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COMMON FIXED POINT THEOREMS IN 2-MENGER SPACES

R. A. RASHWAN AND SHIMAA I. MOUSTAFA*

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt

Abstract. In this paper, we prove some fixed point theorems in 2-Menger space under new contraction conditions.

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1. Introduction

In 1963 Gahler[2, 3] introduced the concept of a 2-metric space. In his paper [3] he claimed that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $d(x, y, z)$ represents the area of a triangle formed by the points x, y and z in X as its vertices.

Definition 1.1. Let X be a non empty set and R denote the set of all real numbers. A function $d : X \times X \times X \to R$ is said to be a 2-metric on X if it satisfies the following properties:

(1): For distinct points $x, y \in X$, there is a point $z \in X$ such that $d(x, y, z) \neq 0$;

(2): $d(x, y, z) = 0$ if any two elements of the triplet $x, y, z \in X$ are equal;

*Corresponding author

E-mail addresses: rr_rashwan54@yahoo.com (R. A. Rashwan), shimaa13620011@yahoo.com (S. I. Moustafa)

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(3): $d(x, y, z) = d(y, x, z) = \ldots$ (symmetry);

(4): $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z, a \in X$.

A nonempty set X together with a 2-metric d is called a 2-metric space.

Definition 1.2.[10] A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point x in X if $d(x_n, x, c) \to 0$ as $n \to \infty$ for all c in X. The point x is called the limit of the sequence $\{x_n\}$ in X.

Definition 1.3.[10] A sequence $\{x_n\}$ in (X, d) is said to be a Cauchy sequence if $d(x_m, x_n, c) \to 0$ as $n \to \infty$ for all c in X.

Definition 1.4.[10] The space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Remark 1.5.[9] We note that, in a usual metric space a convergent sequence is a Cauchy sequence and in a 2-metric space a convergent sequence need not be a Cauchy sequence unless the 2-metric d is continuous on X.

Example 1.1.[11] Let R^2 be the Euclidean space. Let $d(x, y, z)$ denote the area of the triangle formed by joining the three points $x, y, z \in R^2$ .Then (R^2, d) is a 2-metric space.

In fact, it is suitable to look upon the distance concept between two or more objects of the set under considerations as statistical or probabilistic rather than deterministic one. The advantage of a probabilistic approach is that it permits from the initial formulation a greater flexibility rather than that offered by a deterministic approach and the probabilistic metric spaces corresponds to the situations when we do not know the distance between the points or this distance is inexact. So that, in 1989, Chang and Huang [1] gave an analogue of a 2-metric space in probabilistic setting known as a probabilistic 2-metric space.

Definition 1.5.[13] A probabilistic 2-metric space is an order pair $(X, \mathbf{F})$ where X is an arbitrary set and $\mathbf{F}$ is a mapping from X^3 into the set of distribution functions L. The distribution function $F_{x,y,z}(t)$ will denote the value of $F_{x,y,z}$ at the nonnegative real number t. The function $F_{x,y,z}$ are assumed to satisfy the following conditions:

(5): $F_{x,y,z}(0) = 0$ for all $x, y, z \in X$;

(6): $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of the three points x, y, z are equal;

(7): For distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) < 1$ for $t > 0$;

(8): $F_{x,y,z}(t) = F_{x,z,y}(t) = \ldots$ for all $x, y, z \in X$ and $t > 0$;

(13): $F_{x,y,w}(t_1) = 1$,$F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$, for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

A particular type of 2-probabilistic metric space is probabilistic 2-Menger PM-space in which the triangular inequality is postulated with the help of a t-norm.

Definition 1.6. A mapping $t : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is a T-norm if it is associative, commutative, non-decreasing in each co-ordinate and $t(a, 1, 1) = a$ for every $a \in [0, 1]$.

An important T-norm t is $t(a, b, c) = \min\{a, b, c\}$ for all $a, b, c \in [0, 1]$ and it is the unique T-norm such that $t(a, a, a) \geq a$ for every $a \in [0, 1]$.

Definition 1.7. [4] Let X be a non empty set and L the set of all left-continuous distribution functions. A 2-Menger space is a triplet $(X, \mathbf{F}, t)$, where t is a T-norm and $\mathbf{F}$ is a mapping from X^3 into L satisfying the following conditions, (we assign to every $x, y, z \in X$ a distribution function $F_{x,y,z}$ and its value at $t \geq 0$ is represented by $F_{x,y,z}(t)$):

(9): $F_{x,y,z}(0) = 0$;

(10): $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of $x, y, z \in X$ are equal;

(11): $F_{x,y,z}(t) = F_{x,z,y}(t) = \ldots$ for all $x, y, z \in X$ and $t > 0$;

(12): $F_{x,y,z}(t) \geq t(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$ where $t_1, t_2, t_3 > 0, t_1 + t_2 + t_3 = t$ and $x, y, z, w \in X$.

Definition 1.8. Let $(X, \mathbf{F}, t)$ be a Menger space and t be a continuous t-norm.

(i): A sequence $\{x_n\}$ in X is said to converge to a point x in$X$ if and only if for every $\epsilon > 0$, $\lambda > 0$, there exists $M(\epsilon, \lambda)$ such that

$$F_{x_n,x,a}(\epsilon) > 1 - \lambda \text{ for all } n \geq M;$$

(ii): A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if and only if for each $\epsilon > 0$, $\lambda > 0$ there exists an integer $M(\epsilon, \lambda)$ such that

$$F_{x_n,x_m,a}(\epsilon) > 1 - \lambda \text{ for all } n, m \geq M;$$

(iii): A 2-Menger space in which every Cauchy sequence is convergent is said to be complete.

2. Main results

Singh [12] considered a complete and bounded 2-metric space with two metrics ρ and d, one of them is less than or equal the other. He obtained sufficient conditions for the existence of unique fixed point for a mapping which is a contraction with respect to one metric and continuous for the other. Here we extend and generalize this result in probabilistic setting (2-Menger space).

If we define on X two non-negative metrics ρ, d:$X \times X \times X \to R$, we can define for all $x, y, z \in X$ two distribution functions $F_{x,y,z}$ and $F^*_{x,y,z}$ and their values at $t \geq 0$ are $F_{x,y,z}(t)$ and $F^*_{x,y,z}(t)$. We interpret that values as the probability that the distances $\rho(x, y, z)$ and $d(x, y, z)$ be less than t. Before proving our main theorems, we need the following lemmas.

Lemma 2.1. *Let $\{x_n\}$ be a sequence in a 2-Menger space $(X, \boldsymbol{F}, t)$, where t is continuous and $t(a, a, a) \geq a$ for all $a \in [0, 1]$. If there exist a constant $k \in (0, 1)$ such that:*

$$F_{x_{n+1},x_{n+2},a}(kt) \geq F_{x_n,x_{n+1},a}(t), \ \ forall \ n \in N \ and \ a \in X. \tag{2.1}$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof.

$$\begin{aligned} F_{x_{n-1},x_n,a}(\frac{(1-k)\epsilon}{2k}) &\geq F_{x_{n-2},x_{n-1},a}(\frac{(1-k)\epsilon}{2k^2}), \\ &\vdots \\ &\geq F_{x_0,x_1,a}(\frac{(1-k)\epsilon}{2k^n}). \end{aligned}$$

Since $k \in (0,1)$, for $\epsilon > 0$ and $\lambda \in (0,1)$ one can find a positive integer $M(\epsilon,\lambda)$ such that,

$$F_{x_{n-1},x_n,a}(\frac{(1-k)\epsilon}{2k}) > 1-\lambda \ \ forall \ \ n \geq M. \tag{2.2}$$

In this position we must prove that for any $\epsilon > 0$ and $\lambda > 0$ there exist $M \in N$ such that:

$$F_{x_n,x_m,a}(\epsilon) > 1-\lambda \ \ forall \ \ m > n > M. \tag{2.3}$$

Let $m = n+p$, at $p = 1$, (2.3) holds. Assume that (2.3) holds for all $1 < p \leq q$ for some $q \in N$, for $p = q+1$:

$$\begin{aligned} F_{x_n,x_{n+q+1},a}(\epsilon) &\geq F_{x_{n-1},x_{n+q},a}(\frac{\epsilon}{k}), \\ &\geq t(F_{x_{n-1},x_{n+q},x_n}(\frac{(1-k)\epsilon}{2k}), F_{x_{n-1},x_n,a}(\frac{(1-k)\epsilon}{2k}), F_{x_n,x_{n+q},a}(\epsilon)), \\ &> t(1-\lambda, 1-\lambda, 1-\lambda), \\ &\geq 1-\lambda. \end{aligned}$$

Hence (2.3) holds for all $m, n > M$. Therefore, x_n is Cauchy sequence in X.

Lemma 2.2. *Let $\{x_n\}$ be a sequence in a 2-Menger space $(X, \boldsymbol{F}, t)$ with continuous t-norm and two distribution functions F and F^* satisfying $F_{x,y,z}(t) \geq F^*_{x,y,z}(t)$ for all $x, y, z \in X$ and $t \geq 0$. If there exists a constant $k \in (0,1)$ such that:*

$$F^*_{x_{n+1},x_{n+2},a}(kt) \geq F^*_{x_n,x_{n+1},a}(t),$$

for all $n \in N$ and $a \in X$, then $\{x_n\}$ is a Cauchy sequence in X with respect to F.

Proof. Since,

$$\begin{aligned} F_{x_{n+1},x_{n+2},a}(kt) &\geq F^*_{x_{n+1},x_{n+2},a}(kt), \\ &\geq F^*_{x_n,x_{n+1},a}(t), \\ &\geq F^*_{x_{n-1},x_n,a}(\frac{t}{k}), \\ &\vdots \end{aligned}$$

$$F_{x_{n+1},x_{n+2},a}(kt) \geq F^*_{x_0,x_1,a}(\frac{t}{k^n}) \to 1 \ \ as \ \ n \to \infty. \tag{2.4}$$

Then, for all $m > n$, we have:

$$F_{x_n,x_m,a}(kt) \geq t(F_{x_n,x_{n+1},a}(k_1 t), F_{x_{n+1},x_m,a}(k_2 t), F_{x_n,x_m,x_{n+1}}(k_3 t)), \;\; k = k_1 + k_2 + k_3.$$

Letting $n \to \infty$ and using the continuity of t gets,

$$\begin{aligned} \lim_{n\to\infty} F_{x_n,x_m,a}(kt) &\geq t(\lim_{n\to\infty} F_{x_n,x_{n+1},a}(k_1 t), \lim_{n\to\infty} F_{x_{n+1},x_m,a}(k_2 t), \lim_{n\to\infty} F_{x_n,x_m,x_{n+1}}(k_3 t)), \\ &= t(1, \lim_{n\to\infty} F_{x_{n+1},x_m,a}(k_2 t), 1). \end{aligned}$$

Similarly,

$$\lim_{n\to\infty} F_{x_{n+1},x_m,a}(k_2 t) \geq t(1, \lim_{n\to\infty} F_{x_{n+2},x_m,a}(\acute{k}_2 t), 1).$$

Continuing this procedure to obtain,

$$\lim_{n\to\infty} F_{x_{m-2},x_m,a}(k_4 t) \geq t(1, \lim_{n\to\infty} F_{x_{m-1},x_m,a}(\acute{k}_4 t), 1).$$

By using (2.4) we have $\lim_{n\to\infty} F_{x_{m-1},x_m,a}(\acute{k}_4 t) = 1$.

Finally, using the back substitution yields:

$$\lim_{n\to\infty} F_{x_{m-2},x_m,a}(k_4 t) = 1,$$

$$\downarrow$$

$$\lim_{n\to\infty} F_{x_n,x_m,a}(kt) = 1.$$

Thus, $\{x_n\}$ is Cauchy sequence with respect to the distribution function F.

We now prove a fixed point theorem in a complete 2-Menger space with two distribution functions.

Theorem 2.1. *Let $(X, \boldsymbol{F}, t)$ be a complete 2-Menger space with continuous t-norm and $t(a, a, a) \geq a$ for all $a \in [0, 1]$. We define for all $x, y, z \in X$ two distribution functions $F_{x,y,z}$ and $F^*_{x,y,z}$ such that $F_{x,y,z}(t) \geq F^*_{x,y,z}(t)$ for all $x, y, z \in X$ and $t \geq 0$. Suppose that T is a continuous self mapping on X into itself satisfying the following conditions:*

$$\alpha F^*_{x,Tx,a}(kt) + \beta F^*_{y,Ty,a}(kt) \geq (1 - \gamma) F^*_{x,y,a}(t) \tag{2.5}$$

and

$$\frac{\alpha F_{x,Tx,a}(kt)F_{y,Ty,a}(kt)}{F_{Tx,Ty,a}(kt)} + \beta F_{Tx,Ty,a}(kt) \geq \gamma F_{x,y,a}(t), \tag{2.6}$$

for all $x, y, a \in X$, $t \geq 0$ and some $k \in (0,1)$, where, $0 < \alpha, \beta, \gamma < 1$, $\alpha+\beta+\gamma = 1$ and $\frac{1}{2} \leq \alpha + \beta < 1$. Then T has a unique fixed point in X.

Proof. Define a sequence $x_n = Tx_{n-1}$. Using inequality (2.5) with $x = x_n$ and $y = x_{n+1}$:

$$\begin{aligned}
\alpha F^*_{x_n,Tx_n,a}(kt) + \beta F^*_{x_{n+1},Tx_{n+1},a}(kt) &\geq (1-\gamma)F^*_{x_n,x_{n+1},a}(t), \\
\alpha F^*_{x_n,x_{n+1},a}(kt) + \beta F^*_{x_{n+1},x_{n+2},a}(kt) &\geq (1-\gamma)F^*_{x_n,x_{n+1},a}(t), \\
\alpha F^*_{x_n,x_{n+1},a}(t) + \beta F^*_{x_{n+1},x_{n+2},a}(kt) &\geq (1-\gamma)F^*_{x_n,x_{n+1},a}(t), \\
F^*_{x_{n+1},x_{n+2},a}(kt) &\geq \frac{1-\alpha-\gamma}{\beta}F^*_{x_n,x_{n+1},a}(t), \\
F^*_{x_{n+1},x_{n+2},a}(kt) &\geq F^*_{x_n,x_{n+1},a}(t).
\end{aligned}$$

By Lemma (2.1) and (2.2), $\{x_n\}$ is a Cauchy sequence in X w.r.t F and F^*. Since X is complete, then x_n converges to a point $u \in X$.
Under the continuity of T, we have

$$Tu = T\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u,$$

which shows that u is a fixed point of T. For the proof of uniqueness, we use (2.6) with $x = u$ and $y = v$ (another fixed point for T).

$$\begin{aligned}
\frac{\alpha F_{u,Tu,a}(kt)F_{v,Tv,a}(kt)}{F_{Tu,Tv,a}(kt)} + \beta F_{Tu,Tv,a}(kt) &\geq \gamma F_{u,v,a}(t), \\
\alpha + \beta(F_{u,v,a}(kt))^2 &\geq \gamma(F_{u,v,a}(t))^2, \\
(F_{u,v,a}(kt))^2 &\geq \frac{\alpha}{\gamma-\beta} \geq 1.
\end{aligned}$$

Thus, $u = v$ and u is the unique fixed point of T.

Now we extend the above theorem for two continuous mappings.

Theorem 2.2. *Let $(X, \boldsymbol{F}, t)$ be a complete 2-Menger space with continuous t-norm , $t(a,a,a) \geq a$ for all $a \in [0,1]$ and two distribution functions F and F^* such that $F_{x,y,z}(t) \geq$*

$F^*_{x,y,z}(t)$ for all $x, y, z \in X$ and $t \geq 0$. Suppose that S and T are two continuous self mappings on X into itself satisfying the following conditions:

$$\alpha F^*_{x,Sx,a}(kt) + \beta F^*_{y,Ty,a}(kt) \geq (1-\gamma) F^*_{x,y,a}(t) \tag{2.7}$$

and

$$\frac{\alpha F_{x,Sx,a}(t) F_{y,Ty,a}(t)}{F_{Sx,Ty,a}(t)} + \beta F_{Sx,Ty,a}(t) \geq \gamma F_{x,y,a}(t), \tag{2.8}$$

for all $x, y, a \in X$, $t \geq 0$ and some $k \in (0,1)$, where, $0 < \alpha, \beta, \gamma < 1$, $\alpha + \beta + \gamma = 1$ and $\frac{1}{2} \leq \alpha + \beta < 1$. Then S and T have a unique fixed point.

Proof. Define a sequence $\{x_n\}$ as $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, $n = 0, 1, 2, \ldots$. Using inequality (2.7) with $x = x_{2n}$ and $y = x_{2n+1}$ gives:

$$\begin{aligned}
\alpha F^*_{x_{2n},Sx_{2n},a}(kt) + \beta F^*_{x_{2n+1},Tx_{2n+1},a}(kt) &\geq (1-\gamma) F^*_{x_{2n},x_{2n+1},a}(t), \\
\alpha F^*_{x_{2n},x_{2n+1},a}(kt) + \beta F^*_{x_{2n+1},x_{2n+2},a}(kt) &\geq (1-\gamma) F^*_{x_{2n},x_{2n+1},a}(t), \\
\alpha F^*_{x_{2n},x_{2n+1},a}(t) + \beta F^*_{x_{2n+1},x_{2n+2},a}(kt) &\geq (1-\gamma) F^*_{x_{2n},x_{2n+1},a}(t), \\
F^*_{x_{2n+1},x_{2n+2},a}(kt) &\geq \frac{1-\alpha-\gamma}{\beta} F^*_{x_{2n},x_{2n+1},a}(t), \\
F^*_{x_{2n+1},x_{2n+2},a}(kt) &\geq F^*_{x_{2n},x_{2n+1},a}(t),
\end{aligned}$$

which shows $\{x_n\}$ be Cauchy sequence in X w.r.t F and F^*. Since X is complete ,then $\{x_n\}$ and all its subsequences converge to a point $u \in X$.

Under the continuity of S and T, we have:

$$Su = S \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$$

and

$$Tu = T \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = u,$$

which shows that u is a common fixed point of S and T. For the uniqueness, we use (2.8) with $x = u$ and $y = v$ (another common fixed point):

$$\begin{aligned}
\frac{\alpha F_{u,Su,a}(t) F_{v,Tv,a}(t)}{F_{Su,Tv,a}(t)} + \beta F_{Su,Tv,a}(t) &\geq \gamma F_{u,v,a}(t), \\
\frac{\alpha F_{u,u,a}(t) F_{v,v,a}(t)}{F_{u,v,a}(t)} + \beta F_{u,v,a}(t) &\geq \gamma F_{u,v,a}(t), \\
\alpha + \beta (F_{u,v,a}(t))^2 &\geq \gamma (F_{u,v,a}(t))^2, \\
(F_{u,v,a}(t))^2 &\geq \frac{\alpha}{\beta - \gamma} \geq 1
\end{aligned}$$

Thus $u = v$ and u is the unique common fixed point of the two mappings.

Now, we state a common fixed point theorem for two self mappings under new contraction condition which can be proved similarly as theorem (2.2).

Theorem 2.3. *Let S, T be continuous self mappings on a complete 2-Menger space $(X, \boldsymbol{F}, t)$ with continuous t-norm and $t(a, a, a) \geq a$ for all $a \in [0, 1]$. Suppose that there exists functions α_i, $i = 1, 2$ of $X \times X \times X$ into $[0, \infty)$ such that:*

- **(i):** $\gamma \equiv \inf\{\alpha_1(x, y, z) + 2\alpha_2(x, y, z) : x, y, z \in X\} \geq 1$;
- **(ii):** *There exists $k \in (0, 1)$ such that:*

$$F_{Sx,Ty,a}(kt) \geq a_1 F_{x,y,a}(kt) + a_2[F_{Sx,x,a}(t) + F_{Ty,y,a}(t)], \tag{2.9}$$

for all $x, y \in X$ and $t > 0$, where $a_i = \alpha_i(x, y, z)$.

Then S and T have a unique common fixed point in X. Finally, we prove a fixed point theorem for non-self mappings considering two 2-Menger spaces.

Theorem 2.4. *Let $(X, \boldsymbol{F}, t_1)$ and $(Y, \boldsymbol{F}^*, t_2)$ be two complete 2-Menger spaces. If T is a mapping of X into Y and S from Y into X satisfying the inequalities:*

$$F^*_{Tx,TSy,u}(kt) \geq \min\{F_{x,Sy,Su}(t), F^*_{Tx,y,u}(t), F^*_{TSy,y,u}(t)\} \tag{2.10}$$

and

$$F_{Sy,STx,v}(kt) \geq \min\{F^*_{y,Tx,Tv}(t), F_{Sy,x,v}(t), F_{STx,x,v}(t)\} \tag{2.11}$$

for some $k \in (0,1)$ and all $x, v \in X$ and $y, u \in Y$. then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. For a fixed $x \in X$, define two sequences x_n and y_n as:

$$x_n = (ST)^n x \,, \quad y_n = Tx_{n-1} \,\,, \, n = 1, 2, ...$$

Putting $x = x_n$ and $y = y_n$ in inequality (2.11) gives:

$$F_{Sy_n, STx_n, v}(kt) \geq \min\{F^*_{y_n, Tx_n, Tv}(t), F_{Sy_n, x_n, v}(t), F_{STx_n, x_n, v}(t)\}.$$

Since, $Sy_n = STx_{n-1} = ST(ST)^{n-1}x = (ST)^n = x_n$ and $STx_n = ST(ST)^n x = x_{n+1}$,then:

$$\begin{aligned} F_{x_n, x_{n+1}, v}(kt) &\geq \min\{F^*_{y_n, y_{n+1}, Tv}(t), F_{x_n, x_n, v}(t), F_{x_{n+1}, x_n, v}(t)\} \\ &\geq \min\{F^*_{y_n, y_{n+1}, Tv}(t), F_{x_{n+1}, x_n, v}(t)\}. \end{aligned}$$

If $\min\{F^*_{y_n, y_{n+1}, Tv}(t), F_{x_{n+1}, x_n, v}(t)\} = F_{x_{n+1}, x_n, v}(t)$, then $F_{x_n, x_{n+1}, v}(kt) \geq F_{x_{n+1}, x_n, v}(t)$, which contradicts the fact that F is a non-decreasing function and $kt < t$ for all $k \in (0,1)$. Thus,

$$F_{x_n, x_{n+1}, v}(kt) \geq F^*_{y_n, y_{n+1}, Tv}(t). \tag{2.12}$$

Using inequality (2.10) with $x = x_{n-1}$, $y = y_n$ and $u = Tv$ yields:

$$\begin{aligned} F^*_{Tx_{n-1}, TSy_n, Tv}(kt) &\geq \min\{F_{x_{n-1}, Sy_n, STv}(t), F^*_{Tx_{n-1}, y_n, Tv}(t), F^*_{TSy_n, y_n, Tv}(t)\} \\ F^*_{y_n, y_{n+1}, Tv}(kt) &\geq \min\{F_{x_{n-1}, x_n, STv}(t), F^*_{y_n, y_n, Tv}(t), F^*_{y_n, y_{n+1}, Tv}(t)\}. \end{aligned}$$

Therefore,

$$F^*_{y_n, y_{n+1}, Tv}(kt) \geq F_{x_{n-1}, x_n, STv}(t). \tag{2.13}$$

From inequalities (2.12) and (2.13) we find:

$$\begin{aligned} F_{x_n,x_{n+1},v}(kt) &\geq F^*_{y_n,y_{n+1},Tv}(t) \\ &\geq F_{x_{n-1},x_n,STv}(\frac{t}{k}) \\ &\geq F_{x_{n-2},x_{n-1},(ST)^2v}(\frac{t}{k^2}) \\ &\vdots \\ &\geq F_{x_0,x_1,(ST)^nv}(\frac{t}{k^n}) \to 1 \;\; as \;\; n \to \infty. \end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence in X which is complete, thus $\{x_n\}$ converges to $z \in X$. Similarly from inequalities (2.13) and (2.12) we have:

$$\begin{aligned} F^*_{y_n,y_{n+1},Tv}(kt) &\geq F_{x_{n-1},x_n,STv}(t) \\ &\geq F^*_{y_{n-1},y_n,TSTv}(\frac{t}{k}) \\ &\vdots \\ &\geq F^*_{y_0,y_1,(TS)^nTv}(\frac{t}{k^n}) \to 1 \;\; as \;\; n \to \infty. \end{aligned}$$

Then $\{y_n\}$ is a Cauchy sequence in Y which is complete, thus $\{y_n\}$ converges to some $w \in Y$.

Using (2.10) with $x = z$ and $y = y_{n-1}$, we gets:

$$\begin{aligned} F^*_{Tz,TSy_{n-1},u}(kt) &\geq \min\{F_{z,Sy_{n-1},Su}(t), F^*_{Tz,y_{n-1},u}(t), F^*_{TSy_{n-1},y_{n-1},u}(t)\} \\ F^*_{Tz,y_n,u}(kt) &\geq \min\{F_{z,x_{n-1},Su}(t), F^*_{Tz,y_{n-1},u}(t), F^*_{y_n,y_{n-1},u}(t)\}. \end{aligned}$$

Letting $n \to \infty$, gives:

$$F^*_{Tz,w,u}(kt) \geq \min\{F_{z,z,Su}(t), F^*_{Tz,w,u}(t), F^*_{w,w,u}(t)\}$$

Therefore,

$$F^*_{Tz,w,u}(kt) \geq F^*_{Tz,w,u}(t). \; forall \;\; t \geq 0, \tag{2.14}$$

which means that $Tz = w$. By a similar way one can prove $Sw = z$.

Consequently, $STz = Sw = z$ and $TSw = Tz = w$.Thus, z is a fixed point of ST and w is a fixed point of TS.

For uniqueness, let $\acute{z}$ be another fixed point for ST ,i.e., $ST\acute{z} = \acute{z}$.
Putting $y = Tz$ and $x = \acute{z}$ in (2.11) to obtain:

$$\begin{aligned} F_{STz,ST\acute{z},v}(kt) &\geq \min\{F^*_{Tz,T\acute{z},Tv}(t), F_{STz,\acute{z},v}(t), F_{ST\acute{z},\acute{z},v}(t)\} \\ F_{z,\acute{z},v}(kt) &\geq \min\{F^*_{Tz,T\acute{z},Tv}(t), F_{z,\acute{z},v}(t), F_{\acute{z},\acute{z},v}(t)\}. \end{aligned}$$

By (2.10):

$$\begin{aligned} F^*_{T\acute{z},TSTz,Tv}(kt) &\geq \min\{F_{\acute{z},STz,STv}(t), F^*_{T\acute{z},Tz,Tv}(t), F^*_{TSTz,Tz,Tv}(t)\} \\ F^*_{T\acute{z},TSTz,Tv}(kt) &\geq F_{\acute{z},z,STv}(t). \end{aligned}$$

Thus,

$$\begin{aligned} F_{z,\acute{z},v}(kt) &\geq F^*_{Tz,T\acute{z},Tv}(t) \\ &\geq F_{z,\acute{z},STv}(\frac{t}{k}) \\ &\vdots \\ &\geq F_{z,\acute{z},(ST)^n v}(\frac{t}{k^n}) \to 1 \ \ as \ \ n \to \infty. \end{aligned}$$

Therefore $z = \acute{z}$ and z is the unique fixed point of ST in X. Also w is the unique fixed point of TS in Y.

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