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On Bibasic Humbert Hypergeometric Function Φ_1

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Abstract

The main aim of this work is to derive the q-recurrence relations, q-partial derivative relations and summation formula of bibasic Humbert hypergeometric function Φ_1 on two independent bases q and q_1 of two variables and some developments formulae, believed to be new, by using the conception of q-calculus.

Keywords: q-calculus; bibasic Humbert hypergeometric functions; q-derivative.

1 Introduction

The basic analogous of Appell functions were defined and studied by Jackson [7, 8]. In [18, 20, 19], Srivastava defined and investigated bibasic q-Appell functions. In that paper we had shown that an Humbert confluent hypergeometric series with two basis can be reduced to an expression with only one base. We had also given an expansion formula for the Humbert hypergeometric function. For most of the notations and definitions needed in this work, the reader is referred to the papers by Agarwal et al. [1], Andrews [2], Thomas Ernst [3, 4], Ghany [6], Sahai and Verma [13], Jain [10], Purohit [11], Verma and Sahai [22], Yadav et al. [23], Srivastava and Shehata [21], and to the book by Gasper and Rahman [5]. Recently, Shehata investigated the (p, q)-Humbert, (p, q)-Bessel functions in [15, 16]. In [14, 17], Shehata introduced for basic Horn functions H_3, H_4, H_6 and H_7. Motivated by the aforementioned work, we derive the q-recurrence relations, q-derivatives formulas, q-partial derivative relations, and summation formula for these bibasic Humbert confluent hypergeometric function Φ_1 on two independent bases q and q_1. We think these results are not found in the literature.

For $0 < |q| < 1$, $q \in \mathbb{C}$, the q-shifted factorial $(q^a; q)_k$ is defined as

$$\begin{aligned}(q^a;q)_k &= \begin{cases} \prod_{r=0}^{k-1}(1-q^{a+r}), & k \geq 1;\\ 1, & k=0,\end{cases}\\ &= \begin{cases} (1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1}), & k\in\mathbb{N}, q^a \in \mathbb{C}\setminus\{1,q^{-1},q^{-2},\dots,q^{1-k}\};\\ 1, & k=0, a\in\mathbb{C},\end{cases}\end{aligned} \tag{1}$$

where $\mathbb{C}$ and $\mathbb{N}$ are the sets of complex and natural numbers.

Let $\mathbf{f}$ be a function defined on a subset of the real or complex plane. We define the q-derivative also referred to as the Jackson derivative [9] as follows

$$D_q\mathbf{f}(x) = \frac{\mathbf{f}(x) - \mathbf{f}(qx)}{(1-q)x}, x \neq 0. \tag{2}$$

For $n \geq 0$ and $k \geq 0$, the relation is given by (Rainville [12])

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{A}(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathfrak{A}(k,n-k). \tag{3}$$

For $0 < |q| < 1$, $0 < |q_1| < 1$, $q, q_1 \in \mathbb{C}$, we define the bibasic Humbert hypergeometric function Φ_1 on two independent bases q and q_1 as follows

$$\Phi_1(q^a, q_1^b; q^c; q, q_1, x, y) = \sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k, q^c \neq 1, q^{-1}, q^{-2}, \dots. \tag{4}$$

2 Main Results

Here we show that these results can be utilized to derive certain formulae results with the basic analogue of the bibasic Humbert confluent hypergeometric function Φ_1 on two independent bases q and q_1 of two variables.

Theorem 2.1. *The following relations for Φ_1 are true*

$$\begin{aligned}&\Phi_1(q^{a+1}, q_1^b; q^c; q, q_1, x, y)\\&\quad= \Phi_1(q^a, q_1^b; q^c; q, q_1, x, y) + \frac{q^a y}{1-q^c}\Phi_1(q^{a+1}, q_1^b; q^{c+1}; q, q_1, x, y)\\&\qquad+ \frac{q^a}{1-q^a}\Phi_1(q^a, q_1^b; q^c; q, q_1, x, qy) - \frac{q^a}{1-q^a}\Phi_1(q^a, q_1^b; q^c; q, q_1, qx, qy), q^a, \qquad q^c \neq 1,\end{aligned} \tag{5}$$

$$\begin{aligned}&\Phi_1(q^{a+1}, q_1^b; q^c; q, q_1, x, y)\\&\quad= \frac{1}{1-q^a}\Phi_1(q^a, q_1^b; q^c; q, q_1, x, y) - \frac{q^a}{1-q^a}\Phi_1(q^a, q_1^b; q^c; q, q_1, qx, y)\\&\qquad+ \frac{q^a y}{1-q^c}\Phi_1(q^{a+1}, q_1^b; q^{c+1}; q, q_1, qx, y), q^a, \qquad q^c \neq 1,\end{aligned} \tag{6}$$

$$\begin{aligned}&\Phi_1(q^a, q_1^b; q^{c-1}; q, q_1, x, y)\\&\quad= \Phi_1(q^a, q_1^b; q^c; q, q_1, x, y) + \frac{q^{c-1}(1-q^a)y}{(1-q^{c-1})(1-q^c)}\Phi_1(q^{a+1}, q_1^b; q^{c+1}; q, q_1, x, y)\\&\qquad+ \frac{q^{c-1}}{1-q^{c-1}}\Phi_1(q^a, q_1^b; q^c; q, q_1, x, qy) - \frac{q^{c-1}}{1-q^{c-1}}\Phi_1(q^a, q_1^b; q^c; q, q_1, qx, qy), q^c, \qquad q^{c-1} \neq 1,\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\Phi_1(q^a, q_1^b; q^{c-1}; q, q_1, x, y)\\&\quad= \frac{1}{1-q^{c-1}}\Phi_1(q^a, q_1^b; q^c; q, q_1, x, y) - \frac{q^{c-1}}{1-q^{c-1}}\Phi_1(q^a, q_1^b; q^c; q, q_1, qx, y)\\&\qquad+ \frac{q^{c-1}(1-q^a)y}{(1-q^{c-1})(1-q^c)}\Phi_1(q^{a+1}, q_1^b; q^{c+1}; q, q_1, qx, y), q^c, \qquad q^{c-1} \neq 1.\end{aligned} \tag{8}$$

Proof. We first prove identity (5). Using the relations

$$\begin{aligned}(q^a; q)_{\ell+k+1} &= (1-q^a)(q^{a+1}; q)_{\ell+k},\\(q^c; q)_{\ell+k+1} &= (1-q^c)(q^{c+1}; q)_{\ell+k},\end{aligned}$$

we have

$$
\begin{aligned}
&\Phi_1(q^{a+1},q_1^b;q^c;q,q_1,x,y)-\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\\
&\quad=q^a\sum_{\ell,k=0}^{\infty}\left[\frac{1-q^{\ell+k}}{1-q^a}\right]\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k,\\
&\quad=\frac{q^a}{1-q^a}\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k+1}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k+1}(q_1;q_1)_\ell(q;q)_k}x^\ell y^{k+1}+\frac{q^a}{1-q^a}\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell (qy)^k\\
&\qquad-\frac{q^a}{1-q^a}\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}(qx)^\ell (qy)^k,\\
&\quad=\frac{q^a y}{1-q^c}\Phi_1(q^{a+1},q_1^b;q^{c+1};q,q_1,x,y)+\frac{q^a}{1-q^a}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,qy)\\
&\qquad-\frac{q^a}{1-q^a}\Phi_1(q^a,q_1^b;q^c;q,q_1,qx,qy).
\end{aligned}
$$

In a similar way to the proof of equation (5), we obtain the relations (6)-(8) □

Theorem 2.2. *The relations for Φ_1 hold true*

$$
\begin{aligned}
&(1-q^a)\Phi_1(q^{a+1},q_1^b;q^c;q,q_1,x,y)\\
&\quad=(1-q^{a+1-c})\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)+q^{a+1-c}(1-q^{c-1})\Phi_1(q^a,q_1^b;q^{c-1};q,q_1,x,y),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,x,y)\\
&\quad=q^c\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,qx,qy)+(1-q^c)\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\Phi_1(q^{a-1},q_1^b;q^c;q,q_1,x,y)\\
&\quad=q^{a-1}\Phi_1(q^{a-1},q_1^b;q^c;q,q_1,qx,qy)+(1-q^{a-1})\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&(1-q^a)\Phi_1(q^{a+1},q_1^b;q^{c+1};q,q_1,x,y)\\
&\quad=(1-q^{a-c})\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,x,y)+q^{a-c}(1-q^c)\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y),
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
&q^{-c}(1-q^a)\Phi_1(q^{a+1},q_1^b;q^{c+1};q,q_1,x,y)\\
&\quad=(1-q^{a-c})\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,qx,qy)+q^{-c}(1-q^c)\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y).
\end{aligned}
\tag{13}
$$

Proof. From the definition of (4) and using the relationship

$$
\frac{1-q^{c-1}}{(q^{c-1};q)_{\ell+k}}=\frac{1-q^{c+\ell+k-1}}{(q^c;q)_{\ell+k}}=\frac{1}{(q^c;q)_{\ell+k-1}},
$$

we get

$$
\begin{aligned}
&q^{a+1-c}(1-q^{c-1})\Phi_1(q^a,q_1^b;q^{c-1};q,q_1,x,y)\\
&=\sum_{\ell,k=0}^{\infty}\frac{(q^{a+1-c}-q^{a+\ell+k})(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k,\\
&=\sum_{\ell,k=0}^{\infty}\frac{(1-q^{a+\ell+k})(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k-\sum_{\ell,k=0}^{\infty}\frac{(1-q^{a+1-c})(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k,\\
&=(1-q^a)\Phi_1(q^{a+1},q_1^b;q^c;q,q_1,x,y)-(1-q^{a+1-c})\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y).
\end{aligned}
$$

Similarly, we obtain the results (10)-(13). □

Theorem 2.3. *The following relations hold true*

$$
\begin{aligned}
(1-q_1^b)\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)=&\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\\
&-q_1^b\Phi_1(q^a,q_1^b;q^c;q,q_1,q_1x,y),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)=&\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\\
&+\frac{xq_1^b(1-q^a)}{(1-q^c)}\Phi_1(q^{a+1},q_1^{b+1};q^{c+1};q,q_1,x,y),\qquad q^c\neq 1,
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)=&\Phi_1(q^a,q_1^b;q^c;q,q_1,q_1x,y)\\
&+\frac{x(1-q^a)}{(1-q^c)}\Phi_1(q^{a+1},q_1^{b+1};q^{c+1};q,q_1,x,y),\qquad q^c\neq 1.
\end{aligned}
\tag{16}
$$

Proof. Using the relation

$$
(q_1^b;q_1)_{\ell+1}=(1-q_1^b)(q_1^{b+1};q_1)_\ell=(1-q_1^{b+\ell})(q_1^b;q_1)_\ell
$$

and (4), we have

$$
\begin{aligned}
(1-q_1^b)\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)=&\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^\ell y^k\\
&-q_1^b\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}(q_1x)^\ell y^k,\\
=&\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)-q_1^b\Phi_1(q^a,q_1^b;q^c;q,q_1,q_1x,y).
\end{aligned}
\tag{17}
$$

In a similar manner, we get the subsequent results (15)-(16). □

Theorem 2.4. *The q-derivatives relations for Φ_1 hold*

$$D^r_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)=\frac{(q^a;q)_r(q_1^b;q_1)_r}{(1-q_1)^r(q^c;q)_r}\Phi_1(q^{a+r},q_1^{b+r};q^{c+r};q,q_1,x,y), \tag{18}$$

$$D^s_{y,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)=\frac{(q^a;q)_s}{(1-q)^s(q^c;q)_s}\Phi_1(q^{a+s},q_1^{b};q^{c+s};q,q_1,x,y), \tag{19}$$

and

$$D^r_{x,q_1}D^s_{y,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)=\frac{(q^a;q)_{r+s}(q_1^b;q_1)_r}{(1-q_1)^r(1-q)^s(q^c;q)_{r+s}}\Phi_1(q^{a+r+s},q_1^{b+r};q^{c+r+s};q,q_1,x,y). \tag{20}$$

Proof. Using the q-derivative in (3), we get

$$\begin{aligned}D_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)&=\sum_{\ell,k=0}^{\infty}\frac{1-q_1^\ell}{1-q_1}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^{\ell-1}y^k,\\&=\sum_{\ell,k=0}^{\infty}\frac{1}{1-q_1}\frac{(q^a;q)_{\ell+k+1}(q_1^b;q_1)_{\ell+1}}{(q^c;q)_{\ell+k+1}(q_1;q_1)_\ell(q;q)_k}x^{\ell}y^k,\\&=\frac{(1-q^a)(1-q_1^b)}{(1-q_1)(1-q^c)}\sum_{\ell,k=0}^{\infty}\frac{(q^{a+1};q)_{\ell+k}(q_1^{b+1};q_1)_\ell}{(q^{c+1};q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^{\ell}y^k,\\&=\frac{(1-q^a)(1-q_1^b)}{(1-q_1)(1-q^c)}\Phi_1(q^{a+1},q_1^{b+1};q^{c+1};q,q_1,x,y).\end{aligned} \tag{21}$$

and

$$\begin{aligned}D_{y,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)&=\sum_{\ell=0,k=1}^{\infty}\frac{1}{1-q}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_{k-1}}x^{\ell}y^{k-1},\\&=\frac{(1-q^a)}{(1-q)(1-q^c)}\sum_{\ell,k=0}^{\infty}\frac{(q^{a+1};q)_{\ell+k}(q_1^{b};q_1)_\ell}{(q^{c+1};q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^{\ell}y^k,\\&=\frac{(1-q^a)}{(1-q)(1-q^c)}\Phi_1(q^{a+1},q_1^{b};q^{c+1};q,q_1,x,y).\end{aligned} \tag{22}$$

Iterating this q-derivative on Φ_1 for r-times and s-times, we obtain (18) and (19). The q-derivatives given by (20) can be easily obtained. □

Theorem 2.5. *The q-differential recursion relations for Φ_1 hold*

$$xD_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y) = \frac{(1-q_1^b)}{(1-q_1)q_1^b}\Big[\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)-\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\Big], \tag{23}$$

$$xD_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y) = \frac{(1-q_1^b)}{(1-q_1)}\Big[\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y)-\Phi_1(q^a,q_1^b;q^c;q,q_1,q_1x,y)\Big], \tag{24}$$

$$D_{y,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y) = \frac{1}{1-q}\Big[\frac{(1-q^{a-c})}{(1-q^c)}\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,x,y)+q^{a-c}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\Big], q^c\neq 1, \tag{25}$$

and

$$D_{y,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y) = \frac{1}{1-q}\Big[\frac{q^c(1-q^{a-c})}{(1-q^c)}\Phi_1(q^a,q_1^b;q^{c+1};q,q_1,qx,qy)+\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)\Big], q^c\neq 1. \tag{26}$$

Proof. From (21) and (16), we get (23). Similarly, we obtain the results (24)-(26). □

Theorem 2.6. *the following relations for Φ_1 hold:*

$$(1-q^{c-1})\Phi_1(q^a,q_1^b;q^{c-1};q,q_1,x,xy) = (1-q^{c-1})\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy)+(1-q)q^{c-1}xD_{x,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy), \tag{27}$$

$$(1-q^a)\Phi_1(q^{a+1},q_1^b;q^c;q,q_1,x,xy) = (1-q^a)\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy)+(1-q)q^a xD_{x,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy), \tag{28}$$

$$(1-q_1^b)\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y) = (1-q_1^b)\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)+q_1^b(1-q_1)xD_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y), \tag{29}$$

and

$$(1-q_1^b)\Phi_1(q^a,q_1^{b+1};q^c;q,q_1,x,y) = (1-q_1)xD_{x,q_1}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)+(1-q_1^b)\Phi_1(q^a,q_1^b;q^c;q,q_1,q_1x,y). \tag{30}$$

Proof. From (4), we have

$$\begin{aligned}
&(1-q^{c-1})\Phi_1(q^a,q_1^b;q^{c-1};q,q_1,x,xy)\\
&=\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k-1}(q_1;q_1)_\ell(q;q)_k}x^{\ell+k}y^k,\\
&=\sum_{\ell,k=0}^{\infty}\frac{(1-q^{c+\ell+k-1})(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^{\ell+k}y^k,\\
&=\sum_{\ell,k=0}^{\infty}\frac{(1-q^{c-1}+q^{c-1}(1-q^{\ell+k}))(q^a;q)_{\ell+k}(q_1^b;q_1)_\ell}{(q^c;q)_{\ell+k}(q_1;q_1)_\ell(q;q)_k}x^{\ell+k}y^k,\\
&=(1-q^{c-1})\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy)+(1-q)q^{c-1}xD_{x,q}\Phi_1(q^a,q_1^b;q^c;q,q_1,x,xy).
\end{aligned}\tag{31}$$

Similarly, we obtain the results (28)-(30). □

Theorem 2.7. *The summation formula for Φ_1 hold true*

$$\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)=\sum_{\ell}^{\infty}\frac{(q^a;q)_\ell(q_1^b;q_1)_\ell}{(q^c;q)_\ell(q_1;q_1)_\ell}x^\ell\ {}_2\Phi_1(q^{a+\ell},0;q^{c+\ell};q,y).\tag{32}$$

Proof. We start with the definition of Φ_1 and using (3), we have

$$\begin{aligned}
\Phi_1(q^a,q_1^b;q^c;q,q_1,x,y)&=\sum_{\ell,k=0}^{\infty}\frac{(q^a;q)_\ell(q^{a+\ell};q)_k(q_1^b;q_1)_\ell}{(q^c;q)_\ell(q^{c+\ell};q)_k(q_1;q_1)_\ell(q;q)_k}x^\ell y^k,\\
&=\sum_{\ell=0}^{\infty}\frac{(q^a;q)_\ell(q_1^b;q_1)_\ell}{(q^c;q)_\ell(q_1;q_1)_\ell}x^\ell\sum_{k=0}^{\infty}\frac{(q^{a+\ell};q)_k}{(q^{c+\ell};q)_k(q;q)_k}y^k,\\
&=\sum_{\ell=0}^{\infty}\frac{(q^a;q)_\ell(q_1^b;q_1)_\ell}{(q^c;q)_\ell(q_1;q_1)_\ell}x^\ell\ {}_2\Phi_1(q^{a+\ell},0;q^{c+\ell};q,y).
\end{aligned}\tag{33}$$

□

3 Concluding remarks

We conclude with the remark that the technique used here can be employed to yield a variety of interesting results involving the relations of the family for the bibasic Humbert hypergeometric function Φ_1 of two variables. As with the bibasic Humbert hypergeometric function Φ_1, these recursion formulas may find applications in numerous branches of mathematics, mathematical physics, engineering, and associated areas of study.

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References

[1] P. Agarwal, S. Jain & J. Choi (2017). Certain q-series identities. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, *111*(1), 139–146. https://doi.org/10.1007/s13398-016-0281-7.

[2] G. E. Andrews (1972). Summations and transformations for basic Appell series. *Journal of the London Mathematical Society*, *2*(4), 618–622.

[3] T. Ernst (2003). A method for q-calculus. *Journal of Nonlinear Mathematical Physics*, *10*(4), 487–525.

[4] T. Ernst (2012). *A comprehensive treatment of q-calculus*. Springer Science & Business Media, Berlin, Germany.

[5] G. Gasper & M. Rahman (2004). *Basic hypergeometric series*. Cambridge University Press, Cambridge, United Kingdom.

[6] H. A. Ghany (2009). q-derivative of basic hypergeomtric series with respect to parameters. *International Journal of Mathematical Analysis*, *3*(33), 1617–1632.

[7] F. H. Jackson (1942). On basic double hypergeometric functions. *The Quarterly Journal of Mathematics*, *1*, 69–82.

[8] F. Jackson (1944). Basic double hypergeometric functions (II). *The Quarterly Journal of Mathematics*, *1*, 49–61.

[9] F. Jackson (1951). Basic integration. *The Quarterly Journal of Mathematics*, *2*(1), 1–16.

[10] V. Jain (1980). Some expansions involving basic hypergeometric functions of two variables. *Pacific Journal of Mathematics*, *91*(2), 349–361.

[11] S. Purohit (2009). Some recurrence relations for the generalized basic hypergeometric functions. *Bulletin of Mathematical Analysis and Applications*, *1*(1), 22–29.

[12] E. D. Rainville (1960). *Special functions*. Macmillan Publisher, New York.

[13] V. Sahai & A. Verma (2018). Recursion formulas for q-hypergeometric and q-Appell series. *Communications of the Korean Mathematical Society*, *33*(1), 207–236. https://doi.org/10.4134/CKMS.c170121.

[14] A. Shehata (2020). On basic Horn hypergeometric functions h_3 and h_4. *Advances in Difference Equations*, *2020*, Article ID 595. https://doi.org/10.1186/s13662-020-03056-3.

[15] A. Shehata (2020). On the (p,q)-Bessel functions from the view point of the generating function method. *Journal of Interdisciplinary Mathematics*, *23*(7), 1435–1448. https://doi.org/10.1080/09720502.2020.1761046.

[16] A. Shehata (2020). On the-(p,q)-Humbert functions from the view point of the generating function method. *Journal of Function Spaces*, *2020*, Article ID 4794571. https://doi.org/10.1155/2020/4794571.

[17] A. Shehata (2021). On q-Horn hypergeometric functions h_6 and h_7. *Axioms*, *10*(4), Article ID 336. https://doi.org/10.3390/axioms10040336.

[18] B. Srivasatva (1995). A note on certain bibasic q-Appell and Lauricella series. *Glasnik Matematicki*, *30*, 29–36.

[19] B. Srivastava (1998). Some relations between contiguous bibasic q-Appell functions. *Notas de Mathematica*, *173*, 1–6.

[20] B. Srivastava (2007). Bibasic q-Appell series connected with Ramanujan's continued fractions. *Tamkang Journal of Mathematics*, *38*(1), 51–56.

[21] H. Srivastava & A. Shehata (2018). A family of new q-Extensions of the Humbert functions. *European Journal of Mathematical Sciences*, *4*(1), 13–26.

[22] A. Verma & V. Sahai (2020). Some recursion formulas for q-Lauricella series. *Afrika Matematika*, *31*(3–4), 643–686.

[23] R. Yadav, S. Purohit & V. Vyas (2010). On transformations involving generalized basic hypergeometric function of two variables. *Revista Técnica Ingeniería Universidad del Zulia*, *33*(2), 176–182.