - g\mu_{pc}$-closed sets in $\mathcal{X}$ such that $\mathcal{F} \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \backslash \mathcal{A}$. Then, $\mathcal{A} \subseteq \mathcal{X} \backslash \mathcal{F}$. Hence, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{X} \backslash \mathcal{F}$ and so, $\mathcal{F} \subseteq \mathcal{X} \backslash ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$. Therefore, $\mathcal{F} \subseteq \big(\mathcal{X} \backslash ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})\big) \cap \big(ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})\big) = \phi$. □

6. Pairwise $\mu_{pc} - T_k$ and $\mu_{pc} - R_k$ spaces

In this section, we introduce a new type of separation axioms in bispaces called pairwise $\mu_{pc} - T_k, k \in \{0, \frac{1}{2}, 1, 2\}$ and pairwise $\mu_{pc} - R_k, k \in \{0, 1\}$ spaces.

Definition 6.1. A bispace $(\mathcal{X}, \tau_1, \tau_2)$ is said to be:

(1) Pairwise $\mu_{pc} - T_0$ if for each two distinct points $x, y \in \mathcal{X}$, there exists either an $ij - \mu_{pc}$-open set containing x but not y or an $ji - \mu_{pc}$-open set containing y but not x;

(2) Pairwise $\mu_{pc} - T_{\frac{1}{2}}$ if every a $ij - g\mu_{pc}$-closed set is $ji - \mu_{pc}$-closed;

(3) Pairwise $\mu_{pc} - T_1$ if for each two distinct points $x, y \in \mathcal{X}$, there exist an $ij - \mu_{pc}$-open set $\mathcal{U}$ containing x but not y and a $ji - \mu_{pc}$-open set $\mathcal{V}$ containing y but not x;

(4) Pairwise $\mu_{pc} - T_2$ if for each two distinct points $x, y \in \mathcal{X}$, there exist an $ij - \mu_{pc}$-open set $\mathcal{U}$ containing x but not y and a disjoint a $ji - \mu_{pc}$-open set $\mathcal{V}$ containing y but not x.

Proposition 6.1. Every pairwise $\mu_{pc} - T_i$-space is pairwise pT_i for $i = 0, \frac{1}{2}, 1, 2$, but not conversely.

Proof. Obvious. □

Example 6.1.
Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\mathcal{X}, \phi, \{\alpha\}, \{\gamma\}, \{\alpha, \gamma\}, \{\beta, \delta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta, \delta\}\}$ and $\tau_2 = \{\mathcal{X}, \phi, \{\beta\}, \{\gamma\}, \{\delta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\alpha, \gamma\}, \{\gamma, \delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \gamma, \delta\}, \{\beta, \gamma, \delta\}\}$. Let $\mu(\mathcal{A}) = 1 - \mathfrak{Cl}(\mathcal{A})$. Then, $\mathcal{X}$ is a pairwise pT_0-space but it is not pairwise $\mu_{pc} - T_0$.

Example 6.2. Let $\mathcal{X} = \{\alpha, \beta, \gamma, \delta\}$, $\tau_1 = \{\mathcal{X}, \phi, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}, \{\beta, \gamma, \delta\}\}$, $\tau_2 = \{\mathcal{X}, \phi, \{\alpha, \delta\}, \{\alpha, \gamma, \delta\}, \{\alpha, \beta, \delta\}\}$ and $\mu(\mathcal{A}) = 1 - \mathfrak{Cl}(\mathcal{A}) \cap 2 - \mathfrak{Cl}(\mathcal{A})$. Then, $\mathcal{X}$ is a pairwise pT_2-space but it is not pairwise $\mu_{pc} - T_2$.

Theorem 6.1. *A bispace $\mathcal{X}$ is a pairwise $\mu_{pc} - T_{\frac{1}{2}}$ if and only if every singleton set is either $ji - \mu_{pc}$-open or $ij - \mu_{pc}$-closed.*

Proof. Suppose that $x \in \mathcal{X}$ such that $\{x\}$ is not an $ij - \mu_{pc}$-closed set. Then, by Lemma 5.1, $\mathcal{X} \backslash \{x\}$ is an $ij - g\mu_{pc}$-closed set. Since $\mathcal{X}$ is a pairwise $\mu_{pc} - T_{\frac{1}{2}}$-space, $\mathcal{X} \backslash \{x\}$ is a $ji - \mu_{pc}$-closed set and so, $\{x\}$ is $ji - \mu_{pc}$-open. Conversely, suppose that $\mathcal{F}$ is an $ij - g\mu_{pc}$-closed set in $\mathcal{X}$. For any $x \in ji - \mu_{pc}\mathfrak{Cl}(\mathcal{F})$, $\{x\}$ is either a $ji - \mu_{pc}$-open or an $ij - \mu_{pc}$-closed set. If $\{x\}$ is $ji - \mu_{pc}$-open, then $\{x\} \cap \mathcal{F} \neq \phi$ and so, $x \in \mathcal{F}$. If $\{x\}$ is an $ij - \mu_{pc}$-closed set and $x \notin \mathcal{F}$, then $\{x\} \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{F}) \backslash \mathcal{F}$, which contradicts Theorem 5.2 Therefore, $x \in \mathcal{F}$. Hence, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathcal{F}$ is $ji - \mu_{pc}$-closed. Therefore, $\mathcal{X}$ is a pairwise $\mu_{pc} - T_{\frac{1}{2}}$-space. □

Proposition 6.2. A bispace $\mathcal{X}$ is pairwise $\mu_{pc} - T_0$ if and only if for any two distinct points $x, y \in \mathcal{X}$, $ji - \mu_{pc}\mathfrak{Cl}(\{x\}) \neq ij - \mu_{pc}\mathfrak{Cl}(\{y\})$.

Proof. Let $\mathcal{X}$ be a pairwise $\mu_{pc} - T_0$-space and $x, y \in \mathcal{X}$ such that $x \neq y$. Then, there exists an $ij - \mu_{pc}$-open set $\mathcal{U}$ such that $x \in \mathcal{U}$ and $y \notin \mathcal{U}$. Thus, $\{y\} \cap \mathcal{U} = \phi$ and $x \notin ij - \mu_{pc}\mathfrak{Cl}(\{y\})$. Since $x \in ij - \mu_{pc}\mathfrak{Cl}(\{x\})$, $ji - \mu_{pc}\mathfrak{Cl}(\{x\}) \neq ij - \mu_{pc}\mathfrak{Cl}(\{y\})$. Conversely, suppose that $x, y \in \mathcal{X}$ such that $x \neq y$. Then, $ji - \mu_{pc}\mathfrak{Cl}(\{x\}) \neq ij - \mu_{pc}\mathfrak{Cl}(\{y\})$ which implies that either $y \notin ji - \mu_{pc}\mathfrak{Cl}(\{x\})$ or $x \notin ij - \mu_{pc}\mathfrak{Cl}(\{y\})$. If $y \notin ji - \mu_{pc}\mathfrak{Cl}(\{x\})$, there exists a $ji - \mu_{pc}$-open set $\mathcal{U}$ such that $y \in \mathcal{U}$ and $\{x\} \cap \mathcal{U} = \phi$, i.e., $x \notin \mathcal{U}$. If $x \notin ij - \mu_{pc}\mathfrak{Cl}(\{y\})$, there exists an $ij - \mu_{pc}$-open set $\mathcal{V}$ such that $x \in \mathcal{V}$ and $\{y\} \cap \mathcal{V} = \phi$, i.e., $y \notin \mathcal{V}$. In both cases, $\mathcal{X}$ is a pairwise $\mu_{pc} - T_0$-space. □

Theorem 6.2. *A bispace ($\mathcal{X}$ is pairwise $\mu_{pc} - T_1$-space if and only if every singleton set in $\mathcal{X}$ is pairwise μ_{pc}-closed.*

Proof. Let $\mathcal{X}$ be a pairwise $\mu_{pc} - T_1$-space and $x \in \mathcal{X}$. For each $y \neq x$, there exists an $ij - \mu_{pc}$-open set $\mathcal{U}$ containing y but not x. Then, $\{x\} \cap \mathcal{U} = \phi$ and $y \notin ij - \mu_{pc}\mathfrak{Cl}(\{x\})$. Then, $\{x\} = ij - \mu_{pc}\mathfrak{Cl}(\{x\})$, and $\{x\}$ is an $ij - \mu_{pc}$-closed set. Now, for every $y \neq x$, $y \in \mathcal{X}\backslash\{x\}$. So, there exists a $ji - \mu_{pc}$-open set $\mathcal{V}_y$ such that $y \in \mathcal{V}_y$ and $x \notin \mathcal{V}_y$. Therefore, $y \in \mathcal{V}_y \subseteq \mathcal{X}\backslash\{x\}$. Hence, $\mathcal{X}\backslash\{x\}$ is an $ji - \mu_{pc}$-open set and $\{x\}$ is $ji - \mu_{pc}$-closed set. Conversely, let $\{x\} = ji - \mu_{pc}\mathfrak{Cl}(\{x\})$ for every $x \in \mathcal{X}$ and $x, y \in \mathcal{X}$ with $x \neq y$. Then, $\mathcal{X}\backslash ji - \mu_{pc}\mathfrak{Cl}(\{x\})$ is a $ji - \mu_{pc}$-open set containing y but not x. Similarly, if $\{y\} = ij - \mu_{pc}\mathfrak{Cl}(\{y\})$, then $\mathcal{X}\backslash ij - \mu_{pc}\mathfrak{Cl}(\{y\})$ is an $ij - \mu_{pc}$-open set containing x but not y. Thus, $\mathcal{X}$ is a pairwise $\mu_{pc} - T_1$-space. □

Theorem 6.3. *Every pairwise $\mu_{pc} - T_1$-space is pairwise $\mu_{pc} - T_{\frac{1}{2}}$.*

Proof. Let $\mathcal{X}$ be a pairwise $\mu_{pc} - T_1$-space. It suffices to show that a set which is not $ji - \mu_{pc}$-closed is also not $ij - g\mu_{pc}$-closed. Suppose that $\mathcal{A} \subseteq \mathcal{X}$ is not a $ji - \mu_{pc}$-closed set and $x \in ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})\backslash\mathcal{A}$. Then, $\{x\} \subseteq ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})\backslash\mathcal{A}$. Since $\mathcal{X}$ is a pairwise $\mu_{pc} - T_1$-space, $\{x\}$ is $ij - \mu_{pc}$-closed. Therefore, by Theorem 5.2, $\mathcal{A}$ is not $ij - g\mu_{pc}$-closed. □

Corollary 6.1. *Every pairwise $\mu_{pc} - T_{\frac{1}{2}}$-space is pairwise $\mu_{pc} - T_0$.*

Proof. It follows from Theorems 6.2 and 6.3. □

Remark 6.1. From the definitions and previous results, we can get the following diagram of implications: Pairwise $\mu_{pc} - T_2 \Rightarrow$ Pairwise $\mu_{pc} - T_1 \Rightarrow$ Pairwise $\mu_{pc} - T_{\frac{1}{2}} \Rightarrow$ Pairwise $\mu_{pc} - T_0$. The converses of this implications are not true in general. See the following example.

Example 6.3. Let $(\mathcal{X}, \tau_1, \tau_2)$, where $\mathcal{X}$ is an infinite set containing e, $\tau_1 = \{\mathcal{U} \subseteq \mathcal{X} : e \notin \mathcal{U}\} \cup \{\mathcal{X}\}$ and $\tau_2 = \{\phi\} \cup \{\mathcal{U} \subseteq \mathcal{X} : \mathcal{U}^c$ is finite$\}$. Consider the identity map $\mu : ij - \mathfrak{PO}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$. Then $\mathcal{X}$ is a pairwise $\mu_{pc} - T_0$-space but it is not $\mu_{pc} - T_1$.

Definition 6.2. A bispace $\mathcal{X}$ is said to be a pairwise μ_{pc}-symmetric space if for $x, y \in \mathcal{X}$, $x \in ij - \mu_{pc}\mathfrak{Cl}(\{y\})$ implies $y \in ji - \mu_{pc}\mathfrak{Cl}(\{x\})$.

Theorem 6.4. *Let $\mathcal{X}$ be a pairwise μ_{pc}-symmetric space. Then, the following are equivalent:*

(1) *$\mathcal{X}$ is pairwise $\mu_{pc} - T_0$;*
(2) *$\mathcal{X}$ is pairwise $\mu_{pc} - T_{\frac{1}{2}}$;*
(3) *$\mathcal{X}$ is pairwise $\mu_{pc} - T_1$.*

Proof. We only prove (1) $\Rightarrow$ (3). Assume that $x \neq y$ and $x \in \mathcal{U} \subseteq \mathcal{X}\backslash\{y\}$ for some $\mathcal{U} \in ij - \mu_{pc}\mathfrak{O}(X)$. Then, $x \notin ij - \mu_{pc}\mathfrak{Cl}(\{y\})$. Hence, $y \notin ji - \mu_{pc}\mathfrak{Cl}(\{x\})$. Therefore, there exists $\mathcal{V} \in ji - \mu_{pc}\mathfrak{O}(X)$ such that $y \in \mathcal{V} \subseteq \mathcal{X}\backslash\{x\}$ and $\mathcal{X}$ is a pairwise $\mu_{pc} - T_1$-space. □

Definition 6.3. For a subset $\mathcal{A}$ of a bispace $\mathcal{X}$, we define $\mathcal{A}^{ij\mu_{pc}}$ as follows

$$\mathcal{A}^{ij\mu_{pc}} = \bigcap\{\mathcal{U} : \mathcal{A} \subseteq \mathcal{U},\, \mathcal{U} \in ij - \mu_{pc}\mathfrak{O}(\mathcal{X})\}.$$

Theorem 6.5. *Let $\mathcal{A}$ be a subset of a bispace $\mathcal{X}$. Then, $\mathcal{A}^{ij\mu_{pc}} = (\mathcal{A}^{ij\mu_{pc}})^{ij\mu_{pc}}$.*

Proof. The proof is similar to that of Theorem 3.4. □

Lemma 6.1. *A subset $\mathcal{A}$ of a bispace $\mathcal{X}$ is $ij - g\mu_{pc}$-closed if and only if $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{A}^{ij\mu_{pc}}$.*

Proof. Let $\mathcal{A} \subseteq \mathcal{X}$ be an $ij - g\mu_{pc}$-closed set and $x \notin \mathcal{A}^{ij\mu_{pc}}$. Then, there exists $\mathcal{U} \in ij - \mu_{pc}\mathfrak{O}(\mathcal{X})$ such that $x \notin \mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{U}$. Hence, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{U}$. Therefore, $x \notin ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A})$ and so, $ji - \mu_{pc}\mathfrak{Cl}(\mathcal{A}) \subseteq \mathcal{A}^{ij\mu_{pc}}$. The converse is obvious. □

Definition 6.4. Let $\mathcal{X}$ be a bispace. For each $x \in \mathcal{X}$ we define

$$\{x\}^{ij\mu_{pc}} = \bigcap\{\mathcal{U} \in ij - \mu_{pc}\mathfrak{O}(\mathcal{X}) : x \in \mathcal{U}\} = \big\{y : x \in ij - \mu_{pc}\mathfrak{Cl}(\{y\})\big\}.$$

Definition 6.5. A bispace $\mathcal{X}$ is said to be pairwise $\mu_{pc} - R_0$-space if for each $ij - \mu_{pc}$-open set $\mathcal{U}$ and $x \in \mathcal{U}$ implies $ji - \mu_{pc}\mathfrak{Cl}(\{x\}) \subset \mathcal{U}$.

Theorem 6.6. *Let $\mathcal{X}$ be a bispace. The following statements are equivalent:*

(1) *$\mathcal{X}$ is a pairwise $\mu_{pc} - R_0$-space;*
(2) *For any $x \in \mathcal{X}$, $ij - \mu_{pc}(\{x\}) \subseteq \{x\}^{ji\mu_{pc}}$;*
(3) *For any $x, y \in \mathcal{X}$, $y \in \{x\}^{ij\mu_{pc}}$ if and only if $x \in \{y\}^{ji\mu_{pc}}$;*

(4) *For any* $x, y \in \mathcal{X}$, $y \in ij - \mu_{pc}\mathfrak{Cl}(\{x\})$ *if and only if* $x \in ji - \mu_{pc}\mathfrak{Cl}(\{y\})$;

(5) *For any* $ij-\mu_{pc}$*-closed set* $\mathcal{F}$ *and a point* $x \notin \mathcal{F}$, *there exists a* $ji-\mu_{pc}$*-open set* $\mathcal{U}$ *such that* $x \notin \mathcal{U}$ *and* $\mathcal{F} \subset \mathcal{U}$;

(6) *Each* $ij - \mu_{pc}$*-closed set* $\mathcal{F}$ *can be expressed as the intersection of all* $ji - \mu_{pc}$*-open sets containing* $\mathcal{F}$;

(7) *Each* $ij-\mu_{pc}$*-open set* $\mathcal{U}$ *can be expressed as the union of all* $ji-\mu_{pc}$*-closed sets contained in* $\mathcal{U}$;

(8) *For each* $ij - \mu_{pc}$*-closed set* $\mathcal{F}$, $x \notin \mathcal{F}$ *implies* $ji - \mu_{pc}\mathfrak{Cl}(\{x\}) \cap \mathcal{F} = \phi$.

Proof. It is similar to that of Theorem 3.5. □

Theorem 6.7. *Let* $\mathcal{X}$ *be a bispace and* μ *be a pairwise p-operation on* $\mathcal{X}$. *Then,* $\mathcal{X}$ *is pairwise* $\mu_{pc} - T_1$ *if and only if it is both pairwise* $\mu_{pc} - R_0$ *and pairwise* $\mu_{pc} - T_0$.

Proof. Let $\mathcal{X}$ be a pairwise $\mu_{pc} - T_1$-space. By Theorem 6.2, every singleton set $\{x\}$ is pairwise μ_{pc}-closed. Let $x, y \in \mathcal{X}$ with $x \neq y$. Then, $\{x\}$ and $\{y\}$ are pairwise μ_{pc}-closed and hence, $\mathcal{X}\backslash\{x\}$ is an $ij - \mu_{pc}$-open set containing y but not x. This shows that $\mathcal{X}$ is a pairwise $\mu_{pc} - T_0$-space. Again, if $x, y \in X$ with $x \neq y$, then $ij - \mu_{pc}\mathfrak{Cl}(\{x\}) \neq ji - \mu_{pc}\mathfrak{Cl}(\{y\})$. Also, $ij - \mu_{pc}\mathfrak{Cl}(\{x\}) \cap ji - \mu_{pc}\mathfrak{Cl}(\{y\}) = \phi$. Thus, by Theorem 6.6, $\mathcal{X}$ is a pairwise $\mu_{pc} - R_0$-space. □

7. Conclusion

In this paper, we proved that a pairwise pre-R_0-space is equivalent to a pairwise pre-T_1-space (Theorem 3.2. Also, we showed that both a pairwise pre-R_1-space and pairwise pre-T_2-space are equivalent (Theorem 3.3).

In Definition 4.3, in case μ is the identity mapping, the concept of $ij-\mu_p$-open coincides with the concept of ij-preopen. In case $\mu(\mathcal{A}) = j - \mathfrak{Cl}(\mathcal{A})$, the concept of $ij - \mu_p$-open is called $ij - \theta$-preopen set. In case $\mu(\mathcal{A}) = i - \mathfrak{Int}\big(j - \mathfrak{Cl}(\mathcal{A})\big)$, the concept of $ij - \mu_p$-open is called $ij - \delta$-preopen. These special cases will be the object in further study. We hope that the results of our paper will be a starting point for a sufficiently general but simple theory of objects that are suitable for modelling various aspects of computation and useful in modern applications of closed sets to general topology and mathematical analysis. Also, we may once more emphasize the importance of ij-pre-generalized closed sets for the possible application in computer and quantum [8, 15, 19].

REFERENCES

[1] ABD EL-MONSEF, M. E.—KOZAE, A. M.—TAHER, B. M.: *Compactification in bitopological spaces*, Arab. J. Sci. Eng. **22**(1997), no. (1A) 99–105.

[2] AHMAD, B. —HUSSAIN, S.: *Properties of γ-operations in topological spaces*, Aligarh Bull. Math. **22** (2003), no. 1, 45–51.
[3] CALDAS, M.—JAFARI, S.—NOIRI, T.: *Characterizations of pre-R_0, and pre-R_1 topological spaces*, Topology Proc. **25** (2000), 17–30.
[4] CHATTOPADHYAY, A.: *pre-T_0 and pre-T_1 Topological Spaces*, J. Indian Acad. Math. **17** (1995), no. 2, 156–159.
[5] FUKUTAKE, T.: *On generalized closed sets in bitopological spaces*, Bull. Fukuoka Univ. ed. III **35** (1986), 19–28.
[6] JANKOVIC, D. S. : *On functions with α-closed graphs*, Glas. Mat. Ser. III **18** (1983), no. 38, 141–148.
[7] JELIĆ, M.: *A decomposition of pairwise continuity*, J. Inst. Math. Comput. Sci. Math. ser. **3** (1990), 25–29.
[8] KALIMSKY, E. D.—KOPPERMAN, R.—MEYER, P. R.: *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. **36** (1990), 1–17.
[9] KAR, A.—BHATTACHARYYA, P.: *Bitopological preopen sets, precontinuity and preopen mappings*, Indian J. Math. **34** (1992), 295–309.
[10] KASAHARA, S.: *Operation-compact spaces*, Math. Japon. **24** (1979), no. 1, 97–105.
[11] KELLY, J. C.: *Bitopological spaces*, Proc. London. Math. Soc. **13** (1963), no. 3, 71–89.
[12] KHALAF, ALIAS B.—OMAR, ALAN M.: *λ_{pc}-open sets and λ_{pc}-separation axioms in topological spaces*, Int. J. Basic Appl. Sci. **3** (2014), no. 2, 137–145.
[13] KHEDR, F. H.—ABDELHAKIEM, K. M.: *Operations on bitopological spaces*, Fasc. Math. **45** (2010), 47–57.
[14] KHEDR, F. H.—AL-AREEFI, S. M.—-NOIRI, T.: *Precontinuity and semi precontinuity in bitopological spaces*, Indian J. Pure Appl. Math. **23** (1992), no. 9, 625–633.
[15] KONY, T. Y.—KOPPERMAN, R.—MEYER, P. R.: *A topological approach to digital toplogy*, Amer. Math. Monthly **98** (1991), 901–917.
[16] LEVINE, N.: *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo **19** (1970), no. 2, 89–96.
[17] MASHHOUR, A. S.—ABD EL-MONSEF, M. E.—EL-DEEB, S. N.: *On pre-continuous and week precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.
[18] MAKI, H.—UMEHARA, J.—NOIRI, T.: *Every topological space is pre-$T_{\frac{1}{2}}$*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **17** (1996), 33–42.
[19] EL NASCHIE, M. S.: *On the uncertainty of Cantorian geometry and two, slit experiment*, Chaos Solitons Fractals, **9** (1998), no. 3, 517–529.
[20] NOIRI, T.—POPA, V.: *On weakly precontinuous functions in bitopological spaces*, Soochow J. Math. **33** (2007), no. 1, 87–100.
[21] OGATA, H.: *Operations on topological spaces and associated topology*, Math. Japon. **36** (1991), no. 1, 175–184.
[22] RAJESH, N.—MUSTAFA, J. M.: *On (i, j)-ω-preopen sets*, Proyecciones **31** (2012), no. 3, 235–246.

Received January 6, 2023

Department of Mathematics
Faculty of Science
Assiut University
Assiut 71516
EGYPT
E-mail: khedrf@aun.edu.eg
o_sayed@aun.edu.eg
o_r_sayed@yahoo.com.