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FUZZY $c\gamma$-OPEN SETS AND FUZZY $c\gamma$-CONTINUITY IN FUZZIFYING TOPOLOGY

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The concepts of fuzzy $c\gamma$-open sets and fuzzy $c\gamma$-continuity are introduced and studied in fuzzifying topology and by making use of these concepts, some decompositions of fuzzy continuity are introduced.

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1. Introduction. In [8, 9, 10], Ying introduced the concept of fuzzifying topology with the semantic method of continuous valued logic. All the conventions in [8, 9, 10] are good in this paper. Andrijević [3] introduced the concepts of b-open sets in general topology. We note that the concepts of γ-open sets and γ-continuity are considered by Hanafy [4] to fuzzy topology. In [7], the concepts of fuzzy γ-open sets and fuzzy γ-continuity are introduced and studied in fuzzifying topology. In the present paper, we define and study the concepts of $c\gamma$-open sets and $c\gamma$-continuity in fuzzifying topology. The main purpose of the present paper is to obtain decompositions of fuzzy continuity in fuzzifying topology by making use of fuzzy γ-continuity and fuzzy $c\gamma$-continuity.

2. Preliminaries. We present the fuzzy logical and corresponding set theoretical notations due to Ying [8, 9].

For any formulae φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0,1]$. We write $\vDash \varphi$ if $[\varphi] = 1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are

(1) (a) $[\alpha] = \alpha(\alpha \in [0,1])$;

(b) $[\varphi \wedge \psi] := \min([\varphi],[\psi])$;

(c) $[\varphi \to \psi] := \min(1, 1-[\varphi]+[\psi])$.

(2) If $A \in \mathscr{F}(X)$, $[x \in \tilde{A}] := \tilde{A}(x)$.

(3) If X is the universe of discourse, $[\forall x \varphi(x)] := \inf_{x\in X}[\varphi(x)]$.

In addition the following derived formulae are given:

(1) $[\neg\varphi] := [\varphi \to 0] = 1-[\varphi]$;

(2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] := \max([\varphi],[\psi])$;

(3) $[\varphi \leftrightarrow \psi] := [(\varphi \to \psi) \wedge (\psi \to \varphi)]$;

(4) $[\varphi \dot{\wedge} \psi] := [\neg(\varphi \to \neg\psi)] := \max(0, [\varphi]+[\psi]-1)$;

(5) $[\varphi \dot{\vee} \psi] := [\neg\varphi \to \psi] = [\neg(\neg\varphi \dot{\wedge} \neg\psi)] = \min(1, [\varphi]+[\psi])$;

(6) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] = \sup_{x\in X}[\varphi(x)]$;

(7) if $\tilde{A}, \tilde{B} \in \mathscr{F}(X)$, then

(a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x(x \in \tilde{A} \to x \in \tilde{B})] = \inf_{x\in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$;
(b) $[\tilde{A} \equiv \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \wedge (\tilde{B} \subseteq \tilde{A})]$;
(c) $[\tilde{A} \doteq \tilde{B}] := [(\tilde{A} \subseteq \tilde{B}) \dot{\wedge} (\tilde{B} \subseteq \tilde{A})]$,

where $\mathscr{F}(X)$ is the family of all fuzzy sets in X.

We do not often distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying does. We now give the following definitions and results in fuzzifying topology which are used in the sequel.

Definition 2.1 (see [8]). Let X be a universe of discourse, $P(X)$ the family of subsets of X, and $\tau \in \mathscr{F}(P(X))$ satisfy the following conditions:
(1) $\tau(X) = 1$, $\tau(\emptyset) = 1$;
(2) for any A, B, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\tau(\cup_{\lambda\in\wedge} A_\lambda) \geq \bigwedge_{\lambda\in\wedge} \tau(A_\lambda)$.

Then τ is called a fuzzifying topology and (X, τ) is a fuzzifying topological space.

Definition 2.2 (see [8]). The family of fuzzifying closed sets, denoted by $F \in \mathscr{F}(P(X))$, is defined as $A \in F := X \sim A \in \tau$, where $X \sim A$ is the complement of A.

Definition 2.3 (see [8]). Let $x \in X$. The neighborhood system of x, denoted by $N_x \in \mathscr{F}(P(X))$, is defined as $N_x(A) = \sup_{x\in B\subseteq A} \tau(B)$.

Definition 2.4 (see [8, Lemma 5.2]). The closure $\bar{A}$ of A is defined as $\bar{A}(x) = 1 - N_x(X \sim A)$.

In [8, Theorem 5.3], Ying proved that the closure $^- : P(X) \to \mathscr{F}(X)$ is a fuzzifying closure operator (see [8, Definiton 5.3]) since its extension $^- : \mathscr{F}(X) \to \mathscr{F}(X)$, $\bar{\tilde{A}} = \cup_{\alpha\in[0,1]} \alpha \bar{\tilde{A}}_\alpha$, $\tilde{A} \in \mathscr{F}(X)$ satisfies the following Kuratowski closure axioms:
(1) $\models \bar{\emptyset} \equiv \emptyset$;
(2) for any $\tilde{A} \in \mathscr{F}(X)$, $\models \tilde{A} \subseteq \bar{\tilde{A}}$;
(3) for any $\tilde{A}, \tilde{B} \in \mathscr{F}(X)$, $\models \overline{\tilde{A} \cup \tilde{B}} \equiv \bar{\tilde{A}} \cup \bar{\tilde{B}}$;
(4) for any $\tilde{A} \in \mathscr{F}(X)$, $\models \overline{(\bar{\tilde{A}})} \subseteq \bar{\tilde{A}}$,

where $\tilde{A}_\alpha = \{x : \tilde{A}(x) \geq \alpha\}$ is the α-cut of A and $\alpha\tilde{A}(x) = \alpha \wedge \tilde{A}(x)$.

Definition 2.5 (see [9]). For any $A \in P(X)$, the interior of A, denoted by $A^\circ \in \mathscr{F}(P(X))$, is defined as follows: $A^\circ(x) = N_x(A)$.

From [8, Lemma 3.1] and the definitions of $N_x(A)$ and A° for $A \in P(X)$ we have $\tau(A) = \inf_{x\in A} A^\circ(x)$.

Definition 2.6 (see [5]). For any $\tilde{A} \in \mathscr{F}(X)$, $\models (\tilde{A})^\circ \equiv X \sim \overline{(X \sim \tilde{A})}$.

Lemma 2.7 (see [5]). *If* $[\tilde{A} \subseteq \tilde{B}] = 1$, *then*
(1) $\models \bar{\tilde{A}} \subseteq \bar{\tilde{B}}$;
(2) $\models (\tilde{A})^\circ \subseteq (\tilde{B})^\circ$.

Lemma 2.8 (see [5]). *Let* (X, τ) *be a fuzzifying topological space. For any* $\tilde{A}$, $\tilde{B}$;
(1) $\models X^\circ = X$;
(2) $\models (\tilde{A})^\circ \subseteq \tilde{A}$;
(3) $\models (\tilde{A} \cap \tilde{B})^\circ \equiv (\tilde{A})^\circ \cap (\tilde{B})^\circ$;
(4) $\models (\tilde{A})^{\circ\circ} \supseteq (\tilde{A})^\circ$.

Lemma 2.9 (see [5]). *Let (X,τ) be a fuzzifying topological space. For any $\tilde{A} \in \mathcal{F}(X)$,*

(1) $\models X \sim (\tilde{A})^{\circ -} \equiv (X \sim \tilde{A})^{-\circ}$;

(2) $\models X \sim (\tilde{A})^{-\circ} \equiv (X \sim \tilde{A})^{\circ -}$.

Lemma 2.10 (see [2, 5]). *If $[\tilde{A} \subseteq \tilde{B}] = 1$, then*

(1) $\models (\tilde{A})^{\circ -} \subseteq (\tilde{B})^{\circ -}$;

(2) $\models (\tilde{A})^{-\circ} \subseteq (\tilde{B})^{-\circ}$.

Definition 2.11. Let (X,τ) be a fuzzifying topological space.

(1) The family of fuzzifying $c\alpha$-open [6] (resp., $c$semi-open [5], $c$pre-open [2], $c\beta$-open [1]) sets, denoted by $c\alpha\tau$ (resp., $cS\tau$, $cP\tau$, $c\beta\tau$) $\in \mathcal{F}(P(X))$, is defined as follows: $A \in c\alpha\tau$ (resp., $cS\tau$, $cP\tau$, $c\beta\tau$)$:= \forall x(x \in A \cap A^{\circ-\circ}$(resp., $A\cap A^{\circ-}, A\cap A^{-\circ}, A\cap A^{-\circ-}) \to x \in A^{\circ})$.

(2) The family of fuzzifying $c\alpha$-closed [6] (resp., $c$semi-closed [5], $c$pre-closed [2], $c\beta$-closed [1]) sets, denoted by $c\alpha F$ (resp., cSF, cPF, $c\beta F$) $\in \mathcal{F}(P(X))$, is defined as follows: $A \in c\alpha F$ (resp., cSF, cPF, $c\beta F$) $:= X \sim A \in c\alpha\tau$ (resp., $cS\tau$, $cP\tau$, $c\beta\tau$).

Definition 2.12 (see [10]). Let (X,τ) be a fuzzifying topological space.

(1) The family of fuzzifying γ-open sets, denoted by $\gamma\tau \in \mathcal{F}(P(X))$, is defined as follows: $A \in \gamma\tau := \forall x(x \in A \to x \in A^{\circ-} \cup A^{-\circ})$,

(2) The family of fuzzifying γ-closed sets, denoted by $\gamma F \in \mathcal{F}(P(X))$, is defined as follows: $A \in \gamma F := X \sim A \in \gamma\tau$,

(3) Let (X,τ), (Y,U) be two fuzzifying topological spaces. A unary fuzzy predicate $\gamma C \in \mathcal{F}(Y^X)$ called fuzzy γ-continuity, is given as $\gamma C(f) := \forall u\ (u \in U \to f^{-1}(u) \in \gamma\tau)$.

Lemma 2.13 (see [7]). (1) $\models \tau \subseteq \gamma\tau$; (2) $\models F \subseteq \gamma F$.

Definition 2.14 (see [10]). Let (X,τ) and (Y,U) be two fuzzifying topological spaces. A unary fuzzy predicate $C \in \mathcal{F}(Y^X)$ called fuzzy continuity, is given as $C(f) := \forall u\ (u \in U \to f^{-1}(u) \in \tau)$.

3. Fuzzifying $c\gamma$-open sets

Definition 3.1. Let (X,τ) be a fuzzifying topological space.

(1) The family of fuzzifying $c\gamma$-open sets, denoted by $c\gamma\tau \in \mathcal{F}(P(X))$, is defined as $A \in c\gamma\tau := \forall x\ (x \in A \cap (A^{\circ-} \cup A^{-\circ}) \to A^{\circ})$.

(2) The family of fuzzifying $c\gamma$-closed sets, denoted by $c\gamma F \in \mathcal{F}(P(X))$, is defined as $A \in c\gamma F := X \sim A \in c\gamma\tau$.

Lemma 3.2. *For any $\alpha,\beta,\gamma,\delta \in I$, $(1-\alpha+\beta) \wedge (1-\gamma+\delta) \le 1-(\alpha\wedge\gamma)+(\beta\wedge\delta)$.*

Theorem 3.3. *Let (X,τ) be a fuzzifying topological space, then*

(1) $c\gamma\tau(X) = 1$, $c\gamma\tau(\varnothing) = 1$;

(2) $c\gamma\tau(A\cap B) \ge c\gamma\tau(A) \wedge c\gamma\tau(B)$.

Proof. The proof of (1) is straightforward.

(2) From Lemma 3.2, we have

$$
\begin{aligned}
& c\gamma\tau(A) \wedge c\gamma\tau(B) \\
&\quad = \inf_{x\in A} (1-(A^{\circ-}\cup A^{-\circ})(x)+A^{\circ}(x)) \wedge \inf_{x\in B} (1-(B^{\circ-}\cup B^{-\circ})(x)+B^{\circ}(x)) \\
&\quad = \inf_{x\in A\cap B} ((1-(A^{\circ-}\cup A^{-\circ})(x)+A^{\circ}(x)) \wedge (1-(B^{\circ-}\cup B^{-\circ})(x)+B^{\circ}(x))) \\
&\quad \le \inf_{x\in A\cap B} (1-((A^{\circ-}\cup A^{-\circ})\cap(B^{\circ-}\cup B^{-\circ}))(x)+(A^{\circ}\cap B^{\circ})(x)) \\
&\quad \le \inf_{x\in A\cap B} (1-((A\cap B)^{\circ-}\cup(A\cap B)^{-\circ})(x)+(A\cap B)^{\circ}(x)) = c\gamma\tau(A\cap B).
\end{aligned}
\tag{3.1}
$$

□

Theorem 3.4. *Let (X,τ) be a fuzzifying topological space, then*

(1) $c\gamma F(X)=1$, $c\gamma F(\varnothing)=1$;

(2) $c\gamma F(A\cup B)\ge c\gamma F(A)\wedge c\gamma F(B)$.

Proof. From Theorem 3.3 the proof is obtained. □

Theorem 3.5. *Let (X,τ) be a fuzzifying topological space, then*

(1) (a) $\models \tau\subseteq c\alpha\tau$; (b) $\models cP\tau\subseteq c\alpha\tau$; (c) $\models cS\tau\subseteq c\alpha\tau$; (d) $\models c\gamma\tau\subseteq cS\tau$; (e) $\models c\gamma\tau\subseteq cP\tau$; (f) $\models c\beta\tau\subseteq c\gamma\tau$; (g) $\models \tau\subseteq c\gamma\tau$.

(2) (a) $\models F\subseteq c\alpha F$; (b) $\models cPF\subseteq c\alpha F$; (c) $\models cSF\subseteq c\alpha F$; (d) $\models c\gamma F\subseteq cSF$; (e) $\models c\gamma F\subseteq cPF$; (f) $\models c\beta F\subseteq c\gamma F$; (g) $\models F\subseteq c\gamma F$.

Proof. From the properties of the interior and closure operations and [9, Theorem 2.2(3)],

(1) (a) $[A\in\tau]=[A\subseteq A^{\circ}]\le[A\cap A^{\circ-\circ}\subseteq A^{\circ}]=[A\in c\alpha\tau]$;

(b) $[A\in cP\tau]=[A\cap A^{-\circ}\subseteq A^{\circ}]\le[A\cap A^{\circ-\circ}\subseteq A^{\circ}]=[A\in c\alpha\tau]$;

(c) $[A\in cS\tau]=[A\cap A^{\circ-}\subseteq A^{\circ}]\le[A\cap A^{\circ-\circ}\subseteq A^{\circ}]=[A\in c\alpha\tau]$;

(d) $c\gamma\tau(A)=\inf_{x\in A}(1-\max(A^{\circ-}(x),A^{-\circ}(x))+A^{\circ}(x))\le\inf_{x\in A}(1-A^{\circ-}(x)+A^{\circ}(x))=cS\tau(A)$;

(e) $c\gamma\tau(A)=\inf_{x\in A}(1-\max(A^{\circ-}(x),A^{-\circ}(x))+A^{\circ}(x))\le\inf_{x\in A}(1-A^{-\circ}(x)+A^{\circ}(x))=cP\tau(A)$;

(f) $c\beta\tau(A)=\inf_{x\in A}(1-A^{-\circ-}(x)+A^{\circ}(x))\le\inf_{x\in A}(1-\max(A^{\circ-}(x),A^{-\circ}(x))+A^{\circ}(x))=c\gamma\tau(A)$;

(g) $[A\in\tau]=[A\subseteq A^{\circ}]\le[A\cap(A^{-\circ}\cup A^{\circ-})\subseteq A^{\circ}]=[A\in c\gamma\tau]$.

(2) The proof is obtained from (1). □

Remark 3.6. In crisp setting, that is, in case that the underlying fuzzifying topology is the ordinary topology, we have

$$\models A\in\gamma\tau\wedge A\in c\gamma\tau\longrightarrow A\in\tau. \tag{3.2}$$

Of course the implication "→" in (3.2) is either the Lukaciewicz's implication or the Boolean's implication since these implications are identical in crisp setting. But in fuzzifying setting the statement (3.2) may not be true as illustrated by the following counterexample.

Counterexample 3.7. Let $X=\{a,b,c\}$ and let τ be a fuzzifying topology on X defined as follows: $\tau(X)=\tau(\varnothing)=\tau(\{a\})=\tau(\{a,c\})=1$; $\tau(\{b\})=\tau(\{a,b\})=0$; and $\tau(\{c\})=\tau(\{b,c\})=1/8$. From the definitions of the interior and the closure of a subset of X and the interior and the closure of a fuzzy set of X we have the following: $\gamma\tau(\{a,b\}) = 7/8$, $c\gamma\tau(\{a,b\}) = 1/8$.

Theorem 3.8. *Let (X,τ) be a fuzzifying topological space.*

(1) $\models A \in \tau \to (A \in \gamma\tau \wedge A \in c\gamma\tau)$;

(2) *if* $[A \in \gamma\tau] = 1$ *or* $[A \in c\gamma\tau] = 1$, *then* $\models A \in \tau \leftrightarrow (A \in \gamma\tau \wedge A \in c\gamma\tau)$.

Proof. (1) This follows from Theorem 3.5(g) and Lemma 2.13(1).

(2) Assume that $[A \in \gamma\tau] = 1$, then for each $x \in A$, we have $\max(A^{\circ -}(x), A^{-\circ}(x)) = 1$ and so for each $x \in A$, $1 - \max(A^{\circ -}(x), A^{-\circ}(x)) + A^{\circ}(x) = A^{\circ}(x)$. Thus, $[A \in \gamma\tau] \wedge [A \in c\gamma\tau] = [A \in c\gamma\tau] = \inf_{x\in A}(1 - \max(A^{\circ -}(x), A^{-\circ}(x)) + A^{\circ}(x)) = \inf_{x\in A} A^{\circ}(x) = [A \in \tau]$. Now, assume that $[A \in c\gamma\tau] = 1$, then for each $x \in A$, $1 - \max(A^{\circ -}(x), A^{-\circ}(x)) + A^{\circ}(x) = 1$ and so for each $x \in A$, $\max(A^{\circ -}(x), A^{-\circ}(x)) = A^{\circ}(x)$.

Thus,

$$\begin{aligned} &[A \in \gamma\tau] \wedge [A \in c\gamma\tau] \\ &\quad = [A \in \gamma\tau] = \inf_{x\in A} \max\left(A^{\circ -}(x), A^{-\circ}(x)\right) = \inf_{x\in A} A^{\circ}(x) = [A \in \tau]. \end{aligned} \tag{3.3}$$

□

Theorem 3.9. *Let (X,τ) be a fuzzifying topological space. Then* $\models (A \in \gamma\tau \mathbin{\dot\wedge} A \in c\gamma\tau) \to A \in \tau$.

Proof.

$$\begin{aligned} &[A \in \gamma\tau \mathbin{\dot\wedge} A \in c\gamma\tau] \\ &\quad = \inf_{x\in A} \max\left(A^{\circ -}(x), A^{-\circ}(x)\right) \mathbin{\dot\wedge} \inf_{x\in A}\left(1 - \max\left(A^{\circ -}(x), A^{-\circ}(x)\right) + A^{\circ}(x)\right) \\ &\quad = \max\left(0, \inf_{x\in A} \max\left(A^{\circ -}(x), A^{-\circ}(x)\right) + \inf_{x\in A}\left(1 - \max\left(A^{\circ -}(x), A^{-\circ}(x)\right) + A^{\circ}(x)\right) - 1\right) \\ &\quad \le \inf_{x\in A} A^{\circ}(x) = [A \in \tau]. \end{aligned} \tag{3.4}$$

□

4. Fuzzifying $c\gamma$-neighborhood structure

Definition 4.1. Let $x \in X$. The $c\gamma$-neighborhood system of x, denoted by $c\gamma N_x \in \mathcal{F}(P(X))$, is defined as $c\gamma N_x(A) = \sup_{x\in B\subseteq A} c\gamma\tau(B)$.

Theorem 4.2. *A mapping $c\gamma N : X \to \mathcal{F}^N(P(X))$, $x \to c\gamma N_x$, where $\mathcal{F}^N(P(X))$ is the set of all normal fuzzy subsets of $P(X)$, has the following properties:*

(1) $\models A \in c\gamma N_x \to x \in A$;

(2) $\models A \subseteq B \to (A \in c\gamma N_x \to B \in c\gamma N_x)$;

(3) $\models A \in c\gamma N_x \wedge B \in c\gamma N_x \to A \cap B \in c\gamma N_x$.

Conversely, if a mapping $c\gamma N$ satisfies (2) and (3), then $c\gamma N$ assigns a fuzzifying topology on X which is denoted by $\tau_{c\gamma N} \in \mathcal{F}(P(X))$ and defined as

$$A \in \tau_{c\gamma N} := \forall x\ (x \in A \to A \in c\gamma N_x). \tag{4.1}$$

Proof. (1) If $[A \in c\gamma N_x] = \sup_{x\in H\subseteq A} c\gamma\tau(H) > 0$, then there exists $H_\circ$ such that $x \in H_\circ \subseteq A$. Now, we have $[x \in A] = 1$. Therefore, $[A \in c\gamma N_x] \le [x \in A]$ always holds.

(2) The proof is immediate.

(3) From Theorem 3.3(2), we have

$$\begin{aligned}[A\cap B\in c\gamma N_x] &= \sup_{x\in H\subseteq A\cap B} c\gamma\tau(H) = \sup_{\substack{x\in H_1\subseteq A,\\ x\in H_2\subseteq B}} c\gamma\tau(H_1\cap H_2)\\ &\geq \sup_{\substack{x\in H_1\subseteq A,\\ x\in H_2\subseteq B}} c\gamma\tau(H_1)\wedge c\gamma\tau(H_2)\\ &= \sup_{x\in H_1\subseteq A} c\gamma\tau(H_1)\wedge \sup_{x\in H_2\subseteq B} c\gamma\tau(H_2)\\ &= [A\in c\gamma N_x\wedge B\in c\gamma N_x].\end{aligned} \tag{4.2}$$

Conversely, we need to prove that $\tau_{c\gamma N}(A) = \inf_{x\in A} c\gamma N_x(A)$ is a fuzzifying topology. From [8, Theorem 3.2] and since $\tau_{c\gamma N}$ satisfies properties (2) and (3), $\tau_{c\gamma N}$ is a fuzzifying topology. □

Theorem 4.3. *Let (X,τ) be a fuzzifying topological space. Then $\models c\gamma\tau\subseteq\tau_{c\gamma N}$.*

Proof. Let $B\in P(X)$; $\tau_{c\gamma N}(B) = \inf_{x\in B} c\gamma N_x(B) = \inf_{x\in B}\sup_{x\in A\subseteq B} c\gamma\tau(A)\geq c\gamma\tau(B)$. □

5. Fuzzifying $c\gamma$-derived sets, fuzzifying $c\gamma$-closure, and fuzzifying $c\gamma$-interior

Definition 5.1. Let (X,τ) be a fuzzifying topological space. The fuzzifying $c\gamma$-derived set of A, denoted by $c\gamma\text{-d}\in\mathscr{F}(P(X))$, is defined as

$$c\gamma\text{-d}(A) = \inf_{B\cap(A-\{x\})=\varnothing}(1-c\gamma N_x(B)). \tag{5.1}$$

Lemma 5.2. $c\gamma\text{-d}(A)(x) = 1-c\gamma N_x((X\sim A)\cup\{x\})$.

Proof. From Theorem 4.2(2), we have

$$\begin{aligned}c\gamma\text{-d}(A) &= 1-\sup_{B\cap(A-\{x\})=\varnothing} c\gamma N_x(B)\\ &= 1-\sup_{B\subseteq((X\sim A)\cup\{x\})} c\gamma N_x(B)\\ &= 1-c\gamma N_x((X\sim A)\cup\{x\}).\end{aligned} \tag{5.2}$$

□

Theorem 5.3. *For any A, $\models A\in F\tau_{c\gamma N}\leftrightarrow c\gamma\text{-d}(A)\subseteq A$.*

Proof. From Lemma 5.2, we have

$$\begin{aligned}[c\gamma\text{-d}(A)\subseteq A] &= \inf_{x\in X\sim A}(1-c\gamma\text{-d}(A)(x)) = \inf_{x\in X\sim A} c\gamma N_x((X\sim A)\cup\{x\})\\ &= \inf_{x\in X\sim A} c\gamma N_x((X\sim A)) = [X\sim A\in\tau_{c\gamma N}] = [A\in F\tau_{c\gamma N}].\end{aligned} \tag{5.3}$$

□

Definition 5.4. Let (X,τ) be a fuzzifying topological space. The $c\gamma$-closure of A is denoted and defined as follows: $c\gamma\text{-cl}(A)(x) = \inf_{x\notin B\supseteq A}(1-c\gamma F(B))$.

Theorem 5.5. (1) $c\gamma\text{-cl}(A)(x) = 1 - c\gamma N_x(X \sim A)$;
(2) $\models c\gamma\text{-cl}(\varnothing) \equiv \varnothing$;
(3) $\models A \subseteq c\gamma\text{-cl}(A)$.

Proof. (1) $c\gamma\text{-cl}(A)(x) = \inf_{x\notin B\supseteq A}(1 - c\gamma F(B)) = \inf_{x\in X\sim B\subseteq X\sim A}(1 - c\gamma\tau(X \sim B)) = 1 - \sup_{x\in X\sim B\subseteq X\sim A} c\gamma\tau(X \sim B) = 1 - c\gamma N_x(X \sim A)$.

(2) $c\gamma\text{-cl}(\varnothing)(x) = 1 - c\gamma N_x(X \sim \varnothing) = 0$.

(3) It is clear that for any $A \in P(X)$ and any $x \in X$, if $x \notin A$, then $c\gamma N_x(A) = 0$. If $x \in A$, then $c\gamma\text{-cl}(A)(x) = 1 - c\gamma N_x(X \sim A) = 1 - 0 = 1$. Then $[A \subseteq c\gamma\text{-cl}(A)] = 1$. $\square$

Theorem 5.6. *For any x and A;*
(1) $\models c\gamma\text{-cl}(A) \equiv c\gamma\text{-d}(A) \cup A$;
(2) $\models x \in c\gamma\text{-cl}(A) \leftrightarrow \forall B\ (B \in c\gamma N_x \to A \cap B \neq \varphi)$;
(3) $\models A \equiv c\gamma\text{-cl}(A) \leftrightarrow A \in F\tau_{c\gamma N}$.

Proof. (1) Applying Lemma 5.2 and Theorem 5.5(3), we have

$$x \in c\gamma\text{-d}(A) \cup A = \max\left(1 - c\gamma N_x\left((X \sim A) \cup \{x\}\right), A(x)\right) = c\gamma\text{-cl}(A)(x). \tag{5.4}$$

(2) $[\forall B\ (B \in c\gamma N_x \to A \cap B \neq \varnothing)] = \inf_{B\subseteq X\sim A}(1 - c\gamma N_x(B)) = 1 - c\gamma N_x(X \sim A) = [x \in c\gamma\text{-cl}(A)]$.

(3) From Theorem 5.5(1), we have

$$\begin{aligned}[A \equiv c\gamma\text{-cl}(A)] &= \inf_{x\in X\sim A}\left(1 - c\gamma\text{-cl}(A)(x)\right) \\ &= \inf_{x\in X\sim A} c\gamma N_x(X \sim A) = [(X \sim A) \in F\tau_{c\gamma N}] = [A \in \tau_{c\gamma N}].\end{aligned} \tag{5.5}$$

$\square$

Theorem 5.7. *For any A and B, $\models B \doteq c\gamma\text{-cl}(A) \to B \in F\tau_{c\gamma N}$.*

Proof. If $[A \subseteq B] = 0$, then $[B \doteq c\gamma\text{-cl}(A)] = 0$. Now, we suppose $[A \subseteq B] = 1$, then we have $[B \subseteq c\gamma\text{-cl}(A)] = 1 - \sup_{x\in B\sim A} c\gamma N_x(X \sim A)$ and $[c\gamma\text{-cl}(A) \subseteq B] = \inf_{x\in X\sim B} c\gamma N_x(X \sim A)$. So,

$$[B \doteq c\gamma\text{-cl}(A)] = \max\left(0, \inf_{x\in X\sim B} c\gamma N_x(X \sim A) - \sup_{x\in B\sim A} c\gamma N_x(X \sim A)\right). \tag{5.6}$$

If $[B \doteq c\gamma\text{-cl}(A)] > t$, then $\inf_{x\in X\sim B} c\gamma N_x(X \sim A) > t + \sup_{x\in B\sim A} c\gamma N_x(X \sim A)$. For any $x \in X \sim B$, $\sup_{x\in C\subseteq X\sim A} c\gamma\tau(C) > t + \sup_{x\in B\sim A} c\gamma N_x(X \sim A)$, that is, there exists C_x such that $x \in C_x \subseteq X \sim A$ and $c\gamma\tau(C_x) > t + \sup_{x\in B\sim A} c\gamma N_x(X \sim A)$. Now, we want to prove that $C_x \subseteq X \sim B$. If not, then there exists $x' \in B \sim A$ such that $x' \in C_x$. Hence, we can obtain that $\sup_{x\in B\sim A} c\gamma N_x(X \sim A) \geq c\gamma N_{x'}(X \sim A) \geq c\gamma\tau(C_x) > t + \sup_{x\in B\sim A} c\gamma N_x(X \sim A)$. This is a contradiction. Therefore, $F\tau_{c\gamma N}(B) = \tau_{c\gamma N}(X \sim B) = \inf_{x\in X\sim B} c\gamma N_x(X \sim B) \geq \inf_{x\in X\sim B} c\gamma\tau(C_x) > t + \sup_{x\in B\sim A} c\gamma N_x(X \sim A) > t$. Since t is arbitrary, it holds that $[B \doteq \gamma\text{-cl}(A)] \leq [B \in F\tau_{c\gamma N}]$. $\square$

Definition 5.8. Let (X,τ) be a fuzzifying topological space. For any $A \subseteq X$, the $c\gamma$-interior of A is given as follows: $c\gamma\text{-int}(A)(x) = c\gamma N_x(A)$.

Theorem 5.9. *For any x, A, and B,*

(1) $\models B \in \tau_{c\gamma N} \wedge B \subseteq A \to B \subseteq c\gamma\text{-int}(A)$;

(2) $\models A \equiv c\gamma\text{-int}(A) \leftrightarrow A \in \tau_{c\gamma N}$;

(3) $\models x \in c\gamma\text{-int}(A) \leftrightarrow x \in A \wedge x \in (X \sim c\gamma\text{-d}(X \sim A))$;

(4) $\models c\gamma\text{-int}(A) \equiv X \sim c\gamma\text{-cl}(X \sim A)$;

(5) $\models B \dot{\equiv} c\gamma\text{-int}(A) \to B \in \tau_{c\gamma N}$;

(6) (a) $\models c\gamma\text{-int}(X) \equiv X$, (b) $\models c\gamma\text{-int}(A) \subseteq A$.

Proof. (1) If $B \nsubseteq A$, then $[B \in \tau_{c\gamma N} \wedge B \subseteq A] = 0$. If $B \subseteq A$, then

$$\begin{aligned}[B \subseteq c\gamma\text{-int}(A)] &= \inf_{x \in B} c\gamma\text{-int}(A)(x) \\ &= \inf_{x \in B} c\gamma N_x(A) \geq \inf_{x \in B} c\gamma N_x(B) \\ &= [B \in \tau_{c\gamma N}] = [B \in \tau_{c\gamma N} \wedge B \subseteq A].\end{aligned} \tag{5.7}$$

(2)

$$\begin{aligned}[A \equiv c\gamma\text{-int}(A)] &= \min\left(\inf_{x \in A} c\gamma\text{-int}(A)(x), \inf_{x \in X \sim A}(1 - c\gamma\text{-int}(A)(x))\right) \\ &= \inf_{x \in A} c\gamma\text{-int}(A)(x) = \inf_{x \in A} c\gamma N_x(A) = [A \in \tau_{c\gamma N}].\end{aligned} \tag{5.8}$$

(3) If $x \notin A$, then $[x \in c\gamma\text{-int}(A)] = 0 = [x \in A \wedge x \in (X \sim c\gamma\text{-d}(X \sim A))]$. If $x \in A$, then $[x \in c\gamma\text{-d}(X \sim A)] = 1 - c\gamma N_x(A \cup \{x\}) = 1 - c\gamma N_x(A) = 1 - c\gamma\text{-int}(A)(x)$, so that $[x \in A \wedge x \in (X \sim c\gamma\text{-d}(X \sim A))] = [x \in c\gamma\text{-int}(A)]$.

(4) It follows from Theorem 5.5(1).

(5) From (4) and Theorem 5.7, we have

$$[B \dot{\equiv} c\gamma\text{-int}(A)] = [X \sim B \dot{\equiv} c\gamma\text{-cl}(X \sim A)] \leq [X \sim B \in F\tau_{c\gamma N}] = [B \in \tau_{c\gamma N}]. \tag{5.9}$$

(6) (a) It is obtained from (4) above and from Theorem 5.5(2).

(b) It is obtained from (3) above. □

6. Fuzzifying $c\gamma$-continuous functions

Definition 6.1. Let (X, τ) and (Y, U) be two fuzzifying topological spaces. For any $f \in Y^X$, a unary fuzzy predicates $c\gamma C \in \mathcal{F}(Y^X)$, called $c\gamma$-continuity, is given as

$$c\gamma C(f) := \forall u \quad (u \in U \to f^{-1}(u) \in c\gamma\tau). \tag{6.1}$$

Definition 6.2. Let (X, τ) and (Y, U) be two fuzzifying topological spaces. For any $f \in Y^X$, we define the unary fuzzy predicates $\gamma_j \in \mathcal{F}(Y^X)$ where $j = 1, 2, \ldots, 5$ as follows:

(1) $\gamma_1(f) := \forall B\, (B \in F_Y \to f^{-1}(B) \in c\gamma F_X)$, where F_Y is the family of closed subsets of Y and $c\gamma F_X$ is the family of $c\gamma$-closed subsets of X;

(2) $\gamma_2(f) := \forall x \forall u\ (u \in N_{f(x)} \to f^{-1}(u) \in c\gamma N_x)$, where N is the neighborhood system of Y and $c\gamma N$ is the $c\gamma$-neighborhood system of X;

(3) $\gamma_3(f) := \forall x \forall u\ (u \in N_{f(x)} \to \exists v (f(v) \subseteq u \to v \in c\gamma N_x))$;

(4) $\gamma_4(f) := \forall A (f(c\gamma\text{-cl}_X(A)) \subseteq \text{cl}_Y(f(A)))$;

(5) $\gamma_5(f) := \forall B(c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B)))$.

Theorem 6.3. (1) $\models f \in c\gamma C \leftrightarrow f \in \gamma_1$;
(2) $\models f \in c\gamma C \to f \in \gamma_2$;
(3) $\models f \in \gamma_2 \leftrightarrow f \in \gamma_j$ *for* $j = 3,4,5$.

Proof. (1) We prove that $[f \in c\gamma C] = [f \in \gamma_1]$

$$\begin{aligned}
[f \in \gamma_1] &= \inf_{F \in P(Y)} \min\left(1, 1 - F_Y(F) + c\gamma F_X\left(f^{-1}(F)\right)\right) \\
&= \inf_{F \in P(Y)} \min\left(1, 1 - U(Y-F) + c\gamma\tau\left(X \sim f^{-1}(F)\right)\right) \\
&= \inf_{F \in P(Y)} \min\left(1, 1 - U(Y-F) + c\gamma\tau\left(f^{-1}(Y-F)\right)\right) \\
&= \inf_{u \in P(Y)} \min\left(1, 1 - U(u) + c\gamma\tau\left(f^{-1}(u)\right)\right) \\
&= [f \in c\gamma C].
\end{aligned} \tag{6.2}$$

(2) We prove that $\gamma_2(f) \geq c\gamma C(f)$. If $N_{f(x)}(u) \leq c\gamma N_x(f^{-1}(u))$, the result holds. Suppose $N_{f(x)}(u) > c\gamma N_x(f^{-1}(u))$. It is clear that if $f(x) \in A \subseteq u$ then $x \in f^{-1}(A) \subseteq f^{-1}(u)$. Then,

$$\begin{aligned}
N_{f(x)}(u) - c\gamma N_x\left(f^{-1}(u)\right) &= \sup_{f(x) \in A \subseteq u} U(A) - \sup_{x \in B \subseteq f^{-1}(u)} c\gamma\tau(B) \\
&\leq \sup_{f(x) \in A \subseteq u} U(A) - \sup_{f(x) \in A \subseteq u} c\gamma\tau\left(f^{-1}(A)\right) \\
&\leq \sup_{f(x) \in A \subseteq u} \left(U(A) - c\gamma\tau\left(f^{-1}(A)\right)\right).
\end{aligned} \tag{6.3}$$

So, $1 - N_{f(x)}(u) + c\gamma N_x(f^{-1}(u)) \geq \inf_{f(x) \in A \subseteq u}(1 - U(A) + c\gamma\tau(f^{-1}(A)))$ and thus

$$\begin{aligned}
\min\left(1, 1 - N_{f(x)}(u) + c\gamma N_x\left(f^{-1}(u)\right)\right) &\geq \inf_{f(x) \in A \subseteq u} \min\left(1, 1 - U(A) + c\gamma\tau\left(f^{-1}(A)\right)\right) \\
&\geq \inf_{v \in P(Y)} \min\left(1, 1 - U(v) + c\gamma\tau\left(f^{-1}(v)\right)\right) \\
&= c\gamma C(f).
\end{aligned} \tag{6.4}$$

Hence, $\inf_{x \in X} \inf_{u \in P(Y)} \min(1, 1 - N_{f(x)}(u) + c\gamma N_x(f^{-1}(u))) \geq [f \in c\gamma C]$.

(3) (a) We prove that $\models f \in \gamma_2 \leftrightarrow f \in \gamma_3$. Since $c\gamma N_x$ is monotonous (Theorem 4.2(2)), it is clear that $\sup_{v \in P(X), f(v) \subseteq u} c\gamma N_x(v) = \sup_{v \in P(x), v \subseteq f^{-1}(u)} c\gamma N_x(v) = c\gamma N_x(f^{-1}(u))$. Then,

$$\begin{aligned}
\gamma_3(f) &= \inf_{x \in X} \inf_{u \in P(Y)} \min\left(1, 1 - N_{f(x)}(u) + \sup_{v \in P(X), f(v) \subseteq u} c\gamma N_x(v)\right) \\
&= \inf_{x \in X} \inf_{u \in P(Y)} \min\left(1, 1 - N_{f(x)}(u) + c\gamma N_x\left(f^{-1}(u)\right)\right) = \gamma_2(f).
\end{aligned} \tag{6.5}$$

(b) We prove that $\models f \in \gamma_4 \leftrightarrow f \in \gamma_5$.

First, for each $B \in P(Y)$, there exists $A \in P(X)$ such that $f^{-1}(B) = A$ and $f(A) \subseteq B$. So, $[c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \geq [c\gamma\text{-cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))]$. Hence,

$$\begin{aligned}\gamma_5(f) &= \inf_{B\in P(Y)} [c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \\ &\geq \inf_{A\in P(X)} [c\gamma\text{-cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] = \gamma_4(f).\end{aligned} \tag{6.6}$$

Second, for each $A \in P(X)$, there exists $B \in P(Y)$ such that $f(A) = B$ and $f^{-1}(B) \supseteq A$. Hence, $[c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \leq [c\gamma\text{-cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))]$. Thus,

$$\begin{aligned}\gamma_4(f) &= \inf_{A\in P(X)} [c\gamma\text{-cl}_X(A) \subseteq f^{-1}(\text{cl}_Y(f(A)))] \\ &\geq \inf_{B\in P(Y), B=f(A)} [c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] \\ &\geq \inf_{B\in P(Y)} [c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))] = \gamma_5(f).\end{aligned} \tag{6.7}$$

(c) We prove that $\models f \in \gamma_5 \leftrightarrow f \in \gamma_2$; from Theorem 5.5(1),

$$\begin{aligned}\gamma_5(f) &= \forall B(c\gamma\text{-cl}_X(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_Y(B))) \\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-(1-c\gamma N_x(X \sim f^{-1}(B))) + 1 - N_{f(x)}(Y \sim B)) \\ &= \inf_{B\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(Y \sim B) + c\gamma N_x(X \sim f^{-1}(B))) \\ &= \inf_{u\in P(Y)} \inf_{x\in X} \min(1, 1-N_{f(x)}(u) + c\gamma N_x(f^{-1}(u))) = \gamma_2(f).\end{aligned} \tag{6.8}$$

□

Remark 6.4. In the following theorem, we indicate the fuzzifying topologies with respect to which we evaluate the degree to which f is continuous or $c\gamma C$-continuous. Thus, the symbols (τ, U)-$C(f)$, $(\tau_{c\gamma N}, U)$-$C(f)$, $(\tau, U_{c\gamma N})$-$c\gamma C(f)$, and so forth, will be understood.

Applying Theorems 3.5(g) and 4.3, one can deduce the following theorem.

Theorem 6.5. (1) $\models f \in (\tau, U_{c\gamma N})$-$C \to f \in (\tau, U)$-$C$;
(2) $\models f \in (\tau, U)$-$c\gamma C \to f \in (\tau_{c\gamma N}, U)$-$C$;
(3) $\models f \in (\tau, U)$-$C \to f \in (\tau, U)$-$c\gamma C$.

7. Decompositions of fuzzy continuity in fuzzifying topology

Theorem 7.1. *Let (X, τ) and (Y, U) be two fuzzifying topological spaces. For any $f \in Y^X$,*

$$\models C(f) \longrightarrow (\gamma C(f) \wedge c\gamma C(f)). \tag{7.1}$$

Proof. The proof is obtained from Lemma 2.13(1) and Theorem 3.5(g). □

Remark 7.2. In crisp setting, that is, in the case that the underlying fuzzifying topology is the ordinary topology, one can have $\models (\gamma C(f) \wedge c\gamma C(f)) \to C(f)$.

But this statement may not be true in general in fuzzifying topology as illustrated by the following counterexample.

Counterexample 7.3. Let (X,τ) be the fuzzifying topological space defined in Counterexample 3.7. Consider the identity function f from (X,τ) onto (X,σ), where σ is a fuzzifying topology on X defined as follows:

$$\sigma(A)=\begin{cases}1, & A\in\{X,\varnothing,\{a,b\}\},\\ 0, & \text{otherwise}.\end{cases}\tag{7.2}$$

Then, $7/8\wedge 1/8=\gamma C(f)\wedge c\gamma C(f)\nleq C(f)=0$.

Theorem 7.4. *Let (X,τ) and (Y,U) be two fuzzifying topological spaces. For any $f\in Y^X$,*

$$\models C(f)\longrightarrow(\gamma C(f)\longleftrightarrow c\gamma C(f)).\tag{7.3}$$

Proof. $[\gamma C(f)\to c\gamma C(f)]=\min(1,1-\gamma C(f)+c\gamma C(f))\geq\gamma C(f)\wedge c\gamma C(f)$. Also, $[c\gamma C(f)\to\gamma C(f)]=\min(1,1-c\gamma C(f)+\gamma C(f))\geq\gamma C(f)\wedge c\gamma C(f)$. Then from Theorem 7.1 we have $[\gamma C(f)\wedge c\gamma C(f)]\geq C(f)$ and so the result holds. □

Theorem 7.5. *Let (X,τ) and (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$. If $[\gamma\tau(f^{-1}(u))]=1$ or $[c\gamma\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models C(f)\leftrightarrow(\gamma C(f)\wedge c\gamma C(f))$.*

Proof. We need to prove that $C(f)=\gamma C(f)\wedge c\gamma C(f)$. Applying Theorem 3.8(2), we have

$$\begin{aligned}
&\gamma C(f)\wedge c\gamma C(f)\\
&\quad=\inf_{u\in P(Y)}\min(1,1-U(u)+\gamma\tau(f^{-1}(u)))\wedge\inf_{u\in P(Y)}\min(1,1-U(u)+c\gamma\tau(f^{-1}(u)))\\
&\quad=\inf_{u\in P(Y)}\min(1,(1-U(u)+\gamma\tau(f^{-1}(u)))\wedge(1-U(u)+c\gamma\tau(f^{-1}(u))))\\
&\quad=\inf_{u\in P(Y)}\min(1,1-U(u)+(\gamma\tau(f^{-1}(u))\wedge c\gamma\tau(f^{-1}(u))))\\
&\quad=\inf_{u\in P(Y)}\min(1,1-U(u)+\tau(f^{-1}(u)))=C(f).
\end{aligned}\tag{7.4}$$

□

Theorem 7.6. *Let (X,τ) and (Y,U) be two fuzzifying topological spaces and let $f\in Y^X$,*

(1) *if $[\gamma\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models\gamma C(f)\to(c\gamma C(f)\leftrightarrow C(f))$,*
(2) *if $[c\gamma\tau(f^{-1}(u))]=1$ for each $u\in P(Y)$, then $\models c\gamma C(f)\to(\gamma C(f)\leftrightarrow C(f))$.*

Proof. (1) Since $[\gamma\tau(f^{-1}(u))]=1$ and so $[f^{-1}(u)\subseteq((f^{-1}(u))^{\circ-}\cup(f^{-1}(u))^{-\circ})]=1$, then $[(f^{-1}(u)\cap((f^{-1}(u))^{\circ-}\cup(f^{-1}(u))^{-\circ}))\subseteq(f^{-1}(u))^{\circ}]=[f^{-1}(u)\subseteq(f^{-1}(u))^{\circ}]$.

Thus,

$$\begin{aligned} c\gamma C(f) &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+c\gamma\tau\left(f^{-1}(u)\right)\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[\left(f^{-1}(u)\cap\left(\left(f^{-1}(u)\right)^{\circ-}\cup\left(f^{-1}(u)\right)^{-\circ}\right)\right)\subseteq\left(f^{-1}(u)\right)^{\circ}\right]\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[f^{-1}(u)\subseteq\left(f^{-1}(u)\right)^{\circ}\right]\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\tau\left(f^{-1}(u)\right)\right) = C(f). \end{aligned} \tag{7.5}$$

(2) Since $[c\gamma\tau(f^{-1}(u))] = 1$, one can deduce that $((f^{-1}(u))^{\circ-}\cup(f^{-1}(u))^{-\circ}) = (f^{-1}(u))^{\circ}$. So,

$$\begin{aligned} \gamma C(f) &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\gamma\tau\left(f^{-1}(u)\right)\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[f^{-1}(u)\subseteq\left(\left(f^{-1}(u)\right)^{\circ-}\cup\left(f^{-1}(u)\right)^{-\circ}\right)\right]\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\left[f^{-1}(u)\subseteq\left(f^{-1}(u)\right)^{\circ}\right]\right) \\ &= \inf_{u\in P(Y)} \min\left(1, 1-U(u)+\tau\left(f^{-1}(u)\right)\right) = C(f). \end{aligned} \tag{7.6}$$

□

Theorem 7.7. *Let (X,τ), (Y,U), and (Z,V) be three fuzzifying topological spaces. For any $f\in Y^X$ and $g\in Z^Y$,*

(1) $\models c\gamma C(f)\to(C(g)\to c\gamma C(g\circ f))$;

(2) $\models C(g)\to(c\gamma C(g)\to c\gamma C(g\circ f))$.

Proof. (1) We prove that $[c\gamma C(f)]\le[(C(g)\to c\gamma C(g\circ f))]$. If $[C(g)]\le[c\gamma C(g\circ f)]$, then the result holds. If $[C(g)]>[c\gamma C(g\circ f)]$, then we have

$$\begin{aligned} [C(g)]-[c\gamma C(g\circ f)] &= \inf_{v\in P(Z)} \min\left(1, 1-V(v)+U\left(g^{-1}(v)\right)\right) \\ &\quad - \inf_{v\in P(Z)} \min\left(1, 1-V(v)+c\gamma\tau(g\circ f)^{-1}(v)\right) \\ &\le \sup_{v\in P(Z)} \left(U\left(g^{-1}(v)\right)-c\gamma\tau(g\circ f)^{-1}(v)\right) \\ &\le \sup_{u\in P(Y)} \left(U(u)-c\gamma\tau\left(f^{-1}(u)\right)\right). \end{aligned} \tag{7.7}$$

Therefore,

$$\begin{aligned} [C(g)\longrightarrow c\gamma C(g\circ f)] &= \min\left(1, 1-[C(g)]+[c\gamma C(g\circ f)]\right) \\ &\ge \inf_{u\in P(Y)} \min\left(1, 1-U(u)+c\gamma\tau\left(f^{-1}(u)\right)\right) = c\gamma C(f). \end{aligned} \tag{7.8}$$

(2) Since the conjunction $\dot\wedge$ is commutative, from (1) above, one can deduce that

$$\begin{aligned} [C(g)\longrightarrow(c\gamma C(f)\longrightarrow c\gamma C(g\circ f))] &= [\neg(C(g)\,\dot\wedge\, c\gamma C(f)\,\dot\wedge\,\neg c\gamma C(g\circ f))] \\ &= [\neg(c\gamma C(f)\,\dot\wedge\, C(g)\,\dot\wedge\,\neg c\gamma C(g\circ f))] \\ &= [c\gamma C(f)\longrightarrow(C(g)\longrightarrow c\gamma C(g\circ f))] = 1. \end{aligned} \tag{7.9}$$

□

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