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Common fixed points of hybrid pairs of mappings in ordered metric spaces

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Abstract

In this paper, we establish a common fixed point theorem for two pairs of mappings satisfying an almost generalized contractive condition for comparable elements in a partially ordered metric space. Our results generalize and extend the main results in [2] to multivalued mappings. As corollaries we obtain the results in [8, 9], and many others.

Keywords: *Ordered metric spaces, Hybrid pairs, Almost contractive conditions, Common fixed points.*
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1 Introduction

In the literature there is a large number of generalizations of metric space. one of these generalization is the partially ordered metric space, i.e., metric space defined on it a partial order.

Definition 1.1 *Any relation between two elements of a set X is called a binary relation over X. A binary relation is called a partial order if it satisfies the following conditions:*

1) $x \preceq x$ *(reflexivity);*

2) *if* $x \preceq y$ *and* $y \preceq x$ *then* $x = y$ *(antisymmetry);*

3) *if* $x \preceq y$ *and* $y \preceq z$ *then* $x \preceq z$ *(transitivity);*

for all $x, y, z \in X$.

A set with a partial order $\preceq$ is called a partially ordered set

Definition 1.2 *Let* $(X, \preceq)$ *be a partially ordered set and* $x, y \in X$. x *and* y *are said to be comparable elements of* X *if either* $x \preceq y$ *or* $y \preceq x$.

Definition 1.3 *Let* $(X, \preceq)$ *be a partially ordered set. A subset* A *of* X *is said to be well ordered if* $a \preceq b$ *for all* $a, b \in A$.

In recent years, many results appeared related to fixed point theorems for single and multivalued mappings on a complete ordered metric space. The first result in this direction was given by Ran and Reurings [18], where they extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, Nieto and Lpez [17] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Therefore, results in this direction were proved , see for example [16, 1, 4].

Let $(X, d, \preceq)$ be an ordered metric space and $B(X)$ be the class of all nonempty and bounded subsets of X. For $A, B \in B(X)$, functions $D(A, B)$ and $\delta(A, B)$ are defined as follows:

$$D(A, B) = inf\{d(a, b) : a \in A, b \in B\},$$

$$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}.$$

If A consists of a single point a, we write

$$D(A, B) = D(a, B) \ \ and \ \ \delta(A, B) = \delta(a, B).$$

Also in addition, If B consists of a single point b, we write

$$D(A, B) = d(a, b) \ \ and \ \ \delta(A, B) = d(a, b).$$

Obviously, $D(A, B) \leq \delta(A, B)$ for all $A, B \in B(X)$, $D(A, a^*) = 0 \Rightarrow a^* \in A$ and $\delta(A, a^*) = 0 \Rightarrow A = \{a^*\}$, for $\delta(A, a^*) = sup\{d(a, a^*) : a \in A\} = 0 \Rightarrow d(a, a^*) = 0 \ \ \texttt{for all} \ \ a \in A, \ \ \texttt{i.e.}, A = \{a^*\}$. The definition of $\delta(A, B)$ yields the following properties:

$$\delta(A, B) = \delta(B, A)$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$$

$$\delta(A, B) = 0 \ \ iff \ \ A = B = a$$

$$\delta(A, A) = diam(A).$$

Fixed point theory of multivalued functions is a vast chapter of functional analysis. In particular, the function $\delta(A, B)$ has been used in many works in this area. Some of these works are noted in [8, 19, 3].

Definition 1.4 *Let $F : X \to 2^X$, (where, 2^X is the power set of X or the set of all subsets of X) be a set valued mapping, i.e., $X \ni x \mapsto F(x)$ is a subset of X. A point $x \in X$ is said to be a fixed point of the set valued mapping F if $x \in F(x)$.*

Definition 1.5 *Let $I : X \to X$ and $F : X \to B(X)$ be single and set valued mappings on X. A point $x \in X$ is said to be a common fixed point I and F if $\{x\} = \{Ix\} = F(x)$.*

We will use the following relations between two nonempty subsets of a partially ordered set.

Definition 1.6 *[3] Let A and B be two nonempty subsets of a partially ordered set $(X, \preceq)$. The relation between A and B is denoted and defined as follows:*

(i) *$A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$,*

(ii) *$A \prec_2 B$, if for every $b \in B$ there exists $a \in A$ such that $a \preceq b$,*

(iii) *$A \prec_3 B$, if $fA \prec_1 B$ and $A \prec_2 B$.*

Definition 1.7 *A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if*

1) *Given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$, for $n = 1, 2, ...$, and $\{a_n\}$ converges to a.*

2) *Given $\epsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\epsilon$ for $n > N$ where, A_ϵ is the union of all open spheres with centers in A and radius ϵ.*

Lemma 1.8 *[11] If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.9 *[11] Let $\{A_n\}$ be a sequence in $B(X)$ and y a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. The area of common fixed point theory, involving four single valued maps, began with the assumption that all of the maps commuted. Afterward Sessa [20] introduced the notion of weakly commuting mappings, which generalized the concept of commuting mappings. Then Jungck [15] generalized this idea, first to compatible mappings and then to weakly compatible mappings [14]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. In [13, 12] Jungck and Rhoades extended definition of compatibility and weak compatibility to set valued mappings setting as follows:

Definition 1.10 *The mapping $I : X \to X$ and $F : X \to B(X)$ are δ-compatible if*

$$\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such the $IFx_n \in B$, $Fx_n \to \{t\}$ and $Ix_n \to t$, for some $t \in X$.

Definition 1.11 *The mapping $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if for each point u in X such that $Fu = \{Iu\}$, we have $FIu = IFu$.*

Definition 1.12 *[1] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$ be two single valued functions. A mapping f is called weak annihilator of g if $fgx \preceq x$ for all $x \in X$.*

Definition 1.13 *[1] A mapping f defined above is called dominating if $x \preceq fx$ for all $x \in X$.*

The concept of almost contractions were introduced by Berinde[5, 6]. Afterwards, almost contractions and its generalizations were further considered in several works like the paper of Ćirić et al. [9] who proved some fixed point results in ordered metric spaces using almost generalized contractive condition, which is given in the following definition.

Definition 1.14 *Let f and g be two self mappings on a metric space (X, d). f and g are said to satisfy almost generalized contractive condition if there exists $\delta \in [0, 1[$ and $L \geq 0$ such that for all $x, y \in X$:*

$$\begin{aligned} d(fx, gy) \leq & \delta \max \{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}\} \\ & + L \min \{d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}. \end{aligned} \tag{1}$$

Therefore, Aghajani et al. [2] generalized the above definition to four self mappings and proved the following theorem in ordered metric spaces.

Theorem 1.15 *Let $(X, \preceq, d)$ be an ordered complete metric space. Let f, g, S and T be self mappings on X, with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ and dominating mappings f and g are weak annihilators of T and S, respectively. Suppose that f and g satisfy:*

$$d(fx, gy) \leq \delta M(x, y) + LN(x, y), \tag{2}$$

where

$$M(x, y) = \max \{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2}\}$$

$$N(x,y) = \min\{d(fx,Sx), d(gy,Ty), d(Sx,gy), d(fx,Ty)\},$$

for every two comparable elements $x, y \in X$ *and* $\delta \in [0,1[$ *and* $L \geq 0$. *If for a nondecreasing sequence* x_n *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies that* $x_n \preceq u$ *and furthermore (a)* $\{f,S\}$ *and* $\{g,T\}$ *are weakly compatible; (b) one of* $f(X), g(X), S(X)$ *and* $T(X)$ *is a closed subspace of* X, *then* f, g, S *and* T *have a common fixed point.*

Finally, Choudhury and Metiya [7] improved the same almost contraction of Ćirić et al. [9] and established the existence of fixed points of multivalued and single valued mappings in Partially ordered metric space.

Theorem 1.16 *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $T : X \to B(X)$ *be a multivalued mapping such that the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $\{x_0\} \prec_1 Tx_0$,

(ii) *for* $x, y \in X$, $x \preceq y$ *implies* $Tx \prec_1 Ty$,

(iii) *if* $x_n \to x$ *is any sequence inXwhose consecutive terms are comparable, then* $x_n \preceq x$, *for all* n,

(iii)

$$\begin{aligned}\delta(Tx,Ty) \leq & \psi(\max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\} \\ & + L\min\{D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}),\end{aligned} \tag{3}$$

for all comparable $x, y \in X$, *where* $L \geq 0$ *and* $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing and upper semi-continuous function with* $\psi(t) \leq t$ *for each* $t > 0$.

Then T *has a fixed point.*

The main aim of this paper is to define dominating and weak annihilator set valued mappings defined on an ordered set and introduce the corresponding existence theorem of common fixed point for these mappings in partially ordered metric space which generalize the above result of Aghajani et al. [2] and others.

2 Main results

Before we state and prove our main result we need to give the following definitions:

Definition 2.1 *Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X \to B(X)$ be multivalued mapping. F is called dominating if $\{x\} \prec_2 Fx$ for all $x \in X$.*

Definition 2.2 *Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X \to B(X)$, $I : X \to X$ be multivalued and single valued mappings. F is called weak annihilator of I if $FIx \prec_2 \{x\}$ for all $x \in X$.*

Example 2.3 *Let $X = [0, \infty[$ be the set of all nonnegative real numbers. Suppose that "$\leq$" be the usual ordering in R, we define a new ordering "$\preceq$" on X as follows:*

$$x \preceq y \Leftrightarrow y \leq x \qquad \forall x, y \in X.$$

Define the mappings $I : X \to X$ and $F : x \to B(X)$ as,

$$Ix = 2x \text{ and } Fx = [0, \frac{x}{2}] \text{ for all } x \in X.$$

Clearly, for any number $p \in Fx = [0, \frac{x}{2}]$ there exists an element $x \in X$ such that $p \leq x$ or $x \preceq p$, then $\{x\} \prec_2 Fx$ for any $x \in X$.
Also, $Ix = 2x$ for any $x \in X$ and $FIx = [0, x]$, i.e., for any element of the singleton $\{x\}$ one can find the same x in the set FIx such that $x \preceq x \Rightarrow FIx \prec_2 \{x\}$ for all $x \in X$.
Thus, the dominating mapping F is weak annihilator of I.

Example 2.4 *Let $X = [0, 1]$ with the usual order, $\preceq = \leq$, be a partially ordered set. Define the mappings $I : X \to X$ and $F : x \to B(X)$ as,*

$$Ix = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ 2x - 1, & x \in (\frac{1}{2}, 1], \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} 0, & x \in [0, \frac{1}{2}], \\ \{x, 1\}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Obviously, F and I commute at their coincidence points, $\{0, 1\}$, then the pair $\{F, I\}$ is weakly compatible. Also we have $\{x\} \prec_2 Fx$ for all $x \in X$, i.e., F is dominating.
Since, $FIx = 0$ if $x \in [0, \frac{3}{4}]$ and $FIx = \{x, 1\}$ if $x \in (\frac{3}{4}, 1]$ then for any $x \in X$ we can find $y \in FIx$ with $y \preceq x$. Thus, $FIx \prec_2 \{x\}$ for all $x \in X$. This means that F is also weak annihilator of I.

Remark 2.5 *Definition 1.11 is a special case of Definition 2.1 as F is a single valued function.*
Indeed, if F is a single valued then $\{x\} \prec_2 Fx \Rightarrow \{x\} \prec_2 \{Fx\} \Rightarrow x \preceq Fx$ for

all $x \in X$.
Also, Definition 1.10 is a special case of Definition 2.2. Apply the same argument, if F *is a single valued then* $FIx \prec_2 \{x\} \Rightarrow \{FIx\} \prec_2 \{x\} \Rightarrow FIx \preceq x$ *for all* $x \in X$.

Theorem 2.6 *Let* $(X, \preceq, d)$ *be an ordered complete metric space. Let* $I, J : X \to X$ *be single valued and* $F, G : X \to B(X)$ *be multivalued mappings, with* $\cup F(X) \subseteq J(X)$ *and* $\cup G(X) \subseteq I(X)$ *and the dominating set valued mappings* F *and* G *are weak annihilators of* J *and* I*, respectively. Suppose that there exists a non decreasing and upper semi-continuous function* $\psi : [0, \infty) \to [0, \infty)$ *with* $\psi(t) \leq t$ *for each* $t > 0$ *and* $L \geq 0$ *such that for every two comparable elements* $x, y \in X$,

$$\delta(Fx, Gy) \leq \psi(M(x, y)) + LN(x, y), \tag{4}$$

where

$$M(x, y) = \max\{d(Ix, Jy), D(Fx, Ix), D(Gy, Jy), \frac{D(Ix, Gy) + D(Fx, Jy)}{2}\}$$

$$N(x, y) = \min\{D(Fx, Ix), D(Gy, Jy), D(Ix, Gy), D(Fx, Jy)\}.$$

If the pairs $\{F, I\}$ *and* $\{G, J\}$ *are weakly compatible and one of* $I(X)$ *and* $J(X)$ *is a closed subspace of* X*. Furthermore, if* $\{x_n\} \prec_2 A_n$ *for all* n *and* $A_n \to A$ *then* $\{x_n\} \prec_2 A$*, where* $\{x_n\}$ *is a singleton subset of* X *and* $A_n, A \in B(X)$*. Then, the set of common fixed points of* F, G, I *and* J *is well ordered if and only if* F, G, I *and* J *have one and only one common fixed point.*

proof Let x_0 be an arbitrary point in X. Since $\cup F(X) \subseteq J(X)$, then $Fx_0 \subseteq J(X)$ that means, there exists at least one point $x_1 \in X$ such that $y_0 = Jx_1 \in Fx_0$. similarly, by $\cup G(X) \subseteq I(X)$, we can find another point $x_2 \in X$ with $y_1 = Ix_2 \in Gx_1$. Continuing this process we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that:

$$\begin{aligned} y_{2n} &= Jx_{2n+1} \in Fx_{2n} \\ y_{2n+1} &= Ix_{2n+2} \in Gx_{2n+1}. \end{aligned} \tag{5}$$

Since F is dominating and weak annihilators of J then $\{x_{2n}\} \prec_2 Fx_{2n} \prec_2 \{Jx_{2n+1}\} \prec_2 FJx_{2n+1} \prec_2 \{x_{2n+1}\}$, note that, since $y_{2n} \in Fx_{2n}$ and $y_{2n} = Jx_{2n+1} \preceq Jx_{2n+1}$, then for each element in the set $\{Jx_{2n+1}\}$ there exists an element y_{2n} in Fx_{2n} with $y_{2n} \preceq Jx_{2n+1}$, i.e., $Fx_{2n} \prec_2 \{Jx_{2n+1}\}$.

Similarly, Since G is dominating and weak annihilators of I then $\{x_{2n+1}\} \prec_2 Gx_{2n+1} \prec_2 \{Ix_{2n+2}\} \prec_2 GIx_{2n+2} \prec_2 \{x_{2n+2}\}$. Thus we have $x_{2n} \preceq x_{2n+1}$ and $x_{2n+1} \preceq x_{2n+2}$ implies that $x_n \preceq x_{n+1}$ for all $n \geq 0$. Now, we show that the sequence $\{y_n\}$ defined in (2.2) is a Cauchy sequence, for this two cases arise, either $y_n = y_{n+1}$ for some n or $y_n \neq y_{n+1}$ for all n.

Case I. If $y_n = y_{n+1}$ for some n. Without any loss of generality, setting $n = 2m$ and using (2.1) to claim that $y_{2m} = y_{2m+1} \Rightarrow y_{2m+1} = y_{2m+2}$ and so on. For this aim we assume that $d(y_{2m+1}, y_{2m+2}) > 0$ and then obtain a contradiction as follows:

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &\leq \delta(Gx_{2m+1}, Fx_{2m+2}) = \delta(Fx_{2m+2}, Gx_{2m+1}) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1})) + LN(x_{2m+2}, x_{2m+1}), \end{aligned} \tag{6}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) =& \max\{d(Ix_{2m+2}, Jx_{2m+1}), D(Fx_{2m+2}, Ix_{2m+2}), D(Gx_{2m+1}, Jx_{2m+1}), \\ & \frac{D(Ix_{2m+2}, Gx_{2m+1}) + D(Fx_{2m+2}, Jx_{2m+1})}{2}\} \\ \leq& \max\{d(y_{2m+1}, y_{2m}), d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), \\ & \frac{d(y_{2m+1}, y_{2m+1}) + d(y_{2m+2}, y_{2m})}{2}\} \\ =& \max\{0, d(y_{2m+2}, y_{2m+1}), 0, \frac{d(y_{2m+2}, y_{2m})}{2}\} \\ =& d(y_{2m+2}, y_{2m+1}), \end{aligned}$$

and

$$\begin{aligned} N(x_{2m+2}, x_{2m+1}) =& \min\{D(Fx_{2m+2}, Ix_{2m+2}), D(Gx_{2m+1}, Jx_{2m+1}), D(Ix_{2m+2}, Gx_{2m+1}), \\ & D(Fx_{2m+2}, Jx_{2m+1})\} \\ =& \min\{d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), d(y_{2m+1}, y_{2m+1}), d(y_{2m+2}, y_{2m})\} \\ \leq& \min\{d(y_{2m+2}, y_{2m+1}), 0, 0, d(y_{2m+1}, y_{2m})\} \\ =& 0. \end{aligned}$$

Therefore, $d(y_{2m+1}, y_{2m+2}) \leq \psi(d(y_{2m+2}, y_{2m+1}))$ that contradicts with the properties of ψ. Thus, $d(y_{2m+1}, y_{2m+2}) = 0 \Rightarrow y_{2m+1} = y_{2m+2}$. By a similar way one can prove that $y_{2m+2} = y_{2m+3}$ proceeding in this manner, it follows that $y_{2m} = y_{2m+k}$ or $y_n = y_{n+k}$ for each $k \geq 1$. Thus, in this cases $\{y_n\}$ is Cauchy sequence since, we can find an integer $N = n$ with $d(y_m, y_{m+p}) = 0$ for all $m > N$.

Case II. If $y_n \neq y_{n+1}$ for all n. we shall prove that $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$, for this purpose we replace x by x_{2n} and y by x_{2n+1} in condition (2.1),

$$d(y_{2n}, y_{2n+1}) \leq \delta(Fx_{2n}, Gx_{2n+1}) \leq \psi(M(x_{2n}, x_{2n+1})) + LN(x_{2n}, x_{2n+1}), \tag{7}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) =& \max\{d(Ix_{2n}, Jx_{2n+1}), D(Fx_{2n}, Ix_{2n}), D(Gx_{2n+1}, Jx_{2n+1}), \\ & \frac{D(Ix_{2n}, Gx_{2n+1}) + D(Fx_{2n}, Jx_{2n+1})}{2}\} \\ \leq & \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}\} \\ =& \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2}\}, \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) =& \min\{D(Fx_{2n}, Ix_{2n}), D(Gx_{2n+1}, Jx_{2n+1}), D(Ix_{2n}, Gx_{2n+1}), D(Fx_{2n}, Jx_{2n+1})\} \\ \leq & \min\{d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n})\} \\ =& 0. \end{aligned}$$

Since, $\frac{d(y_{2n-1}, y_{2n+1})}{2} \leq \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}$, it follows that

$$d(y_{2n}, y_{2n+1}) \leq \psi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}). \tag{8}$$

If $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$, then it follows from inequality (2.5) that:

$$d(y_{2n}, y_{2n+1}) \leq \psi(d(y_{2n}, y_{2n+1}))$$

which implies that $d(y_{2n}, y_{2n+1}) = 0$, contradicting our assumption that $y_n \neq y_{n+1}$ for all n. Thus

$$d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n}). \tag{9}$$

Again applying the considered contractive condition with $x = x_{2n}$ and $y = x_{2n-1}$, we have

$$d(y_{2n-1}, y_{2n}) \leq \delta(Gx_{2n-1}, Fx_{2n}) = \delta(Fx_{2n}, Gx_{2n-1}) \leq \psi(M(x_{2n}, x_{2n-1})) + LN(x_{2n}, x_{2n-1}), \tag{10}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n-1}) =& \max\{d(Ix_{2n}, Jx_{2n-1}), D(Fx_{2n}, Ix_{2n}), D(Gx_{2n-1}, Jx_{2n-1}), \\ & \frac{D(Ix_{2n}, Gx_{2n-1}) + D(Fx_{2n}, Jx_{2n-1})}{2}\} \\ \leq & \max\{d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n-2}), \frac{d(y_{2n-1}, y_{2n-1}) + d(y_{2n}, y_{2n-2})}{2}\} \\ =& \max\{d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1}), \frac{d(y_{2n}, y_{2n-2})}{2}\} \\ =& \max\{d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1})\}, \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n-1}) =& \min\{D(Fx_{2n}, Ix_{2n}), D(Gx_{2n-1}, Jx_{2n-1}), D(Ix_{2n}, Gx_{2n-1}), D(Fx_{2n}, Jx_{2n-1})\} \\ \leq& \min\{d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n-2}), d(y_{2n-1}, y_{2n-1}), d(y_{2n}, y_{2n-2})\} \\ =& 0. \end{aligned}$$

If $d(y_{2n-1}, y_{2n-2}) \leq d(y_{2n}, y_{2n-1})$, we obtain a contradiction, then

$$d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1}). \tag{11}$$

Therefore, $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all n and $\{d(y_n, y_{n+1})\}$ is a monotone decreasing and bounded sequence of nonnegative real numbers. Then there exists a $r > 0$ such that, $d(y_n, y_{n+1}) \to r$ as $n \to \infty$.
From (2.4), $d(y_{2n}, y_{2n+1}) \leq \psi(d(y_{2n-1}, y_{2n}))$. Then passing to the upper limit and using the upper semi-continuity of ψ yields $r \leq \limsup_{n\to\infty\psi(d(y_{2n-1},y_{2n}))\leq\psi(r)}$, a contradiction unless $r = 0$. Thus

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{12}$$

Next, we claim that $\{y_n\}$ is a Cauchy sequence. Suppose the contrary, then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that for all k,

$$\begin{aligned} &n_k > m_k > k, \\ &d(y_{2m_k}, y_{2n_k}) \geq \epsilon \text{ and} \\ &d(y_{2m_k}, y_{2n_k-1}) < \epsilon. \end{aligned}$$

By the triangle inequality, $\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) < \epsilon + d(y_{2n_k-1}, y_{2n_k})$. Taking the limit as $k \to \infty$ and using (2.9) tends to

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = 0. \tag{13}$$

Also we have
$d(y_{2m_k-1}, y_{2n_k}) \leq d(y_{2m_k-1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k})$ and
$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2n_k})$.
Taking the limit at $k \to \infty$ in the above inequalities and using (2.9) and (2.10), we get

$$\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) = \epsilon. \tag{14}$$

By a similar way we can prove that

$$\lim_{k\to\infty} d(y_{2m_k-1}, y_{2n_k+1}) = \epsilon. \tag{15}$$

Also we have
$d(y_{2m_k}, y_{2n_k+1}) \leq d(y_{2m_k}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1})$ and

$d(y_{2m_k-1}, y_{2n_k+1}) \leq d(y_{2m_k-1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k+1})$.
Taking the limit at $k \to \infty$ in the above inequalities and using (2.9), (2.10) and (2.12), we get

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k+1}) = \epsilon. \quad (16)$$

Now in a position to apply the contractive condition and use the previous limits, to obtain

$$d(y_{2m_k}, y_{2n_k+1}) \leq \delta(Fx_{2m_k}, Gx_{2n_k+1}) \leq \psi(M(x_{2m_k}, x_{2n_k+1})) + LN(x_{2m_k}, x_{2n_k+1}), \quad (17)$$

where

$$\begin{aligned} M(x_{2m_k}, x_{2n_k+1}) &= \max\{d(Ix_{2m_k}, Jx_{2n_k+1}), D(Fx_{2m_k}, Ix_{2m_k}), D(Gx_{2n_k+1}, Jx_{2n_k+1}), \\ &\quad \frac{D(Ix_{2m_k}, Gx_{2n_k+1}) + D(Fx_{2m_k}, Jx_{2n_k+1})}{2}\} \\ &\leq \max\{d(y_{2m_k-1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k-1}), d(y_{2n_k+1}, y_{2n_k}), \\ &\quad \frac{d(y_{2m_k-1}, y_{2n_k+1}) + d(y_{2m_k}, y_{2n_k})}{2}\} \\ &\to \max\{\epsilon, 0, 0, \frac{\epsilon+\epsilon}{2}\} = \epsilon, \end{aligned}$$

and

$$N(x_{2m_k}, x_{2n_k+1}) \to 0 \text{ as } k \to \infty.$$

Passing to the upper limit in Eq. (2.14), $\epsilon \leq \limsup_{k\to\infty\psi(M(x_{2m_k}, x_{2n_k+1}))\leq\psi(\epsilon)}$, which contradicting with the fact that $\epsilon > 0$. Then we deduce that $\{y_n\}$ is Cauchy sequence in X.
Since X is complete then there exists a point p in X such that $\{y_n\}$ converges to this point. Therefore, $y_{2n} = Jx_{2n+1} \to p$ and $y_{2n+1} = Ix_{2n+2} \to p$. Since $Jx_{2n+1} \in Fx_{2n}$ for all n and $Jx_{2n+1} \to p$ then by Def. (1.5) $Fx_{2n} \to \{p\}$. Also. $Gx_{2n+1} \to \{p\}$. Assume that $I(X)$ is closed then there is an element u in X with $p = Iu$.
Also, $\{x_{2n+1}\} \prec_2 Gx_{2n+1} \to \{p\} \Rightarrow \{x_{2n+1}\} \prec_2 \{p\} \Rightarrow x_{2n+1} \preceq p$ for all n, and $\{x_{2n+1}\} \prec_2 \{p\} = \{Iu\} \prec_2 GIu \prec_2 \{u\} \Rightarrow x_{2n+1} \preceq u$ for all n. Using inequality (2.1) with $x = u$ and $y = x_{2n+1}$ implies $Fu = p$ as follows:

$$\delta(Fu, Gx_{2n+1}) \leq \psi(M(u, x_{2n+1})) + LN(u, x_{2n+1}), \quad (18)$$

where

$$\begin{aligned} M(u, x_{2n+1}) &= \max\{d(Iu, Jx_{2n+1}), D(Fu, Iu), D(Gx_{2n+1}, Jx_{2n+1}), \frac{D(Iu, Gx_{2n+1}) + D(Fu, Jx_{2n+1}}{2} \\ &\to D(Fu, p) \text{ as } n \to \infty, \end{aligned}$$

and

$$N(u, x_{2n+1}) = \min\{D(Fu, Iu), D(Gx_{2n+1}, Jx_{2n+1}), D(Iu, Gx_{2n+1}), D(Fu, Jx_{2n+1})\} \to 0 \ \texttt{as} \ n \to$$

Taking the limit when $n \to \infty$ in (2.5), we get $\delta(Fu, p) \le \psi(D(Fu, p)) \le \psi(\delta(Fu, p)) \Rightarrow \delta(Fu, p) = 0$. Then, $Fu = \{p\}$. (Note that , $D(Fu, p) \le \delta(Fu, p) \Rightarrow D(Fu, p) \le \delta(Fu, p) \le \alpha D(Fu, p) \Rightarrow D(Fu, p) = 0 \Rightarrow p \in Fu$, and $\delta(Fu, p) \le \alpha D(Fu, p) \le \alpha\delta(Fu, p) \Rightarrow \delta(Fu, p) = 0 \Rightarrow Fu = \{p\}$). Thus, we have $Fu = \{Iu\} = \{p\}$. By Using the weak compatibility of the pair $\{F, I\}$ we get $Fp = FIu = IFu = \{Iu\}$. We in the position to show that $Fp = \{p\}$, for this we set $x = p$ and $y = x_{2n+1}$ in (2.1) and then take the limit at n tends to infinity as follows:

$$\delta(Fp, Gx_{2n+1}) \le \psi(M(p, x_{2n+1})) + LN(p, x_{2n+1}), \tag{19}$$

where

$$\begin{aligned} M(p, x_{2n+1}) =& \max\{d(Ip, Jx_{2n+1}), D(Fp, Ip), D(Gx_{2n+1}, Jx_{2n+1}), \frac{D(Ip, Gx_{2n+1}) + D(Fp, Jx_{2n+1})}{2} \\ &\to D(Fp, p) \ \texttt{as} \ n \to \infty \end{aligned}$$

and

$$\begin{aligned} N(p, x_{2n+1}) =& \min\{D(Fp, Ip), D(Gx_{2n+1}, Jx_{2n+1}), D(Ip, Gx_{2n+1}), D(Fp, Jx_{2n+1})\} \\ =& 0 \end{aligned}$$

This implies

$$Fp = \{Tp\} = \{p\}. \tag{20}$$

Since $\cup F(X) \subseteq J(X)$, then $Fu \subseteq J(X)$, i.e., there exists $\nu \in X$ with $J\nu \in Fu = \{Iu\} = \{p\}$. Thus, we have $J\nu = Iu = p$. Now we show that $G\nu = \{J\nu\} = \{p\}$. Using inequality (2.1) with $x = u$ and $y = \nu$ gets:

$$\delta(J\nu, G\nu) \le \delta(Fu, G\nu) \le \psi(M(u, \nu)) + LN(u, \nu), \tag{21}$$

where

$$\begin{aligned} M(u, \nu) =& \max\{d(Iu, J\nu), D(Fu, Iu), D(G\nu, J\nu), \frac{D(Iu, G\nu) + D(Fu, J\nu)}{2}\} \\ =& \max\{0, 0, D(G\nu, J\nu), \frac{D(J\nu, G\nu)}{2}\} \\ =& D(G\nu, J\nu), \end{aligned}$$

and

$$\begin{aligned} N(u, \nu) =& \min\{D(Fu, Iu), D(G\nu, J\nu), D(Iu, G\nu), D(Fu, J\nu)\} \\ =& \min\{0, D(G\nu, J\nu), D(Iu, G\nu), 0\} \\ =& 0. \end{aligned}$$

So $\delta(J\nu, G\nu) \leq \psi(D(G\nu, J\nu))$. Thus, $G\nu = \{J\nu\} = \{p\}$.
By Using the weak compatibility of the pair $\{G, J\}$ we get $Gp = GJ\nu = JG\nu = \{Jp\}$. Putting $x = p$ and $y = p$ in (2.1) and then take the limit at n tends to infinity yields:

$$\delta(p, Gp) \leq \psi(M(p,p)) + LN(p,p), \tag{22}$$

where

$$\begin{aligned} M(p,p) =& \max\{d(Ip, Jp), D(Fp, Ip), D(Gp, Jp), \frac{D(Ip, Gp) + D(Fp, Jp)}{2}\} \\ =& \max\{D(p, Gp), 0, 0, \frac{D(p, Gp) + D(p, Gp)}{2}\} \\ =& D(Gp, p), \end{aligned}$$

and

$$\begin{aligned} N(p,p) =& \min\{D(Fp, Ip), D(Gp, Jp), D(Ip, Gp), D(Fp, Jp)\} \\ =& 0 \end{aligned}$$

This implies

$$Gp = \{Jp\} = \{p\}. \tag{23}$$

From Eqs. (2.7) and (2.10) we say that p is a common fixed point of F, G, I and J.
Now suppose that the set of common fixed points of the four mappings is well comparable and there is another common fixed point $q \in X$ with $p \neq q$.Then we have:

$$d(p,q) \leq \delta(Fp, Gq) \leq \psi(M(p,q)) + LN(p,q), \tag{24}$$

where

$$\begin{aligned} M(p,q) =& \max\{d(Ip, Jq), D(Fp, Ip), D(Gq, Jq), \frac{D(Ip, Gq) + D(Fp, Jq)}{2}\} \\ \leq& \max\{d(p,q), 0, 0, \frac{d(p,q) + d(p,q)}{2}\} \\ =& d(p,q), \end{aligned}$$

and

$$\begin{aligned} N(p,p) =& \min\{D(Fp, Ip), D(Gq, Jq), D(Ip, Gq), D(Fp, Jq)\} \\ =& 0. \end{aligned}$$

Then, $d(p,q) \leq \psi(d(p,q))$. i.e., $p = q$. Conversely, if F, G, I and J have only one common fixed point then the set of the common fixed point of these four mappings being singleton which is well ordered.

Remark 2.7 *Let $F : X \to B(X)$ be multi valued mapping defined on the set X and I_X be the identity mapping on the same set X. If F is dominating and weak annihilator of I_X, then we have $\{x\} \prec_2 Fx$ and $Fx \prec_2 \{x\}$ for all $x \in X$, this means, $x \preceq y$ for all $y \in Fx$ and there exist $z \in Fx$ with $z \preceq x$. Thus we have $x \preceq z$ and $z \preceq x$ for some z. Then $x = z \in Fx$. If we consider A_n and A to be singleton subsets of X, i.e. $A_n = \{y_n\}$ and $A = \{y\}$, then the relation $\{x_n\} \prec_2 A_n$ is equivalent to this one $x_n \preceq y_n$, $A_n \to A \Leftrightarrow y_n \to y$ and $\{x_n\} \prec_2 A \Rightarrow \{x_n\} \prec_2 \{y\} \Rightarrow x \preceq y$.*

Takin $\psi(t) = \alpha t$, $\alpha \in [0, 1[$ and considering F and G be single valued mappings in Theorem 2.1, we have the following result

Corollary 2.8 *Let $(X, \preceq, d)$ be an ordered complete metric space. Let $F, G, I, J : X \to X$ be single valued mappings, with $F(X) \subseteq J(X)$ and $G(X) \subseteq I(X)$ and the dominating mappings F and G are weak annihilators of J and I, respectively. Suppose that there exists an $\alpha \in [0, 1[$ and $L \geq 0$ such that for every two comparable elements $x, y \in X$,*

$$d(Fx, Gy) \leq \alpha M(x, y) + LN(x, y), \tag{25}$$

where

$$M(x, y) = \max\{d(Ix, Jy), d(Fx, Ix), d(Gy, Jy), \frac{d(Ix, Gy) + d(Fx, Jy)}{2}\}$$
$$N(x, y) = \min\{d(Fx, Ix), d(Gy, Jy), d(Ix, Gy), d(Fx, Jy)\}.$$

If the pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible and one of $I(X)$ and $J(X)$ is a closed subspace of X. Furthermore, if $x_n \preceq y_n$ for all n and $y_n \to y$ then $x_n \preceq y$. Then, the set of common fixed points of F, G, I and J is well ordered if and only if F, G, I and J have one and only one common fixed point.

The following result is a natural consequence of Theorem 2.1 when $F = G$ and $I = J = I_X$.

Corollary 2.9 *Let $(X, \preceq, d)$ be an ordered complete metric space. Let $F : X \to B(X)$ be multivalued mappings with $x \in Fx$ for all $x \in X$. Suppose that there exists a non decreasing and upper semi-continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) \leq t$ for each $t > 0$ and $L \geq 0$ such that for every two comparable elements $x, y \in X$,*

$$\delta(Fx, Fy) \leq \psi(M(x, y)) + LN(x, y), \tag{26}$$

where

$$M(x, y) = \max\{d(x, y), D(Fx, x), D(Fy, y), \frac{D(x, Fy) + D(Fx, y)}{2}\}$$
$$N(x, y) = \min\{D(Fx, x), D(Fy, y), D(x, Fy), D(Fx, y)\}.$$

If $x_n \preceq y_n$ for all n and $y_n \to y$ then $x_n \preceq y$. Then, the set of fixed points of F is well ordered if and only if F has one and only one fixed point.

3 Open Problem

In this section we should present open problems

1 Is Theorem 2.1 true in ordered cone metric space.

2 Is Theorem 2.1 true for hybrid pairs of occasionally weakly compatible mappings.

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