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# Common Fixed Point Theorems For Four Weakly Compatible Mappings In Menger Spaces

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**Abstract**

In this paper, we show that some results of Sharma and Deshpande [19] are not valid, we also give supporting example. Finally, we consider the concept of weakly compatible mappings to improve the main result of Pathak [9].

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*Key words and phrases:* T-norm; Menger space; Common fixed point; Compatible maps; Weak-compatible maps

## 1 Introduction

Jungck [3] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Sessa [18] defined weak commutativity and proved common fixed point theorem for weakly commuting mappings. Further, Jungck [4] introduced the notion of compatibility, which is more general than that of weak commutativity, then various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors. In 1998, Jungck and Rhoades [5] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true. Finally, Sharma and Choubey [17] proved common fixed point theorems for weakly compatible mappings on a complete Menger spaces without using the condition of continuity.

Menger [7] introduced the notion of probabilistic metric spaces, which is a generalization of metric spaces. The study of these spaces was performed

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extensively by Schweizer and Sklar [15]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis.

Recently, some fixed point theorems in Menger spaces have been proved by many authors, Radu [10, 11], Stojakovic [13, 14], Dedeic and Sarapa [2], Murthy and Stojakovic [1] and others under various contractive conditions.

In this paper, we point out that[[19], Theorem (3.1)] is false unless some additional conditions are imposed. An example is given to justify our claim. Finally, we improve the results of Pathak [9] by replacing the condition of compatibility of type (P) by weak-compatibility in complete Menger space.

# 2 Preliminaries

The following definitions appear in [15].

**Definition 2.1** *A real valued function* f *on the set of real numbers is called a distribution function if it is non-decreasing, left continuous with* $\inf_{u\in R}\ \mathrm{f}(u) = 0$ *and* $\sup_{u\in R} \mathrm{f}(u) = 1$.

*The Heaviside function* $H$ *is a distribution function defined by*

$$H(u) = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0. \end{cases}$$

**Definition 2.2** *Let* $X$ *be a non-empty set and let* $L$ *denote the set of all distribution functions defined on* $X$. *An ordered pair* $(X, \Im)$ *is called a probabilistic metric space where* $\Im$ *is a mapping from* $X \times X$ *into* $L$ *if for every pair* $(x, y) \in X$ *a distribution function* $F(x, y)$ *or* $F_{x,y}$ *assumed to satisfy the following conditions:*

**(1)** $F_{x,y}(u) = H(u)$ *if and only if* $x = y$.

**(2)** $F_{x,y}(u) = F_{y,x}(u)$.

**(3)** $F_{x,y}(0) = 0$.

**(4)** *If* $F_{x,y}(u_1) = 1$ *and* $F_{y,z}(u_2) = 1$, *then* $F_{x,z}(u_1 + u_2) = 1$ *for all* $x, y, z$ *in* $X$ *and* $u_1, u_2 \geq 0$.

Every metric space $(X, d)$ can be realized as a probabilistic metric space by taking $\Im : X \times X \to L$ defined by $F_{x,y}(u) = H(u - d(x, y))$ for all $x$, $y$ in $X$. So probabilistic metric spaces provide a wider framework than that of the metric spaces and are better suited in many situations.

**Definition 2.3** *A t-norm is a function* $t : [0, 1] \times [0, 1] \to [0, 1]$ *satisfying the following conditions:*

**(T1)** $t(a, 1) = a$, $t(0, 0) = 0$,

**(T2)** $t(a, b) = t(b, a)$,

**(T3)** $t(c, d) \geq t(a, b)$ *for* $c \geq a, d \geq b$,

**(T4)** $t(t(a, b), c) = t(a, t(b, c))$ *for all* $a, b, c \in [0, 1]$.

**Definition 2.4** *A Menger probabilistic metric space* $(X, \Im, t)$ *is an ordered triple, where* $t$ *is a t-norm, and* $(X, \Im)$ *is a probabilistic metric space satisfying the following condition:* $F_{x,z}(u_1 + u_2) \geq t(F_{x,y}(u_1), F_{y,z}(u_2))$ *for all* $x, y, z$ *in* $X$ *and* $u_1, u_2 \geq 0$.

**Definition 2.5** *A sequence* $\{x_n\}$ *in* $(X, \Im, t)$ *is said to converge to a point* $x \in X$ *if for every* $\varepsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* $N(\varepsilon, \lambda)$ *such that* $F_{x_n,x}(\varepsilon) > 1 - \lambda$ *for all* $n \geq N(\varepsilon, \lambda)$.

**Definition 2.6** *A sequence* $\{x_n\}$ *in* $(X, \Im, t)$ *is said to be a Cauchy sequence if for every* $\varepsilon > 0$ *and* $\lambda > 0$*, there exists a positive integer* $N(\varepsilon, \lambda)$ *such that* $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ *for all* $n, m \geq N(\varepsilon, \lambda)$.

**Definition 2.7** *A Menger space* $(X, \Im, t)$ *with continuous t-norm is said to be complete if every Cauchy sequence in* $X$ *converges to a point in* $X$.

**Definition 2.8** *A coincidence point (or simply coincidence) of two mappings is a point in their domain having the same image point under both mappings.*

*Formally, given two mappings* $\mathbf{f}, \mathbf{g} : \boldsymbol{X} \to \boldsymbol{Y}$ *we say that a point* $x$ *in* $X$ *is a coincidence point of* f *and* g *if* $f(x) = g(x)$.

**Definition 2.9 ([5])** *A pair of mappings* $A$ *and* $S$ *is called a weakly compatible pair if they commute at a coincidence point.*

**Example 2.1** *Define the pair* $A$, $S : [0, 3] \to [0, 3]$ *by*

$$A(x) = \begin{cases} x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases} \qquad , \qquad S(x) = \begin{cases} 3 - x, & x \in [0,1) \\ 3, & x \in [1,3]. \end{cases}$$

Then for any $x \in [1, 3]$, $ASx = SAx$, showing that $A$, $S$ are weakly compatible maps on $[0, 3]$.

**Definition 2.10** *An PM-space* $(X, \Im)$ *is said to be a simple space if and only if there exists a metric* $d$ *on* $X$ *and a distribution function* $G$ *satisfying* $G(0) = 0$*, such that for every* $x$*,* $y$ *in* $X$

$$F_{x,y}(u) = \begin{cases} G\left(\frac{u}{d(x,y)}\right), & x \neq y \ ; \\ H(u), & x = y \qquad \text{for all } x, y \in X. \end{cases}$$

*Furthermore, we say that* $(X, \Im)$ *is the simple space generated by the metric space* $(X, d)$ *and the distribution function* $G$.

**Theorem 2.1 ([15])** *A simple space is a Menger space under any choice of* $T$ *satisfying* $(T1)$, $(T2)$, $(T3)$ *and* $(T4)$.

# 3 Common Fixed Point Theorems

Rashwan and Hedar [12] proved the following Theorem:

**Theorem 3.1** *Let A, B, S and T be self mappings on a complete Menger space $(X, F, t)$, where $t$ is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, satisfying the conditions:*

**(3.1.1)** $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

**(3.1.2)** *there exists $k \in (0, 1)$ such that*
$F_{Ap,Bq}(ku) \geq t(F_{Ap,Sp}(u), t(F_{Bq,Tq}(u), t(F_{Ap,Tq}(\alpha u), F_{Bq,Sp}(2u - \alpha u))))$
*for all $x, y \in X, u > 0$ and $\alpha \in (0, 2)$,*

**(3.1.3)** *one of $A, B, S$ and $T$ is continuous,*

**(3.1.4)** *the pairs $\{A, S\}$ and $\{B, T\}$ are compatible.*

*Then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

In [19] Sharma and Deshpande improved Theorem (3.1) by replacing the condition of compatibility by weak-compatibility, relaxing the continuity requirement of maps and relaxing the completeness of the space $(X, F, t)$ .They proved the following theorem:

**Theorem 3.2** *Let A, B, S and T be self mappings on a Menger space $(X, F, t)$ where $t$ is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, satisfying the conditions:*

**(3.2.1)** $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

**(3.2.2)** *there exists $k \in (0, 1)$ such that*
$F_{Ax,By}(ku) \geq t(F_{Ax,Sx}(u), t(F_{By,Ty}(u), t(F_{Ax,Ty}(\alpha u), F_{By,Sx}(2u - \alpha u))))$
*for all $x, y \in X, u > 0$ and $\alpha \in (0, 2)$.*
*If*

**(3.2.3)** *one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of $X$,*
*then*

*(i) A and S have a coincidence point, and*

*(ii) B and T have a coincidence point.*

*Further if*

**(3.2.4)** *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,*
*then*

*(iii) A, B, S and T have a unique common fixed point in X.*

Now, we provide an example to demonstrate that this claim is false unless some additional conditions are imposed.
The next example shows that there exists four mappings $A$, $B$, $S$ and $T$ satisfying all the properties (3.2.1) $\to$ (3.2.4) but they have more than one common fixed point.

**Example 3.1** *Let $X = (0, 3]$ with the Euclidean metric $d$ and let $\Im : X \times X \to L$ be defined as:*

$$F_{x,y}(u) = \begin{cases} G\left(\frac{u}{d(x,y)}\right), & x \neq y \ ; \\ H(u), & x = y \qquad \text{for all } x, y \in X, \end{cases}$$

where $G(x)$ is any distribution function such that $G(0) = 0$, then, $(X, \Im)$ is a simple space. By using Theorem (2.1), the space $(X, \Im, t)$ will be Menger space with "$t = \min$".
Define the mappings $A$, $B$, $S$ and $T$ as following:

$$A(x) = \begin{cases} \frac{1}{2}, & x \in (0,1) \\ 3, & x \in [1,3] \end{cases} \quad , \quad B(x) = \begin{cases} 1-x, & x \in (0,1) \\ 3, & x \in [1,3] \end{cases}$$

$$S(x) = \begin{cases} 1-x, & x \in (0,1) \\ 3, & x \in [1,3] \end{cases} \quad , \quad T(x) = \begin{cases} x, & x \in (0,1) \\ 3, & x \in [1,3]. \end{cases}$$

We see that, $X = (0, 3]$ is not complete. Since $A(X) = \{\frac{1}{2}, 3\}$, $B(X) = \{(0, 1) \cup \{3\}\}$, $T(X) = \{(0, 1) \cup \{3\}\}$ and $S(X) = \{(0, 1) \cup \{3\}\}$, then we can say that $A(X) \subseteq S(X)$, $B(X) \subseteq T(X)$ and $A(X)$ is complete subspace of $X$.

We have $ASx = SAx$ for all $x \in [1, 3]$ or $x = \frac{1}{2}$, then $\{A, S\}$ commute at coincidence points, and also, $BTx = TBx$ for all $x \in [1, 3]$ or $x = \frac{1}{2}$, then $\{B, T\}$ commute at coincidence points, i.e, $\{A, S\}$ and $\{B, T\}$ are weakly compatible pairs.
Hence, these mappings satisfy the conditions (3.2.1), (3.2.3) and (3.2.4). Moreover, $A$, $B$, $S$ and $T$ satisfying (3.2.2), for $k = \frac{3}{4}$, $\alpha = \frac{3}{2}$ and $u \geq 0$ as follows:

**1-** If $x \in (0, 1), y \in (0, 1), x \neq y, x \neq \frac{1}{2}$ and $y \neq \frac{1}{2}$.
Left hand side of (3.2.2) (L.H.S) is $F_{Ax,By}(ku) = F_{\frac{1}{2},1-y}(\frac{3}{4}u) = G(\frac{\frac{3}{4}u}{|\frac{1}{2}-y|})$.
Right hand side of (3.2.2) (R.H.S) equal,
$t(F_{Ax,Sx}(u), t(F_{By,Ty}(u), t(F_{Ax,Ty}(\frac{3}{2}u), F_{By,Sx}(\frac{1}{2}u))))$
$= \min\{F_{\frac{1}{2},1-x}(u), F_{1-y,y}(u), F_{\frac{1}{2},y}(\frac{3}{2}u), F_{1-y,1-x}(\frac{1}{2}u)\}$
$= \min\{G(\frac{u}{|\frac{1}{2}-x|}), G(\frac{u}{|1-2y|}), G(\frac{\frac{3}{2}u}{|\frac{1}{2}-y|}), G(\frac{\frac{1}{2}u}{|x-y|})\}$
$\because |1-2y| = 2|\frac{1}{2}-y| > \frac{2}{3}|\frac{1}{2}-y| \Rightarrow \frac{u}{|1-2y|} \leq \frac{\frac{3}{2}u}{|\frac{1}{2}-y|}$, (equal sign holds at$u = 0$), since $G(x)$ is non decreasing distribution function , then, $G(\frac{u}{|1-2y|}) \leq G(\frac{\frac{3}{2}u}{|\frac{1}{2}-y|})$.

R.H.S $= \min\{G(\frac{u}{|\frac{1}{2}-x|}), G(\frac{u}{|1-2y|}), G(\frac{\frac{1}{2}u}{|x-y|})\}$.
Now we consider many cases,

**Case(i)** If $x > y$ and $x < \frac{1}{2}$:
$x > y \Rightarrow -x < -y \Rightarrow \frac{1}{2} - x < \frac{1}{2} - y \Rightarrow |\frac{1}{2} - x| < |\frac{1}{2} - y| \Rightarrow \frac{u}{|\frac{1}{2}-x|} \geq \frac{u}{|1-2y|} \Rightarrow G(\frac{u}{|\frac{1}{2}-x|}) \geq G(\frac{u}{|1-2y|})$.
Also, $x < \frac{1}{2} \Rightarrow x - y < \frac{1}{2} - y \Rightarrow |x - y| < |\frac{1}{2} - y| < |1 - 2y| \Rightarrow \frac{u}{|x-y|} \geq \frac{u}{|\frac{1}{2}-y|} \Rightarrow \frac{\frac{1}{2}u}{|x-y|} \geq \frac{u}{|1-2y|} \Rightarrow G(\frac{\frac{1}{2}u}{|x-y|}) \geq G(\frac{u}{|1-2y|})$.
Hence, R.H.S $= G(\frac{u}{|1-2y|})$, and we have $\frac{\frac{3}{4}u}{|\frac{1}{2}-y|} = \frac{\frac{3}{2}u}{|1-2y|} \geq \frac{u}{|1-2y|}$.
Then, R.H.S $= G(\frac{u}{|1-2y|}) \leq G(\frac{\frac{3}{4}u}{|\frac{1}{2}-y|}) =$ L.H.S.

**Case(ii)** If $x > y$ , $x > \frac{1}{2}$ and $y < \frac{1}{2}$:
$x < \frac{1}{2} \Rightarrow x - y > \frac{1}{2} - y$.Since $y < \frac{1}{2}$ and $x > y$, then $|x - y| > |\frac{1}{2} - y| \Rightarrow \frac{\frac{1}{2}u}{|x-y|} \leq \frac{\frac{1}{2}u}{|\frac{1}{2}-y|} \Rightarrow G(\frac{\frac{1}{2}u}{|x-y|}) \leq G(\frac{u}{|1-2y|})$.
Also, $y < \frac{1}{2} \Rightarrow y - x < \frac{1}{2} - x \Rightarrow x - y > x - \frac{1}{2} \Rightarrow |x - y| > |x - \frac{1}{2}| \Rightarrow \frac{\frac{1}{2}u}{|x-y|} \leq \frac{u}{|x-y|} \leq \frac{u}{|x-\frac{1}{2}|} \Rightarrow G(\frac{\frac{1}{2}u}{|x-y|}) \leq G(\frac{u}{|x-\frac{1}{2}|})$.
Hence, $R.H.S = G(\frac{\frac{1}{2}u}{|x-y|})$, and we have $\frac{\frac{1}{2}u}{|x-y|} \leq \frac{\frac{1}{2}u}{|\frac{1}{2}-y|} \leq \frac{\frac{3}{4}u}{|\frac{1}{2}-y|} \Rightarrow R.H.S = G(\frac{\frac{1}{2}u}{|x-y|}) \leq L.H.S = G(\frac{\frac{3}{4}u}{|\frac{1}{2}-y|})$.

**Case(iii)** If we take any value for $x$ and $y$ such that $x > y$ , $x > \frac{1}{2}$ and $y > \frac{1}{2}$, we note that $|1 - 2y| \geq |\frac{1}{2} - x| \geq 2|x - y|$. Then for any $x > y > \frac{1}{2}$ we have $\frac{u}{|1-2y|} \leq \frac{u}{|\frac{1}{2}-x|} \leq \frac{u}{2|x-y|} \Rightarrow G(\frac{u}{|1-2y|}) \leq G(\frac{u}{|\frac{1}{2}-x|}) \leq G(\frac{u}{2|x-y|}) \Rightarrow R.H.S = G(\frac{u}{|1-2y|}), i.e, L.H.S \geq R.H.S$.

**Case(iv)** If $x < y$:
As in the previous cases one can verify that, $R.H.S \leq L.H.S$.

**2-** If $x = \frac{1}{2}$ and $y = \frac{1}{2}$.
L.H.S $= F_{Ax,By}(\frac{3}{4}u) = F_{\frac{1}{2},\frac{1}{2}}(\frac{3}{4}u) = H(u)$.
R.H.S $= \min\{F_{Ax,Sx}(u), F_{By,Ty}(u), F_{Ax,Ty}(\frac{3}{2}u), F_{By,Sx}(\frac{1}{2}u)\}$
$= \min\{F_{\frac{1}{2},\frac{1}{2}}(u), F_{\frac{1}{2},\frac{1}{2}}(u), F_{\frac{1}{2},\frac{1}{2}}(\frac{3}{2}u), F_{\frac{1}{2},\frac{1}{2}}(\frac{1}{2}u)\}$
$= H(u)$.

**3-** If $x = \frac{1}{2}$ and $y \in (0, 1)$.
L.H.S $= F_{\frac{1}{2},1-y}(\frac{3}{4}u) = G(\frac{\frac{3}{4}u}{|\frac{1}{2}-y|})$.
R.H.S $= \min\{F_{\frac{1}{2},\frac{1}{2}}(u), F_{1-y,y}(u), F_{\frac{1}{2},y}(\frac{3}{2}u), F_{1-y,\frac{1}{2}}(\frac{1}{2}u)\}$
$= \min\{G(\frac{u}{|1-2y|}), G(\frac{\frac{3}{2}u}{|\frac{1}{2}-y|}), G(\frac{\frac{1}{2}u}{|\frac{1}{2}-y|})\}$.
Since,
$\frac{\frac{3}{2}u}{|\frac{1}{2}-y|} \geq \frac{\frac{1}{2}u}{|\frac{1}{2}-y|} = \frac{u}{|1-2y|} \Rightarrow R.H.S \leq L.H.S$.

**4-** If $y = \frac{1}{2}$ and $x \in (0, 1)$.
$L.H.S = F_{\frac{1}{2},\frac{1}{2}}(\frac{3}{4}u) = H(u)$.

$R.H.S = \min\{F_{\frac{1}{2},1-x}(u), F_{\frac{1}{2},\frac{1}{2}}(u), F_{\frac{1}{2},\frac{1}{2}}(\frac{3}{2}u), F_{\frac{1}{2},1-x}(\frac{1}{2}u)\}$
$= G(\frac{\frac{1}{2}u}{|\frac{1}{2}-x|})$.
If $u > 0 \Rightarrow L.H.S = 1 > R.H.S$
If $u = 0 \Rightarrow L.H.S = 0 = G(0) = R.H.S$.

**5-** If $x \in [1,3], y \in [1,3]$.
L.H.S = R.H.S.

**6-** If $x \in [1,3]$ and $y \in (0,1), y \neq 0$.
L.H.S= $F_{3,1-y}(\frac{3}{4}u) = G(\frac{\frac{3}{4}u}{|2+y|})$
R.H.S= $\min\{F_{3,3}(u), F_{1-y,y}(u), F_{3,y}(\frac{3}{2}u), F_{1-y,3}(\frac{1}{2}u)\}$
$= \min\{H(u), G(\frac{u}{|1-2y|}), G(\frac{\frac{3}{2}u}{|3-y|}), G(\frac{\frac{1}{2}u}{|2+y|})\}$
If $u = 0$ R.H.S $= 0 =$ L.H.S.
Now suppose $u > 0$, we have two cases:

**Case(i)** If $y > \frac{1}{2}$:
$y > \frac{1}{2} \Rightarrow y+2 > \frac{1}{2}+2 = 3-\frac{1}{2} > 3-y \Rightarrow |y+2| > |3-y| \Rightarrow 2|y+2| > |y+2| > |3-y| > \frac{2}{3}|3-y| \Rightarrow \frac{\frac{1}{2}u}{|y+2|} < \frac{\frac{3}{2}u}{|3-y|} \Rightarrow G(\frac{\frac{1}{2}u}{|y+2|}) < G(\frac{\frac{3}{2}u}{|3-y|})$.
Also, $2y - 1 = 2(y-\frac{1}{2}) < 2(y+2) \Rightarrow 2|y-\frac{1}{2}| < 2|y+2| \Rightarrow \frac{u}{|2y-1|} > \frac{\frac{1}{2}u}{|y+2|} \Rightarrow G(\frac{u}{|2y-1|}) > G(\frac{\frac{1}{2}u}{|y+2|})$. Hence, R.H.S= $G(\frac{\frac{1}{2}u}{|y+2|}) < G(\frac{u}{|2y-1|}) =$ L.H.S.

**Case(ii)** If $y < \frac{1}{2}$:
$\frac{2}{3}(3-y) < \frac{2}{3}(3+y) < \frac{2}{3}(3+\frac{1}{2}) = 2+\frac{1}{3} < 2(2-\frac{1}{2}) < 2(2-y) < 2(2+y)$,
then $\frac{2}{3}|3-y| < 2|2+y| \Rightarrow \frac{\frac{3}{2}u}{|3-y|} > \frac{\frac{1}{2}u}{|2+y|} \Rightarrow G(\frac{\frac{3}{2}u}{|3-y|}) > G(\frac{\frac{1}{2}u}{|2+y|})$.
Also, $2|y+2| > |1-2y| \Rightarrow \frac{\frac{1}{2}u}{|y+2|} < \frac{u}{|1-2y|} \Rightarrow G(\frac{\frac{1}{2}u}{|y+2|}) < G(\frac{u}{|1-2y|})$.
Hence, R.H.S= $G(\frac{\frac{1}{2}u}{|y+2|}) \leq$ L.H.S.

**7-** If $x \in [1,3], y = \frac{1}{2}$.
L.H.S= $F_{3,\frac{1}{2}}(\frac{3}{4}u) = G(\frac{\frac{3}{4}u}{\frac{5}{2}}) = G(\frac{3}{10}u)$
R.H.S= $\min\{F_{3,3}(u), F_{\frac{1}{2},\frac{1}{2}}(u), F_{3,\frac{1}{2}}(\frac{3}{2}u), F_{\frac{1}{2},3}(\frac{1}{2}u)\} = \min\{H(u), H(u), G(\frac{3}{5}u), G(\frac{1}{5}u)\} = G(\frac{1}{5}u) \leq L.H.S.$

**8-** If $x \in (0,1), y \in [1,3]$and$x \neq \frac{1}{2}$.
L.H.S= $F_{\frac{1}{2},3}(\frac{3}{4}u) = G(\frac{\frac{3}{4}u}{\frac{5}{2}}) = G(\frac{3}{10}u)$
R.H.S= $\min\{F_{\frac{1}{2},1-x}(u), F_{3,3}(u), F_{\frac{1}{2},3}(\frac{3}{2}u), F_{3,1-x}(\frac{1}{2}u)\}$
$= \min\{G(\frac{u}{|\frac{1}{2}-x}|), H(u), G(\frac{3}{5}u), G(\frac{\frac{1}{2}u}{|2+x|})\}$.
$\because 2|2+x| > |2+x| > |\frac{1}{2}-x| \Rightarrow G(\frac{u}{|\frac{1}{2}-x|}) \geq G(\frac{\frac{1}{2}u}{|2+x|})$.
Also,$\frac{5}{3} < 2(2+x) \Rightarrow G(\frac{3}{5}u) \geq G(\frac{\frac{1}{2}u}{|2+x|})$. i.e, $R.H.S = G(\frac{\frac{1}{2}u}{|2+x|}) \leq L.H.S.$

**9-** If $y \in [1,3], x = \frac{1}{2}$.
L.H.S= $H(u) \geq$ R.H.S.

Then $A, B, S$ and $T$ satisfying (3.2.1) $\rightarrow$ (3.2.4) but they have more than one common fixed point.

The following Lemma is used in the sequel.

**Lemma 3.1 ([20, 16])** *Let $\{x_n\}$ be a sequence in a Menger space $(X, \Im, t)$, where $t$ is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$ . If there exists a constant $k \in (0,1)$ such that,*
*$F_{x_n,x_{n+1}}(kx) \geq F_{x_{n-1},x_n}(x)$ for all $x > 0$ and $n \in N$, then $\{x_n\}$ is Cauchy sequence in $X$.*

Now we prove a common fixed point theorem for four weakly compatible maps on a complete Menger space.

**Theorem 3.3** *Let $A$, $B$, $S$ and $T$ be self mappings on a complete Menger space $(X, F, t)$ where $t(x,y) = \min(x,y)$ for all $x, y \in [0,1]$, satisfying the following conditions:*

**(3.3.1)** *$A(X)$, $B(X)$ are closed sets of $X$ and $A(X) \subset T(X), B(X) \subset S(X)$,*

**(3.3.2)** *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,*

**(3.3.3)** $[F_{Ax,By}(ku)]^2 \geq$

$$\min\Big\{[F_{Sx,Ty}(u)]^2, F_{Sx,Ax}(u)F_{Ty,By}(u), F_{Sx,Ty}(u)F_{Sx,Ax}(u),$$
$$F_{Sx,Ty}(u)F_{Ty,By}(u), F_{Sx,Ty}(u)F_{Sx,By}(2u), F_{Sx,Ty}(u)F_{Ty,Ax}(u),$$
$$F_{Sx,By}(2u)F_{Ty,Ax}(u), F_{Sx,Ax}(u)F_{Ty,Ax}(u), F_{Sx,By}(2u)F_{Ty,By}(u)\Big\}$$

*for all $x$, $y \in X$, $x > 0$ and $u > 0$, where $k \in (0,1)$. Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** since $A(X) \subset T(X)$ for any arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$ and $B(X) \subset S(X)$ implies that for this point $x_1$ we can find a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2}, \qquad n = 0, 1, 2, ..., . \end{aligned} \tag{3.1}$$

Now we prove that the sequence defined by (3.1) is a Cauchy sequence. By Lemma (3.1) it is sufficient to show that $F_{y_{2n},y_{2n+1}}(ku) \geq F_{y_{2n-1},y_{2n}}(u)$ for all $u > 0$ where $k \in (0,1)$. Suppose that $F_{y_{2n},y_{2n+1}}(ku) < F_{y_{2n-1},y_{2n}}(u)$ and using (3.3.3), we have

$[F_{y_{2n},y_{2n+1}}(ku)]^2 = [F_{Ax_{2n},Bx_{2n+1}}(ku)]^2 \geq$
$\min\Big\{[F_{Sx_{2n},Tx_{2n+1}}(u)]^2, F_{Sx_{2n},Ax_{2n}}(u)F_{Tx_{2n+1},Bx_{2n+1}}(u), F_{Sx_{2n},Tx_{2n+1}}(u)F_{Sx_{2n},Ax_{2n}}(u),$
$F_{Sx_{2n},Tx_{2n+1}}(u)F_{Tx_{2n+1},Bx_{2n+1}}(u), F_{Sx_{2n},Tx_{2n+1}}(u)F_{Sx_{2n},Bx_{2n+1}}(2u),$
$F_{Sx_{2n},Tx_{2n+1}}(u)F_{Tx_{2n+1},Ax_{2n}}(u), F_{Sx_{2n},Bx_{2n+1}}(2u)F_{Tx_{2n+1},Ax_{2n}}(u),$
$F_{Sx_{2n},Ax_{2n}}(u)F_{Tx_{2n+1},Ax_{2n}}(u), F_{Sx_{2n},Bx_{2n+1}}(2u)F_{Tx_{2n+1},Bx_{2n+1}}(u)\Big\},$
$\geq \min\Big\{[F_{y_{2n-1},y_{2n}}(u)]^2, F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n+1}}(u), F_{y_{2n-1},y_{2n}}(u)F_{y_{2n-1},y_{2n}}(u),$
$F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n+1}}(u), F_{y_{2n-1},y_{2n}}(u)F_{y_{2n-1},y_{2n+1}}(2u), F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n}}(u),$
$F_{y_{2n-1},y_{2n+1}}(2u)F_{y_{2n},y_{2n}}(u), F_{y_{2n-1},y_{2n}}(u)F_{y_{2n},y_{2n}}(u), F_{y_{2n-1},y_{2n+1}}(2u)F_{y_{2n},y_{2n+1}}(u)\Big\},$
$\geq \min\Big\{[F_{y_{2n},y_{2n+1}}(ku)]^2, F_{y_{2n},y_{2n+1}}(ku)F_{y_{2n},y_{2n+1}}(u), [F_{y_{2n},y_{2n+1}}(ku)]^2),$
$F_{y_{2n},y_{2n+1}}(ku)F_{y_{2n},y_{2n+1}}(u), F_{y_{2n},y_{2n+1}}(ku)t\{F_{y_{2n-1},y_{2n}}(u), F_{y_{2n},y_{2n+1}}(u)\},$
$F_{y_{2n},y_{2n+1}}(ku), t\{F_{y_{2n-1},y_{2n}}(u), F_{y_{2n},y_{2n+1}}(u)\},$
$F_{y_{2n},y_{2n+1}}(ku), t\{F_{y_{2n-1},y_{2n}}(u), F_{y_{2n},y_{2n+1}}(u)\}F_{y_{2n},y_{2n+1}}(u)\Big\}.$

Since $k \in (0,1)$ then $u > ku$ for any $u > 0$ and $F_{y_{2n},y_{2n+1}}(u) > F_{y_{2n},y_{2n+1}}(ku)$ and also
$t\{F_{y_{2n-1},y_{2n}}(u), F_{y_{2n},y_{2n+1}}(u)\} \geq t\{F_{y_{2n},y_{2n+1}}(ku), F_{y_{2n},y_{2n+1}}(u)\} \geq$
$t\{F_{y_{2n},y_{2n+1}}(ku), F_{y_{2n},y_{2n+1}}(ku)\} \geq F_{y_{2n},y_{2n+1}}(ku).$

Hence,
$[F_{y_{2n},y_{2n+1}}(ku)]^2 \geq \min\{[F_{y_{2n},y_{2n+1}}(ku)]^2, F_{y_{2n},y_{2n+1}}(ku)\}.$
Since $F_{x,y}(u)$ for any $x, y \in X, u > 0$ is a nondecreasing and $\inf F_{x,y}(u) = 0, \sup F_{x,y}(u) = 1$, then $[F_{y_{2n},y_{2n+1}}(ku)]^2 \geq \{[F_{y_{2n},y_{2n+1}}(ku)]^2\}$, which is a contradiction. Thus $F_{y_{2n},y_{2n+1}}(ku) \geq F_{y_{2n-1},y_{2n}}(u)$ and $\{y_n\}$ is Cauchy sequence in $X$.

Since the Menger space $(X, F, t)$ is complete, then the sequence $\{y_n\}$ converges to a point $z$ in $X$and the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$ of $\{y_{2n}\}$ also converge to $z$.

Since $A(x) \subset B(x)$, there exists $p \in X$ , such that $z = Tp$. by using (3.3.3) we have

$[F_{Ax_{2n},Bp}(ku)]^2 \geq$
$\min\Big\{[F_{Sx_{2n},Tp}(u)]^2, F_{Sx_{2n},Ax_{2n}}(u)F_{Tp,Bp}(u),$
$F_{Sx_{2n},Tp}(u)F_{Sx_{2n},Ax_{2n}}(u), F_{Sx_{2n},Tp}(u)F_{Tp,Bp}(u),$
$F_{Sx_{2n},Tp}(u)F_{Sx_{2n},Bp}(2u), F_{Sx_{2n},Tp}(u)F_{Tp,Ax_{2n}}(u),$

$F_{Sx_{2n},Bp}(2u)F_{Tp,Ax_{2n}}(u), F_{Sx_{2n},Ax_{2n}}(u)F_{Tp,Ax_{2n}}(u), F_{Sx_{2n},Bp}(2u)F_{Tp,Bp}(u)\Big\}.$

Taking limits as $n \to \infty$ gives,

$[F_{z,Bp}(ku)]^2 \geq \min\Big\{[F_{z,z}(u)]^2, F_{z,z}(u)F_{z,Bp}(u), F_{z,z}(u)F_{z,z}(u),$

$F_{z,z}(u)F_{z,Bp}(u), F_{z,z}(u)F_{z,Bp}(2u), F_{z,z}(u)F_{z,z}(u), F_{z,Bp}(2u)F_{z,z}(u),$

$F_{z,z}(u)F_{z,z}(u), F_{z,Bp}(2u)F_{z,Bp}(u)\Big\},$

$\geq \min\{[F_{z,Bp}(u)]^2, F_{z,Bp}(u)\} \geq [F_{z,Bp}(u)]^2.$

Which means that $Bp = z$ then we have, $Bp = Tp = z$.

By a similar way, since $B(X) \subset S(X)$, there exists $q \in X$ such that $z = Sq$. Again by using (3.3.3)

$[F_{Aq,Bx_{2n+1}}(ku)]^2 \geq$

$\min\Big\{[F_{Sq,Tx_{2n+1}}(u)]^2, F_{Sq,Aq}(u)F_{Tx_{2n+1},Bx_{2n+1}}(u),$

$F_{Sq,Tx_{2n+1}}(u)F_{Sq,Aq}(u), F_{Sq,Tx_{2n+1}}(u)F_{Tx_{2n+1},Bx_{2n+1}}(u),$

$F_{Sq,Tx_{2n+1}}(u)F_{Sq,Bx_{2n+1}}(2u), F_{Sq,Tx_{2n+1}}(u)F_{Tx_{2n+1},Aq}(u),$

$F_{Sq,Bx_{2n+1}}(2u)F_{Tx_{2n+1},Aq}(u), F_{Sq,Aq}(u)F_{Tx_{2n+1},Aq}(u),$

$F_{Sq,Bx_{2n+1}}(2u)F_{Tx_{2n+1},Bx_{2n+1}}(u)\Big\}.$

Taking limits as $n \to \infty$ gives,

$[F_{Aq,z}(ku)]^2 \geq$

$\min\Big\{[F_{z,z}(u)]^2, F_{z,Aq}(u)F_{z,z}(u), F_{z,z}(u)F_{z,Aq}(u),$

$F_{z,z}(u)F_{z,z}(u), F_{z,z}(u)F_{z,z}(2u), F_{z,z}(u)F_{z,Aq}(u), F_{z,z}(2u)F_{z,Aq}(u),$

$F_{z,Aq}(u)F_{z,Aq}(u), F_{z,z}(2u)F_{z,z}(u)\Big\},$

$\geq \min\{[F_{z,Aq}(u)]^2, F_{z,Aq}(u)\} \geq [F_{z,Aq}(u)]^2.$

This yields $Aq = z$ then we have,$Aq = Sq = z$.

Since $\{B, T\}$ are weakly compatible then they commute at their coincidence point $p$. i.e, $BTp = TBp$ or $Bz = Tz$

Now we show that $z$ is a fixed point of $B$.

By using (3.3.3), $[F_{Aq,Bz}(ku)]^2 \geq$

$\min\Big\{[F_{Sq,Tz}(u)]^2, F_{Sq,Aq}(u)F_{Tz,Bz}(u), F_{Sq,Tz}(u)F_{Sq,Aq}(u),$

$F_{Sq,Tz}(u)F_{Tz,Bz}(u), F_{Sq,Tz}(u)F_{Sq,Bz}(2u), F_{Sq,Tz}(u)F_{Tz,Aq}(u), F_{Sq,Bz}(2u)F_{Tz,Aq}(u),$

$F_{Sq,Aq}(u)F_{Tz,Aq}(u), F_{Sq,Bz}(2u)F_{Tz,Bz}(u)\Big\},$

$[F_{z,Bz}(ku)]^2 \geq \min\Big\{[F_{z,Bz}(u)]^2, F_{z,z}(u)F_{Bz,Bz}(u), F_{z,Bz}(u)F_{z,z}(u),$

$F_{z,Bz}(u)F_{Bz,Bz}(u), F_{z,Bz}(u)F_{z,Bz}(2u), F_{z,Bz}(u)F_{Bz,z}(u), F_{z,Bz}(2u)F_{Bz,z}(u),$

$F_{z,z}(u)F_{Bz,z}(u), F_{z,Bz}(2u)F_{Bz,Bz}(u)\Big\}, \geq \min\{[F_{z,Bz}(u)]^2, F_{z,Bz}(u)\},$
$\geq [F_{z,Bz}(u)]^2.$
which implies that $Bz = Tz = z$
Similarly, since $\{A, S\}$ are weakly compatible then they commute at their coincidence point $q$. i.e, $ASq = SAq$ or $Az = Sz$.
Now we show that $z$ is a fixed point of $A$. By using (3.3.3),

$$[F_{Az,z}(ku)]^2 \geq \min\Big\{[F_{Az,z}(u)]^2, F_{Az,Az}(u)F_{z,z}(u), F_{Az,z}(u)F_{Az,Az}(u),$$
$$F_{Az,z}(u)F_{z,z}(u), F_{Az,z}(u)F_{Az,z}(2u), F_{Az,z}(u)F_{z,Az}(u), F_{Az,z}(2u)F_{z,Az}(u),$$
$$F_{Az,Az}(u)F_{z,Az}(u), F_{Az,z}(2u)F_{z,z}(u)\Big\},$$
$$\geq [F_{Az,z}(u)]^2 \text{ which means that } Az = Sz = z.$$

Thus $z$ is a common fixed point of $A$, $B$, $S$ and $T$. Finally in order to prove the uniqueness of $z$, suppose that $z$, $w$ are common fixed points of $A$, $B$, $S$ and $T$. We prove the converse by putting $x = z$, $y = w$ in (3.3.3).

$$[F_{Az,Bw}(ku)]^2 \geq \min\Big\{[F_{Sz,Tw}(u)]^2, F_{Sz,Az}(u)F_{Tw,Bw}(u), F_{Sz,Tw}(u)F_{Sz,Az}(u),$$
$$F_{Sz,Tw}(u)F_{Tw,Bw}(u), F_{Sz,Tw}(u)F_{Sz,Bw}(2u), F_{Sz,Tw}(u)F_{Tw,Az}(u), F_{Sz,Bw}(2u)F_{Tw,Az}(u),$$
$$F_{Sz,Az}(u)F_{Tw,Az}(u), F_{Sz,Bw}(2u)F_{Tw,Bw}(u)\Big\},$$
$$[F_{z,w}(ku)]^2 \geq \min\Big\{[F_{z,w}(u)]^2, F_{z,z}(u)F_{w,w}(u), F_{z,w}(u)F_{z,z}(u),$$
$$F_{z,w}(u)F_{w,w}(u), F_{z,w}(u)F_{z,w}(2u), F_{z,w}(u)F_{w,z}(u), F_{z,w}(2u)F_{w,z}(u),$$
$$F_{z,z}(u)F_{w,z}(u), F_{z,w}(2u)F_{w,w}(u)\Big\} \geq [F_{z,w}(u)]^2.$$
which means that $z = w$. This complete the proof of the theorem. □

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