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FUZZIFYING PROXIMITY AND STRONG FUZZIFYING UNIFORMITY

BY

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Abstract. We introduce the concept of a fuzzifying proximity and study some properties of fuzzifying proximities - in particular we show how a fuzzifying proximity on a set X naturally induces a fuzzifying topology on the same set. Besides, the concept of a strong fuzzifying uniformity (which is a certain modification of Ying's concept of a fuzzifying uniformity ([4])) is introduced. Some relations between fuzzifying proximities, strong fuzzifying uniformities and corresponding fuzzifying topologies are established. In particular, we show that the fuzzifying topology induced by the fuzzifying proximity and the fuzzifying topology induced by the strong fuzzifying uniformity are coincide.

Introduction

Ying [3] used the semantic method of continuous valued logic to initiate the study of the so-called fuzzifying topology and elementally develop topology in the framework of fuzzy sets from a completely different direction. Briefly speaking, a fuzzifying topology on a set X assigns to every crisp subset of X a certain degree of begin open, other than being definitely open or not. Furthermore, in 1993 Ying introduced the concept of a fuzzifying uniform spaces ([4]) and established some fundamental properties of it. In the framework of the fuzzifying topology we introduce and study the concept of fuzzifying proximity. Also, a fuzzifying topology induced by the fuzzifying proximity is introduced (Theorem 2.1). Since the α-level of the fuzzifying uniformity due to Ying may not be a classical uniformity (Counterexample 3.1), we introduce a type of fuzzifying

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uniformity stronger than the fuzzifying uniformity due to Ying which satisfies that the α-level of it is a classical uniformity (Theorem 3.1). Besides, a fuzzifying topology indued by the strong fuzzifying uniformity is introduce (Theorem 3.2). Some relations between fuzzifying proximities, strong fuzzifying uniformities and corresponding fuzzifying topologies are established. In particular, we show that the fuzzifying topology indued by the fuzzifying proximity and the fuzzifying topology induced by the strong fuzzifying uniformity are coincide (Theorem 3.5).

1. Preliminaries

In this section we present the fuzzy logical and corresponding set theoretical notations due to Ying [3]. For any formulae φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval [0, 1]. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. The original formulae of fuzzy logical and corresponding set theoretical notations are:

(1) (a) $[\alpha] = \alpha (\alpha \in [0,1])$; (b) $[\varphi \wedge \psi] := \min([\varphi], [\psi])$;
(c) $[\varphi \to \psi] := \min(1, 1 - [\varphi] + [\psi])$.
(2) If $\tilde{A} \in \mathcal{F}(X)$, then $[x \in \tilde{A}] := \tilde{A}(x)$.
(3) If X is the universe of discourse, $[\forall x \varphi(x)] := \inf_{x\in X} [\varphi(x)]$.

In addition the following derived formulae are given,

(1) $[\neg\varphi] := [\varphi \to 0] := 1 - [\varphi]$; (2) $[\varphi \vee \psi] := [\neg(\neg\varphi \wedge \neg\psi)] := \max([\varphi], [\psi])$;
(3) $[\varphi \leftrightarrow \psi] := [\varphi \to \psi] \wedge [\psi \to \varphi]$; (4) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x\in X} [\varphi(x)]$;
(5) if $\tilde{A}$, $\tilde{B} \in \mathcal{F}(X)$, then
(a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x (x \in \tilde{A} \to x \in \tilde{B})] := \inf_{x\in X} \min(1,\ 1 - \tilde{A}(x) + \tilde{B}(x))$;
(b) $[\tilde{A} \equiv \tilde{B}] := [\tilde{A} \subseteq \tilde{B}] \wedge [\tilde{B} \subseteq \tilde{A}]$,
where $\mathcal{F}(X)$ is the family of all fuzzy sets in X.

We do often not distinguish the connectives and their truth value functions and state strictly our results on formalization as Ying did ([3, 4]). We give now the following definitions and results in fuzzifying topology which are useful in the rest of the present paper.

Definition 1.1.([3]) Let X be a universe of discourse, $P(X)$ the family of all subsets of X and $\tau \in \mathcal{F}(P(X))$, i.e., $\tau : P(X) \to [0,1]$ satisfy the following conditions:

(1) $\models X \in \tau$, $\models \phi \in \tau$;
(2) for any A, B, $\models (A \in \tau) \wedge (B \in \tau) \to A \cap B \in \tau$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\models \forall\lambda(\lambda \in \Lambda \to A_\lambda \in \tau) \to (\bigcup_{\lambda\in\Lambda} A_\lambda) \in \tau$.
Then τ is called a fuzzifying topology and (X, τ) is a fuzzifying topological spaces.

Remark 1.1.([3]) The conditions in Definiton 1.1 may be rewritten respectively as follows:
(1) $\tau(X) = 1$, $\tau(\phi) = 1$;
(2) for any A, B, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for any $\{A_\lambda : \lambda \in \Lambda\}$, $\tau(\bigcup_{\lambda\in\Lambda} A_\lambda) \geq \wedge_{\lambda\in\Lambda}\tau(A_\lambda)$.

Definition 1.2.([3]) The family of fuzzifying closed sets, denoted by $F \in \mathcal{F}(P(X))$, is defined as $A \in F := X \sim A \in \tau$, where $X \sim A$ is the complement of A.

Definition 1.3.([3]) A fuzzy set $\tilde{A} \in \mathcal{F}(X)$ is called normal if there exists $x \in X$ such that $\tilde{A}(x) = 1$.

Definition 1.4.([4]) Let X be a set and $u \in \mathcal{F}^N(P(X \times X))$, i.e., $u : P(X \times X) \to [0, 1]$ and normal. If for any U, $V \subseteq X \times X$,
(FU1) $\models (U \in u) \to (\Delta \subseteq U)$;
(FU2) $\models (U \in u) \to (U^{-1} \in u)$;
(FU3) $\models (U \in u) \to (\exists V)((V \in u) \wedge (V \circ V \subseteq U))$;
(FU4) $\models (U \in u) \wedge (V \in u) \to (U \cap V \in u)$;
(FU5) $\models (U \in u) \wedge (U \subseteq V) \to (V \in u)$,
then u is called a fuzzifying uniformity and (X, u) is called a fuzzifying uniform space.

2. Fuzzifying Proximity Space

In this section the concept of fuzzifying proximity spaces is established and some of its properties are discussed. Also, a fuzzifying topology induced by the fuzzifying proximity is introduced.

Definition 2.1. Let X be a set and $\delta \in \mathcal{F}(P(X) \times P(X))$, i.e., $\delta : P(X) \times P(X) \to [0, 1]$. If for any A, B, $C \in P(X)$, the following axioms are satisfied:

(FP1) $\models \neg(X, \phi) \in \delta$;

(FP2) $\models (A, B) \in \delta \leftrightarrow (B, A) \in \delta$;

(FP3) $\models (A, B \cup C) \in \delta \leftrightarrow (A, B) \in \delta \vee (A, C) \in \delta$;

(FP4) for every A, $B \subseteq X$ there exists $C \subseteq X$ such that

$\models ((A, C) \in \delta \vee (B, X \sim C) \in \delta) \rightarrow (A, B) \in \delta$;

(FP5) $\models \{x\} \equiv \{y\} \leftrightarrow (\{x\}, \{y\}) \in \delta$,

then δ is called a fuzzifying proximity on X and (X, δ) is called a fuzzifying proximity space.

Theorem 2.1. *Let (X, δ) be a fuzzifying proximity space. Then, we have*

(1) $\models (A, B) \in \delta \wedge B \subseteq C \rightarrow (A, C) \in \delta$;

(2) $\models (A \cap B) \neq \phi \rightarrow (A, B) \in \delta$;

(3) $\models \neg\delta(A, \phi)$.

Proof. (1) If $[B \subseteq C] = 0$, then the result holds. Suppose that $[B \subseteq C] = 1$. Then, we have $\delta(A, C) = \delta(A, B \cup C) = \delta(A, B) \vee \delta(A, C) \geq \delta(A, B)$.

(2) If $[A \cap B \neq \phi] = 0$, then the result holds. Suppose $[A \cap B \neq \phi] = 1$. Then there exists $x \in A \cap B$. Thus, we obtain $1 = \delta(\{x\}, \{x\}) = \delta(\{x\}, \{x\}) \wedge [\{x\} \subseteq A] \leq \delta(\{x\}, A) = \delta(A, \{x\}) = \delta(A, \{x\}) \wedge [\{x\} \subseteq B] = \delta(A, B)$.

(3) $\delta(A, \phi) = \delta(\phi, A) = \delta(\phi, A) \wedge [A \subseteq X] \leq \delta(\phi, X) = 0$. Hence, $[\neg(A, \phi) \in \delta] = 1$.

Proposition 2.1. *For every $\alpha \in (0, 1]$, δ_a is a proximity on X, where δ_α is the α-level of δ, i.e., $\delta_\alpha = \{(A, B) : \delta(A, B) \geq \alpha\}$.*

Proof. Let α be a fixed value in (0,1].

(P1) Since by (FP1) $\delta(X, \phi) = 0$, then we have $\delta(X, \phi) < \alpha$. So, $(X, \phi) \notin \delta_\alpha$.

(P2) Suppose $(A, B) \in \delta_\alpha$. Then $\delta(A, B) \geq \alpha$ and by (FP2) $\delta(A, B) = \delta(B, A) \geq \alpha$. Hence, we obtain $(B, A) \in \delta_\alpha$.

(P3) Using (FP3) we have $(A, B \cup C) \in \delta_\alpha$ if and only if $\delta(A, B \cup C) \geq \alpha$ if and only if $\delta(A, B) \vee \delta(A, C) \geq \alpha$ if and only if $\delta(A, B) \geq \alpha$ or $\delta(A, C) \geq \alpha$ if and only if $(A, B) \in \delta_\alpha$ or $(A, C) \in \delta_\alpha$.

(P4) Let $(A, B) \notin \delta_\alpha$. Then we have $\delta(A, B) < \alpha$ and by (FP4) there exists $C \in P(X)$ such that $\delta(A, B) \geq \delta(A, C) \vee \delta(B, X \sim C)$. Hence, $\delta(A, C) \vee$

$\delta(B, X \sim C) < \alpha$ which implies that $\delta(A, C) < \alpha$ and $\delta(B, X \sim C) < \alpha$. So, $(A, C) \notin \delta_\alpha$ and $(B, X \sim C) \notin \delta_\alpha$.

(P5) Suppose $x = y$. Then we have $[\{x\} \equiv \{y\}] = 1$. So, by (FP5) we have $\delta(\{x\}, \{y\}) = 1$. Hence, $(\{x\}, \{y\}) \in \delta_c$.

Definition 2.2. Let (X, δ) be a fuzzifying proximity space. For each $\alpha \in (0, 1]$ and $A \subseteq X$, we define the interior operation induced by δ_α, denoted by $\text{int}_{\delta_\alpha}$:$P(X) \to P(X)$, as follows:

$$\text{int}_{\delta_\alpha}(A) = \bigcup_{B \in P(X), (B, X \sim A) \neq \delta_\alpha} B,$$

Proposition 2.2. *For every $\alpha \in (0, 1]$, the family $\tau_{\delta_\alpha} = \{A : A \subseteq X$ and $\text{int}_{\delta_\alpha}(A) = A\}$ is a topology on X.*

Proof. Let α be a fixed value in (0,1].

(1) Since $\text{int}_{\delta_\alpha}(X) = \bigcup_{B \in P(X), (B, \phi) \neq \delta_\alpha} B = X$ and $\text{int}_{\delta_\alpha}(\phi) = \bigcup_{B \in P(X), (B, X) \notin \delta_\alpha} B = \phi$, then $X \in \tau_{\delta_\alpha}$ and $\phi \in \tau_{\delta_\alpha}$.

(2) Let $A, C \in \tau_{\delta_\alpha}$. Then, we obtain that

$$A \cap C = \text{int}_{\delta_\alpha}(A) \cap \text{int}_{\delta_\alpha}(C) = \bigcup_{B \in P(X), (B, X \sim A) \notin \delta_\alpha} B \cap \bigcup_{G \in P(X), (G, X \sim C) \notin \delta_\alpha} G$$
$$= \bigcup_{B \cap G \in P(X), (B \cap G, X \sim A \cap C) \notin \delta_\alpha} (B \cap G) = \bigcup_{H \in P(X), (H, X \sim A \cap C) \notin \delta_\alpha} H = \text{int}_{\delta_\alpha}(A \cap C).$$

Hence, $A \cap C \in \tau_{\delta_\alpha}$.

(3) Let $\{A_\lambda : \lambda \in \Lambda\} \subseteq \tau_{\delta_\alpha}$. Now, $\bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} \text{int}_{\delta_\alpha}(A_\lambda) \subseteq \text{int}_{\delta_\alpha}(\bigcup_{\lambda \in \Lambda} A_\lambda)$, because $\text{int}_{\delta_\alpha}$ is monotone (Indeed, If $A \subseteq C$, then $\text{int}_{\delta_\alpha}(A) = \bigcup_{B \in P(X), (B, X \sim A) \notin \delta_\alpha} B \subseteq \bigcup_{B \in P(X), (B, X \sim C) \notin \delta_\alpha} B = \text{int}_{\delta_\alpha}(C)$). Also, $\text{int}_{\delta_\alpha}(\bigcup_{\lambda \in \Lambda} A_\lambda) \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$ because $\text{int}_{\delta_\alpha}(A) = \bigcup_{B \in P(X), (B, X \sim A) \notin \delta_\alpha} B = \bigcup_{B \in P(X), B \cap (X \sim A) = \phi} B = \bigcup_{B \in P(X), B \subseteq A} B = A$ for any $A \in P(X)$. Hence $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau_{\delta_\alpha}$.

Theorem 2.1. *Let (X, δ) be a fuzzifying proximity space. The mapping $\tau_\delta : P(X) \to [0, 1]$ defined by: $\tau_\delta(A) = \sup_{\alpha \in (0,1), A \in \tau_{\delta_\alpha}} \alpha$ is a fuzzifying topology and is called the fuzzifying topology induced by the fuzzifying proximity δ.*

Proof. (1) $\tau_\delta(X) = \sup_{\alpha\in(0,1], X\in\tau_{\delta_\alpha}} \alpha = 1, \tau_\delta(\phi) = \sup_{\alpha\in(0,1],\phi\in\tau_{\delta_\alpha}} \alpha = 1.$

(2) $\tau_\delta(A\cap B) = \sup_{\alpha\in(0,1], A\cap B\in\tau_{\delta_\alpha}} \alpha \geq \sup_{\alpha\in(0,1], A\in\tau_{\delta_\alpha}, B\in\tau_{\delta_\alpha}} \alpha = \sup_{\alpha\in(0,1], A\in\tau_{\delta_\alpha}} \alpha \wedge \sup_{\alpha\in(0,1], B\in\tau_{\delta_\alpha}} \alpha = \tau_\delta(A)\wedge\tau_\delta(B).$

(3) Let $\{A_\lambda : \lambda\in\Lambda\}\subseteq P(X)$. Then we have

$$\tau_\delta(\bigcup_{\lambda\in\Lambda} A_\lambda) = \sup_{\alpha\in(0,1],\bigcup_{\lambda\in\Lambda} A_\lambda\in\tau_{\delta_\alpha}} \alpha \geq \sup_{\alpha\in(0,1], A_{\lambda\in\tau_{\delta_\alpha}},\lambda\in\Lambda} \alpha = \inf_{\lambda\in\Lambda} \sup_{\alpha\in(0,1], A_\lambda\in\tau_{\delta_\alpha}} \alpha$$
$$= \inf_{\lambda\in\Lambda} \tau_\delta(A_\lambda).$$

Definition 2.3. Let (X,δ) and (Y,δ^*) be a fuzzifying proximity spaces. A unary fuzzy predicate $PC\in\mathcal{F}(Y^X)$, i.e., $PC : Y^X\to[0,1]$ and Y^X is the set of all functions from X to Y, is called a fuzzifying proximal continuity and is defined as follows: $f\in PC := \forall(A,B)((A,B)\in\delta\to(f(A),f(B))\in\delta^*)$. Intuitively, the degree to which f is proximal continuous is

$$PC(f) = \inf_{(A,B)\subseteq P(X)\times P(X)} \min(1, 1-\delta(A,B)+\delta^*(f(A),f(B))).$$

Theorem 2.2.

(1) $\models \forall f(f\in PC\to\forall\alpha(\alpha\in(0,1]\to(1-\alpha)\vee H^\alpha(f)))$, *where*

$$H^\alpha(f) = \left\{\begin{array}{ll} 1 & iff : (X,\delta_\alpha)\to(Y,\delta^*_{PC(f)-(1-\alpha)}) \textit{ is proximal continuous when} \\ & PC(f)\geq 1-\alpha \\ 0 & \textit{otherwise} \end{array}\right\};$$

(2) *if* $PC(f)=1$, *then* $\models\forall f(f\in PC\leftrightarrow\forall\alpha(\alpha\in(0,1]\to(1-\alpha)\vee H^\alpha(f)))$*;*

(3) $\models\forall f(\forall\alpha(\alpha\in(0,1]\to(1-\alpha)\wedge H^\alpha(f))\to f\in PC)$*.*

Proof. (1) Suppose $PC(f)=\gamma$. Then for each $(A,B)\in P(X)\times P(X)$ such that $(A,B)\in\delta_a$ we have, $1-\delta(A,B)+\delta^*(f(A),f(B))\geq\gamma$, $\delta^*(f(A),f(B))\geq \gamma+\delta(A,B)-1\geq\gamma+\alpha-1=\gamma-(1-\alpha)$. If $\gamma<1-\alpha$, then $\gamma\leq(1-\alpha)\vee H^\alpha(f)$ and if $\gamma\geq 1-\alpha$, then $(1-\alpha)\vee H^\alpha(f)=(1-\alpha)\vee 1\geq\gamma$. Hence, $\inf_{\alpha\in(0,1]}((1-\alpha)\vee H^\alpha(f))\geq\gamma$.

(2) If $PC(f)=1$, then $PC(f)\geq\inf_{\alpha\in(0,1]}((1-\alpha)\vee H^\alpha(f))$. Using (1) above we obtain that $PC(f)=\inf_{\alpha\in(0,1]}((1-\alpha)\vee H^\alpha(f))$.

(3) If $1-\alpha \leq \gamma$, then $(1-\alpha)\vee H^{\alpha}(f) \leq \gamma$ and if $1-\alpha > \gamma$, then $(1-\alpha)\wedge 0 \leq \gamma$. Hence, $\inf_{\alpha\in[0,1)}((1-\alpha)\wedge H^{\alpha}(f)) \leq \gamma$.

Corollary 2.1. *$PC(f) = 1$ if and only if for each $\alpha \in (0,1]$, $f : (X, \delta_{\alpha}) \to (Y, \delta^{*}_{\alpha})$ is proximal continuous.*

Proof. From Theorem 2.2 (2), $PC(f) = 1 = \inf_{\alpha\in(0,1]}((1-\alpha)\vee H^{\alpha}(f))$. Then for each $\alpha \in (0,1]$, $(1-\alpha)\vee H^{\alpha}(f) = 1$ implies $H^{\alpha}(f) = 1$. This implies that $f : (X, \delta_{\alpha}) \to (Y, \delta^{*}_{\alpha})$ is proximal continuous. Conversely, if for each $\alpha \in (0,1]$ then function $f : (X, \delta_{\alpha}) \to (Y, \delta^{*}_{\alpha})$ is proximal continuous, then $H^{\alpha}(f) = 1$. So $PC(f) = \inf_{\alpha\in(0,1]}((1-\alpha)\vee 1) = 1$.

3. Strong Fuzzifying Uniform Spaces

In this section we give a conuterexample to illustrate that there exists some α-level of the fuzzifying uniformity in sense of Ying [4] which is not a uniformity. Then we introduce and study a type of fuzzifying uniformity stonger than Ying's one with respect to which each α-level is a uniformity. Besides, a fuzzifying topology induced by the strong fuzzifying uniformity is introduced. Further, the connections between the fuzzifying proximity, the strong fuzzifying uniformity and the corresponding fuzzifying topologies are considered.

Counterexample 3.1. Consider the subsets U and V_{α} of $[0,1]\times[0,1]$ defined as follows: $U = [0,1]\times[0,1] - \{(0,1)\}$ and $V_{\alpha} = \Delta \cup \{(\eta,\delta) : \eta \in (\alpha,1)$ and $\delta \in (0,1]\}\cup\{(\delta,\zeta) : \delta \in (0,1]$ and $\zeta \in (\alpha,1)\}$ for each $\alpha \in (0,1)$. Define the fuzzifying uniformity in sense of Ying on [0,1] as follows:

$$u(H) = \left\{\begin{array}{ll} 1, & \text{if } H \in \{[0,1]\times[0,1], U, U^{-1}, U\cap U^{-1}\}; \\ 1-\alpha, & \text{if } H = V_{\alpha} \text{ or if } H \supseteq V_{\alpha} \text{ and } H \not\supseteq V_{\alpha^*} \text{ where } \alpha > \alpha^*; \\ 0, & \text{otherwise} \end{array}\right\}.$$

We note that $V_{\alpha} = V_{\alpha}^{-1}$, if $\alpha_1 \geq \alpha_2$. Then $V_{\alpha_1} \subseteq V_{\alpha_2}$ and $V_{\alpha} \subseteq U\cap U^{-1}$. Also, for each $\alpha \in (0,1)$, $V_{\alpha}\circ V_{\alpha} \subseteq U\cap U^{-1}$ because $(0,1) \notin V_{\alpha}\circ V_{\alpha}$ and $(1,0) \notin V_{\alpha}\circ V_{\alpha}$. For each $H \supseteq V_{\alpha}$ and $H \not\supseteq V^{*}_{\alpha}$, where $\alpha < \alpha^{*}$ we have $u(H) = u(H^{-1})$. Now, the 1-level of u denoted by $u_1 = \{[0,1]\times[0,1], U, U^{-1}, U\cap U^{-1}\}$. There is no subset $G \in u_1$ such that $G\circ G \subseteq U$. So u_1 is not uniformity.

In the following we introduce a fuzzifying uniform space stronger than Ying's one.

Definition 3.1. Let X be a set and $u \in \mathcal{F}^{\mathcal{N}}(P(X\times X))$, i.e., $u : P(X\times X) \to [0,1]$ and normal. If for any $U, V \subseteq X \times X$,

(FU1) $\models (U \in u) \to (\Delta \subseteq U)$;

(FU2) $\models (U \in u) \to (U^{-1} \in u)$;

(FU3)* $\models (U \in u) \to (\exists V)((V \in H \subset P(X \times X))) \wedge (V \in u) \wedge (V \circ V \subseteq U)$;
where $\subset$ stands for "a finite subset of";

(FU4) $\models (U \in u) \wedge (V \in u) \to (U \cap V \in u)$;

(FU5) $\models (U \in u) \wedge (U \subseteq V) \to (V \in u)$,

then u is called strong fuzzifying uniformity and (X, u) is called strong fuzzifying uniform space.

Remark 3.1. In Counterexample 3.1 u is a fuzzifying uniformity in the sense of Ying but it is not a strong fuzzifying uniformity.

Theorem 3.1. *Let (X, u) be a strong fuzzifying uniform space. Then for each $\alpha \in (0, 1]$, the α-level of u denoted by u_α is a classical uniformity on X.*

Proof. Let $\alpha \in (0, 1]$. Since u is normal, then there exists $U \in P(X \times X)$ such that $u(U) = 1 \geq \alpha$. Thus $U \in u_\alpha$ and so $u_\alpha \neq \phi$.

(U1) Let $U \in u_\alpha$. Then $u(U) \geq \alpha$ and so from (FU1), we have $[\Delta \subseteq U] = 1$.

(U2) Let $U \in u_\alpha$. Then from condition (FU2), $u(U^{-1}) \geq u(U) \geq \alpha$. Then $U^{-1} \in u_\alpha$.

(U3) Let $U \in u_\alpha$. Then from condition (FU3)*, $\sup_{V \in H \subset P(X\times X)}(u(V) \wedge [V \circ V \subseteq U]) \geq u(U) \geq \alpha$.

Then there exists $V_\bullet \in H$ such that $u(V_\bullet) \wedge [V_\bullet \circ V_\bullet \subseteq U] \geq \alpha$. Hence, we obtain that $V_\bullet \in u_\alpha$ and $[V_\bullet \circ V_\bullet \subseteq U] = 1$.

(U4) Let $U, V \in u_\alpha$. From condition (FU4), $u(U \cap V) \geq u(U) \wedge u(V) \geq \alpha$. So, $U \cap V \in u_\alpha$.

(U5) Let $U \in u_\alpha$ and $[U \subseteq V] = 1$. Using (FU5) $u(V) \geq u(V) \wedge [U \subseteq V] = u(U) \geq \alpha$. So, $V \in u_\alpha$.

Theorem 3.2. *Let* (X,u) *be a strong fuzzifying uniform space. The fuzzy set* $\tau_u \in \mathcal{F}(P(X))$*, defined by:* $\tau_u(A) = \sup_{\alpha\in(0,1], A\in\tau_{u\alpha}} \alpha$*, is a fuzzifying topology. It is called the fuzzifying topology induced by the strong fuzzifying uniformity* u*.*

Proof. (1) Since $X, \phi \in \tau_{u_\alpha}$ for each $\alpha \in (0,1]$, then we have $\tau_u(X) = \sup_{\alpha\in(0,1], X\in\tau_{u_\alpha}} \alpha = 1$ and $\tau_u(\phi) = \sup_{\alpha\in(0,1],\phi\in\tau_{u_\alpha}} \alpha = 1$.

(2) $\tau_u(A\cap B) = \sup_{\alpha\in(0,1], A\cap B\in\tau_{u_\alpha}} \alpha \geq \sup_{\alpha\in(0,1], A\in\tau_{u_\alpha}} \alpha \wedge \sup_{\alpha\in(0,1], B\in\tau_{u_\alpha}} \alpha = \tau_u(A)\wedge\tau_u(B)$.

(3) $\tau_u(\bigcup_{\lambda\in\Lambda} A_\lambda) = \sup_{\alpha\in(0,1], \bigcup_{\lambda\in\Lambda} A_\lambda\in\tau_{u_\alpha}} \alpha \geq \sup_{\alpha\in(0,1], A_\lambda\in\tau_{u_\alpha}, \lambda\in\Lambda} \alpha = \inf_{\lambda\in\Lambda} \sup_{\alpha\in(0,1], A_\lambda\in\tau_{u_\alpha}} \alpha = \inf_{\lambda\in\Lambda} \tau_u(A_\lambda)$.

Theorem 3.3. *Let* δ_{u_α} *be the proximity induced by the uniformity* u_α*. Then the mapping* $\delta_u : P(X)\times P(X) \to [0,1]$*, defined by* $\delta_u(A,B) = \sup_{\alpha\in(0,1],(A,B)\in\delta_{u_\alpha}} \alpha$*, is a fuzzifying proximity. It is called the fuzzifying proximity induced by the strong fuzzifying uniformity* u*.*

Proof. (FP1) $\delta_u(X,\phi) = \sup_{\alpha\in(0,1],(X,\phi)\in\delta_{u_\alpha}} \alpha = 0$.

(FP2) It is clear that $\delta_u(A,B) = \delta_u(B,A)$, because $(A,B) \in \delta_{u_\alpha}$ if and only if $(B,A) \in \delta_{u_\alpha}$.

(FP3) $\delta_u(A, B\cup C) = \sup_{\alpha\in(0,1],(A,B\cup C)\in\delta_{u_\alpha}} \alpha = \sup_{\alpha\in(0,1],(A,B)\in\delta_{u_\alpha} \text{ or } (A,C)\in\delta_{u_\alpha}} \alpha$ $= \sup_{\alpha\in(0,1](A,B)\in\delta_{u_\alpha}} \alpha \vee \sup_{\alpha\in(0,1],(A,C)\in\delta_{u_\alpha}} \alpha = \delta_u(A,B)\vee\delta_u(A,C)$.

(FP4) Assume $(A,B) \notin \delta_{u_\alpha}$. Then there exists $C \in P(X)$ such that $(A,C) \notin \delta_{u_\alpha}$ and $(B, X\sim C) \notin \delta_{u_\alpha}$. So, for every $C \in P(X)$ such that $(A,C) \in \delta_{u_\alpha}$ or $(B, X\sim C) \in \delta_{u_\alpha}$ implies $(A,B) \in \delta_{u_\alpha}$. Therefore, $\delta_u(A,B) = \sup_{\alpha\in(0,1],(A,B)\in\delta_{u_\alpha}} \alpha \geq \sup_{\alpha\in(0,1],(A,C)\in\delta_{u_\alpha} \text{ or } (B,X\sim C)\in\delta_{u_\alpha}} \alpha = \sup_{\alpha\in(0,1],(A,C)\in\delta_{u_\alpha}} \alpha \vee \sup_{\alpha\in(0,1],(B,X\sim C)\in\delta_{u_\alpha}} \alpha = \delta_u(A,C)\wedge\delta_u(B, X\sim C)$.

(FP5) Suppose $[\{x\} \equiv \{y\}] = 1$. Then $x = y$. So, $(\{x\},\{y\}) \in \delta_{u_\alpha}$ for any $\alpha \in (0,1]$. Therefore, $\delta_u(\{x\},\{y\}) = \sup_{\alpha\in(0,1],(\{x\},\{y\})\in\delta_{u_\alpha}} \alpha = 1$. Again, assume $[\{x\} \equiv \{y\}] = 0$. Then $x \neq y$. So, $(\{x\},\{y\}) \notin \delta_{u_\alpha}$ for any $\alpha \in (0,1]$. Hence, $\delta_u(\{x\},\{y\}) = \sup_{\alpha\in(0,1],(\{x\},\{y\})\in\delta_{u_\alpha}} \alpha = 0$.

Theorem 3.4. *The mapping* $u_\delta : P(X \times X) \to [0,1]$ *defined by* $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha$ *is a strong fuzzifying uniformity. It is called the strong fuzzifying uniformity induced by the fuzzifying proximity* δ.

Proof. (FU1) If $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha > 0$, then there exists $\alpha_\bullet \in (0,1]$ such that $\alpha_\bullet > 0$ and $U \in u_{\delta_{\alpha_\bullet}}$. So, $\Delta \subseteq U$. Hence, $[\Delta \subseteq U] = 1 \geq u_\delta(U)$.

(FU2) $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha = \sup_{\alpha\in(0,1],U^{-1}\in u_{\delta_\alpha}} \alpha = u_\delta(U^{-1})$.

(FU3)* $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha$. Now for each $U \in u_{\delta_\alpha}$ there exists $V \in P(X \times X)$ such that $V \circ V \subseteq U$ and $V \in u_{\delta_\alpha}$. So, $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha \leq \sup_{V\in H\subset P(X\times X),V\circ V\subseteq U} \sup_{\alpha\in(0,1],V\in u_{\delta_\alpha}} \alpha = \sup_{V\in H\subset P(X\times X),V\circ V\subseteq U} u_\delta(V)$.

(FU4) $u_\delta(U \cap V) = \sup_{\alpha\in(0,1],U\cap V\in u_{\delta_\alpha}} \alpha \geq \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha},V\in u_{\delta_\alpha}} \alpha \geq \sup_{\alpha\in(0,1],V\in u_{\delta_\alpha}} \alpha \wedge \sup_{\alpha\in(0,1],V\in u_{\delta_\alpha}} \alpha = u_\delta(U) \wedge u_\delta(V)$.

(FU5) Suppose $[U \subseteq V] = 1$. Then $u_\delta(U) = \sup_{\alpha\in(0,1],U\in u_{\delta_\alpha}} \alpha \leq \sup_{\alpha\in(0,1],V\in u_{\delta_\alpha}} \alpha = u_\delta(V)$.

Theorem 3.5. *Let* (X,u) *be a strong fuzzifying uniform space,* (X,δ) *be a fuzzifying proximity space,* u_δ *be the strong fuzzifying uniformity induced by* δ *and* δ_u *be the fuzzifying proximity induced by* u. *Then* $\models \tau_{u_\delta} \equiv \tau_{\delta_u}$.

Proof. $\tau_{u_\delta}(A) = \sup_{\alpha\in(0,1],A\in\tau_{u_{\delta_\alpha}}} \alpha = \sup_{\alpha\in(0,1],A\in\tau_{\delta_{u_\alpha}}} \alpha = \tau_{\delta_u}(A)$.

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