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*Research Article*

# Fractional Hybrid Differential Equations and Coupled Fixed-Point Results for $\alpha$-Admissible $F(\psi_1, \psi_2)$ – Contractions in $M$ – Metric Spaces

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In this paper, we investigate the existence of a unique coupled fixed point for $\alpha-$admissible mapping which is of $F(\psi_1, \psi_2)-$contraction in the context of $M-$metric space. We have also shown that the results presented in this paper would extend many recent results appearing in the literature. Furthermore, we apply our results to develop sufficient conditions for the existence and uniqueness of a solution for a coupled system of fractional hybrid differential equations with linear perturbations of second type and with three-point boundary conditions.

## 1. Introduction

Fixed-point theory is an outstanding source which gives responsible techniques for the existence of fixed points for self-mappings under different conditions. One of the newest branches of fixed-point theory concerned with the study of coupled fixed points, brought by Guo and Lakshmikantham [1]. In [2], Bhaskar and Lakshmikantham established some fixed and coupled fixed-point theorems for contractions in two variables defined on partially ordered metric spaces with applications to ordinary differential equations. Thereafter, these results were extended by several authors (see [3–6]).

Inspired by the notion of partial metric (or, $p-$metric) which is one of the vital generalizations of the standard metric, Asadi et al. [7] proposed the concept of $M-$metric which refines the $p-$metric and produces useful basic topological concepts. For some fixed-point results and various contractive definitions that have been employed in $M-$metric space, we refer the reader to [8–12].

In [13] (see also, [14–16]), Monfared et al. established some fixed-point results for $\alpha-$admissible mappings which are $F(\psi, \varphi)-$contractions in complete $M-$metric spaces. Now, we state one of their main results.

**Theorem 1.** *Let $(X, \mu)$ be a complete $M-$metric space and $T: X \longrightarrow X$ be an $\alpha-$admissible mapping. Suppose that the following condition is satisfied:*

$$[\psi(\mu(Tx, Ty)) + l]^{\alpha(x,Tx)\alpha(y,Ty)} \leq F(\psi(\mu(x, y)), \varphi(\mu(x, y))) + l, \quad (1)$$

*for all $x, y \in X$ and $l \geq 1$, where $F \in \mathscr{C}$, $\psi$ is an altering distance function, and $\varphi$ is an ultra-altering distance function. Suppose that either*

*(a) $T$ is continuous*

*(b) If $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \longrightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$, $\forall n$, then $\alpha(x, Tx) \geq 1$*

*If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $T$ has a fixed point.*

Hybrid differential equations have been of great interest as they include several dynamic systems as special cases. The papers [17, 18] discussed the existence and uniqueness results and some fundamental differential inequalities for first-order hybrid differential equations with perturbations of 1st and 2nd type, respectively.

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of arbitrary order. Indeed, it is found to be more realistic in describing and modeling several natural phenomena than the classical one. In fact, fractional differential equations (FDEs) play a major role in modeling many real-life problems such as physical phenomena, computer networking, medicine (the modeling of human tissue), mechanics (theory of viscoelasticity), electrical engineering (transmission of ultrasound waves) and many others (see [19–21]).

Fractional hybrid differential equations (FHDEs) can be employed in modeling and describing nonhomogenous physical phenomena that take place in their form. FHDEs have been studied using a Riemann–Liouville differential operator of order $\alpha > 0$ in many literature studies (see [22–26]).

In [27], Shaob et al. used Bashiri fixed-point theorem [22] to prove the existence only of a solution to a three-point boundary value problem for a coupled system of FHDEs in Banach spaces.

In line with the above studies, our purpose in this paper is to introduce the notion of $\alpha$–admissible mapping with two variables and generalize Theorem 1 to coupled fixed-point version. Then, we apply our main results to prove the existence and uniqueness of a solution to the following system of FHDEs involving Riemann–Liouville fractional derivative:

$$\begin{gathered} \mathscr{D}^{\alpha}[x(t) - f(t, x(t))] = g\left(t, y(t), I^{\beta} y(t)\right), \\ x^{(i)}(0) = \left.\frac{\partial^i f(t, x(t))}{\partial t^i}\right|_{t=0} = 0, \\ x(\tau) = \delta x(\eta), \end{gathered} \tag{2}$$

$$\begin{gathered} \mathscr{D}^{\alpha}[y(t) - f(t, y(t))] = g\left(t, x(t), I^{\beta} x(t)\right), \\ y^{(i)}(0) = \left.\frac{\partial^i f(t, y(t))}{\partial t^i}\right|_{t=0} = 0, \\ y(\tau) = \delta y(\eta), \end{gathered} \tag{3}$$

for all $i = 0, 1, \ldots, n-2$, $t \in J = [o, \tau]$, $\tau > 0$, $\alpha \in (n-1, n]$, $\beta > 0$, $0 < \eta < \tau$, $\delta \neq (\tau/\eta)^{\alpha-1}$, $f \in C(J \times \mathbb{R})$, and $g \in C(J \times \mathbb{R}^2)$.

## 2. Preliminaries

In 1994, Matthews [28] introduced the notion of a $p$–metric space as a part of the study of denotational semantics of dataflow networks. In $p$–metric spaces, self-distance of an arbitrary point need not be equal to zero.

*Definition 1* (see [28]). A $p$–metric on a nonempty set $X$ is a mapping $p$: $X \times X \longrightarrow [0, \infty)$ such that, for all $x, y, z \in X$,

$(p_1)$ $p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y$
$(p_2)$ $p_{(x,x)} \leq p(x, y)$
$(p_3)$ $p(x, y) = p(y, x)$
$(p_4)$ $p(x, y) \leq p(x, z) + p(z, y) - p_{(z,z)}$

Then, $(X, p)$ is called a $p$–metric space.

Notice that, every metric space can be defined to be $p$–metric space with zero self-distance. After that, Asadi et al. generalized the above definition by relaxing the axiom $(p_2)$ as follows.

*Definition 2* (see [7]). For a nonempty set $X$, a function $\mu$: $X \times X \longrightarrow [0, \infty)$ is called an $M$–metric if it fulfils the following:

$(m_1)$ $\mu(x, x) = \mu(y, y) = \mu(x, y) \Leftrightarrow x = y$
$(m_2)$ $m_{xy} \leq \mu(x, y)$, where $m_{xy} = \min\{\mu(x, x), \mu(y, y)\}$
$(m_3)$ $\mu(x, y) = \mu(y, x)$
$(m_4)$ $(\mu(x, y) - m_{xy}) \leq (\mu(x, z) - m_{xz}) + (\mu(z, y) - m_{zy})$

Then, the pair $(X, \mu)$ is called an $M$–metric space.

**Lemma 1** *(see [7]). Every $p$–metric is an $M$–metric.*

*Here, we give an example to show that the converse might not be held.*

*Example 1* (see [7]). Let $X = \{1, 2, 3\}$ and define

$$\begin{aligned} &\mu(1, 2) = \mu(2, 1) = 10, \\ &\mu(1, 1) = 1, \\ &\mu(2, 2) = 9, \\ &\mu(1, 3) = \mu(3, 1) = \mu(3, 2) = \mu(2, 3) = 7, \\ &\mu(3, 3) = 5. \end{aligned} \tag{4}$$

So $\mu$ is $M$–metric but it is not $p$–metric for $\mu(2, 2) \nleq \mu(2, 3)$. Also, $\mu$ is not metric for self-distances are not zero.

Thus, the class of $M$–metric spaces is effectively larger than that of both ordinary metric and $p$–metric spaces.

*Notation 1.* Let $(X, \mu)$ be an $M$–metric space; then define

$$\begin{aligned} &\mu^{w}(x, y) = \mu(x, y) - 2m_{xy} + M_{xy}, \\ &\text{where } M_{xy} = \max\{\mu(x, x), \mu(y, y)\}. \end{aligned} \tag{5}$$

Hence, $\mu^w$ is an ordinary metric induced by the $M$–metric $\mu$.

Each $M$–metric $\mu$ on $X$ generates a $T_0$ topology $\tau_\mu$ on $X$ formed by the set

$$\left\{B_{\mu}(x, \varepsilon): x \in X, \varepsilon > 0\right\}, \tag{6}$$

where

$$B_\mu(x,\varepsilon)=\{y\in X\colon \mu(x,y)<m_{xy}+\varepsilon\}. \tag{7}$$

The notions of convergent sequence, Cauchy sequence, and complete $M-$metric space $(X,\mu)$ are given as follows:

(1) A sequence $\{x_n\}$ in $(X,\mu)$ converges to a point $x\in X$ if

$$\lim_{n\longrightarrow\infty}\left(\mu(x_n,x)-m_{x_nx}\right)=0. \tag{8}$$

(2) A sequence $\{x_n\}$ in $(X,\mu)$ is called $\mu-$Cauchy if

$$\begin{aligned}&\lim_{n,m\longrightarrow\infty}\left(\mu(x_n,x_m)-m_{x_nx_m}\right),\\&\lim_{n,m\longrightarrow\infty}\left(M_{x_nx_m}-m_{x_nx_m}\right)\end{aligned} \tag{9}$$

exist and are finite.

(3) $(X,\mu)$ is said to be complete if every $\mu-$Cauchy sequence $\{x_n\}$ in it converges, with respect to $\tau_\mu$, to a point $x\in X$, and

$$\lim_{n,m\longrightarrow\infty}\left(\mu(x_n,x_m)-m_{x_nx_m}\right)=\lim_{n,m\longrightarrow\infty}\left(M_{x_nx_m}-m_{x_nx_m}\right)=0. \tag{10}$$

**Lemma 2** (see [7]). *Let $(X,\mu)$ be an $M-$metric space; then,*

*(1) $\{x_n\}$ is a $\mu-$Cauchy sequence in $(X,\mu)$ if and only if it is Cauchy sequence in the metric space $(X,\mu^w)$.*

*(2) $(X,\mu)$ is complete if and only if $(X,\mu^w)$ is complete. Furthermore,*

$$\begin{aligned}\lim_{n\longrightarrow\infty}\mu^w(x_n,x)=0\Longleftrightarrow&\lim_{n\longrightarrow\infty}\left(\mu(x_n,x)-m_{x_nx}\right)\\=&\lim_{n\longrightarrow\infty}\left(M_{x_nx}-m_{x_nx}\right)=0.\end{aligned} \tag{11}$$

**Lemma 3** (see [7]). *Assume that $x_n\longrightarrow x$ and $y_n\longrightarrow y$ in an $M-$metric space $(X,\mu)$; then,*

$$\lim_{n\longrightarrow\infty}\left(\mu(x_n,y_n)-m_{x_ny_n}\right)=\left(\mu(x,y)-m_{xy}\right). \tag{12}$$

As a consequence of Lemma 3, we have

$$\begin{aligned}x_n\longrightarrow x,\quad &\text{in } (X,\mu)\Longrightarrow\lim_{n\longrightarrow\infty}\left(\mu(x_n,y)-m_{x_ny}\right)=\left(\mu(x,y)-m_{xy}\right),\\ x_n\longrightarrow x \text{ and } x_n\longrightarrow y,\quad &\text{in } (X,\mu)\Longrightarrow\lim_{n\longrightarrow\infty}\left(\mu(x_n,x_n)-m_{x_nx_n}\right)=\left(\mu(x,y)-m_{xy}\right).\end{aligned} \tag{13}$$

*Definition 3* (see [29]). A mapping $F\colon[0,\infty)^2\longrightarrow R$ is called a $C-$class function if it is continuous and satisfies the following axioms:

(1) $F(s,t)\le s$

(2) $F(s,t)=s$ implies that either $s=0$ or $t=0$ for all $t\in[0,\infty)$

Let $\mathscr{C}$ denote the $C-$class functions.

*Definition 4* (see [20, 21]). The fractional integral of order $\alpha>0$ of a function $x\colon[0,\infty)\longrightarrow\mathbb{R}$ is given by

$$I^\alpha x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\mathrm{d}s, \tag{14}$$

provided that the right side is pointwise defined on $[0,\infty)$.

*Definition 5* (see [20, 21]). The fractional derivative of order $\alpha>0$ of a continuous function $x\colon[0,\infty)\longrightarrow\mathbb{R}$ is given by

$$\mathscr{D}^\alpha x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t\frac{x(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \tag{15}$$

where $n=[\alpha]+1$, provided that the right side is pointwise defined on $[0,\infty)$.

**Lemma 4** (see [30]). *Riemann–Liouville fractional integral and derivative have the following properties:*

*(1) $I^\alpha I^\beta x(t)=I^{\alpha+\beta}x(t)$ and $\mathscr{D}^\alpha I^\beta x(t)=I^{\beta-\alpha}x(t)$, for all $\beta\ge\alpha>0$, $x\in L[0,1]$*

*(2) $I^\alpha\mathscr{D}^\alpha x(t)=x(t)+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n}$, where $n=[\alpha]+1$ and $x,\mathscr{D}^\alpha x\in C[0,1]\cap L[0,1]$*

*(3) $I^\alpha\colon C[0,1]\longrightarrow C[0,1]$, $\alpha>0$*

## 3. Fixed-Point Results

First, we introduce the following concepts that generalize the corresponding ones used in [13] and will be beneficial in the sequel.

*Definition 6* Let $T\colon X\times X\longrightarrow X$ and $\alpha\colon X\times X\longrightarrow[0,\infty)$; then, $T$ is called an $\alpha-$admissible mapping if

$$\begin{aligned}&\alpha(x,u)\ge 1,\\&\alpha(y,v)\ge 1\Longrightarrow\alpha(T(x,y),T(u,v))\ge 1,\quad\forall(x,y),(u,v)\in X^2.\end{aligned} \tag{16}$$

Note that, if equation (16) holds, then we have $\alpha(T(y,x),T(v,u))\ge 1$ too. Consider the following classes of functions:

$$\Psi_1 = \{\psi\colon [0,\infty)^2 \longrightarrow [0,\infty), \psi \text{ is continuous, strictly increasing and } \psi(t_1,t_2) = 0 \Longrightarrow t_1 = t_2 = 0\},$$
$$\Psi_2 = \{\psi\colon [0,\infty)\times[0,\infty) \longrightarrow [0,\infty), \psi \text{ is continuous and } \psi(t_1,t_2) = 0 \Longrightarrow t_1 = t_2 = 0\}, \tag{17}$$
$$\Phi = \left\{\varphi\colon [0,\infty) \longrightarrow [0,\infty), \varphi(s+t) \le \varphi(s) + \varphi(t) \text{ and } \varphi\left(\frac{t}{2}\right) \le \frac{\varphi(t)}{2}\ \forall s,t \ge 0\right\}.$$

**Theorem 2.** *Let $(X,\mu)$ be a complete $M-$metric space and $T\colon X\times X \longrightarrow X$ be an $\alpha-$admissible mapping for which there exist $F \in \mathscr{C}$, $\phi \in \Phi$, $\psi_1 \in \Psi_1$, and $\psi_2 \in \Psi_2$ such that $\psi_1(t.t) \le \phi(t)$ and for all $(x,y),(u,v) \in X^2$ with $\alpha(x,u) \ge 1, \alpha(y,v) \ge 1$; we have*

$$[\phi(\mu(T(x,y),T(u,v))) + l]^{\max\{\alpha(x,u),\alpha(y,v)\}} \le F\left(\psi_1\left(\frac{K(x,u)+K(y,v)}{2}\right), \psi_2\left(\frac{K(x,u)+K(y,v)}{2}\right)\right) + l, \tag{18}$$

*where*

$$K(x,u) = \left(\frac{\mu(u,T(u,v))[1+\mu(x,T(x,y))]}{1+\mu(x,u)}, \mu(x,u)\right),$$
$$K(y,v) = \left(\frac{\mu(v,T(v,u))[1+\mu(y,T(y,x))]}{1+\mu(y,v)}, \mu(y,v)\right). \tag{19}$$

*Suppose that either*

*(a) $T$ is continuous.*

*(b) For a convergent sequence $\{x_n\}$ in $(X,\mu)$, we have*

$$\{x_n\} \longrightarrow x, \quad \alpha(x_n,x_{n+1}) \ge 1 \Longrightarrow \alpha(x_n,x) \ge 1,\ \forall n,$$
$$x_n \longrightarrow x, \quad x_n \longrightarrow y \Longrightarrow \alpha(x,y) \ge 1. \tag{20}$$

*If there exist $x_0,y_0 \in X$ such that $\alpha(x_0,T(x_0,y_0)) \ge 1$ and $\alpha(y_0,T(y_0,x_0)) \ge 1$, then $T$ has a coupled fixed point.*

*Proof.* Starting with $x_0, y_0 \in X$, define the sequences $\{x_n\},\{y_n\} \subset X$ by

$$x_{n+1} = T(x_n,y_n),$$
$$y_{n+1} = T(y_n,x_n), \tag{21}$$
$$\forall n \in \mathbb{N}_0.$$

By induction methodology for $n \in \mathbb{N}_0$, we shall prove that

$$\alpha(x_n,x_{n+1}) \ge 1,$$
$$\alpha(y_n,y_{n+1}) \ge 1, \forall n. \tag{22}$$

Indeed, we have $\alpha(x_0,x_1) \ge 1$ and $\alpha(y_0,y_1) \ge 1$. Suppose that (22) holds for some $n$ and we are going to prove it for $n+1$. Since $T$ is $\alpha-$admissible mapping, then by (21), we obtain $\alpha(x_{n+1},x_{n+2}) \ge 1$ and $\alpha(y_{n+1},y_{n+2}) \ge 1$. Thus, (22) holds for all $n$. From (18)–(22), we have

$$\phi(\mu(x_n,x_{n+1})) + l \le [\phi(\mu(T(x_{n-1},y_{n-1}),T(x_n,y_n))) + l]^{\max\{\alpha(x_{n-1},x_n),\alpha(y_{n-1},y_n)\}}$$
$$\le F\left(\psi_1\left(\frac{K(x_{n-1},x_n)+K(y_{n-1},y_n)}{2}\right), \psi_2\left(\frac{K(x_{n-1},x_n)+K(y_{n-1},y_n)}{2}\right)\right) + l, \tag{23}$$

where

$$K(x_{n-1},x_n) = \left(\frac{\mu(x_n,T(x_n,y_n))[1+\mu(x_{n-1},T(x_{n-1},y_{n-1}))]}{1+\mu(x_{n-1},x_n)}, \mu(x_{n-1},x_n)\right)$$
$$= (\mu(x_n,x_{n+1}),\mu(x_{n-1},x_n)), \tag{24}$$
$$K(y_{n-1},y_n) = (\mu(y_n,y_{n+1}),\mu(y_{n-1},y_n)).$$

Hence,

$$\phi(\mu(x_n, x_{n+1})) \le F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right), \quad (25)$$

where $w_n = \mu(x_n, x_{n+1}) + \mu(y_n, y_{n+1})$. Similarly, we have

$$\phi(\mu(y_n, y_{n+1})) \le F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right). \quad (26)$$

Adding (25) and (26) and using properties on $F$ and $\phi$, we obtain

$$\begin{aligned}\psi_1\left(\frac{w_n}{2}, \frac{w_n}{2}\right) &\le \phi\left(\frac{w_n}{2}\right) \le F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right) \\ &\le \psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right).\end{aligned} \quad (27)$$

Since $\psi_1$ is strictly increasing, then $w_n \le w_{n-1}$, $\forall n$. Hence, the sequence $\{w_n\}$ is monotone decreasing and bounded as follows. Therefore, there exist some $w \ge 0$ such that

$$\lim_{n \longrightarrow \infty} w_n = w. \quad (28)$$

Now, we shall prove that $w = 0$. Assume that $w > 0$. Using the properties of $\psi_1, \psi_2$, and $F$ and letting $n \longrightarrow \infty$ in (27) yield that

$$\psi_1\left(\frac{w}{2}, \frac{w}{2}\right) \le F\left(\psi_1\left(\frac{w}{2}, \frac{w}{2}\right), \psi_2\left(\frac{w}{2}, \frac{w}{2}\right)\right) < \psi_1\left(\frac{w}{2}, \frac{w}{2}\right), \quad (29)$$

which is contradiction. Thus, $w = 0$ and

$$\lim_{n \longrightarrow \infty} \mu(x_n, x_{n+1}) = \lim_{n \longrightarrow \infty} \mu(y_n, y_{n+1}) = 0. \quad (30)$$

In what follows, we prove that $\{x_n\}$ and $\{y_n\}$ are $\mu$–Cauchy sequences in $(X, \mu)$. Since we have

$$\begin{aligned}&0 \le m_{x_n, x_{n+1}} \le \mu(x_n, x_{n+1}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \\ &\Longrightarrow \lim_{n \longrightarrow \infty} m_{x_n, x_{n+1}} = \min\left\{\lim_{n \longrightarrow \infty} \mu(x_n, x_n), \lim_{n \longrightarrow \infty} \mu(x_{n+1}, x_{n+1})\right\} = 0 \\ &\Longrightarrow \lim_{n \longrightarrow \infty} \mu(x_n, x_n) = 0,\end{aligned} \quad (31)$$

then

$$\begin{aligned}\lim_{n,m \longrightarrow \infty} m_{x_n, x_m} &= \min\left\{\lim_{n \longrightarrow \infty} \mu(x_n, x_n), \lim_{m \longrightarrow \infty} \mu(x_m, x_m)\right\} = 0, \\ \lim_{n,m \longrightarrow \infty} M_{x_n, x_m} &= \max\left\{\lim_{n \longrightarrow \infty} \mu(x_n, x_n), \lim_{m \longrightarrow \infty} \mu(x_m, x_m)\right\} = 0.\end{aligned} \quad (32)$$

That is,

$$\lim_{n \longrightarrow \infty} \left(M_{x_n, x_m} - m_{x_n, x_m}\right) = 0. \quad (33)$$

On the other hand, we have

$$\mu(x_n, x_m) - m_{x_n, x_m} \le \mu(x_n, x_{n+1}) - m_{x_n, x_{n+1}} + \mu(x_{n+1}, x_{n+1}) - m_{x_{n+1}, x_{n+2}} + \cdots + \mu(x_{m-1}, x_m) - m_{x_{m-1}, x_m} \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \quad (34)$$

Therefore, (33) and (34) imply that $\{x_n\}$ is an $\mu$–Cauchy sequence. In a similar way, we can show that $\{y_n\}$ is also a $\mu$–Cauchy sequence. By the completeness of the space $(X, \mu)$, there exist $x, y \in X$ such that

$$\begin{aligned}\lim_{n \longrightarrow \infty} \left(\mu(x_n, x) - m_{x_n, x}\right) &= \lim_{n \longrightarrow \infty} \left(\mu(y_n, y) - m_{y_n, y}\right) = 0, \\ \lim_{n \longrightarrow \infty} \left(M_{x_n, x} - m_{x_n, x}\right) &= \lim_{n \longrightarrow \infty} \left(M_{y_n, y} - m_{y_n, y}\right) = 0.\end{aligned} \quad (35)$$

With respect to the sequence $\{x_n\}$, we obtain

$$\begin{aligned}&\mu(x_n, x_n) \longrightarrow 0 \Longrightarrow m_{x_n, x} \longrightarrow 0 \Longrightarrow \mu(x_n, x), \\ &M_{x_n, x} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\end{aligned} \quad (36)$$

but

$$M_{x_n, x} = \max\{\mu(x_n, x_n), \mu(x, x)\} \longrightarrow \mu(x, x). \quad (37)$$

Thus, the uniqueness of the limit implies that

$$\mu(x, x) = 0. \quad (38)$$

Now, suppose that $(a)$ holds. According to Lemma 2, since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a complete $M$–metric space $(X, \mu)$, then they converge to some $x, y$ in the metric space $(X, \mu^w)$. Also, as $F$ is continuous, $F(x_n, y_n)$ converges to $F(x, y)$ in $(X, \mu^w)$, that is, $\lim_{n \longrightarrow \infty} \mu^w(F(x_n, y_n), F(x, y)) = 0$ which is equivalent to

$$\begin{aligned}&\mu(F(x_n, y_n), F(x, y)) - m_{F(x_n, y_n), F(x, y)} \longrightarrow 0, \\ &M_{F(x_n, y_n), F(x, y)} - m_{F(x_n, y_n), F(x, y)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (39)$$

Also, we have

$$\begin{aligned}&\mu(x_{n+1}, x_{n+1}) \longrightarrow 0 \Longrightarrow m_{F(x_n, y_n), F(x, y)} \\ &\longrightarrow 0 \Longrightarrow M_{F(x_n, y_n), F(x, y)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\end{aligned} \quad (40)$$

but

$$M_{F(x_n,y_n),F(x,y)} = \max\{\mu(x_{n+1},x_{n+1}),\mu(F(x,y),F(x,y))\} \longrightarrow \mu(F(x,y),F(x,y)). \tag{41}$$

Thus, the uniqueness of the limit implies that

$$\mu(F(x,y),F(x,y)) = 0. \tag{42}$$

By Lemma 3, we obtain

$$\mu(x_{n+1},F(x,y)) - m_{x_{n+1},F(x,y)} \longrightarrow \mu(x,F(x,y)) - m_{x,F(x,y)} = \mu(x,F(x,y)). \tag{43}$$

Compared with (39), we obtain

$$\mu(x,F(x,y)) = 0. \tag{44}$$

From (38), (42), and (44), we obtain

$$x = F(x,y). \tag{45}$$

Proceeding as above, one can obtain

$$y = F(y,x). \tag{46}$$

Suppose that $(b)$ holds, then $\alpha(x_n,x) \geq 1$ and $\alpha(y_n,y) \geq 1$. Setting $(x,y) = (x_n,y_n)$ and $(u,v) = (x,y)$ in (18), we obtain

$$[\phi(\mu(T(x_n,y_n),T(x,y))) + l]^{\max\{\alpha(x_n,x),\alpha(y_n,y)\}} \leq F\left(\psi_1\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right),\psi_2\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right)\right) + l, \tag{47}$$

where

$$\begin{aligned} K(x_n,x) &= \left(\frac{\mu(x,T(x,y))[1+\mu(x_n,x_{n+1})]}{1+\mu(x_n,x)},\mu(x_n,x)\right) \longrightarrow (\mu(x,T(x,y)),0), \\ K(y_n,y) &= \left(\frac{\mu(y,T(y,x))[1+\mu(y_n,y_{n+1})]}{1+\mu(y_n,y)},\mu(y_n,y)\right) \longrightarrow (\mu(y,T(y,x)),0), \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{48}$$

That is,

$$\phi(\mu(x_{n+1},T(x,y))) \leq F\left(\psi_1\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right),\psi_2\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right)\right). \tag{49}$$

In a similar way, one can obtain

$$\phi(\mu(y_{n+1},T(y,x))) \leq F\left(\psi_1\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right),\psi_2\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right)\right). \tag{50}$$

Adding (49) and (50) and using properties on $F$ and $\phi$, we obtain

$$\psi_1\left(\frac{\mu(x_{n+1},T(x,y))+\mu(y_{n+1},T(y,x))}{2},\frac{\mu(x_{n+1},T(x,y))+\mu(y_{n+1},T(y,x))}{2}\right) \leq \psi_1\left(\frac{K(x_n,x)+K(y_n,y)}{2}\right). \tag{51}$$

Taking limits at $n \longrightarrow \infty$ yields

$$\psi_1\left(\frac{\mu(x,T(x,y))+\mu(y,T(y,x))}{2},\frac{\mu(x,T(x,y))+\mu(y,T(y,x))}{2}\right) \leq \psi_1\left(\frac{\mu(x,T(x,y))+\mu(y,T(y,x))}{2},0\right). \tag{52}$$

Therefore, we have $\mu(x, T(x, y)) + \mu(y, T(y, x)) = 0$. Again from (18) and taking into account that $\alpha(x, x), \alpha(y, y) \geq 1$, we obtain that $\mu(T(x, y), T(x, y)) = \mu(T(y, x), T(y, x)) = 0$. Consequently, $x = T(x, y)$ and $y = T(y, x)$.

For the uniqueness of the coupled fixed point in Theorem 2, we consider the following condition:

$$\begin{aligned} &\text{if } (x, y) \text{and} (u, v) \text{are two coupled fixed points of } T,\\ &\text{then } \alpha(x, u) \geq 1,\\ &\text{or } \alpha(y, v) \geq 1. \end{aligned} \tag{53}$$

□

**Theorem 3.** *Adding condition (53) to the hypotheses of Theorem 2, we obtain that T has a unique coupled fixed point.*

*Proof.* Theorem 2 asserts that $T$ has at least one coupled fixed point. Assume that $(x, y)$ and $(u, v)$ are two coupled fixed points of $T$, then $\alpha(x, u) \geq 1$ or $\alpha(y, v) \geq 1$. Now, we apply (18) and use the properties of $\phi$, $\psi_1$, $\psi_2$, and $F$ to obtain

$$\begin{aligned} \psi_1\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right) &\leq \phi\left(\frac{\mu(x,x)+\mu(y,y)}{2}\right)\\ &\leq F\left(\psi_1\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right), \psi_2\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right)\right)\\ &\leq \psi_1\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right). \end{aligned} \tag{54}$$

Hence, we have

$$\begin{aligned} &F\left(\psi_1\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right), \psi_2\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right)\right)\\ &= \psi_1\left(\frac{\mu(x,x)+\mu(y,y)}{2}, \frac{\mu(x,x)+\mu(y,y)}{2}\right) \Longrightarrow \mu(x,x) = \mu(y,y) = 0. \end{aligned} \tag{55}$$

Similarly, we have

$$\begin{aligned} \mu(u, u) &= \mu(v, v) = 0,\\ \mu(x, u) &= \mu(y, v) = 0. \end{aligned} \tag{56}$$

Hence, by $(m_1)$ $x = u$ and $y = v$, i.e., $(x, y)$ is the unique coupled fixed point of $T$.

If we define $F(s, t) = s - t$ and

$$\alpha(x, y) = \begin{cases} 1, & x \prec y;\\ 0, & \text{otherwise}, \end{cases} \tag{57}$$

then we get the following corollary which is a generalization of the main results in [31]. □

**Corollary 1.** *Let $(X, \mu)$ be an ordered complete M−metric space and $T\colon X \times X \longrightarrow X$ be an increasing mapping for which there exist $\phi \in \Phi$, $\psi_1 \in \Psi_1$, and $\psi_2 \in \Psi_2$ such that $\psi_1(t, t) \leq \phi(t)$ and for all $(x, y), (u, v) \in X^2$ with $x \prec u$ and $y \prec v$; we have*

$$\phi(\mu(T(x, y), T(u, v))) \leq \psi_1\left(\frac{K(x, u) + K(y, v)}{2}\right) - \psi_2\left(\frac{K(x, u) + K(y, v)}{2}\right), \tag{58}$$

*where*

$$\begin{aligned} K(x, u) &= \left(\frac{\mu(u, T(u, v))[1 + \mu(x, T(x, y))]}{1 + \mu(x, u)}, \mu(x, u)\right),\\ K(y, v) &= \left(\frac{\mu(v, T(v, u))[1 + \mu(y, T(y, x))]}{1 + \mu(y, v)}, \mu(y, v)\right). \end{aligned} \tag{59}$$

*Suppose that either*

*(a) T is continuous.*

*(b) For a convergent sequence $\{x_n\}$ in $(X, \mu)$, we have*

$$\begin{aligned} &\{x_n\} \longrightarrow x,\\ &x_n \prec x_{n+1} \Longrightarrow x_n \prec x, \quad \forall n,\\ &x_n \longrightarrow x,\\ &x_n \longrightarrow y \Longrightarrow x \prec y. \end{aligned} \tag{60}$$

*If there exist $x_0, y_0 \in X$ such that $x_0 \prec T(x_0, y_0)$ and $y_0 \prec T(y_0, x_0)$, then $T$ has a coupled fixed point.*

*Now, we introduce the following classes of functions $\Psi$ and $\Phi$ by*

$$\begin{aligned} \Psi &= \{\psi\colon [0,\infty) \longrightarrow [0,\infty), \psi \text{ is continuous, strictly increasing and } \psi(t) > 0 \text{ for } t > 0\},\\ \Phi &= \{\varphi\colon [0,\infty) \longrightarrow [0,\infty), \varphi \text{ is continuous and } \varphi(t) > 0 \text{ for } t > 0\}. \end{aligned} \tag{61}$$

*If we consider $\phi(t) = \psi(t)$, $\psi_1(s,t) = \psi(t)$ for some $\psi \in \Psi$ and $\psi_2(s,t) = \varphi(t)$ for some $\varphi \in \Phi$, then we obtain an extension of the main result in [13].*

**Corollary 2.** *Let $(X, \mu)$ be a complete $M-$metric space and $T\colon X \times X \longrightarrow X$ be an $\alpha-$admissible mapping such that*

$$[\psi(\mu(T(x,y), T(u,v))) + l]^{\max\{\alpha(x,u),\alpha(y,v)\}} \leq F\left(\psi\left(\frac{\mu(x,u) + \mu(y,v)}{2}\right), \varphi\left(\frac{\mu(x,u) + \mu(y,v)}{2}\right)\right) + l, \tag{62}$$

*for all $(x,y), (u,v) \in X^2$ with $\alpha(x,u) \geq 1, \alpha(y,v) \geq 1$, where $F \in \mathscr{C}$, $\psi \in \Psi$, and $\varphi \in \Phi$. Suppose that either*

*(a) $T$ is continuous.*

*(b) For a convergent sequence $\{x_n\}$ in $(X, \mu)$, we have*

$$\begin{aligned} &\{x_n\} \longrightarrow x,\\ &\alpha(x_n, x_{n+1}) \geq 1 \Longrightarrow \alpha(x_n, x) \geq 1, \quad \forall n,\\ &x_n \longrightarrow x,\\ &x_n \longrightarrow y \Longrightarrow \alpha(x, y) \geq 1. \end{aligned} \tag{63}$$

*If there exist $x_0, y_0 \in X$ such that $\alpha(x_0, T(x_0, y_0)) \geq 1$ and $\alpha(y_0, T(y_0, x_0)) \geq 1$, then $T$ has a coupled fixed point.*

*Remark 1.* Notice that in [32, 33], it was shown that each coupled fixed-point theorem can be observed from the analogue of single/standard fixed-point theorems. On the other hand, for the usage of it in application, the coupled fixed-point theorem can be used to handle the problem. Therefore, in this paper, we consider the coupled fixed-point results, Theorem 2 and Theorem 3.

## 4. Fractional Differential Equations

In this section, we present sufficient conditions for the existence and uniqueness of the solution of coupled systems (2) and (3). Before starting and proving the main results, we need to fix the analytical framework of our considered problem.

Consider the complete $M-$metric space $(X, \mu)$, where $X = C(J, \mathbb{R})$ and $\mu$ is defined by

$$\mu(x,y) = \sup_{t\in J}|x(t) - y(t)|, \quad \forall x, y \in X. \tag{64}$$

In addition, define the operator $T\colon X \times X \longrightarrow X$ as

$$T(x,y) = Ax(t) + By(t), \tag{65}$$

where

$$\begin{aligned} Ax(t) &= f(t, x(t)) + t^{\alpha-1}\frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}},\\ By(t) &= I^\alpha g\left(t, y(t), I^\beta y(t)\right) + t^{\alpha-1}\frac{\delta I^\alpha g(\eta, y(\eta), I^\beta y(\eta)) - I^\alpha g(\tau, y(\tau), I^\beta y(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \end{aligned} \tag{66}$$

Now, we claim that whenever $(x,y) \in X^2$ is a coupled fixed point of the operator $T$, it follows that $x(t)$ and $y(t)$ solve (2) and (3).

**Lemma 5.** *Let $n - 1 < \alpha \leq n$, $0 < \eta < \tau$, $\delta \neq (\tau/\eta)^{\alpha-1}$, and $h \in L(0,\tau)$; then, the boundary value problem*

$$\begin{aligned} &\mathscr{D}^\alpha[x(t) - f(t, x(t))] = h(t), \quad \forall t \in J,\\ &x^{(i)}(0) = \left.\frac{\partial^i f(t, x(t))}{\partial t^i}\right|_{t=0} = 0, \quad x(\tau) = \delta x(\eta), \quad \forall i = 0, 1, \ldots, n-2, \end{aligned} \tag{67}$$

*has the integral representation of solution*

$$x(t) = f(t, x(t)) + t^{\alpha-1}\frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^{\alpha}h(t) + t^{\alpha-1}\frac{\delta I^{\alpha}h(\eta) - I^{\alpha}h(\tau)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \tag{68}$$

*Proof.* Applying the operator $I^{\alpha}$ on both sides of (67) and using Lemma 4, we obtain

$$x(t) - f(t, x(t)) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} = I^{\alpha}h(t), \quad \text{at } t = 0 \Longrightarrow x(0) = 0, f(0,0) = 0 \Longrightarrow c_n = 0. \tag{69}$$

Also, we have

$$\begin{aligned} &\acute{x}(t) - \frac{\mathrm{d}f(t, x(t))}{\mathrm{d}t} + c_1(\alpha-1)t^{\alpha-2} + c_2(\alpha-2)t^{\alpha-3} + \cdots + c_{n-1}(\alpha-n+1)t^{\alpha-n} \\ &= I^{\alpha-1}h(t), \quad \text{at } t = 0 \Longrightarrow \acute{x}(0) = 0, \left.\frac{\mathrm{d}f(t,x(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f(t,x(t))}{\partial t}\right|_{t=0} = 0 \Longrightarrow c_{n-1} = 0, \\ &\vdots \\ &x^{(n-2)}(t) - \frac{\mathrm{d}^{n-2}f(t, x(t))}{\mathrm{d}t^{n-2}} + c_1(\alpha-1)\ldots(\alpha-n+2)t^{\alpha-n+1} + c_2(\alpha-2)\ldots(\alpha-n+1)t^{\alpha-n} \\ &= I^{\alpha-n+2}h(t), \quad \text{at } t = 0 \Longrightarrow x^{(n-2)}(0) = 0, \left.\frac{\mathrm{d}^{n-2}f(t, x(t))}{\mathrm{d}t^{n-2}}\right|_{t=0} = \left.\frac{\partial^{n-2}f(t, x(t))}{\partial t^{n-2}}\right|_{t=0} = 0 \Longrightarrow c_2 = 0. \end{aligned} \tag{70}$$

Hence, we obtain

$$x(t) - f(t, x(t)) + c_1 t^{\alpha-1} = I^{\alpha}h(t). \tag{71}$$

At $t = \tau$ and $\eta$, we have

$$x(\tau) - f(\tau, x(\tau)) + c_1 \tau^{\alpha-1} = I^{\alpha}h(\tau), \tag{72}$$

$$\delta x(\eta) - \delta f(\eta, x(\eta)) + c_1 \delta\eta^{\alpha-1} = \delta I^{\alpha}h(\eta). \tag{73}$$

By subtracting (73) from (72), we obtain

$$c_1 = \frac{f(\tau, x(\tau)) + I^{\alpha}h(\tau) - \delta[f(\eta, x(\eta)) + I^{\alpha}h(\eta)]}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \tag{74}$$

Consequently, the general solution of (67) is

$$\begin{aligned} x(t) = {} & f(t, x(t)) + t^{\alpha-1}\frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} \\ & + I^{\alpha}h(t) + t^{\alpha-1}\frac{\delta I^{\alpha}h(\eta) - I^{\alpha}h(\tau)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \end{aligned} \tag{75}$$

Consider the following coupled system of fractional hybrid integral equations (in short, FHIE):

$$x(t) = f(t, x(t)) + t^{\alpha-1}\frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^{\alpha}g\left(t, y(t), I^{\beta}y(t)\right) + t^{\alpha-1}\frac{\delta I^{\alpha}g\left(\eta, y(\eta), I^{\beta}y(\eta)\right) - I^{\alpha}g\left(\tau, y(\tau), I^{\beta}y(\tau)\right)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}, \tag{76}$$

$$y(t) = f(t, y(t)) + t^{\alpha-1}\frac{\delta f(\eta, y(\eta)) - f(\tau, y(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^{\alpha}g\left(t, x(t), I^{\beta}x(t)\right) + t^{\alpha-1}\frac{\delta I^{\alpha}g\left(\eta, x(\eta), I^{\beta}x(\eta)\right) - I^{\alpha}g\left(\tau, x(\tau), I^{\beta}x(\tau)\right)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \tag{77}$$

**Lemma 6.** *Assume that the function $\rho: J \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\rho(t, x) = x(t) - f(t, x(t))$ satisfies the following:*

$$\left.\mathscr{D}^{i}\rho(t, x)\right|_{t=0} = 0 \Longrightarrow x = 0, \quad \forall\, i = 0, 1, \ldots, n-2. \tag{78}$$

*Then, $(x, y) \in C^2(J, \mathbb{R})$ is a solution of FHDE systems (2) and (3) if and only if $(x, y)$ is a solution of FHIE systems (76) and (77).*

*Proof.* Let $x$ and $y$ be a solution of (2) and (3). Then, by Lemma 5, we gain that the general solution of (2) has the integral form presented in (76) and the solution of (3) has the form presented in (77). Thus, $x$ and $y$ satisfy (76) and (77).

Conversely, let $x$ and $y$ fulfill (76) and (77). Then, applying $\mathscr{D}^\alpha$ on both sides of (76) and using the relation $\mathscr{D}^\alpha t^\lambda = ((\Gamma(\lambda+1))/(\Gamma(\lambda-\alpha+1)))t^{\lambda-\alpha}$ if $\lambda > -1, \lambda \geq \alpha > 0$, and $\mathscr{D}^\alpha t^\lambda = 0$ if $\lambda < \alpha$ (Remark 2.1 in [34]) yield

$$\begin{aligned}\mathscr{D}^\alpha[x(t)-f(t,x(t))] &= \mathscr{D}^\alpha t^{\alpha-1}\frac{\delta f(\eta,x(\eta))-f(\tau,x(\tau))}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\\ &\quad+\mathscr{D}^\alpha I^\alpha g\left(t,y(t),I^\beta y(t)\right)+\mathscr{D}^\alpha t^{\alpha-1}\frac{\delta I^\alpha g\left(\eta,y(\eta),I^\beta y(\eta)\right)-I^\alpha g\left(\tau,y(\tau),I^\beta y(\tau)\right)}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\\ &\Longrightarrow \mathscr{D}^\alpha[x(t)-f(t,x(t))] = g\left(t,y(t),I^\alpha y(t)\right).\end{aligned} \tag{79}$$

So $x(t)$ satisfies the differential equation in (2). To see that it also satisfies the boundary conditions in the same equation, fix $i = 0, 1, \ldots, n-2$ and apply $\mathscr{D}^i$ in (76):

$$\begin{aligned}\mathscr{D}^i x(t) &= \mathscr{D}^i f(t,x(t)) + I^{\alpha-i}g\left(t,y(t),I^\beta y(t)\right)+\frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}t^{\alpha-1-i}\\ &\quad\cdot\left[\frac{\delta f(\eta,x(\eta))-f(\tau,x(\tau))}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}+\frac{\delta I^\alpha g\left(\eta,y(\eta),I^\beta y(\eta)\right)-I^\alpha g\left(\tau,y(\tau),I^\beta y(\tau)\right)}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\right].\end{aligned} \tag{80}$$

Substituting $t = 0$ in (80) and taking into account that $\alpha - 1 - i > 0$ yield

$$x^{(i)}(t)\Big|_{t=0} - \frac{\mathrm{d}^i f(t,x(t))}{\mathrm{d}t^i}\Big|_{t=0} = 0 \Longrightarrow x^{(i)}(0) = \frac{\partial^i f(t,x(t))}{\partial t^i}\Big|_{t=0} = 0. \tag{81}$$

Again, putting $t = \tau$ and $t = \eta$ in (76) implies

$$x(\tau) - \delta x(\eta) = 0. \tag{82}$$

Thus, $x(t)$ satisfies (2). A completely dual calculation reveals that $y(t)$ also satisfies (3).

As a consequence of Lemma 6, the coupled fixed point of the operator $T$ coincides with the solution of (76) and (77) and then with the solution of (2) and (3). □

**Theorem 4.** *Assume that $f: J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g: J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions and there exist two functions $\varphi_0, \varphi_1: J \longrightarrow \mathbb{R}$ with bounds $\|\varphi_0\|$ and $\|\varphi_1\|$, respectively, such that*

$$\begin{aligned}|f(t,x)-f(t,y)| &\leq \varphi_0(t)|x-y|,\\ |g(t,x,u)-g(t,y,v)| &\leq \varphi_1(t)(|x-y|+|u-v|).\end{aligned} \tag{83}$$

*Moreover, define these functions $\phi \in \Phi$, $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$, $F \in \mathscr{C}$, and $\alpha: X^2 \longrightarrow [0,\infty)$ as*

$$\begin{aligned}\phi(s) &= s,\\ \psi_1(s,t) &= (\rho+1)t,\\ \psi_2(s,t) &= t,\\ F(s,t) &= s-t,\\ \alpha(x,y) &= 1,\\ &\forall s,t \in [0,\infty),\\ &x,y \in X,\end{aligned} \tag{84}$$

*where*

$$\rho = 2\max\left\{\|\varphi_0\|\left[1+\frac{\delta+1}{\left|1-\delta(\eta/\tau)^{\alpha-1}\right|}\right],\frac{\|\varphi_1\|\tau^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{\tau^\beta}{\Gamma(\beta+1)}\right]\left[1+\frac{\delta+1}{\left|1-\delta(\eta/\tau)^{\alpha-1}\right|}\right]\right\} > 0. \tag{85}$$

*Then, problems (2) and (3) have a unique solution.*

*Proof.* We check that the hypothesis of Theorems 2 and Theorem 3 is satisfied. For $(x, y), (u, v) \in X^2$, we have

$$\begin{aligned}
\mu(T(x,y),T(u,v)) &= \sup_{t\in J}|T(x,y)(t)-T(u,v)(t)| \\
&\le \sup_{t\in J}\Bigg[\varphi_0(t)\Bigg(|x(t)-u(t)|+\left|\frac{t^{\alpha-1}}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\right|[\delta|x(\eta)-u(\eta)|+|u(\tau)-x(\tau)|]\Bigg) \\
&\quad + I^{\alpha}\Big[\varphi_1(t)\Big(|y(t)-v(t)|+\left|I^{\beta}y(t)-I^{\beta}v(t)\right|\Big)\Big]+\left|\frac{t^{\alpha-1}}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\right| \\
&\quad \cdot\Big[\delta I^{\alpha}\Big[\varphi_1(t)\Big(\left|y(\eta)-v(\eta)\right|+\left|I^{\beta}y(\eta)-I^{\beta}v(\eta)\right|\Big)\Big]+I^{\alpha}\Big[\varphi_1(t)\Big(|y(\tau)-v(\tau)|+\left|I^{\beta}y(\tau)-I^{\beta}v(\tau)\right|\Big)\Big]\Big]\Bigg] \\
&\le \|\varphi_0\|\left(\mu(x,u)+\left|\frac{t^{\alpha-1}}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\right|[\delta\mu(x,u)+\mu(x,u)]\right)+\|\varphi_1\|\Bigg(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left[\mu(y,v)+\frac{t^{\beta}}{\Gamma(\beta+1)}\mu(y,v)\right] \\
&\quad +\left|\frac{t^{\alpha-1}}{\tau^{\alpha-1}-\delta\eta^{\alpha-1}}\right|\left[\delta\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}\left[\mu(y,v)+\frac{\eta^{\beta}}{\Gamma(\beta+1)}\mu(y,v)\right]+\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left[\mu(y,v)+\frac{\tau^{\beta}}{\Gamma(\beta+1)}\mu(y,v)\right]\right]\Bigg) \\
&\le \|\varphi_0\|\left[1+\frac{\delta+1}{\left|1-\delta(\eta/\tau)^{\alpha-1}\right|}\right]\mu(x,u)+\frac{\|\varphi_1\|\tau^{\alpha}}{\Gamma(\alpha+1)}\left[1+\frac{\tau^{\beta}}{\Gamma(\beta+1)}\right]\left[1+\frac{\delta+1}{\left|1-\delta(\eta/\tau)^{\alpha-1}\right|}\right]\mu(y,v) \\
&\le (\rho+1)\left[\frac{\mu(x,u)+\mu(y,v)}{2}\right]-\left[\frac{\mu(x,u)+\mu(y,v)}{2}\right].
\end{aligned} \tag{86}$$

Thus, for any $s \ge 0$, we obtain

$$\phi(\mu(T(x,y),T(u,v))) \le F\left(\psi_1\left(s,\frac{\mu(x,u)+\mu(y,v)}{2}\right),\right. \left.\cdot\,\psi_2\left(s,\frac{\mu(x,u)+\mu(y,v)}{2}\right)\right). \tag{87}$$

Therefore, the operator $T$ satisfies condition (18) of Theorem 2. With simple calculations, we can derive that the other hypothesis of Theorems 2 and Theorem 3 holds. So, the operator $T$ has a unique fixed point, or equivalently, systems (2) and (3) have a unique solution in $X^2$.

Now, we present an illustrated example to justify our results. □

*Example 2.* Consider the following system of two FHDEs with three-point boundary conditions:

$$\begin{aligned}
&\mathscr{D}^{\alpha}[x(t)-f(t,x(t))] = g\big(t,y(y),I^{\beta}y(t)\big), \\
&x^{(i)}(0) = \left.\frac{\partial^i f(t,x(t))}{\partial t^i}\right|_{t=0} = 0, \\
&x(\tau) = \delta x(\eta),
\end{aligned} \tag{88}$$

$$\begin{aligned}
&\mathscr{D}^{\alpha}[y(t)-f(t,y(t))] = g\big(t,x(t),I^{\beta}x(t)\big), \\
&y^{(i)}(0) = \left.\frac{\partial^i f(t,y(t))}{\partial t^i}\right|_{t=0} = 0, \\
&y(\tau) = \delta y(\eta),
\end{aligned} \tag{89}$$

where

$$\begin{aligned}
\alpha &= \frac{5}{2}, \\
\beta &= \frac{9}{2}, \\
\eta &= \frac{1}{2}, \\
\tau &= \delta = 1, \\
f(t,x) &= x+\sqrt{x^2+1}-e^{t|x|}, \\
g(t,x,u) &= x+\sin u.
\end{aligned} \tag{90}$$

By computation, we can show that

$$\begin{aligned}
|f(t,x)-f(t,y)| &= \Big|x+\sqrt{x^2+1}-e^{t|x|} -\Big(x+\sqrt{x^2+1}-e^{t|x|}\Big)\Big| \\
&\le |x-y|+\left|\sqrt{x^2+1}-\sqrt{y^2+1}-1\right| \\
&\le 2|x-y|, \\
|g(t,x,u)-g(t,y,v)| &= |x+\sin u-y+\sin v| \\
&\le |x-y|+|u-v|.
\end{aligned} \tag{91}$$

Applying Theorem 4, we conclude that problem (89) has one solution.

## 5. Concluding Remarks

In this work, we proved some coupled fixed-point results for $\alpha-$admissible mappings which are $F(\psi_1, \psi_2)$-contractions in a larger structure such as $M-$metric spaces. Furthermore, we applied aforesaid fixed-point results to investigate the existence of a unique solution for a coupled system of higher-order fractional hybrid differential equations which are equipped with three-point boundary conditions. The respective results have been verified by providing a suitable example.

In fact, the results dealing with solutions of the general systems of fractional differential equations are useful in applications to various problems which are simply modelled by means of these systems.

It is believed that several recent studies (see, for example, [35–42]) on fractional calculus and its widespread applications will possibly motivate further research studies on mathematical modeling and analysis of applied problems along the lines which we have developed in this article.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors have read and agreed to the published version of the manuscript.

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