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Coupled fixed point theorems by altering distances between points in partially ordered metric spaces

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Abstract

In this paper, some coupled coincidence and common fixed point theorems for two self mappings have been derived which satisfy certain inequality involving a function of two variables that measures the distance between points in ordered metric spaces. For particular choices of the function several generalizations of many fixed point theorems which contain altering distance functions may be obtained. Our results can be applied directly to study multidimensional fixed point theorems which cover the concepts of coupled , tripled, quadruple fixed point etc.

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1 Introduction

Fixed point theory in metric and partially ordered metric spaces has a vast literature. The theory of fixed points has become an important tool in non-linear functional analysis. In particular, there has been a number of works on fixed points involving altering distance functions, see for example [3, 4, 7, 8, 13, 18, 19]. Jha et. al [9] deals with survey work on some fixed point theorems by altering distances between points in metric spaces. In [11], Marr defined the concept of convergence in partially ordered metric space. Also, he tried to obtain a relation between metric space and partially ordered metric space, and claimed that the fixed point theorems in metric spaces are the particular cases of fixed point results in partially ordered metric spaces. On the other hand, the first result on existence of fixed points in partially ordered sets was given by Ran and Reurings [16] who extended the Banach contraction principle in partially ordered sets and presented some applications to linear and nonlinear matrix equations.

Definition 1.1. [11] A partially ordered space is a set X with a binary relation $\preceq$, which satisfy the three conditions:

(1) $x \preceq x$ for all $x \in X$;

(2) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in X$;

(3) $x \preceq y$ and $y \preceq x$ implies $x = y$ for all $x, y \in X$.

Definition 1.2. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff:

(1) (X, d) is a metric space, and

(2) $(X, \preceq)$ is partially ordered.

Bhaskar and lakshmikantham [2] introduced the notion of a mixed monotone property and a coupled fixed point for contractive operator of the form $F : X \times X \to X$, where X is a partially ordered set and then established some existence and uniqueness for fixed and coupled fixed points of F (x is a fixed point of F if $F(x, x) = x$). They also illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem. In [5] lakshmikantham and $\acute{C}iric$ extended these results by defining the mixed g-monotone property and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered metric spaces.

Definition1.3. [5] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \;\; g(x_1) \preceq g(x_2) \;\text{ implies }\; F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \;\; g(y_1) \preceq g(y_2) \;\text{ implies }\; F(x, y_1) \succeq F(x, y_2).$$

If g is the identity mapping, we obtain the Bhaskar and lakshmikantham's notion of a mixed monotone property of the mapping F.

Definition 1.4. [5] An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), F(y, x) = g(y).$$

Also, if g is the identity mapping, then (x, y) is called a coupled fixed point of the mapping F.

Let $C(F, g)$ be the set of all coupled coincidence points of F and g.

Definition 1.5. Let $F : X \times X \to X$ and $g : X \to X$ be two single valued mappings. The mappings F and g are said to be weakly compatible or $w-$ compatible if they commute at their coincidence points, i.e., if $(x, y) \in C(F, g)$ then $g(F(x, y)) = F(gx, gy)$.

Definition 1.6. The mapping g is said to be F-weakly commuting at $(x, y) \in X \times X$ if $g^2x = F(gx, gy)$ and $g^2y = F(gy, gx)$.

Delbosco [6] and Skof [20] obtained fixed point theorems for self mappings of complete metric spaces by altering the distances between the points with the use of a function $\phi : R^+ \to R^+$ satisfying the following properties:

1) ϕ is continuous and increasing,

2) $\phi(t) = 0 \;\Leftrightarrow\; t = 0$,

3) $\phi(t) \geq Mt^{\mu}$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constants.

The set of all such functions ϕ is denoted by Φ. In [[20], Corol. 2] the following theorem was proved:

Theorem 1.1. *[20] Let F be a self mapping of a complete metric space (X, d) and $\phi \in \Phi$ such that for every $x, y \in X$,*

$$\phi(d(Fx, Fy)) \leq a\phi(d(x, y)) + b\phi(d(x, Fx)) + c\phi(d(y, Fy)), \tag{1.1}$$

where, $0 \leq a + b + c < 1$. Then F has a unique fixed point.

Then, Khan et al. [10] generalized the above result by assuming stronger condition than (1.1) without using definition (3) above.

Theorem 1.2. *[10] Let F be a self mapping of a complete metric space (X, d) and $\phi : R^+ \to R^+$ an increasing, continuous function satisfying property (2). Furthermore, let a, b, c be three decreasing functions from $R^+ \setminus \{0\}$ into $[0, 1)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition:*

$$\begin{aligned}\phi(d(Fx, Fy)) \leq\ & a(d(x, y))\phi(d(x, y)) + b(d(x, y))\{\phi(d(x, Fx)) + \phi(d(y, Fy))\} \\ & + c(d(x, y)) \min\{\phi(d(x, Fy)), \phi(d(y, Fx))\},\end{aligned} \tag{1.2}$$

where, $x, y \in X$ and $x \neq y$. Then F has a unique fixed point.

Rashwan and Sadeek [15] established a common fixed point theorem in complete metric spaces which generalizes some results in [10]. In [[12], Corol. 2], Naidu and Visakhapatnam obtained the following result theorems that generalize Theorem 1.2.

Theorem 1.3. *[12] Let (X, d) be a metric space, F be a self mapping on X, $\theta : R^+ \to [0, 1]$ be a monotonically decreasing function with $\theta(t) < 1$ $\forall t \in (0, \infty)$ and ρ be a nonnegative real valued function on $X \times X$ with the following two properties:*

(i) *$\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ is convergent whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in X such that $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is convergent;*

(i) *for any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X, the sequence $\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ converges to zero iff the sequence $\{d(x_n, y_n)\}_{n=1}^{\infty}$ converges to zero.*

Suppose that

$$\rho(Fx, Fy) \leq \theta(d(x, y)) \max\{\rho(x, y), \frac{1}{2}[\rho(x, Fx) + \rho(y, Fy)], [\rho(x, Fy)\rho(Fx, y)]^{\frac{1}{2}}\}, \tag{1.3}$$

for all $x, y \in X$. Then for any $x \in X$, $\{F^n x\}$ is Cauchy. For any $x_0 \in X$, the limit of $\{F^n x_0\}$, if it exists, is the unique fixed point of F.

The following lemma of Babu and Sailaja [1] will be used in the sequel.

Lemma 1.1. *[1] Suppose (X, d) is a metric space, let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$, $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k - 1}, x_{n_k}) < \epsilon$. By using triangle inequality one can get:*

$$d(x_{m_k}, x_{n_k}), d(x_{m_k+1}, x_{n_k+1}) \to \epsilon$$

and

$$d(x_{m_k-1}, x_{n_k}), d(x_{m_k}, x_{n_k-1}) \to \epsilon.$$

Motivated by the result of Naidu and Visakhapatnam [12], we generalize Theorem 1.3 for two single valued mappings $F : X \times X \to X$ and $g : X \to X$ having the mixed monotone property and involving generalized altering distance functions and obtain coupled coincidence and coupled common fixed point results for these mappings in partially ordered metric space. Under special choices of the mapping ρ we can obtain a generalization of the result of Khan et al. [10] and many others.

2 Main Result

Throughout this paper, unless otherwise stated, N is the set of all positive integers and R^+ is the set of all nonnegative real numbers.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space, $F : X \times X \to X$ be a mapping having the $g-$mixed monotone property, where $g : X \to X$, $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Suppose that for any $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$ we have*

$$\begin{aligned}\rho\big(F(x,y),F(u,v)\big) \leq \frac{1}{2}\big[\theta_1\big(d(gx,gu)\big)+\theta_2\big(d(gy,gv)\big)\big] & \\ \max\Big\{\rho(gx,gu),\rho(gy,gv),\frac{\rho\big(gx,F(x,y)\big)+\rho\big(gu,F(u,v)\big)}{2},\frac{\rho\big(gy,F(y,x)\big)+\rho\big(gv,F(v,u)\big)}{2}, & \\ \big[\rho(gx,F(u,v))\rho(gu,F(x,y))\big]^{\frac{1}{2}},\big[\rho(gy,F(v,u))\rho(gv,F(y,x))\big]^{\frac{1}{2}}\Big\}, & \end{aligned} \tag{2.1}$$

where $\theta_1, \theta_2 : R^+ \to [0,1]$ are decreasing functions satisfying

$$\theta_1(t)+\theta_2(t)\begin{cases} <1, & t>0 \\ =1, & t=0. \end{cases}$$

and ρ is a nonnegative real valued function on $X \times X$ has the following properties

(i) *$\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ is convergent whenever $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in X such that $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is convergent.*

(ii) *$\{\rho(x_n, y_n)\}_{n=1}^{\infty}$ converges to $\rho(x, y)$ whenever $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in X such that $\{d(x_n, y_n)\}_{n=1}^{\infty}$ converges to $d(x, y)$.*

(iii) *for any sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X, the sequence $\{\rho(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to zero iff the sequence $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to zero too.*

From properties (ii) and (iii) of ρ, respectively, we note that $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ iff $x = y$ for any $x, y \in X$.
Assume that X has the following properties:

1. *if a non-decreasing sequence $x_n \to x \in X$, then $x_n \preceq x$ for all n;*

2. *if a non-increasing sequence $y_n \to y \in X$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exists $(x, y) \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$.

Proof. For the existence points $x_0, y_0 \in X$ and using $F(X \times X) \subseteq g(X)$, we can find $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Also for $x_1, y_1 \in X$ there exist $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way, we construct two sequences $\{gx_n\}_{n\geq 0}$ and $\{gy_n\}_{n\geq 0}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \ \forall n \geq 0.$$

Since F has the $g-$mixed monotone property, then $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ yield

$$\begin{aligned}
&gx_0 \preceq gx_1 \text{ and } gy_0 \succeq gy_1 \\
&F(x_0, y_0) \preceq F(x_1, y_1) \text{ and } F(y_0, x_0) \succeq F(y_1, x_1) \\
&\Rightarrow gx_1 \preceq gx_2 \text{ and } gy_1 \succeq gy_2 \\
&\vdots \\
&\Rightarrow gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1} \ \forall n \geq 0.
\end{aligned}$$

If $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n then $gx_n = F(x_n, y_n)$ and $gy_n = F(gy_n, gx_n)$, i.e., (x_n, y_n) is a coupled coincidence point of F and g and this complete the proof. So from now on, we assume that either $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n. Since $gx_{n-1} \preceq gx_n$ and $gy_{n-1} \succeq gy_n$, then from (2.1), we have

$$\begin{aligned}
\rho(gx_n, gx_{n+1}) &= \rho\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\
&\leq \frac{1}{2}\big[\theta_1\big(d(gx_{n-1}, gx_n)\big) + \theta_2\big(d(gy_{n-1}, gy_n)\big)\big] \max\Big\{\rho(gx_{n-1}, gx_n), \rho(gy_{n-1}, gy_n), \\
&\frac{\rho\big(gx_{n-1}, F(x_{n-1}, y_{n-1})\big) + \rho\big(gx_n, F(x_n, y_n)\big)}{2}, \frac{\rho\big(gy_{n-1}, F(y_{n-1}, x_{n-1})\big) + \rho\big(gy_n, F(y_n, x_n)\big)}{2}, \\
&\big[\rho(gx_{n-1}, F(x_n, y_n))\rho(gx_n, F(x_{n-1}, y_{n-1}))\big]^{\frac{1}{2}}, \big[\rho(gy_{n-1}, F(y_n, x_n))\rho(gy_n, F(y_{n-1}, x_{n-1}))\big]^{\frac{1}{2}}\Big\} \\
&\leq \frac{1}{2}\big[\theta_1\big(d(gx_{n-1}, gx_n)\big) + \theta_2\big(d(gy_{n-1}, gy_n)\big)\big] \\
&\max\Big\{\rho(gx_{n-1}, gx_n), \rho(gy_{n-1}, gy_n), \frac{\rho(gx_{n-1}, gx_n) + \rho(gx_n, gx_{n+1})}{2}, \frac{\rho(gy_{n-1}, gy_n) + \rho(gy_n, gy_{n+1})}{2}, \\
&\qquad \big[\rho(gx_{n-1}, gx_{n+1})\rho(gx_n, gx_n)\big]^{\frac{1}{2}}, \big[\rho(gy_{n-1}, gy_{n+1})\rho(gy_n, gy_n)\big]^{\frac{1}{2}}\Big\} \\
&\leq \frac{1}{2}\big[\theta_1\big(d(gx_{n-1}, gx_n)\big) + \theta_2\big(d(gy_{n-1}, gy_n)\big)\big] \\
&\max\Big\{\rho(gx_{n-1}, gx_n), \rho(gy_{n-1}, gy_n), \frac{\rho(gx_{n-1}, gx_n) + \rho(gx_n, gx_{n+1})}{2}, \frac{\rho(gy_{n-1}, gy_n) + \rho(gy_n, gy_{n+1})}{2}\Big\}.
\end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}
\rho(gy_{n+1}, gy_n) &\leq \frac{1}{2}\big[\theta_1\big(d(gy_n, gy_{n-1})\big) + \theta_2\big(d(gx_n, gx_{n-1})\big)\big] \\
&\max\Big\{\rho(gy_n, gy_{n-1}), \rho(gx_n, gx_{n-1}), \frac{\rho(gy_n, gy_{n+1}) + \rho(gy_{n-1}, gy_n)}{2}, \frac{\rho(gx_n, gx_{n+1}) + \rho(gx_{n-1}, gx_n)}{2}\Big\}.
\end{aligned} \tag{2.3}$$

Note that, if

$$\max\left\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n),\frac{\rho\big(gx_{n-1},gx_n\big)+\rho\big(gx_n,gx_{n+1}\big)}{2},\frac{\rho\big(gy_{n-1},gy_n\big)+\rho\big(gy_n,gy_{n+1}\big)}{2}\right\}$$
$$=\frac{\rho\big(gx_{n-1},gx_n\big)+\rho\big(gx_n,gx_{n+1}\big)}{2}$$
$$\Rightarrow \rho(gx_{n-1},gx_n)\leq\frac{\rho\big(gx_{n-1},gx_n\big)+\rho\big(gx_n,gx_{n+1}\big)}{2}$$
$$\Rightarrow \rho(gx_{n-1},gx_n)\leq \rho(gx_n,gx_{n+1}).$$

But we have from (2.2) and by using the properties on θ_1 and θ_2

$$\rho(gx_n,gx_{n+1})\leq\frac{1}{2}\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n)\}\big)\big]$$
$$\frac{\rho\big(gx_{n-1},gx_n\big)+\rho\big(gx_n,gx_{n+1}\big)}{2}$$
$$\Rightarrow \rho(gx_n,gx_{n+1})<\rho\big(gx_{n-1},gx_n\big).$$

By a similar way, we can get antonym inequalities if we assume that
$\max\left\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n),\frac{\rho\left(gx_{n-1},gx_n\right)+\rho\left(gx_n,gx_{n+1}\right)}{2},\frac{\rho(gy_{n-1},gy_n)+\rho\left(gy_n,gy_{n+1}\right)}{2}\right\}$
$=\frac{\rho\left(gy_{n-1},gy_n\right)+\rho\left(gy_n,gy_{n+1}\right)}{2}$.
Hence

$$\rho(gx_n,gx_{n+1})\leq\frac{1}{2}\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)\big]$$
$$\max\big\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n)\big\}$$
$$\rho(gy_n,gy_{n+1})\leq\frac{1}{2}\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)\big]$$
$$\max\big\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n)\big\}.$$

That is

$$\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\leq\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)\big]$$
$$\max\big\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n)\big\}$$
$$\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\leq\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)\big]$$
$$\big[\rho(gx_{n-1},gx_n)+\rho(gy_{n-1},gy_n)\big]. \tag{2.4}$$

Note that for any two real numbers a,b, $\max\{a,b\}\leq a+b$.

Therefore, we have $\{\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\}_{n=1}^{\infty}$ is decreasing sequence of positive real numbers bounded below by zero. Then, there is some $s\geq 0$ such that

$$\lim_{n\to\infty}\big[\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\big]=s.$$

Now we claim that $s=0$, for this purpose we assume $s>0$ and discuss two cases to obtain a contradiction.

Case(I) If one of the two equalities $x_n=x_{n-1}$ or $y_n=y_{n-1}$ holds, say $x_n=x_{n-1}$.

In this case we have $\rho(x_n,x_{n-1})=0$, $\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}=zero$ and

$\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)=1$. Then Eq. (2.4) yields

$$\begin{aligned}
&\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\le\\
&\quad\big[\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)\big]\\
&\quad\max\big\{\rho(gx_{n-1},gx_n),\rho(gy_{n-1},gy_n)\big\}\\
&\quad\le\rho(gy_{n-1},gy_n)\\
&\quad\le\frac{1}{2}\big[\theta_1\big(\min\{d(gx_{n-2},gx_{n-1}),d(y_{n-2},y_{n-1})\}\big)+\theta_2\big(\min\{d(gx_{n-2},gx_{n-1}),d(y_{n-2},y_{n-1})\}\big)\big]\\
&\quad\max\big\{\rho(gx_{n-2},gx_{n-1}),\rho(gy_{n-2},gy_{n-1})\big\}\\
&\quad\le\frac{1}{2}\big[\theta_1\big(\min\{d(gx_{n-2},gx_{n-1}),d(y_{n-2},y_{n-1})\}\big)+\theta_2\big(\min\{d(gx_{n-2},gx_{n-1}),d(y_{n-2},y_{n-1})\}\big)\big]\\
&\quad\big[\rho(gx_{n-2},gx_{n-1})+\rho(gy_{n-2},gy_{n-1})\big]\\
&\quad\le\frac{1}{2}\big[\rho(gx_{n-2},gx_{n-1})+\rho(gy_{n-2},gy_{n-1})\big].
\end{aligned}$$

Taking the limit as $n\to\infty$ above, implies $s\le\frac{1}{2}s$, which contradicts with $s>0$.

Case(II) If $x_n\ne x_{n-1}$ and $y_n\ne y_{n-1}$, then

$$\theta_1\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)+\theta_2\big(\min\{d(gx_{n-1},gx_n),d(y_{n-1},y_n)\}\big)<1$$

and

$$\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})<\rho(gx_{n-1},gx_n)+\rho(gy_{n-1},gy_n).$$

Again by taking limit of both sides as $n\to\infty$ we get a contradiction.

Therefore, $s=0$ and $\lim_{n\to\infty}\big[\rho(gx_n,gx_{n+1})+\rho(gy_n,gy_{n+1})\big]=0$.

$$\lim_{n\to\infty}\rho(gx_n,gx_{n+1})=\lim_{n\to\infty}\rho(gy_n,gy_{n+1})=0. \tag{2.5}$$

By virtue of property (iii) of ρ, we have

$$\lim_{n\to\infty}d(gx_n,gx_{n+1})=\lim_{n\to\infty}d(gy_n,gy_{n+1})=0. \tag{2.6}$$

Now we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X,d). Suppose the contrary, that $\{gx_n\}$ and $\{gy_n\}$ are not Cauchy sequences. Then, Lemma 1.1 implies that there exist $\epsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers (for all positive integer k, $m_k>n_k$) such that

$$d(gx_{m_k},gx_{n_k}),d(gx_{m_k+1},gx_{n_k+1}),d(gx_{n_k},gx_{m_k+1}),d(gx_{n_k+1},gx_{m_k})\to\epsilon$$

and

$$d(gy_{m_k},gy_{n_k}),d(gy_{m_k+1},gy_{n_k+1}),d(gy_{n_k},gy_{m_k+1}),d(gy_{n_k+1},gy_{m_k})\to\epsilon.$$

Using property (i) of ρ yields

$$\begin{aligned}
&\rho(gx_{m_k},gx_{n_k}),\rho(gx_{m_k+1},gx_{n_k+1}),\rho(gx_{n_k},gx_{m_k+1}),\rho(gx_{n_k+1},gx_{m_k}),\\
&\rho(gy_{m_k},gy_{n_k}),\rho(gy_{m_k+1},gy_{n_k+1}),\rho(gy_{n_k},gy_{m_k+1}),\rho(gy_{n_k+1},gy_{m_k})\to\tau>0.
\end{aligned}\tag{2.7}$$

Since $m_k \geq n_k$, so $gx_{m_k} \succeq gx_{n_k}$ and $gy_{n_k} \preceq gx_{m_k}$, now we can use Eq. (2.1).

$$\rho(x_{m_k+1}, x_{n_k+1}) = \rho\big(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k})\big) \leq$$
$$\frac{1}{2}\big[\theta_1\big(d(gx_{m_k}, gx_{n_k})\big) + \theta_2\big(d(gy_{m_k}, gy_{n_k})\big)\big] \max\Big\{\rho(gx_{m_k}, gx_{n_k}), \rho(gy_{m_k}, gy_{n_k}),$$
$$\frac{\rho\big(gx_{m_k}, F(x_{m_k}, y_{m_k})\big) + \rho\big(gx_{n_k}, F(x_{n_k}, y_{n_k})\big)}{2}, \frac{\rho\big(gy_{m_k}, F(y_{m_k}, x_{m_k})\big) + \rho\big(gy_{n_k}, F(y_{n_k}, x_{n_k})\big)}{2},$$
$$\big[\rho(gx_{m_k}, F(x_{n_k}, y_{n_k}))\rho(gx_{n_k}, F(x_{m_k}, y_{m_k}))\big]^{\frac{1}{2}}, \big[\rho(gy_{m_k}, F(y_{n_k}, x_{n_k}))\rho(gy_{n_k}, F(y_{m_k}, x_{m_k}))\big]^{\frac{1}{2}}\Big\}$$
$$\leq \frac{1}{2}\big[\theta_1\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big) + \theta_2\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big)\big]$$
$$\max\Big\{\rho(gx_{m_k}, gx_{n_k}), \rho(gy_{m_k}, gy_{n_k}), \frac{\rho\big(gx_{m_k}, gx_{m_k+1}\big) + \rho\big(gx_{n_k}, gx_{n_k+1}\big)}{2},$$
$$\frac{\rho\big(gy_{m_k}, gy_{m_k+1}\big) + \rho\big(gy_{n_k}, gy_{n_k+1}\big)}{2},$$
$$\big[\rho(gx_{m_k}, gx_{n_k+1})\rho(gx_{n_k}, gx_{m_k+1})\big]^{\frac{1}{2}}, \big[\rho(gy_{m_k}, gy_{n_k+1})\rho(gy_{n_k}, gy_{m_k+1})\big]^{\frac{1}{2}}\Big\}. \tag{2.8}$$

Taking the limit superior as k tends to infinity and using ((2.5) and (2.7)) to get

$$\tau \leq \frac{1}{2}\limsup_{k\to\infty}\big[\theta_1\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big) + \theta_2\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big)\big]$$
$$\max\{\tau, \tau, 0, 0, \tau, \tau\}$$
$$\tau < \tau.$$

By a similar way we can get same inequality for $\{gy_n\}$. This is a contradiction. We deduce that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Note that, $\theta_1\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big) + \theta_2\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big) \leq 1 \;\; \forall \;\; k \in \mathbb{N} \Rightarrow$

$$\limsup_{k\to\infty}\big[\theta_1\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big) + \theta_2\big(\min\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k})\}\big)\big] \leq 1$$

Since $g(X)$ is complete, so there exist points gx and gy in $g(X)$ such that

$$gx_n \to gx \text{ and } gy_n \to gy \text{ as } n \to \infty. \tag{2.9}$$

Now we show that (x, y) is coupled coincidence point for F and g. For this purpose we shall use (2.1) with $u = x_n$ and $v = y_n$, then take limit superiors on both sides as n tends to infinity and use the property (ii) of ρ.

$$\rho(F(x, y), gx_{n+1}) + \rho(F(y, x), gy_{n+1}) \leq \big[\theta_1\big(\min\{d(gx, gx_n), d(gy, gy_n)\}\big) + \theta_2\big(\min\{d(gx, gx_n), d(gy, y_n)\}\big)\big]$$
$$\max\big\{\rho(gx, gx_n), \rho(gy, gy_n), \frac{\rho(gx, F(x, y)) + \rho(gx_n, gx_{n+1})}{2}, \frac{\rho(gy, F(y, x)) + \rho(gy_n, gy_{n+1})}{2},$$
$$[\rho(gx, gx_{n+1})\rho(gx_n, F(x, y))]^2, [\rho(gy, gy_{n+1})\rho(gy_n, F(y, x))]^2\big\},$$
$$\rho(F(x, y), gx) + \rho(F(y, x), gy) \leq \limsup_{n\to\infty}\big[\theta_1\big(\min\{d(gx, gx_n), d(gy, gy_n)\}\big) + \theta_2\big(\min\{d(gx, gx_n), d(gy, y_n)\}\big)\big]$$
$$\max\big\{\frac{\rho(gx, F(x, y))}{2}, \frac{\rho(gy, F(y, x))}{2}\big\}$$
$$\leq [\frac{\rho(gx, F(x, y)) + \rho(gy, F(y, x))}{2}].$$

Note that, since $d(F(x,y),gx_{n+1}) \to d(F(x,y),gx)$, then property (ii) of ρ implies

$$\rho(F(x,y),gx_{n+1}) \to \rho(F(x,y),gx) \text{ as } n \to \infty.$$

From the above inequality we have $\rho(F(x,y),gx)+\rho(F(y,x),gy) = 0 \Rightarrow \rho(F(x,y),gx) = \rho(F(y,x),gy) = 0 \Rightarrow F(x,y) = gx$ and $F(x,y) = gx$. Hence (x,y) is a coupled coincidence point for F and g. □

Remark 2.1. We can replace the conditions on the set X by the continuity of the commutative mappings F and g.
Indeed, $g(gx_{n+1}) = g(F(x_n,y_n)) = F(gx_n,gy_n)$ and $g(gy_{n+1}) = g(F(y_n,x_n)) = F(gy_n,gx_n)$. Taking limit at $n \to \infty$ and using (2.9) implies, $g(gx) = F(gx,gy)$ and $g(gy) = F(gy,gx)$. That is (gx,gy) is a coincidence point for F and g.

The conditions of Theorem 2.1 are not enough to prove the existence of coupled common fixed point for the mappings F and g. We apply an additional conditions to obtain the following common fixed point theorem.

Theorem 2.2. *By adding to the hypotheses of Theorem 2.1 one of the following conditions,*

(a) *F and g are $w-$compatible, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x,y) \in C(F,g)$, $u,v \in X$ and g is continuous at u and v. Also consider that $g^n x \preceq g^{n+1}x$ and $g^n y \succeq g^{n+1}y$ for all n.*

(b) *g is F-weakly commuting for some $(x,y) \in X \times X$, $g^2 x = gx$ and $g^2 y = gy$.*

(c) *g is continuous at (x,y) for some $(x,y) \in C(F,g)$ and there exist $u,v \in X$ with $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$.*

(d) *$g(C(F,g))$ is singleton subset of $C(F,g)$.*

then F and g have a coupled common fixed point.

Proof. Suppose that (a) holds. Since Theorem 2.1 ensure the existence of at least one coupled coincidence point for F and g, say $(x,y) \in C(F,g)$, then the two limits $\lim_{n\to\infty} g^n x$ and $\lim_{n\to\infty} g^n y$ exist and equal u and v (res.), for some $u,v \in X$. By using the continuity of g at u and v, we obtain

$$u = \lim_{n\to\infty} g^{n+1}x = \lim_{g^n x \to u} g(g^n x) = gu \tag{2.10}$$

and

$$v = \lim_{n\to\infty} g^{n+1}y = \lim_{g^n y \to v} g(g^n y) = gv. \tag{2.11}$$

From F and g are w-compatible, $gx = F(x,y)$ and $gy = F(y,x)$, we can get

$$g(g(x)) = g(F(x,y)) = F(gx,gy) \tag{2.12}$$

and

$$g(g(y)) = g(F(y,x)) = F(gy,gx).$$

Thus,

$$(gx,gy) \in C(F,g).$$

Again by the w-compatibility of F and g and Eq. (2.12), we obtain that

$$g(g^2x) = g(F(gx, gy)) = F(g^2x, g^2y).$$

Similarly, we have

$$g(g^2y) = F(g^2y, g^2x).$$

So,

$$(g^2x, g^2y) \in C(F, g)$$

Continuing this process, we can get that

$$g^n x = F(g_{n-1}x, g^{n-1}y), \ \ g^n y = F(g^{n-1}y, g^{n-1}x) \tag{2.13}$$

and

$$(g^n x, g^n y) \in C(F, g) \ \texttt{for all} \ n \geq 1. \tag{2.14}$$

Since, $\{g^n x\}$ is non-decreasing sequence and $g^n x \to u = gu \in X$, then $g^n x \preceq gu \ \forall \ n$, also we have $g^n y \preceq gv \ \forall \ n$. Put $x = g^{n-1}x$ and $y = g^{n-1}y$ in Eq. (2.1) yields

$$\begin{aligned}\rho\big(g^n x, F(u,v)\big) = \rho\big(F(g^{n-1}x, g^{n-1}y), F(u,v)\big) \leq \frac{1}{2}\big[\theta_1\big(d(g^n x, gu)\big) + \theta_2\big(d(g^n y, gv)\big)\big] \max\Big\{\rho(g^n x, gu), \rho(g^n y, gv),\\ \frac{\rho\big(g^n x, F(g^{n-1}x, g^{n-1}y)\big) + \rho\big(gu, F(u,v)\big)}{2}, \frac{\rho\big(g^n y, F(g^{n-1}y, g^{n-1}x)\big) + \rho\big(gv, F(v,u)\big)}{2},\\ \big[\rho(g^n x, F(u,v))\rho(gu, F(g^{n-1}x, g^{n-1}y))\big]^{\frac{1}{2}}, \big[\rho(g^n y, F(v,u))\rho(gv, F(g^{n-1}y, g^{n-1}x))\big]^{\frac{1}{2}}\Big\}\end{aligned} \tag{2.15}$$

Similarly,

$$\begin{aligned}\rho\big(g^n y, F(v,u)\big) = \rho\big(F(g^{n-1}y, g^{n-1}x), F(v,u)\big) \leq \frac{1}{2}\big[\theta_1\big(d(g^n y, gv)\big) + \theta_2\big(d(g^n x, gu)\big)\big] \max\Big\{\rho(g^n y, gv), \rho(g^n x, gu),\\ \frac{\rho\big(g^n y, F(g^{n-1}y, g^{n-1}x)\big) + \rho\big(gv, F(v,u)\big)}{2}, \frac{\rho\big(g^n x, F(g^{n-1}x, g^{n-1}y)\big) + \rho\big(gu, F(u,v)\big)}{2},\\ \big[\rho(g^n y, F(v,u))\rho(gv, F(g^{n-1}y, g^{n-1}x))\big]^{\frac{1}{2}}, \big[\rho(g^n x, F(u,v))\rho(gu, F(g^{n-1}x, g^{n-1}y))\big]^{\frac{1}{2}}\Big\}\end{aligned} \tag{2.16}$$

Adding Equations (2.15) and (2.16) and taking limit superior, imply

$$\begin{aligned}\rho\big(u, F(u,v)\big) + \rho\big(v, F(v,u)\big) &\leq \limsup_{n\to\infty}\big[\theta_1\big(\min\{d(g^n x, gu), d(g^n y, gv)\}\big) + \theta_2\big(\min\{d(g^n x, gu), d(g^n y, gv)\}\big)\big]\\ &\max\Big\{\rho(u, gu), \rho(v, gv), \frac{0 + \rho\big(gu, F(u,v)\big)}{2}, \frac{0 + \rho\big(gv, F(v,u)\big)}{2},\\ &\quad \big[\rho(u, F(u,v))\rho(gu, u)\big]^{\frac{1}{2}}, \big[\rho(v, F(v,u))\rho(gv, v)\big]^{\frac{1}{2}}\Big\}\\ &\leq \frac{1}{2}\limsup_{n\to\infty}\big[\theta_1\big(\min\{d(g^n x, gu), d(g^n y, gv)\}\big) + \theta_2\big(\min\{d(g^n x, gu), d(g^n y, gv)\}\big)\big]\\ &\max\Big\{0, 0, \rho\big(gv, F(v,u)\big), \rho\big(gu, F(u,v)\big)\Big\}.\end{aligned}$$

This occurs only if $\rho\big(u, F(u,v)\big) = \rho\big(v, F(v,u)\big) = 0$. That is, $u = F(u,v)$ and $v = F(v,u)$. Hence (u,v) is a coupled common fixed point of F and g.

Suppose that (b) holds. Then for some $(x,y) \in X \times X$, $gx = g(gx) = F(gx,gy)$ and $gy = g(gy) = F(gy,gx)$. Hence (gx,gy) is a coupled common fixed point of F and g.

Suppose that (c) holds. It follows from the continuity of g at x,y that $x = \lim_{n\to\infty} g^{n+1}u = \lim_{g^n u \to x} g(g^n u) = gx = F(x,y)$ and $y = gy = F(y,x)$. Hence (x,y) is a coupled common fixed point of F and g.

Finally, suppose that (d) holds. Say $C(F,g) = \{(x,y),(u,v),...\}$, we have $g(c(F,g)) = \{(gx,gy)\}$ and $g\big(C(F,g)\big) = \{(x,y)\}$ for some $(x,y) \in C(F,g)$, i.e., $x = gx = F(x,y)$ and $y = gy = F(y,x)$. Hence (x,y) is a coupled common fixed point of F and g. $\square$

From Theorem 2.1, one can obtain the following corollary which is a generalization of Theorem 2 in [14].

Corollary 2.1. *Let $(X,\preceq,d)$ be a partially ordered metric space and $F : X \times X \to X$ be a mapping having the $g-$mixed monotone property, where $g : X \to X$ and $F(X \times X) \subseteq g(X)$. Suppose that for any $x,y,u,v \in X$ with $gx \preceq gu$ and $gy \succeq gv$ we have*

$$\begin{aligned}\psi\big(d(F(x,y),F(u,v))\big) \le\ & a_1\big(d(gx,gu)\big)\Big[\psi\big(d(gx,gu)\big) + c_1\big[\psi\big(d(gx,F(u,v))\big)\psi\big(d(gu,F(x,y))\big)\big]^{\frac{1}{2}}\Big] \\ & + b_1\big(d(gx,gu)\big)\big[\psi\big(d(gx,F(x,y))\big) + \psi\big(d(gu,F(u,v))\big)\big] \\ & + a_2\big(d(gy,gv)\big)\Big[\psi\big(d(gy,gv)\big) + c_2\big[\psi\big(d(gy,F(v,u))\big)\psi\big(d(gv,F(y,x))\big)\big]^{\frac{1}{2}}\Big] \\ & + b_2\big(d(gy,gv)\big)\big[\psi\big(d(gy,F(y,x))\big) + \psi\big(d(gv,F(v,u))\big)\big],\end{aligned} \tag{2.17}$$

where ψ is an altering function and a_i, b_i, $i = 1,2$ are decreasing functions from $[0,\infty)$ into $[0,1)$ such that $a_1(t) + a_2(t) + b_1(t) + b_2(t) < \frac{1}{4}$ for every $t > 0$, c_i are constants in $[0,1]$ such that $a_1(t)(1+c_1) + a_2(t)(1+c_2) < 1$ $\forall t > 0$. If X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x \in X$, then $x_n \preceq x$ for all n and*

(ii) *if a non-increasing sequence $y_n \to y \in X$, then $y_n \succeq y$ for all n.*

and there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$, then F and g have a coupled coincidence point in X, that is, there exists $(x,y) \in X$ such that $F(x,y) = gx$ and $F(y,x) = gy$.

Theorem 2.3. *Let $(X,\preceq)$ be a partially ordered set and d be a metric on X, that is, $(X,\preceq,d)$ is an ordered metric space. Let $F : X \times X \to X$ be a mapping having the $g-$mixed monotone property, where $g : X \to X$ and $F(X \times X) \subseteq g(X)$. Suppose that for any $x,y,u,v \in X$ with $gx \preceq gu$ and*

$gy \succeq gv$ we have

$$\begin{aligned}\psi\Big(d\big(F(x,y),F(u,v)\big)\Big) \le\ & a_1\big(d(gx,gu)\big)\psi\big(d(gx,gu)\big)\\ &+ b_1\big(d(gx,gu)\big)\frac{\psi\big(d(gx,F(x,y)\big)+\psi\big(d(gu,F(u,v)\big)}{2}\\ &+ c_1\big(d(gx,gu)\big)\min\big\{\psi\big(d(gx,F(u,v)\big),\psi\big(d(gu,F(x,y)\big)\big\}\\ &+ a_2\big(d(gy,gv)\big)\psi\big(d(gy,gv)\big)\\ &+ b_2\big(d(gy,gv)\big)\frac{\psi\big(d(gy,F(y,x)\big)+\psi\big(d(gv,F(v,u)\big)}{2}\\ &+ c_2\big(d(gy,gv)\big)\min\big\{\psi\big(d(gy,F(v,u)\big),\psi\big(d(gv,F(y,x)\big)\big\},\end{aligned} \tag{2.18}$$

where ψ is an altering function and a_i, b_i, c_i, $i = 1, 2$ are decreasing functions from $[0, \infty)$ into $[0, 1)$ such that $a_1(t) + a_2(t) + b_1(t) + b_2(t) + c_1(t) + c_2(t) < 1$ for every $t > 0$ and $a_1(t) + a_2(t) + b_1(t) + b_2(t) + c_1(t) + c_2(t) = 1$ if $t = 0$. If X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x \in X$, then $x_n \preceq x$ for all n and*

(ii) *if a non-increasing sequence $y_n \to y \in X$, then $y_n \succeq y$ for all n.*

and there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exists $(x, y) \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$.

Proof. For the existence points $x_0, y_0 \in X$ and using $F(X \times X) \subseteq g(X)$, we can find $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Also for $x_1, y_1 \in X$ there exist $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way, we construct two sequences $\{gx_n\}_{n\ge 0}$ and $\{gy_n\}_{n\ge 0}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \texttt{ and } gy_{n+1} = F(y_n, x_n)\ \forall n \ge 0.$$

Since F has the $g-$mixed monotone property, $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then we obtain

$$\begin{aligned}&gx_0 \preceq gx_1 \texttt{ and } gy_0 \succeq gy_1\\ &F(x_0, y_0) \preceq F(x_1, y_1) \texttt{ and } F(y_0, x_0) \preceq F(y_1, x_1)\\ &\Rightarrow gx_1 \preceq gx_2 \texttt{ and } gy_1 \preceq gy_2\\ &\vdots\\ &\Rightarrow gx_n \preceq gx_{n+1} \texttt{ and } gy_n \preceq gy_{n+1}\ \forall n \ge 0.\end{aligned}$$

If $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some n then $gx_n = F(x_n, y_n)$ and $gy_n = F(gy_n, gx_n)$, i.e., (x_n, y_n) is a coupled coincidence point of F and g and this complete the proof. So from now on, we assume that either $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n. Since $gx_n \preceq gx_{n+1}$, then from (2.18), we have

$$\begin{aligned}\psi\big(d(gx_n, gx_{n+1})\big) = \psi\bigg(d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big)\bigg) &\le a_1\big(d(gx_{n-1}, gx_n)\big)\psi\big(d(gx_{n-1}, gx_n)\big)\\ &+ b_1\big(d(gx_{n-1}, gx_n)\big)\frac{\psi\big(d(gx_{n-1}, F(x_{n-1}, y_{n-1}))\big) + \psi\big(d(gx_n, F(x_n, y_n))\big)}{2}\\ &+ c_1\big(d(gx_{n-1}, gx_n)\big)\min\big\{\psi\big(d(gx_{n-1}, F(x_n, y_n))\big), \psi\big(d(gx_n, F(x_{n-1}, y_{n-1}))\big)\big\}\\ &+ a_2\big(d(gy_{n-1}, gy_n)\big)\psi\big(d(gy_{n-1}, gy_n)\big)\\ &+ b_2\big(d(gy_{n-1}, gy_n)\big)\frac{\psi\big(d(gy_{n-1}, F(y_{n-1}, x_{n-1}))\big) + \psi\big(d(gy_n, F(y_n, x_n))\big)}{2}\\ &+ c_2\big(d(gy_{n-1}, gy_n)\big)\min\big\{\psi\big(d(gy_{n-1}, F(y_n, x_n))\big), \psi\big(d(gy_n, F(y_{n-1}, x_{n-1}))\big)\big\}\end{aligned}$$

$$\begin{aligned}&\le a_1\big(d(gx_{n-1}, gx_n)\big)\psi\big(d(gx_{n-1}, gx_n)\big)\\ &+ b_1\big(d(gx_{n-1}, gx_n)\big)\frac{\psi\big(d(gx_{n-1}, gx_n) + \psi\big(d(gx_n, gx_{n+1})\big)}{2}\\ &+ c_1\big(d(gx_{n-1}, gx_n)\big)\min\big\{\psi\big(d(gx_{n-1}, gx_{n+1})\big), \psi\big(d(gx_n, gx_n)\big)\big\}\\ &+ a_2\big(d(gy_{n-1}, gy_n)\big)\psi\big(d(gy_{n-1}, gy_n)\big)\\ &+ b_2\big(d(gy_{n-1}, gy_n)\big)\frac{\psi\big(d(gy_{n-1}, gy_n)\big) + \psi\big(d(gy_n, gy_{n+1})\big)}{2}\\ &+ c_2\big(d(gy_{n-1}, gy_n)\big)\min\big\{\psi\big(d(gy_{n-1}, gy_{n+1})\big), \psi\big(d(gy_n, gy_n)\big)\big\}\\ &\le a_1\big(d(gx_{n-1}, gx_n)\big)\psi\big(d(gx_{n-1}, gx_n)\big)\\ &+ b_1\big(d(gx_{n-1}, gx_n)\big)\frac{\psi\big(d(gx_{n-1}, gx_n) + \psi\big(d(gx_n, gx_{n+1})\big)}{2}\\ &+ a_2\big(d(gy_{n-1}, gy_n)\big)\psi\big(d(gy_{n-1}, gy_n)\big)\\ &+ b_2\big(d(gy_{n-1}, gy_n)\big)\frac{\psi\big(d(gy_{n-1}, gy_n)\big) + \psi\big(d(gy_n, gy_{n+1})\big)}{2}.\end{aligned} \tag{2.19}$$

By a similar way we have

$$\begin{aligned}\psi\big(d(gy_n, gy_{n+1})\big) &\le a_1\big(d(gy_{n-1}, gy_n)\big)\psi\big(d(gy_{n-1}, gy_n)\big)\\ &+ b_1\big(d(gy_{n-1}, gy_n)\big)\frac{\psi\big(d(gy_{n-1}, gy_n) + \psi\big(d(gy_n, gy_{n+1})\big)}{2}\\ &+ a_2\big(d(gx_{n-1}, gx_n)\big)\psi\big(d(gx_{n-1}, gx_n)\big)\\ &+ b_2\big(d(gx_{n-1}, gx_n)\big)\frac{\psi\big(d(gx_{n-1}, gx_n)\big) + \psi\big(d(gx_n, gx_{n+1})\big)}{2}.\end{aligned} \tag{2.20}$$

Adding the two above equations (2.19) and (2.20) implies

$$\begin{aligned}&\psi\big(d(gx_n, gx_{n+1})\big) + \psi\big(d(gy_n, gy_{n+1})\big)\\ &\le \left(a_1\big(d(gx_{n-1}, gx_n)\big) + a_2\big(d(gx_{n-1}, gx_n)\big) + \frac{b_1\big(d(gx_{n-1}, gx_n)\big) + b_2\big(d(gx_{n-1}, gx_n)\big)}{2}\right)\psi\big(d(gx_{n-1}, gx_n)\big)\\ &+ \left(a_2\big(d(gy_{n-1}, gy_n)\big) + a_1\big(d(gy_{n-1}, gy_n)\big) + \frac{b_2\big(d(gy_{n-1}, gy_n)\big) + b_1\big(d(gy_{n-1}, gy_n)\big)}{2}\right)\psi\big(d(gy_{n-1}, gy_n)\big)\\ &+ \left(\frac{b_1\big(d(gx_{n-1}, gx_n)\big) + b_2\big(d(gx_{n-1}, gx_n)\big)}{2}\right)\psi\big(d(gx_n, gx_{n+1})\big)\\ &+ \left(\frac{b_2\big(d(gy_{n-1}, gy_n)\big) + b_1\big(d(gy_{n-1}, gy_n)\big)}{2}\right)\psi\big(d(gy_n, gy_{n+1})\big)\end{aligned}$$

Consider, $\rho_{n-1} = \min\big\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\big\}$. Since, a_1, a_2, b_1 and b_2 are all decreasing functions, then we have

$$\begin{aligned} \rho_{n-1} &\leq d(gx_{n-1}, gx_n) \\ &\leq d(gy_{n-1}, gy_n) \\ \Rightarrow a_1(\rho_{n-1}) &\geq a_1\big(d(gx_{n-1}, gx_n)\big) \\ &\geq a_1\big(d(gy_{n-1}, gy_n)\big). \end{aligned} \tag{2.21}$$

The same argument can also be for a_2, b_1 and b_2.

$$\begin{aligned} \psi\big(d(gx_n,gx_{n+1})\big) &+ \psi\big(d(gy_n, gy_{n+1})\big) \\ &\leq \left(a_1(\rho_{n-1}) + a_2(\rho_{n-1}) + \frac{b_1(\rho_{n-1}) + b_2(\rho_{n-1})}{2}\right)\psi\big(d(gx_{n-1}, gx_n)\big) \\ &+ \left(a_2(\rho_{n-1}) + a_1(\rho_{n-1}) + \frac{b_2(\rho_{n-1}) + b_1(\rho_{n-1})}{2}\right)\psi\big(d(gy_{n-1}, gy_n)\big) \\ &+ \left(\frac{b_1(\rho_{n-1}) + b_2(\rho_{n-1})}{2}\right)\psi\big(d(gx_n, gx_{n+1})\big) \\ &+ \left(\frac{b_2(\rho_{n-1}) + b_1(\rho_{n-1})}{2}\right)\psi\big(d(gy_n, gy_{n+1})\big). \end{aligned}$$

Hence,

$$\begin{aligned} \psi\big(d(gx_n,gx_{n+1})\big) &+ \psi\big(d(gy_n, gy_{n+1})\big) \\ &\leq \big(a_1(\rho_{n-1}) + a_2(\rho_{n-1}) + b_1(\rho_{n-1})/2 + b_2(\rho_{n-1})/2\big)\psi\big(d(gx_{n-1}, gx_n)\big) \\ &+ \big(a_2(\rho_{n-1}) + a_1(\rho_{n-1}) + b_2(\rho_{n-1})/2 + b_1(\rho_{n-1})/2\big)\psi\big(d(gy_{n-1}, gy_n)\big) \\ &+ \big(b_1(\rho_{n-1})/2 + b_2(\rho_{n-1})/2\big)\Big[\psi\big(d(gx_n, gx_{n+1})\big) + \psi\big(d(gy_n, gy_{n+1})\big)\Big] \end{aligned}$$

$$\begin{aligned} \psi\big(d(gx_n,gx_{n+1})\big) &+ \psi\big(d(gy_n, gy_{n+1})\big) \\ &\leq \frac{a_1(\rho_{n-1}) + a_2(\rho_{n-1}) + b_1(\rho_{n-1})/2 + b_2(\rho_{n-1})/2}{1 - b_1(\rho_{n-1})/2 - b_2(\rho_{n-1})/2}\psi\big(d(gx_{n-1}, gx_n)\big) \\ &+ \frac{a_2(\rho_{n-1}) + a_1(\rho_{n-1}) + b_2(\rho_{n-1})/2 + b_1(\rho_{n-1})/2}{1 - b_1(\rho_{n-1})/2 - b_2(\rho_{n-1})/2}\psi\big(d(gy_{n-1}, gy_n)\big) \\ &. \end{aligned}$$

$$\begin{aligned} \psi\big(d(gx_n,gx_{n+1})\big) &+ \psi\big(d(gy_n, gy_{n+1})\big) \\ &\leq \frac{a_1(\rho_{n-1}) + a_2(\rho_{n-1}) + b_1(\rho_{n-1})/2 + b_2(\rho_{n-1})/2}{1 - b_1(\rho_{n-1})/2 - b_2(\rho_{n-1})/2}\Big[\psi\big(d(gx_{n-1}, gx_n)\big) + \psi\big(d(gy_{n-1}, gy_n)\big)\Big]. \end{aligned}$$

Let $\theta(\rho_{n-1}) = \frac{a_1(\rho_{n-1})+a_2(\rho_{n-1})+b_1(\rho_{n-1})/2+b_2(\rho_{n-1})/2}{1-b_1(\rho_{n-1})/2-b_2(\rho_{n-1})/2} < 1$ and $\tau_n = \psi\big(d(gx_n, gx_{n+1})\big)+\psi\big(d(gy_n, gy_{n+1})\big)$. Then we have

$$\begin{aligned} \tau_n &\leq \theta(\rho_{n-1})\tau_{n-1} \\ &< \tau_{n-1}. \end{aligned} \tag{2.22}$$

Note that $\theta(\rho_{n-1})$ can not equal zero for all n. For, if $\theta(\rho_{n-1}) = 0 \Rightarrow \psi\big(d(gx_n, gx_{n+1})\big) + \psi\big(d(gy_n, gy_{n+1})\big) = 0$. i.e., $d(gx_n, gx_{n+1})$ and $d(gy_n, gy_{n+1})$ equal to zero, but we assumed from before that at least one of them don't equal zero.

That means, $\{\tau_n\}_{n\geq 0}$ is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that $\lim_{n\to\infty} \tau_n = \delta$. We claim that $\delta = 0$. Taking the limit as $n \to \infty$ of both sides of (2.22), we get

$$\delta \leq \theta(\rho_{n-1})\delta$$
$$\Rightarrow \delta = 0 \text{ as } 0 < \theta(\rho_{n-1}) < 1.$$

Therefore,

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \left[\psi\big(d(gx_n, gx_{n+1})\big) + \psi\big(d(gy_n, gy_{n+1})\big)\right] = 0. \tag{2.23}$$

By virtue of the fact that ψ is continuous and $\psi(\epsilon) = 0 \Leftrightarrow \epsilon = 0$, we have

$$\begin{aligned}
&\psi\big(\lim_{n\to\infty} d(gx_n, gx_{n+1})\big) = \lim_{n\to\infty} \psi\big(d(gx_n, gx_{n+1})\big) = \psi(0) = 0\\
&\Rightarrow \lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0\\
&\text{and}\\
&\psi\big(\lim_{n\to\infty} d(gy_n, gy_{n+1})\big) = \lim_{n\to\infty} \psi\big(d(gy_n, gy_{n+1})\big) = 0\\
&\Rightarrow \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0.
\end{aligned} \tag{2.24}$$

Now we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X, d). Suppose the contrary, that $\{gx_n\}$ and $\{gy_n\}$ are not Cauchy sequences. Then, Lemma 1.1 implies that there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers (with for all positive integer k, $m_k > n_k$) such that

$$d(gx_{m_k}, gx_{n_k}), d(gx_{m_k+1}, gx_{n_k+1}), d(gy_{m_k}, gy_{n_k}), d(gy_{m_k+1}, gy_{n_k+1}) \to \epsilon \tag{2.25}$$

and

$$\limsup_{k\to\infty} d(gx_{n_k}, gx_{m_k+1}), \limsup_{k\to\infty} d(gx_{n_k+1}, gx_{m_k}), \limsup_{k\to\infty} d(gy_{n_k}, gy_{m_k+1}), \limsup_{k\to\infty} d(gy_{n_k+1}, gy_{m_k}) \leq \epsilon. \tag{2.26}$$

Since $m_k \geq n_k$, so $gx_{m_k} \succeq gx_{n_k}$ and $gy_{n_k} \preceq gx_{m_k}$. Also, by (2.18) we have

$$\begin{aligned}
\psi\big(d(gx_{m_k+1}, gx_{n_k+1})\big) &= \psi\Big(d\big(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k})\big)\Big) \leq a_1\big(d(gx_{m_k}, gx_{n_k})\big)\psi\big(d(gx_{m_k}, gx_{n_k})\big)\\
&+ b_1\big(d(gx_{m_k}, gx_{n_k})\big)\frac{\psi\big(d(gx_{m_k}, F(x_{m_k}, y_{m_k}))\big) + \psi\big(d(gx_{n_k}, F(x_{n_k}, y_{n_k}))\big)}{2}\\
&+ c_1\big(d(gx_{m_k}, gx_{n_k})\big) \min\left\{\psi\big(d(gx_{m_k}, F(x_{n_k}, y_{n_k}))\big), \psi\big(d(gx_{n_k}, F(x_{m_k}, y_{m_k}))\big)\right\}\\
&+ a_2\big(d(gy_{m_k}, gy_{n_k})\big)\psi\big(d(gy_{m_k}, gy_{n_k})\big)\\
&+ b_2\big(d(gy_{m_k}, gy_{n_k})\big)\frac{\psi\big(d(gy_{m_k}, F(y_{m_k}, x_{m_k}))\big) + \psi\big(d(gy_{n_k}, F(y_{n_k}, x_{n_k}))\big)}{2}\\
&+ c_2\big(d(gy_{m_k}, gy_{n_k})\big) \min\left\{\psi\big(d(gy_{m_k}, F(y_{n_k}, x_{n_k}))\big), \psi\big(d(gy_{n_k}, F(y_{m_k}, x_{m_k}))\big)\right\}\\
&\leq a_1\big(d(gx_{m_k}, gx_{n_k})\big)\psi\big(d(gx_{m_k}, gx_{n_k})\big)\\
&+ b_1\big(d(gx_{m_k}, gx_{n_k})\big)\frac{\psi\big(d(gx_{m_k}, gx_{m_k+1}) + \psi\big(d(gx_{n_k}, gx_{n_k+1})\big)}{2}\\
&+ c_1\big(d(gx_{m_k}, gx_{n_k})\big) \min\left\{\psi\big(d(gx_{m_k}, gx_{n_k+1})\big), \psi\big(d(gx_{n_k}, gx_{m_k+1})\big)\right\}\\
&+ a_2\big(d(gy_{m_k}, gy_{n_k})\big)\psi\big(d(gy_{m_k}, gy_{n_k})\big)\\
&+ b_2\big(d(gy_{m_k}, gy_{n_k})\big)\frac{\psi\big(d(gy_{m_k}, gy_{m_k+1})\big) + \psi\big(d(gy_{n_k}, gy_{n_k+1})\big)}{2}\\
&+ c_2\big(d(gy_{m_k}, gy_{n_k})\big) \min\left\{\psi\big(d(gy_{m_k}, gy_{n_k+1})\big), \psi\big(d(gy_{n_k}, gy_{m_k+1})\big)\right\}
\end{aligned}$$

$$\begin{aligned}\psi\big(d(gx_{m_k+1},gx_{n_k+1})\big) \le\ & a_1(\rho)\psi\big(d(gx_{m_k},gx_{n_k})\big)\\ &+ b_1(\rho)\frac{\psi\big(d(gx_{m_k},gx_{m_k+1}) + \psi\big(d(gx_{n_k},gx_{n_k+1})\big)}{2}\\ &+ c_1(\rho)\min\big\{\psi\big(d(gx_{m_k},gx_{n_k+1})\big),\psi\big(d(gx_{n_k},gx_{m_k+1})\big)\big\}\\ &+ a_2(\rho)\psi\big(d(gy_{m_k},gy_{n_k})\big)\\ &+ b_2(\rho)\frac{\psi\big(d(gy_{m_k},gy_{m_k+1})\big) + \psi\big(d(gy_{n_k},gy_{n_k+1})\big)}{2}\\ &+ c_2(\rho)\min\big\{\psi\big(d(gy_{m_k},gy_{n_k+1})\big),\psi\big(d(gy_{n_k},gy_{m_k+1})\big)\big\}.\end{aligned} \tag{2.27}$$

Where, $\rho = \min\big\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k})\big\}$. Taking the limit superior as k tends to infinity, using ((2.24) to (2.26)) and all properties of ψ in above, we get

$$\begin{aligned}\psi(\epsilon) \le\ & a_1(\rho)\psi(\epsilon) + b_1(\rho)\frac{\psi(0)+\psi(0)}{2} + c_1(\rho)\min\{\psi(\epsilon),\psi(\epsilon)\}\\ &+ a_2(\rho)\psi(\epsilon) + b_2(\rho)\frac{\psi(0)+\psi(0)}{2} + c_2(\rho)\min\{\psi(\epsilon),\psi(\epsilon)\}\\ &\le \big[a_1(\rho)+a_2(\rho)+c_1(\rho)+c_2(\rho)\big]\psi(\epsilon)\end{aligned}$$

By a similar way we can get same inequality for $\{gy_n\}$. This occurs only if $\epsilon = 0$, this a contradiction because we have $\epsilon > 0$. We deduce that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.
Since $g(X)$ is complete, so there exist points gx and gy in $g(X)$ such that

$$gx_n \to gx \ \texttt{and}\ gy_n \to gy \ \text{ as } \ n \to \infty. \tag{2.28}$$

Now we show that (x,y) is coupled coincidence point for F and g. Using triangle inequality and the fact, for any sequence of real numbers $\{a_n\}$, $\alpha \le a_n \forall n \Rightarrow \alpha \le \liminf_{n\to\infty} a_n$ give

$$\begin{aligned}d(F(x,y),gx) &\le d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx)\\ &\le \liminf_{n\to\infty}\big[d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx)\big]\\ &\le \limsup_{n\to\infty}\big[d(F(x,y),gx_{n+1}) + d(gx_{n+1},gx)\big]\\ &\le \limsup_{n\to\infty} d(F(x,y),gx_{n+1})\\ \psi\big(d(F(x,y),gx)\big) &\le \limsup_{n\to\infty}\psi\big(d(F(x,y),gx_{n+1})\big)\end{aligned}$$

and

$$\psi\big(d(F(y,x),gy)\big) \le \limsup_{n\to\infty}\psi\big(d(F(y,x),gy_{n+1})\big).$$

Hence

$$\psi\big(d(F(x,y),gx)\big)+\psi\big(d(F(y,x),gy)\big) \le \limsup_{n\to\infty}\psi\big(d(F(x,y),gx_{n+1})\big)+\limsup_{n\to\infty}\psi\big(d(F(y,x),gy_{n+1})\big). \tag{2.29}$$

Suppose that X has the properties (i) and (ii), i.e., $\{gx_n\}$ being increasing and $gx_n \to gx$ for all n implies $gx_n \preceq gx\ \forall n$. Also $\{gy_n\}$ being decreasing and $gy_n \to gy$ for all n implies $gy_n \succeq y\ \forall n$. Now consider

$$\begin{aligned}
\psi\big(d(F(x,y),gx_{n+1})\big)+\psi\big(d(F(y,x),gy_{n+1})\big) &\le a_1\big(d(gx,gx_n)\big)\psi\big(d(gx,gx_n)\big)\\
&+b_1\big(d(gx,gx_n)\big)\frac{\psi\big(d(gx,F(x,y))\big)+\psi\big(d(gx_n,gx_{n+1})\big)}{2}\\
&+c_1\big(d(gx,gx_n)\big)\min\{\psi\big(d(gx,gx_{n+1})\big),\psi\big(d(gx_n,F(x,y))\big)\}\\
&+a_2\big(d(gy,gy_n)\big)\psi\big(d(gy,gy_n)\big)\\
&+b_2\big(d(gy,gy_n)\big)\frac{\psi\big(d(gy,F(y,x))\big)+\psi\big(d(gy_n,gy_{n+1})\big)}{2}\\
&+c_2\big(d(gy,gy_n)\big)\min\{\psi\big(d(gy,gy_{n+1})\big),\psi\big(d(gy_n,F(y,x))\big)\}\\
&+a_1\big(d(gy,gy_n)\big)\psi\big(d(gy,gy_n)\big)\\
&+b_1\big(d(gy,gy_n)\big)\frac{\psi\big(d(gy,F(y,x))\big)+\psi\big(d(gy_n,gy_{n+1})\big)}{2}\\
&+c_1\big(d(gy,gy_n)\big)\min\{\psi\big(d(gy,gy_{n+1})\big),\psi\big(d(gy_n,F(y,x))\big)\}\\
&+a_2\big(d(gx,gx_n)\big)\psi\big(d(gx,gx_n)\big)\\
&+b_2\big(d(gx,gx_n)\big)\frac{\psi\big(d(gx,F(x,y))\big)+\psi\big(d(gx_n,gx_{n+1})\big)}{2}\\
&+c_2\big(d(gx,gx_n)\big)\min\{\psi\big(d(gx,gx_{n+1})\big),\psi\big(d(gx_n,F(x,y))\big)\}
\end{aligned}$$

Setting $\rho=\min\{d(gx,gx_n),d(gy,gy_n)\}$, then taking the upper limit as n tends to infinity and using (2.24 and 2.26) with the continuity of ψ and $\psi(0)=0$ give us

$$\begin{aligned}
&\limsup_{n\to\infty}\psi\big(d(F(x,y),gx_{n+1})\big)+\limsup_{n\to\infty}\psi\big(d(F(y,x),gy_{n+1})\big)\le\\
&\quad\limsup_{n\to\infty}b_1\big(d(gx,gx_n)\big)\frac{\psi\big(d(gx,F(x,y))\big)}{2}+\limsup_{n\to\infty}b_2\big(d(gy,gy_n)\big)\frac{\psi\big(d(gy,F(y,x))\big)}{2}\\
&\quad+\limsup_{n\to\infty}b_1\big(d(gy,gy_n)\big)\frac{\psi\big(d(gy,F(y,x))\big)}{2}+\limsup_{n\to\infty}b_2\big(d(gx,gx_n)\big)\frac{\psi\big(d(gx,F(x,y))\big)}{2}\\
&\quad\le\limsup_{n\to\infty}\left[\frac{b_1\big(d(gx,gx_n)\big)+b_2\big(d(gx,gx_n)\big)}{2}\right]\psi\big(d(gx,F(x,y))\big)\\
&\quad+\limsup_{n\to\infty}\left[\frac{b_1\big(d(gy,gy_n)\big)+b_2\big(d(gy,gy_n)\big)}{2}\right]\psi\big(d(gy,F(y,x))\big)
\end{aligned}$$

By using (2.29) with the help of, for any sequence of real numbers $\{a_n\}$, $a_n\le\alpha\forall n\Rightarrow\limsup_{n\to\infty}a_n\le\alpha$ we have

$$\begin{aligned}
\frac{b_1\big(d(gx,gx_n)\big)+b_2\big(d(gx,gx_n)\big)}{2}&\le\frac{b_1(\rho)+b_2(\rho)}{2}\\
\limsup_{n\to\infty}\frac{b_1\big(d(gx,gx_n)\big)+b_2\big(d(gx,gx_n)\big)}{2}&\le\frac{b_1(\rho)+b_2(\rho)}{2}.
\end{aligned}\tag{2.30}$$

and

$$\psi\big(d(F(x,y),gx)\big)+\psi\big(d(F(y,x),gy)\big)\le\frac{b_1(\rho)+b_2(\rho)}{2}\Big[\psi\big(d(gx,F(x,y))\big)+\psi\big(d(gy,F(y,x))\big)\Big]$$

This shows that $\psi\big(d(F(x,y),gx)\big)+\psi\big(d(F(y,x),gy)\big)=0\Rightarrow\psi\big(d(F(x,y),gx)\big)=\psi\big(d(F(y,x),gy)\big)=0\Rightarrow d(F(x,y),gx)=d(F(y,x),gy)=0\Rightarrow F(x,y)=gx$ and $F(y,x)=gy$. Hence (x,y) is a coupled coincidence point for F and g. □

If we consider $a_1(t)+a_2(t)+b_1(t)+b_2(t)+c_1(t)+c_2(t)<\frac{1}{2}$ for every $t>0$ in Theorem 2.3, we get the following remark.

Remark 2.2. We can prove Theorem 2.3 by another way, directly from Theorem 2.1 as follows

Proof. Let $\psi \circ d$ in Theorem 2.3 equal ρ. Define $\theta_1, \theta_2 : R^+ \to [0,1]$ as $\theta_1(t) = a_1(t) + b_1(t) + c_1(t)$ and $\theta_2(t) = a_2(t) + b_2(t) + c_2(t)$ if $t > 0$ and $\theta_1(0) + \theta_2(0) = 1$. Then ρ is a nonnegative real valued function on $X \times X$ having properties (i) to (iii) as in Theorem 2.1 and θ_1, θ_2 are decreasing functions on R^+ with $\theta_1(t) + \theta_2(t) < 1$ for all $t > 0$. Under this considerations, contraction condition (2.18) yields to

$$\begin{aligned}\psi\big(d(F(x,y),F(u,v))\big) \leq{}& \big[a_1(d(gx,gu)) + b_1(d(gx,gu)) + c_1(d(gx,gu))\big]\\ &\max\Big\{\psi\big(d(gx,gu)\big), \frac{\psi\big(d(gx,F(x,y))\big) + \psi\big(d(gu,F(u,v))\big)}{2},\\ &\min\big\{\psi\big(d(gx,F(u,v))\big), \psi\big(d(gu,F(x,y))\big)\big\}\Big\}\\ &+ \big[a_2(d(gy,gv)) + b_2(d(gy,gv)) + c_2(d(gy,gv))\big]\\ &\max\Big\{\psi\big(d(gy,gv)\big), \frac{\psi\big(d(gy,F(y,x))\big) + \psi\big(d(gv,F(v,u))\big)}{2},\\ &\min\big\{\psi\big(d(gy,F(v,u))\big), \psi\big(d(gv,F(y,x)\big)\big\}\Big\}\end{aligned}$$

$$\begin{aligned}\rho\big(F(x,y),F(u,v)\big) \leq{}& \frac{1}{2}\theta_1\big(d(gx,gu)\big)\max\Big\{\rho(gx,gu), \frac{\rho(gx,F(x,y)) + \rho(gu,F(u,v))}{2},\\ &\min\big\{\rho(gx,F(u,v)), \rho(gu,F(x,y))\big\}\Big\}\\ &+ \frac{1}{2}\theta_2\big(d(gy,gv)\big)\max\Big\{\rho(gy,gv), \frac{\rho(gy,F(y,x)) + \rho(gv,F(v,u))}{2},\\ &\min\big\{\rho(gy,F(v,u)), \rho(gv,F(y,x))\big\}\Big\}\\ \leq{}& \frac{1}{2}\big[\theta_1\big(d(gx,gu)\big) + \theta_2\big(d(gy,gv)\big)\big]\\ &\max\Big\{\rho(gx,gu), \frac{\rho(gx,F(x,y)) + \rho(gu,F(u,v))}{2}, \min\big\{\rho(gx,F(u,v)), \rho(gu,F(x,y))\big\},\\ &\rho(gy,gv)), \frac{\rho(gy,F(y,x)) + \rho(gv,F(v,u))}{2}, \min\big\{\rho(gy,F(v,u)), \rho(gv,F(y,x))\big\}\Big\}\\ \leq{}& \frac{1}{2}\big[\theta_1(d(gx,gu)) + \theta_2(d(gy,gv))\big]\\ &\max\Big\{\rho(gx,gu), \rho(gy,gv)), \frac{\rho(gx,F(x,y)) + \rho(gu,F(u,v))}{2}, \frac{\rho(gy,F(y,x)) + \rho(gv,F(v,u))}{2},\\ &\big[\rho(gx,F(u,v)), \rho(gu,F(x,y))\big]^{\frac{1}{2}}, \big[\rho(gy,F(v,u)), \rho(gv,F(y,x))\big]^{\frac{1}{2}}\Big\}.\end{aligned}$$

All conditions of Theorem 2.1 hold, then Theorem 2.3 follows from Theorem 2.1. □

Here, we derive some consequences for Theorem 2.1.

Corollary 2.2. *The result of Theorem 2.3 remains valid even If we assume that the functions a_i, b_i and $c_i, i = 1,2$ are increasing instead of decreasing.*

Proof. The proof of this corollary is same as that one of Theorem 2.3 under some changes. If we consider ρ_{n-1} in Equation (2.21) as

$$\rho_{n-1} = \max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}$$

Therefore, we have

$$\begin{aligned}\rho_{n-1} &\geq d(gx_{n-1}, gx_n)\\ &\geq d(gy_{n-1}, gy_n)\\ \Rightarrow a_1(\rho_{n-1}) &\geq a_1\big(d(gx_{n-1}, gx_n)\big)\\ &\geq a_1\big(d(gy_{n-1}, gy_n)\big).\end{aligned}$$

Doing the same argument for ρ in Equations (2.27) and (2.30) and complete the proof as in Theorem 2.3 implies same result. □

Now, a consequence of Theorem 2.3 by taking $F(x, y) = fx$ where $f : X \to X$, is the following corollary which is the ordered version of Khan et al.'s Theorem 1.2.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing given mapping such that*

$$\begin{aligned}\psi\big(d(fx, fu)\big) \leq a\big(d(x, u)\big)\psi\big(d(x, u)\big) + b\big(d(x, u)\big)\frac{\psi\big(d(x, fx)\big) + \psi\big(d(u, fu)\big)}{2}\\ + c\big(d(x, u)\big)\min\big\{\psi\big(d(x, fu)\big), \psi\big(d(u, fx)\big)\big\},\end{aligned} \tag{2.31}$$

for $x, y \in X$ with $x \preceq u$, where ψ is an altering function and a, b, c are decreasing functions from $[0, \infty)$ into $[0, 1)$ such that $a(t) + b(t) + c(t) < 1$ for every $t > 0$. Assume either f is continuous or X has the following property, if a nondecreasing sequence $x_n \to x \in X$, then $x_n \preceq x$ for all n. If there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has a unique fixed point.

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