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Fuzzy pretopogenous structure based on way below relation

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Abstract. This paper aims to define fuzzy pretopogenous structure based on way below relation (or an L-fuzzifying pretopogenous structure (LFPT structure, for short)) and study some of its properties. Also, the concepts of L-fuzzifying pre-neighborhood, L-fuzzifying pre-interior, and L-fuzzifying pre-closure operators are established and we used these concepts to build an L-fuzzifying topology. Furthermore, a natural link is established between L-fuzzifying pretopogenous and L-fuzzifying topology. Finally, the maps between L-fuzzifying pretopogenous structures and initial fuzzifying structures are investigated.

1. Introduction and preliminary concepts

Today topology and the many related theories, have and will have a fundamental play in applied sciences. Every representation of real entities in a mathematical language necessarily implies a topological study of its goodness; it is a problem of linguistic translation continuity. Some mathematicians developed the notion of order relation between subsets of a set. The authors [7] introduced the term of topogenous order $<$ on a set X, a binary relation on 2^X. Zadeh produced the innovative concept of fuzzy set in his acclaimed [31]. Since that milestone, mathematicians have struggled to extend fundamental mathematical structures such as groups, rings, vector spaces, topologies, uniformities, and proximities to a fuzzy framework. As well, Badard [3] defined fuzzy pretopological spaces and studied their representation. In [17], the authors, in their attempt to find a unified theory of fuzzy topologies, fuzzy proximities, and fuzzy uniformities, introduced the fuzzy syntopogenous structures. The concept of a fuzzy syntopogenous structure on a set X is based on the basic term of order on the family of all fuzzy sets in X. It was shown that the fuzzy topologies, the fuzzy proximities, and the fuzzy uniformities are special cases of these structures. In [18], the authors continued with the investigation of fuzzy syntopogenous structures. The concept of fuzzifying syntopogenous structures was developed in [19, 23]. The L-fuzzy topologies were investigated and described with algebraic and analytic methods (for example [8, 13, 20, 21, 26, 27, 29, 30, 32, 33]). The authors [9, 10, 25] established the notions of L-fuzzy topogenous orders and investigated some of their properties. However, Ju-Mok and Kim [16] explained the relation between L-fuzzy topogenous orders and topological structures. The new notion of fuzzy topogenous has been introduced by using $(L, \leq, \odot, *)$, where $(L, \leq, \odot, *)$ is a strictly two-sided commutative quantale lattice with a strong negation "$*$" in [24]. The author [22] introduced and studied the concept of smooth pretopogenous structures and gave some particular construction of them. The way below relation was defined in [11], and in this article some of its

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properties were studied too. Also, Bancerek [4, 5] introduced the way below relation and stated several propositions in topics such as continuous lattices, directed powers, and topological spaces. However, as far as we are aware of there exists no analysis of the relationships between the fuzzy structure of a pretopogenous order and relations such as the way below relation. So far they have remained as two divergent fields of research. Here we shall conduct a substantial analysis of their mutual relationships. This achievement produces a basically theoretical article which is nonetheless necessary to provide a strong foundation of this novel aspect of topological fuzzy set theory. Both disciplines should be promoted with this pioneering analysis, which may also foster the inspection of other relationships among different types of topological structures.

The rest of this paper is organized as follows. This section contains some necessary concepts and properties. In Section 2, the notion of L-fuzzifying pretopogenous order is established and some of its properties are studied. Furthermore, the concepts of L-fuzzifying pre-neighborhood, L-fuzzifying pre-interior and L-fuzzifying pre-closure operators are investigated. In Section 3, we build an L-fuzzifying topology using an L-fuzzifying pre-interior operator, L-fuzzifying pre-closure operator, and L-fuzzifying pretopogenous order. Also, we create an L-fuzzifying pretopogenous order using L-fuzzifying topology. In Section 4, the concept of L-fuzzifying pretopogenous continuous functions is given and some results are discussed. In Section 5, maps between L-fuzzifying pretopogenous structures and initial fuzzifying structures are studied. In Section 6, the links between L-fuzzifying preproximity, L-preuniformity and L-fuzzifying pretopogenous on $\mathcal{X}$ are investigated. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research.

In this paper we adopt the standard terminology from lattice theory (which can be consulted in monographs like [6, 11, 12]). We assume that $(L, \leq, \wedge, \vee, ')$ is a completely distributive complete lattice whose smallest element is $\perp$ and whose largest element is $\top$. All other requirements or restrictions on L will be made explicit when required.

In this context, a first fundamental concept is given in our next definition.

Definition 1.1. ([4, 5, 11]) Let L be a complete lattice. We say that x is way below y, in symbols $x \ll y$, if and only if for any directed subset $\mathcal{D} \subseteq L$ the relation $y \leq \sup \mathcal{D}$ always implies the existence of $d \in \mathcal{D}$ with $x \leq d$.

Some immediate facts ensue from this notion:

Proposition 1.2. ([4, 5, 11]) *In a complete lattice L one has the following statements for all $u, x, y, z \in L$:*

(1) $x \ll y$ implies $x \leq y$;
(2) $u \leq x \ll y \leq z$ implies $u \ll z$;
(3) $x \ll z$ and $y \ll z$ together imply $x \vee y \ll z$;
(4) $0 \ll x$.

A second fundamental notion is given in the next definition.

Definition 1.3. ([8]) Let $\mathcal{X}$ be a nonempty set, L be a complete lattice and $\tau : 2^{\mathcal{X}} \longrightarrow L$ be a function that satisfies the following conditions:

(1) $\tau(\mathcal{X}) = \tau(\phi) = \top$;
(2) $\tau(\mathcal{A} \cap \mathcal{B}) \geq \tau(\mathcal{A}) \wedge \tau(\mathcal{B})$, for all $\mathcal{A}, \mathcal{B} \subseteq 2^{\mathcal{X}}$;
(3) for each $\{\mathcal{A}_i : i \in \Gamma\} \subseteq 2^{\mathcal{X}}$, $\tau\left(\bigcup_{i\in\Gamma} \mathcal{A}_i\right) \geq \inf_{i\in\Gamma} \tau(\mathcal{A}_i)$.

Then τ is called an L-fuzzifying topology on $\mathcal{X}$ and the pair $(\mathcal{X}, \tau)$ is called an L-fuzzifying topological space.

Henceforth, $(\mathcal{X}, \tau)$ will denote an L-fuzzifying topological space, with $\mathcal{X}$ being the universe of discourse.

Definition 1.4. ([33]) Let $(\mathcal{X}, \tau_1)$ and $(\mathcal{Y}, \tau_2)$ be two L-fuzzifying topological spaces. A function $f : (\mathcal{X}, \tau_1) \longrightarrow (\mathcal{Y}, \tau_2)$ is called L-fuzzifying continuous if for all $\mathcal{B} \in 2^{\mathcal{Y}}$, $\tau_2(\mathcal{B}) \leq \tau_1(f^{-1}(\mathcal{B}))$.

Associated with Definition 1.3 a concept exists whose definition is stated below in Definition 1.5, under the assumption that the lattice is completely distributive.

Definition 1.5. ([2]) A function $\mathcal{U} : 2^{X\times X} \longrightarrow L$ is called an L-fuzzifying preuniform structure on X if it satisfies the following axioms:
PU1: For any $u \in 2^{X\times X}$, if $\mathcal{U}(u) \neq \bot$, then $\Delta \subseteq u$.
PU2: If $\mathcal{U}(u) \ll r$ and $u \subseteq v$, then $\mathcal{U}(v) \ll r$, where $r \in L\setminus\{\bot\}$.
The pair $(X, \mathcal{U})$ is called an L-fuzzifying preuniform space.

2. L-fuzzifying pretopogenous order

Along this section, L represents a completely distributive lattice with order reversing involution denoted by $'$.

Now, we give the basic definition of the L-fuzzifying pretopogenous space using the way below relation.

Definition 2.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2 \in 2^X$ and $\{\mathcal{A}_i : i \in \Gamma\}, \{\mathcal{B}_i : i \in \Gamma\} \subseteq 2^X$. Then the function $\eta : 2^X \times 2^X \longrightarrow L$ is said to be an L-fuzzifying pretopogenous order on X (LFPT order on X, for short) if it satisfies the following axioms:

PT1 : $\eta(X, X) = \eta(\phi, \phi) = \top$;
PT2 : If $\overline{\eta}(\mathcal{A}, \mathcal{B}) \ll \top$, then $\mathcal{A} \subseteq \mathcal{B}$, where $\overline{\eta}$ is the negation of η;
The pair (X, η) is said to be an L-fuzzifying pretopogenous space on X (LFPT space on X, for short).
We will use the following additional properties:
PT3 : If $\mathcal{A} \subseteq \mathcal{A}_1, \mathcal{B}_1 \subseteq \mathcal{B}$ implies $\eta(\mathcal{A}_1, \mathcal{B}_1) \leq \eta(\mathcal{A}, \mathcal{B})$, then η is said to be of type **I**;
PT4 : If $\eta(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}) = \eta(\mathcal{A}_1, \mathcal{B}) \wedge \eta(\mathcal{A}_2, \mathcal{B})$ and
PT5 : $\eta(\mathcal{A}, \mathcal{B}_1 \cap \mathcal{B}_2) = \eta(\mathcal{A}, \mathcal{B}_1) \wedge \eta(\mathcal{A}, \mathcal{B}_2)$, then η is said to be of type **D**;
PT6 : If $\eta\left(\bigcup_{i\in\Gamma}\mathcal{A}_i, \mathcal{B}\right) = \inf_{i\in\Gamma} \eta(\mathcal{A}_i, \mathcal{B})$, then η is said to be perfect;
PT7 : If in addition to PT6, $\eta\left(\mathcal{A}, \bigcap_{i\in\Gamma}\mathcal{B}_i\right) = \inf_{i\in\Gamma} \eta(\mathcal{A}, \mathcal{B}_i)$, then η is said to be biperfect.

The next technical result will help us give some important facts about Definition 2.1.

Proposition 2.2. *Let η be an LFPT order on X. Then the mapping $\eta^s : 2^X \times 2^X \longrightarrow L$ defined by $\eta^s(\mathcal{A}, \mathcal{B}) = \eta(X - \mathcal{B}, X - \mathcal{A})$, satisfies the following:*

(1) η^s is an LFPT order on X;
*(2) If η is of type **I**, then so is η^s;*
*(3) If η is of type **D**, then so is η^s;*
(4) If η is biperfect, then so is η^s;
(5) $(\eta^s)^s = \eta$.

Proof. (1) (PT1) $\eta^s(X, X) = \eta(X - X, X - X) = \eta(\phi, \phi) = \top$. Similarly, $\eta^s(\phi, \phi) = \top$.
(PT2) Suppose that $\overline{\eta^s}(\mathcal{A}, \mathcal{B}) \ll \top$. Then $\overline{\eta}(X - \mathcal{B}, X - \mathcal{A}) \ll \top$. So, $X - \mathcal{B} \subseteq X - \mathcal{A}$. Hence $\mathcal{A} \subseteq \mathcal{B}$. Therefore, η^s is an LFPT order on X.
(2) Suppose that η is of type **I**, $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B}_1 \subseteq \mathcal{B}$. Then we have $X - \mathcal{B} \subseteq X - \mathcal{B}_1$ and $X - \mathcal{A}_1 \subseteq X - \mathcal{A}$. So, $\eta^s(\mathcal{A}_1, \mathcal{B}_1) = \eta(X - \mathcal{B}_1, X - \mathcal{A}_1) \leq \eta(X - \mathcal{B}, X - \mathcal{A}) = \eta^s(\mathcal{A}, \mathcal{B})$. Therefore η^s is of type **I**.
(3) Suppose η is of type **D**. Then we have:
(PT4) $\eta^s(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}) = \eta(X - \mathcal{B}, X - (\mathcal{A}_1 \cup \mathcal{A}_2))$
$= \eta(X - \mathcal{B}, (X - \mathcal{A}_1) \cap (X - \mathcal{A}_2))$
$= \eta(X - \mathcal{B}, X - \mathcal{A}_1) \wedge \eta(X - \mathcal{B}, X - \mathcal{A}_2)$
$= \eta^s(\mathcal{A}_1, \mathcal{B}) \wedge \eta^s(\mathcal{A}_2, \mathcal{B})$ and

(PT5) $\eta^s(\mathcal{A}, \mathcal{B}_1 \cap \mathcal{B}_2) = \eta(\mathcal{X} - (\mathcal{B}_1 \cap \mathcal{B}_2), \mathcal{X} - \mathcal{A})$
$= \eta((\mathcal{X} - \mathcal{B}_1) \cup (\mathcal{X} - \mathcal{B}_2), \mathcal{X} - \mathcal{A})$
$= \eta(\mathcal{X} - \mathcal{B}_1, \mathcal{X} - \mathcal{A}) \wedge \eta(\mathcal{X} - \mathcal{B}_2, \mathcal{X} - \mathcal{A})$
$= \eta^s(\mathcal{A}, \mathcal{B}_1) \wedge \eta^s(\mathcal{A}, \mathcal{B}_2)$.

Therefore η^s is of type **D**.

(4) Suppose η is biperfect. Then we have:

(PT6) $\eta^s\left(\bigcup_{i\in\Gamma} \mathcal{A}_i, \mathcal{B}\right) = \eta\left(\mathcal{X} - \mathcal{B}, \mathcal{X} - \bigcup_{i\in\Gamma} \mathcal{A}_i\right)$
$= \eta\left(\mathcal{X} - \mathcal{B}, \bigcap_{i\in\Gamma}(\mathcal{X} - \mathcal{A}_i)\right)$
$= \inf_{i\in\Gamma} \eta(\mathcal{X} - \mathcal{B}, \mathcal{X} - \mathcal{A}_i)$
$= \inf_{i\in\Gamma} \eta^s(\mathcal{A}_i, \mathcal{B})$ and

(PT7) $\eta^s\left(\mathcal{A}, \bigcap_{i\in\Gamma} \mathcal{B}_i\right) = \eta\left(\mathcal{X} - \bigcap_{i\in\Gamma} \mathcal{B}_i, \mathcal{X} - \mathcal{A}\right)$
$= \eta\left(\bigcup_{i\in\Gamma}(\mathcal{X} - \mathcal{B}_i), \mathcal{X} - \mathcal{A}\right)$
$= \inf_{i\in\Gamma} \eta(\mathcal{X} - \mathcal{B}_i, \mathcal{X} - \mathcal{A})$
$= \inf_{i\in\Gamma} \eta^s(\mathcal{A}, \mathcal{B}_i)$.

Therefore η^s is biperfect.

(5) $(\eta^s)^s(\mathcal{A}, \mathcal{B}) = \eta^s(\mathcal{X} - \mathcal{B}, \mathcal{X} - \mathcal{A}) = \eta(\mathcal{A}, \mathcal{B})$. □

Our next goal is to show that symmetrical is helpful to simplify the verification of certain properties of LFPT order.

Definition 2.3. Let η be an LFPT order on $\mathcal{X}$. If $\eta = \eta^s$, then η is said to be symmetrical.

Another natural property that holds true concerns the composition of LFPT orders.

Theorem 2.4. *Let $\eta_1, \eta_2 : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \longrightarrow L$ be two LFPT orders on $\mathcal{X}$. Define the composition of η_1 and η_2 on $\mathcal{X}$ by $\eta(\mathcal{A}, \mathcal{B}) = \sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}))$, $\eta = \eta_1 \circ \eta_2$. Then η has the following properties:*

(1) η is an LFPT order on $\mathcal{X}$;
*(2) If both η_1 and η_2 are of type **I**, then so is η;*
*(3) If both η_1 and η_2 are of type **D**, then so is η;*
(4) If both η_1 and η_2 are perfect (resp. biperfect), then η is perfect (resp. biperfect);
(5) $\eta^s = \eta_2^s \circ \eta_1^s$.

Proof. (1) (PT1) $\eta(\mathcal{X}, \mathcal{X}) = \eta_1 \circ \eta_2(\mathcal{X}, \mathcal{X}) = \sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{X}, C) \wedge \eta_2(C, \mathcal{X})) = \top$. Similarly, $\eta(\phi, \phi) = \top$.

(PT2) Suppose that $\overline{\eta}(\mathcal{A}, \mathcal{B}) \ll \top$. Then, we have $\overline{\sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}))} \ll \top$. Hence $\inf_{C\in 2^{\mathcal{X}}}(\overline{\eta_1}(\mathcal{A}, C) \vee \overline{\eta_2}(C, \mathcal{B})) \ll \top$. So, there exist C_1 such that $\overline{\eta_1}(\mathcal{A}, C_1) \vee \overline{\eta_2}(C_1, \mathcal{B}) \ll \top$. Since $\overline{\eta_1}(\mathcal{A}, C_1) \leq \overline{\eta_1}(\mathcal{A}, C_1) \vee \overline{\eta_2}(C_1, \mathcal{B})$, then $\overline{\eta_1}(\mathcal{A}, C_1) \ll \top$ which implies $\mathcal{A} \subseteq C_1$. Similarly, $C_1 \subseteq \mathcal{B}$. Therefore, $\mathcal{A} \subseteq \mathcal{B}$ and η is an LFPT order on $(\mathcal{X})$.

(2) Assume η_1 and η_2 are of type **I**, $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $\eta_1(\mathcal{A}_1, C) \leq \eta_1(\mathcal{A}, C)$ and $\eta_2(C, \mathcal{B}_1) \leq \eta_2(C, \mathcal{B})$. Hence $\eta_1(\mathcal{A}_1, C) \wedge \eta_2(C, \mathcal{B}_1) \leq \eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B})$. So, $\sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{A}_1, C) \wedge \eta_2(C, \mathcal{B}_1)) \leq \sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}))$. Therefore, $\eta(\mathcal{A}_1, \mathcal{B}_1) \leq \eta(\mathcal{A}, \mathcal{B})$ and η is of type **I**.

(3) Suppose η_1 and η_2 are of type **D**. Then we have:

(PT4) $\eta(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}) = \sup_{C\in 2^{\mathcal{X}}}(\eta_1(\mathcal{A}_1 \cup \mathcal{A}_2, C) \wedge \eta_2(C, \mathcal{B}))$

$$= \sup_{C\in 2^{X}}((\eta_1(\mathcal{A}_1, C) \wedge \eta_1(\mathcal{A}_2, C)) \wedge \eta_2(C, \mathcal{B}))$$
$$= \sup_{C\in 2^{X}}((\eta_1(\mathcal{A}_1, C) \wedge \eta_2(C, \mathcal{B})) \wedge (\eta_1(\mathcal{A}_2, C) \wedge \eta_2(C, \mathcal{B})))$$
$$= \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}_1, C) \wedge \eta_2(C, \mathcal{B})) \wedge \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}_2, C) \wedge \eta_2(C, \mathcal{B}))$$
$$= \eta(\mathcal{A}_1, \mathcal{B}) \wedge \eta(\mathcal{A}_2, \mathcal{B});$$

(PT5) $\eta(\mathcal{A}, \mathcal{B}_1 \cap \mathcal{B}_2) = \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_1 \cap \mathcal{B}_2))$
$$= \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge (\eta_2(C, \mathcal{B}_1) \wedge \eta_2(C, \mathcal{B}_2)))$$
$$= \sup_{C\in 2^{X}}((\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_1)) \wedge (\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_2)))$$
$$= \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_1)) \wedge \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_2))$$
$$= \eta(\mathcal{A}, \mathcal{B}_1) \wedge \eta(\mathcal{A}, \mathcal{B}_2).$$

Hence η is of type **D**.

(4) Suppose η_1 and η_2 are perfect. Then

(PT6) $\eta\left(\bigcup_{i\in\Gamma}\mathcal{A}_i, \mathcal{B}\right) = \sup_{C\in 2^{X}}\left(\eta_1\left(\bigcup_{i\in\Gamma}\mathcal{A}_i, C\right) \wedge \eta_2(C, \mathcal{B})\right)$
$$= \sup_{C\in 2^{X}}\left(\inf_{i\in\Gamma} \eta_1(\mathcal{A}_i, C) \wedge \eta_2(C, \mathcal{B})\right)$$
$$= \inf_{i\in\Gamma} \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}_i, C) \wedge \eta_2(C, \mathcal{B}))$$
$$= \inf_{i\in\Gamma} \eta(\mathcal{A}_i, \mathcal{B}).$$

Therefore, η is perfect.

Again, suppose that η_1 and η_2 are biperfect. Then:

(PT7) $\eta\left(\mathcal{A}, \bigcap_{i\in\Gamma}\mathcal{B}_i\right) = \sup_{C\in 2^{X}}\left(\eta_1(\mathcal{A}, C) \wedge \eta_2\left(C, \bigcap_{i\in\Gamma}\mathcal{B}_i\right)\right)$
$$= \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge \inf_{i\in\Gamma} \eta_2(C, \mathcal{B}_i))$$
$$= \inf_{i\in\Gamma} \sup_{C\in 2^{X}}(\eta_1(\mathcal{A}, C) \wedge \eta_2(C, \mathcal{B}_i))$$
$$= \inf_{i\in\Gamma} \eta(\mathcal{A}, \mathcal{B}_i).$$

Hence η is biperfect.

(5) $\eta^s(\mathcal{A}, \mathcal{B}) = \eta(X - \mathcal{B}, X - \mathcal{A})$
$$= \sup_{C\in 2^{X}}(\eta_1(X - \mathcal{B}, C) \wedge \eta_2(C, X - \mathcal{A}))$$
$$= \sup_{C\in 2^{X}}(\eta_1^s(X - C, \mathcal{B}) \wedge \eta_2^s(\mathcal{A}, X - C)), \text{ put } X - C = \mathcal{D}$$
$$= \sup_{\mathcal{D}\in 2^{X}}(\eta_2^s(\mathcal{A}, \mathcal{D}) \wedge \eta_1^s(\mathcal{D}, \mathcal{B})) = \eta_2^s \circ \eta_1^s(\mathcal{A}, \mathcal{B}). \quad \square$$

Our next theorem identifies some properties of the set of all pre-neighborhoods inherited of an LFPT order.

Theorem 2.5. *Let η be an LFPT order on X. Then the mapping $\mathcal{N}_\eta : 2^X \times (L\setminus\{\top\}) \longrightarrow 2^X$ defined by $\mathcal{N}_\eta(\mathcal{A}, r) = \left\{\mathcal{B} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{B}) \ll r'\right\}$, where r' is the complement of r, satisfies the following statements:*

(1) If $\mathcal{B} \in \mathcal{N}_\eta(\mathcal{A}, r)$, then $\mathcal{A} \subseteq \mathcal{B}$;
(2) If η is symmetric, then $\mathcal{B} \in \mathcal{N}_\eta(\mathcal{A}, r)$ if and only if $\mathcal{A}^c \in \mathcal{N}_\eta(\mathcal{B}^c, r)$, where $\mathcal{A}^c = X - \mathcal{A}$;
*(3) If η is of type **I** and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{N}_\eta(\mathcal{B}, r) \subseteq \mathcal{N}_\eta(\mathcal{A}, r)$;*
*(4) If η is of type **D** and $\mathcal{A}, \mathcal{B} \in \mathcal{N}_\eta(C, r)$, then $\mathcal{A} \cap \mathcal{B} \in \mathcal{N}_\eta(C, r)$.*

Proof. (1) Since $\mathcal{B} \in \mathcal{N}_\eta(\mathcal{A}, r)$, then $\overline{\eta}(\mathcal{A}, \mathcal{B}) \ll r' \leq \top$. Hence $\overline{\eta}(\mathcal{A}, \mathcal{B}) \ll \top$. Therefore, $\mathcal{A} \subseteq \mathcal{B}$.

(2) Suppose $\mathcal{B} \in \mathcal{N}_\eta(\mathcal{A}, r)$. Hence $\overline{\eta}(\mathcal{A}, \mathcal{B}) \ll r'$. Since η is symmetrical, then $\overline{\eta}(\mathcal{X} - \mathcal{B}, \mathcal{X} - \mathcal{A}) = \overline{\eta}(\mathcal{A}, \mathcal{B}) \ll r'$. So, $\mathcal{A}^c \in \mathcal{N}_\eta(\mathcal{B}^c, r)$. Similarly, if $\mathcal{A}^c \in \mathcal{N}_\eta(\mathcal{B}^c, r)$, then $\mathcal{B} \in \mathcal{N}_\eta(\mathcal{A}, r)$.

(3) Suppose $C \in \mathcal{N}_\eta(\mathcal{B}, r)$. Then $\overline{\eta}(\mathcal{B}, C) \ll r'$. Since $\mathcal{A} \subseteq \mathcal{B}$ and $C \subseteq C$, then $\eta(\mathcal{B}, C) \leq \eta(\mathcal{A}, C)$. Hence $\overline{\eta}(\mathcal{A}, C) \leq \overline{\eta}(\mathcal{B}, C)$. Thus $\overline{\eta}(\mathcal{A}, C) \ll r'$. Therefore $C \in \mathcal{N}_\eta(\mathcal{A}, r)$.

(4) Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{N}_\eta(C, r)$. Then $\overline{\eta}(C, \mathcal{A}) \ll r'$ and $\overline{\eta}(C, \mathcal{B}) \ll r'$. So, $(\overline{\eta}(C, \mathcal{A}) \vee \overline{\eta}(C, \mathcal{B})) \ll r'$. Since η is of type **D**, then $\eta(C, \mathcal{A} \cap \mathcal{B}) = \eta(C, \mathcal{A}) \wedge \eta(C, \mathcal{B})$. Therefore, $\overline{\eta}(C, \mathcal{A} \cap \mathcal{B}) = (\overline{\eta}(C, \mathcal{A}) \vee \overline{\eta}(C, \mathcal{B})) \ll r'$ and so $\mathcal{A} \cap \mathcal{B} \in \mathcal{N}_\eta(C, r)$. □

Note: $\mathcal{N}_\eta(\mathcal{A}, r)$ is called the set of pre-neighborhoods of $\mathcal{A}$.

The following theorem aims to give some properties of an L-fuzzifying pre-interior operator inherited from an LFPT order.

Theorem 2.6. *Let η be an LFPT order on $\mathcal{X}$ and define the mapping $\mathcal{I}_\eta : 2^{\mathcal{X}} \times (L \backslash \{\top\}) \longrightarrow 2^{\mathcal{X}}$ by $\mathcal{I}_\eta(\mathcal{A}, r) = \bigcup \left\{\mathcal{B} \in 2^{\mathcal{X}} : \overline{\eta}(\mathcal{B}, \mathcal{A}) \ll r'\right\}$. Then the mapping $\mathcal{I}_\eta$ is called an L-fuzzifying pre-interior operator and it satisfies the following statements:*

(1) $\mathcal{I}_\eta(\mathcal{X}, r) = \mathcal{X}$;
(2) $\mathcal{I}_\eta(\mathcal{A}, r) \subseteq \mathcal{A}$;
*(3) If η is of type **I** and $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{I}_\eta(\mathcal{A}, r) \subseteq \mathcal{I}_\eta(\mathcal{B}, r)$;*
*(4) If η is of type **D**, then $\mathcal{I}_\eta(\mathcal{A} \cap \mathcal{B}, r) = \mathcal{I}_\eta(\mathcal{A}, r) \cap \mathcal{I}_\eta(\mathcal{B}, r)$;*
(5) If $r \leq s$, then $\mathcal{I}_\eta(\mathcal{A}, r) \supseteq \mathcal{I}_\eta(\mathcal{A}, s)$.

Proof. (1) Since $\eta(\mathcal{X}, \mathcal{X}) = \top$, then $\overline{\eta}(\mathcal{X}, \mathcal{X}) = \bot \ll r'$. Hence $\mathcal{I}_\eta(\mathcal{X}, r) = \mathcal{X}$.

(2) As $\overline{\eta}(\mathcal{B}, \mathcal{A}) \ll r'$, then $\overline{\eta}(\mathcal{B}, \mathcal{A}) \ll \top$. So, $\mathcal{B} \subseteq \mathcal{A}$. Therefore, $\mathcal{I}_\eta(\mathcal{A}, r) \subseteq \mathcal{A}$.

(3) Since $C \subseteq C$, $\mathcal{A} \subseteq \mathcal{B}$ and η is of type **I**, then we obtain that $\eta(C, \mathcal{A}) \leq \eta(C, \mathcal{B})$. Hence $\overline{\eta}(C, \mathcal{B}) \leq \overline{\eta}(C, \mathcal{A})$. Also, since $\overline{\eta}(C, \mathcal{A}) \ll r'$ implies $\overline{\eta}(C, \mathcal{B}) \ll r'$, then $\mathcal{I}_\eta(\mathcal{A}, r) = \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{A}) \ll r'\right\} \subseteq \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{B}) \ll r'\right\} = \mathcal{I}_\eta(\mathcal{B}, r)$.

(4)
$$
\begin{aligned}
\mathcal{I}_\eta(\mathcal{A} \cap \mathcal{B}, r) &= \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta(C, \mathcal{A}) \wedge \eta(C, \mathcal{B})} \ll r'\right\} \\
&= \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{A}) \vee \overline{\eta}(C, \mathcal{B}) \ll r'\right\} \\
&= \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{A}) \ll r' \wedge \overline{\eta}(C, \mathcal{B}) \ll r'\right\} \\
&= \left(\bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{A}) \ll r'\right\}\right) \bigcap \left(\bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{B}) \ll r'\right\}\right) \\
&= \mathcal{I}_\eta(\mathcal{A}, r) \cap \mathcal{I}_\eta(\mathcal{B}, r).
\end{aligned}
$$

(5) Since $\mathcal{I}_\eta(\mathcal{A}, s) = \bigcup \left\{C \in 2^{\mathcal{X}} : \overline{\eta}(C, \mathcal{A}) \ll s'\right\}$ and $s' \leq r'$, $\overline{\eta}(C, \mathcal{A}) \ll r'$. So, $\mathcal{I}_\eta(\mathcal{A}, r) \supseteq \mathcal{I}_\eta(\mathcal{A}, s)$. □

The following theorem aims to give some properties of an L-fuzzifying pre-closure operator inherited from an LFPT order.

Theorem 2.7. *Let η be an LFPT order on $\mathcal{X}$ and define the mapping $C_\eta : 2^{\mathcal{X}} \times (L \backslash \{\top\}) \longrightarrow 2^{\mathcal{X}}$ by $C_\eta(\mathcal{A}, r) = \bigcap \left\{\mathcal{F} \in 2^{\mathcal{X}} : \overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r'\right\}$. Then the mapping C_η is called an L-fuzzifying pre-closure operator and it satisfies the following:*

(1) $\mathcal{A} \subseteq C_\eta(\mathcal{A}, r)$;
(2) $C_\eta(\phi, r) = \phi$;
*(3) If η is of type **I** and $\mathcal{A} \subseteq \mathcal{B}$, then $C_\eta(\mathcal{A}, r) \subseteq C_\eta(\mathcal{B}, r)$;*
*(4) If η is of type **D**, then $C_\eta(\mathcal{A} \cup \mathcal{B}, r) = C_\eta(\mathcal{A}, r) \cap C_\eta(\mathcal{B}, r)$;*
*(5) If η is of type **I** and **D**, then $C_\eta(\mathcal{A} \cup \mathcal{B}, r) = C_\eta(\mathcal{A}, r) \cup C_\eta(\mathcal{B}, r)$;*
(6) If $r \leq s$, then $C_\eta(\mathcal{A}, r) \subseteq C_\eta(\mathcal{A}, s)$.

Proof. (1) $C_\eta(\mathcal{A}, r) = \bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r'\right\} \supseteq \bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{F}) \ll \top\right\}$
$\supseteq \bigcap \left\{\mathcal{F} \in 2^X : \mathcal{A} \subseteq \mathcal{F}\right\} = \mathcal{A}$.

(2) Since $\eta(\phi, \phi) = \top$, then $\overline{\eta}(\phi, \phi) = \bot \ll r'$. Therefore, $C_\eta(\phi, r) = \phi$.

(3) Suppose $\mathcal{A} \subseteq \mathcal{B}$ and η is of type **I**. Then $\overline{\eta}(\mathcal{A}, \mathcal{F}) \leq \overline{\eta}(\mathcal{B}, \mathcal{F})$. Also, if $\overline{\eta}(\mathcal{B}, \mathcal{F}) \ll r'$, then $\overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r'$. So, $C_\eta(\mathcal{A}, r) \subseteq C_\eta(\mathcal{B}, r)$.

(4) $C_\eta(\mathcal{A} \cup \mathcal{B}, r) = \bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A} \cup \mathcal{B}, \mathcal{F}) \ll r'\right\}$
$= \bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{F}) \vee \overline{\eta}(\mathcal{B}, \mathcal{F}) \ll r'\right\}$
$= \bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r' \wedge \overline{\eta}(\mathcal{B}, \mathcal{F}) \ll r'\right\}$
$= \left(\bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r'\right\}\right) \bigcap \left(\bigcap \left\{\mathcal{F} \in 2^X : \overline{\eta}(\mathcal{B}, \mathcal{F}) \ll r'\right\}\right)$
$= C_\eta(\mathcal{A}, r) \bigcap C_\eta(\mathcal{B}, r)$.

(5) Follows from (3) and (4) above.

(6) Since $\overline{\eta}(\mathcal{A}, \mathcal{F}) \ll s'$ and $s' \leq r'$, then $\overline{\eta}(\mathcal{A}, \mathcal{F}) \ll r'$. Hence $C_\eta(\mathcal{A}, r) \subseteq C_\eta(\mathcal{A}, s)$. □

3. LFPT orders and *L*-fuzzifying topologies

This section is devoted to building an *L*-fuzzifying topology using an L-fuzzifying pre-interior operator, L-fuzzifying preclosure operator, and *L*-fuzzifying pretopogenous order. Also, we create an *L*-fuzzifying pretopogenous order using *L*-fuzzifying topology.

Theorem 3.1. *Let η be an LFPT order $\mathcal{X}$ of type **I** and **D**. Then the map $\tau_\eta : 2^X \longrightarrow L$ defined by $\tau_\eta(\mathcal{A}) = \sup \left\{r \in (L \backslash \{\top\}) : \mathcal{I}_\eta(\mathcal{A}, r) = \mathcal{A}\right\}$ is an L-fuzzifying topology on $\mathcal{X}$.*

Proof. (1) Since $\mathcal{I}_\eta(\mathcal{X}, r) = \mathcal{X}$ and $\mathcal{I}_\eta(\phi, r) = \phi$ for all $r \in (L \backslash \{\top\})$, then $\tau_\eta(\mathcal{X}) = \tau_\eta(\phi) = \top$.

(2)
$$\begin{aligned}\tau_\eta(\mathcal{A}) \wedge \tau_\eta(\mathcal{B}) &= \left(\sup \left\{r_1 \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A}, r_1) = \mathcal{A}\right\}\right) \bigwedge \left(\sup \left\{r_2 \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{B}, r_2) = \mathcal{B}\right\}\right)\\
&= \sup \left\{r_1 \wedge r_2 \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A}, r_1) = \mathcal{A} \ \textit{and} \ \mathcal{I}_\eta(\mathcal{B}, r_2) = \mathcal{B}\right\}\\
&\leq \sup \left\{r_1 \wedge r_2 \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A}, r_1) \cap \mathcal{I}_\eta(\mathcal{B}, r_2) = \mathcal{A} \cap \mathcal{B}\right\}\\
&\leq \sup \left\{r \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A} \cap \mathcal{B}, r) \supseteq \mathcal{A} \cap \mathcal{B}\right\}, \text{ where } r = r_1 \wedge r_2\\
&= \sup \left\{r \in L \backslash \{\top\} | \left\{\mathcal{I}_\eta(\mathcal{A} \cap \mathcal{B}, r) = \mathcal{A} \cap \mathcal{B}\right\}\right\} = \tau_\eta(\mathcal{A} \cap \mathcal{B}).\end{aligned}$$

(3)
$$\begin{aligned}\inf_{i \in \Gamma} \tau_\eta(\mathcal{A}_i) &= \inf_{i \in \Gamma} \sup \left\{r_i \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A}_i, r_i) = \mathcal{A}_i\right\}\\
&= \sup \left\{\inf_{i \in \Gamma} r_i \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\mathcal{A}_i, r_i) = \mathcal{A}_i\right\}\\
&\leq \sup \left\{\inf_{i \in \Gamma} r_i \in L \backslash \{\top\} \middle| \bigcup_{i \in \Gamma} \mathcal{I}_\eta(\mathcal{A}_i, r_i) = \bigcup_{i \in \Gamma} \mathcal{A}_i\right\}\\
&\leq \sup \left\{\inf_{i \in \Gamma} r_i \in L \backslash \{\top\} \middle| \mathcal{I}_\eta(\bigcup_{i \in \Gamma} \mathcal{A}_i, \inf_{i \in \Gamma} r_i) \supseteq \bigcup_{i \in \Gamma} \mathcal{A}_i\right\}\\
&= \sup \left\{r \in L \backslash \{\top\} \middle| \mathcal{I}_\eta\left(\bigcup_{i \in \Gamma} \mathcal{A}_i, r\right) = \bigcup_{i \in \Gamma} \mathcal{A}_i\right\}\\
&= \tau_\eta\left(\bigcup_{i \in \Gamma} \mathcal{A}_i\right), \text{ where } r = \inf_{i \in \Gamma} r_i\end{aligned}$$

Thus τ_η is an *L*-fuzzifying topology on *X*. □

Theorem 3.2. *Let η is an LFPT order on $\mathcal{X}$ of type **I** and **D**. Define a map $\tau_\eta : 2^X \longrightarrow L$ by*

$$\tau_\eta(\mathcal{A}) = \sup\left\{r \in (L\setminus\{\top\}) : C_\eta(\mathcal{A}^c, r) = \mathcal{A}^c\right\}.$$

Then τ_η is an L-fuzzifying topology on $\mathcal{X}$.

Proof. (1) Obvious.

(2) $\tau_\eta(\mathcal{A}) \wedge \tau_\eta(\mathcal{B}) = \left(\sup\left\{r_1 \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c, r_1) = \mathcal{A}^c\right\}\right) \bigwedge \left(\sup\left\{r_2 \in L\setminus\{\top\} \middle| C_\eta(\mathcal{B}^c, r_2) = \mathcal{B}^c\right\}\right)$

$$= \sup\left\{r_1 \wedge r_2 \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c, r_1) = \mathcal{A}^c \ \ \textit{and} \ \ C_\eta(\mathcal{B}^c, r_2) = \mathcal{B}^c\right\}$$

$$\leq \sup\left\{r_1 \wedge r_2 \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c, r_1) \cup C_\eta(\mathcal{B}^c, r_2) = \mathcal{A}^c \cup \mathcal{B}^c\right\}$$

$$\leq \sup\left\{r \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c, r) \cup C_\eta(\mathcal{B}^c, r) \subseteq \mathcal{A}^c \cup \mathcal{B}^c\right\}, \text{ where } r = r_1 \wedge r_2.$$

$$\leq \sup\left\{r \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c \cup \mathcal{B}^c, r) \subseteq \mathcal{A}^c \cup \mathcal{B}^c\right\}$$

$$= \sup\left\{r \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}^c \cup \mathcal{B}^c, r) = \mathcal{A}^c \cup \mathcal{B}^c\right\}$$

$$= \sup\left\{r \in L\setminus\{\top\} \middle| C_\eta((\mathcal{A} \cap \mathcal{B})^c, r) = (\mathcal{A} \cap \mathcal{B})^c\right\} = \tau_\eta(\mathcal{A} \cap \mathcal{B}).$$

(3)= Suppose there exists a family $\left\{\mathcal{A}_i \in 2^{\mathcal{X}} \,|i \in \Gamma\right\}$.

$$\inf_{i\in\Gamma}\tau_\eta(\mathcal{A}_i) = \inf_{i\in\Gamma}\ \sup\left\{r_i \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}_i^c, r_i) = \mathcal{A}_i^c\right\}$$

$$= \sup\left\{\inf_{i\in\Gamma} r_i \in L\setminus\{\top\} \middle| C_\eta(\mathcal{A}_i^c, r_i) = \mathcal{A}_i^c\right\}$$

$$\leq \sup\left\{\inf_{i\in\Gamma} r_i \in L\setminus\{\top\} \middle| \bigcap_{i\in\Gamma} C_\eta(\mathcal{A}_i^c, r_i) = \bigcap_{i\in\Gamma}\mathcal{A}_i^c\right\}$$

$$\leq \sup\left\{\inf_{i\in\Gamma} r_i \in L\setminus\{\top\} \middle| C_\eta\left(\bigcap_{i\in\Gamma}\mathcal{A}_i^c, \inf_{i\in\Gamma} r_i\right) \subseteq \bigcap_{i\in\Gamma}\mathcal{A}_i^c\right\}$$

$$= \sup\left\{r \in L\setminus\{\top\} \middle| C_\eta\left(\left(\bigcup_{i\in\Gamma}\mathcal{A}_i\right)^c, r\right) = \left(\bigcup_{i\in\Gamma}\mathcal{A}_i\right)^c\right\}\Bigg\}$$

$$= \tau_\eta\left(\bigcup_{i\in\Gamma}\mathcal{A}_i\right), \text{ where } r = \inf_{i\in\Gamma} r_i$$

Thus τ_η is an L-fuzzifying topology on $\mathcal{X}$. □

Theorem 3.3. *Let η be an LFPT order of type **D** on $\mathcal{X}$. Then the map $\tau_\eta : 2^{\mathcal{X}} \longrightarrow L$ defined by $\tau_\eta(\mathcal{A}) = \inf_{x\in\mathcal{A}} (\eta(\{x\}, \mathcal{A}))$ is an L-fuzzifying topology on $\mathcal{X}$.*

Proof. (1) Since $\eta(\mathcal{X}, \mathcal{X}) = \eta(\phi, \phi) = \top$, then $\tau_\eta(\mathcal{X}) = \tau_\eta(\phi) = \top$.

(2) Since η is of type **D**, then

$$\begin{aligned}\tau_\eta(\mathcal{A} \cap \mathcal{B}) &= \inf_{x\in\mathcal{A}\cap\mathcal{B}} \eta(\{x\}, \mathcal{A} \cap \mathcal{B}) \\ &= \inf_{x\in\mathcal{A}\cap\mathcal{B}} (\eta(\{x\}, \mathcal{A}) \wedge \eta(\{x\}, \mathcal{B})) \\ &\geq \left(\inf_{x\in\mathcal{A}} \eta(\{x\}, \mathcal{A})\right) \wedge (\inf_{x\in\mathcal{B}} \eta(\{x\}, \mathcal{B})) \\ &= \tau_\eta(\mathcal{A}) \wedge \tau_\eta(\mathcal{B}).\end{aligned}$$

(3) $$\begin{aligned}\tau_\eta\left(\bigcup_{i\in\Gamma}\mathcal{A}_i\right) &= \inf_{x\in\bigcup_{i\in\Gamma}\mathcal{A}_i} \eta\left(\{x\}, \bigcup_{i\in\Gamma}\mathcal{A}_i\right) \\ &= \inf_{i\in\Gamma}\ \inf_{x\in\mathcal{A}} \eta\left(\{x\}, \bigcup_{i\in\Gamma}\mathcal{A}_i\right) \\ &\geq \inf_{i\in\Gamma}\ \inf_{x\in\mathcal{A}} \eta(\{x\}, \mathcal{A}_i)\end{aligned}$$

$= \inf_{i\in\Gamma} \tau_\eta(\mathcal{A}_i)$.

Hence τ is an L-fuzzifying topology. $\square$

Theorem 3.4. *Let $(\mathcal{X}, \tau)$ be an L-fuzzifying topological space. Define the function $\eta_\tau : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \longrightarrow L$ as follows:*

$$\eta_\tau(\mathcal{A},\mathcal{B}) = \begin{cases} \sup\limits_{\mathcal{D}\in\Phi(\mathcal{A},\mathcal{B}^c)} \tau(\mathcal{D}), & \Phi(\mathcal{A},\mathcal{B}^c) \neq \phi; \\ \bot, & \Phi(\mathcal{A},\mathcal{B}^c) = \phi, \end{cases}$$

where $\Phi : 2^{\mathcal{X}} \times 2^{\mathcal{X}} \longrightarrow 2^{(2^{\mathcal{X}})}$ is defined as $\Phi(\mathcal{A},\mathcal{B}^c) = \left\{\mathcal{D} \in 2^{\mathcal{X}} : \mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{B} \in 2^{\mathcal{X}}\right\}, \forall \mathcal{A},\mathcal{B} \in 2^{\mathcal{X}}$. Then η_τ is an LFPT order on $\mathcal{X}$.

Proof. (PT1) Since $\Phi(\mathcal{X},\phi) = \left\{\mathcal{D} \in 2^{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{D} \subseteq \mathcal{X}\right\} = \{\mathcal{X}\} \neq \phi$ and $\tau(\mathcal{X}) = \top$, then $\eta_\tau(\mathcal{X},\mathcal{X}) = \top$. Similarly, $\eta_\tau(\phi,\phi) = \top$.

(PT2) Suppose $\overline{\eta}_\tau(\mathcal{A},\mathcal{B}) \ll \top$. Then $\eta_\tau(\mathcal{A},\mathcal{B}) > \bot$. So, there exist $\mathcal{D} \in 2^{\mathcal{X}}$ such that $\mathcal{D} \in \Phi(\mathcal{A},\mathcal{B}^c)$ and $\tau(\mathcal{D}) \geq \bot$. Thus $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{B}$. Therefore η_τ is an L-fuzzifying pretopogenous order.

(PT3) Suppose that $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B}^c \subseteq \mathcal{B}_1^c$. If $\eta_\tau(\mathcal{A}_1,\mathcal{B}_1) = \bot$, then $\Phi(\mathcal{A}_1,\mathcal{B}_1^c) = \phi$. Hence $\Phi(\mathcal{A},\mathcal{B}^c) = \phi$ and so $\eta_\tau(\mathcal{A},\mathcal{B}) = \bot$. Therefore $\eta_\tau(\mathcal{A}_1,\mathcal{B}_1) \leq \eta_\tau(\mathcal{A},\mathcal{B})$. If $\eta_\tau(\mathcal{A}_1,\mathcal{B}_1) \neq \bot$, then there exist $\mathcal{D} \in 2^{\mathcal{X}}$ such that $\mathcal{A}_1 \subseteq \mathcal{D} \subseteq \mathcal{B}_1$. So, $\mathcal{A} \subseteq \mathcal{A}_1 \subseteq \mathcal{D} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}$. Thus $\Phi(\mathcal{A}_1,\mathcal{B}_1^c) \subseteq \Phi(\mathcal{A},\mathcal{B}^c)$. Therefore, $\eta_\tau(\mathcal{A}_1,\mathcal{B}_1) = \sup\limits_{\mathcal{D}\in\Phi(\mathcal{A}_1,\mathcal{B}_1^c)} \tau(\mathcal{D}) \leq \sup\limits_{\mathcal{D}\in\Phi(\mathcal{A},\mathcal{B}^c)} \tau(\mathcal{D}) = \eta_\tau(\mathcal{A},\mathcal{B})$ and η_τ is of type **I**.

(PT4) Since $\mathcal{A} \subseteq \mathcal{A} \cup \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{C}$, then $\eta_\tau(\mathcal{A} \cup \mathcal{B},\mathcal{C}) \leq \eta_\tau(\mathcal{A},\mathcal{C})$. Similarly, $\eta_\tau(\mathcal{A} \cup \mathcal{B},\mathcal{C}) \leq \eta_\tau(\mathcal{B},\mathcal{C})$. Hence $\eta_\tau(\mathcal{A} \cup \mathcal{B},\mathcal{C}) \leq \eta_\tau(\mathcal{A},\mathcal{C}) \wedge \eta_\tau(\mathcal{B},\mathcal{C})$. Also,

$$\begin{aligned} \eta_\tau(\mathcal{A},\mathcal{C}) \wedge \eta_\tau(\mathcal{B},\mathcal{C}) &= \sup_{\mathcal{D}\in\Phi(\mathcal{A},\mathcal{C}^c)} \tau(\mathcal{D}) \wedge \sup_{\mathcal{H}\in\Phi(\mathcal{B},\mathcal{C}^c)} \tau(\mathcal{H}) \\ &= \sup_{\mathcal{D}\in\Phi(\mathcal{A},\mathcal{C}^c)} \sup_{\mathcal{H}\in\Phi(\mathcal{B},\mathcal{C}^c)} (\tau(\mathcal{D}) \wedge \tau(\mathcal{H})) \\ &\leq \sup_{\mathcal{D}\in\Phi(\mathcal{A},\mathcal{C}^c)} \sup_{\mathcal{H}\in\Phi(\mathcal{B},\mathcal{C}^c)} (\tau(\mathcal{D} \cup \mathcal{H})) \\ &\leq \sup_{\mathcal{D}\cup\mathcal{H}\in\Phi(\mathcal{A}\cup\mathcal{B},\mathcal{C}^c)} \tau(\mathcal{D} \cup \mathcal{H}) \\ &= \sup_{\mathcal{F}\in\Phi(\mathcal{A}\cup\mathcal{B},\mathcal{C}^c)} (\tau(\mathcal{F})) \\ &= \eta_\tau(\mathcal{A} \cup \mathcal{B},\mathcal{C}). \end{aligned}$$

(PT5) Since $\mathcal{C} \subseteq \mathcal{C}$ and $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A}$, then we have $\eta_\tau(\mathcal{C},\mathcal{A} \cap \mathcal{B}) \leq \eta_\tau(\mathcal{C},\mathcal{A})$. Similarly, we obtain $\eta_\tau(\mathcal{C},\mathcal{A} \cap \mathcal{B}) \leq \eta_\tau(\mathcal{C},\mathcal{B})$. So, $\eta_\tau(\mathcal{C},\mathcal{A} \cap \mathcal{B}) \leq \eta_\tau(\mathcal{C},\mathcal{A}) \wedge \eta_\tau(\mathcal{C},\mathcal{B})$. Furthermore

$$\begin{aligned} \eta_\tau(\mathcal{C},\mathcal{A}) \wedge \eta_\tau(\mathcal{C},\mathcal{B}) &= \sup_{\mathcal{D}\in\Phi(\mathcal{C},\mathcal{A}^c)} \tau(\mathcal{D}) \wedge \sup_{\mathcal{H}\in\Phi(\mathcal{C},\mathcal{B}^c)} \tau(\mathcal{H}) \\ &= \sup_{\mathcal{D}\in\Phi(\mathcal{C},\mathcal{A}^c)} \sup_{\mathcal{H}\in\Phi(\mathcal{C},\mathcal{B}^c)} (\tau(\mathcal{D}) \wedge \tau(\mathcal{H})) \\ &\leq \sup_{\mathcal{D}\in\Phi(\mathcal{C},\mathcal{A}^c)} \sup_{\mathcal{H}\in\Phi(\mathcal{C},\mathcal{B}^c)} \tau(\mathcal{D} \cap \mathcal{H}) \\ &\leq \sup_{\mathcal{D}\cap\mathcal{H}\in\Phi(\mathcal{C},\mathcal{A}^c\cup\mathcal{B}^c)} \tau(\mathcal{D} \cap \mathcal{H}) \\ &= \sup_{\mathcal{M}\in\Phi(\mathcal{C},(\mathcal{A}\cap\mathcal{B})^c)} \tau(\mathcal{M}) \\ &= \eta_\tau(\mathcal{C},\mathcal{A} \cap \mathcal{B}). \end{aligned}$$

Therefore η_τ is of type **D**. $\square$

4. LFPT continuity

To conclude the theoretical contribution of this paper, in this section we define and investigate the concept of LFPT continuity. This notion is formalized as follows.

Definition 4.1. Let $(\mathcal{X}, \eta_1)$ and $(\mathcal{Y}, \eta_2)$ be two LFPT spaces. A function $f : (\mathcal{X}, \eta_1) \longrightarrow (\mathcal{Y}, \eta_2)$ is said to be an LFPT continuous if $\eta_2(\mathcal{A}, \mathcal{B}) \leq \eta_1(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B})), \forall \mathcal{A}, \mathcal{B} \in 2^{\mathcal{Y}}$.

If f is surjective, then we say that f is an LFPT map if and only if $f^{-1}(\eta_2)$ is coarser than η_1, i.e., $f^{-1}(\eta_2)(\mathcal{A}, \mathcal{B}) \leq \eta_1(\mathcal{A}, \mathcal{B})$.

A technical characterization gives an alternative view of the concept above.

Theorem 4.2. *Let $(\mathcal{X}, \eta_1)$ and $(\mathcal{Y}, \eta_2)$ be two LFPT spaces on $\mathcal{X}$ and $\mathcal{Y}$, respectively and $f : (\mathcal{X}, \eta_1) \longrightarrow (\mathcal{Y}, \eta_2)$ be a function. Then the following are equivalent:*

(1) f is an LFPT continuous function;

(2) $\eta_2^s(\mathcal{A}, \mathcal{B}) \leq \eta_1^s(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B})), \forall \mathcal{A}, \mathcal{B} \in 2^{\mathcal{Y}}$;

(3) If $\mathcal{U}$ is a pre-neighborhood of $f(\mathcal{D})$, then $f^{-1}(\mathcal{U})$ is a pre-neighborhood of $\mathcal{D}$ for every $\mathcal{U} \in 2^{\mathcal{Y}}$, $\mathcal{D} \in 2^{\mathcal{X}}$ and f is surjective.

Proof. (1) ⇒ (2):
$$\begin{aligned}\eta_2^s(\mathcal{A}, \mathcal{B}) &= \eta_2(\mathcal{Y} - \mathcal{B}, \mathcal{Y} - \mathcal{A})\\ &\leq \eta_1(f^{-1}(\mathcal{Y} - \mathcal{B}), f^{-1}(\mathcal{Y} - \mathcal{A}))\\ &= \eta_1(\mathcal{X} - f^{-1}(\mathcal{B}), \mathcal{X} - f^{-1}(\mathcal{A}))\\ &= \eta_1^s(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B})).\end{aligned}$$

(2) ⇒ (3): Suppose $\mathcal{U}$ is a pre-neighborhood of $f(\mathcal{D})$. Then $\overline{\eta_2}(f(\mathcal{D}), \mathcal{U}) \ll r'$ which implies $(\overline{\eta_2})^s(\mathcal{Y} - \mathcal{U}, \mathcal{Y} - f(\mathcal{D})) \ll r'$. Since $(\overline{\eta_1})^s(\mathcal{X} - f^{-1}(\mathcal{U}), \mathcal{X} - f^{-1}(f(\mathcal{D}))) \leq (\overline{\eta_2})^s(\mathcal{Y} - \mathcal{U}, \mathcal{Y} - f(\mathcal{D}))$, then $(\overline{\eta_1})^s(\mathcal{X} - f^{-1}(\mathcal{U}), \mathcal{X} - \mathcal{D}) \ll r'$. Hence $\overline{\eta_1}(\mathcal{D}, f^{-1}(\mathcal{U})) \ll r'$. Therefore $f^{-1}(\mathcal{U})$ is a pre-neighborhood of $\mathcal{D}$.

(3) ⇒ (1): Since $\overline{\eta_2}(f(\mathcal{D}), \mathcal{U}) \ll r'$, $\overline{\eta_1}(\mathcal{D}, f^{-1}(\mathcal{U})) \ll r'$. Thus $\overline{\eta_1}(\mathcal{D}, f^{-1}(\mathcal{U})) \leq \overline{\eta_2}(f(\mathcal{D}), \mathcal{U})$. So $\eta_2(f(\mathcal{D}), \mathcal{U}) \leq \eta_1(\mathcal{D}, f^{-1}(\mathcal{U}))$. Put $\mathcal{A} = f(\mathcal{D})$ and $\mathcal{B} = \mathcal{U}$. Hence $\eta_2(\mathcal{A}, \mathcal{B}) \leq \eta_1(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B}))$. □

Another natural property that holds true concerns the composition of LFPT continuous functions.

Theorem 4.3. *Let $(\mathcal{X}, \eta_1), (\mathcal{Y}, \eta_2)$ and $(\mathcal{Z}, \eta_3)$ be three LFPT spaces on $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. If $f : (\mathcal{X}, \eta_1) \longrightarrow (\mathcal{Y}, \eta_2)$ and $g : (\mathcal{Y}, \eta_2) \longrightarrow (\mathcal{Z}, \eta_3)$ are LFPT continuous, then $g \circ f : (\mathcal{X}, \eta_1) \longrightarrow (\mathcal{Z}, \eta_3)$ is an LFPT continuous.*

Proof. Since both f and g are LFPT continuous functions, then for all $\mathcal{A}, \mathcal{B} \in 2^{\mathcal{Z}}$ we have
$$\begin{aligned}\eta_3(\mathcal{A}, \mathcal{B}) &\leq \eta_2(g^{-1}(\mathcal{A}), g^{-1}(\mathcal{B}))\\ &\leq \eta_1(f^{-1}(g^{-1}(\mathcal{A})), f^{-1}(g^{-1})(\mathcal{B}))\\ &= \eta_1((f^{-1} \circ g^{-1})(\mathcal{A}), (f^{-1} \circ g^{-1})(\mathcal{B}))\\ &= \eta_1((g \circ f)^{-1}(\mathcal{A}), (g \circ f)^{-1}(\mathcal{B})).\end{aligned}$$
Hence $g \circ f$ is an LFPT continuous function. □

An important characterization of LFPT continuous function is given as follows.

Theorem 4.4. *Let $(\mathcal{X}, \eta_1)$ and $(\mathcal{Y}, \eta_2)$ be two LFPT spaces and $f : (\mathcal{X}, \eta_1) \longrightarrow (\mathcal{Y}, \eta_2)$ be an LFPT continuous function. Then the following statements hold:*

(1) $f\left(C_{\eta_1}(\mathcal{A}, r)\right) \subseteq C_{\eta_2}(f(\mathcal{A}), r)$, for each $\mathcal{A} \in 2^{\mathcal{X}}$;

(2) $C_{\eta_1}\left(f^{-1}(\mathcal{B}), r\right) \subseteq f^{-1}\left(C_{\eta_2}(\mathcal{B}, r)\right)$, for each $\mathcal{B} \in 2^{\mathcal{Y}}$;

(3) $f^{-1}\left(\mathcal{I}_{\eta_2}(C, r)\right) \subseteq \mathcal{I}_{\eta_1}\left(f^{-1}(C), r\right)$, for each $C \in 2^{\mathcal{Y}}$;

(4) $f : (\mathcal{X}, \tau_{\eta_1}) \longrightarrow (\mathcal{Y}, \tau_{\eta_2})$ is an L-fuzzifying continuous function.

Proof. (1) Since f is an LFPT continuous function, then $\overline{\eta_2}(f(\mathcal{A}), f(\mathcal{B})) \geq \overline{\eta_1}(\mathcal{A}, \mathcal{B})$. So, $\overline{\eta_2}(f(\mathcal{A}), f(\mathcal{B})) \ll r'$ implies $\overline{\eta_1}(\mathcal{A}, \mathcal{B}) \ll r'$. Therefore
$$\begin{aligned}f\left(C_{\eta_1}(\mathcal{A}, r)\right) &= f\left(\bigcap\left\{\mathcal{B} \in 2^{\mathcal{X}} : \overline{\eta_1}(\mathcal{A}, \mathcal{B}) \ll r'\right\}\right)\\ &\subseteq \bigcap\left\{f(\mathcal{B}) \in 2^{\mathcal{Y}} : \overline{\eta_2}(f(\mathcal{A}), f(\mathcal{B})) \ll r'\right\}\\ &= C_{\eta_2}(f(\mathcal{A}), r).\end{aligned}$$

(2) Since $f\left(C_{\eta_1}(f^{-1}(\mathcal{B}),r)\right)\subseteq C_{\eta_2}\left(f\left(f^{-1}(\mathcal{B})\right),r\right)\subseteq C_{\eta_2}(\mathcal{B},r)$, then
$C_{\eta_2}\left(f^{-1}(\mathcal{B}),r\right)\subseteq f^{-1}\left(f\left(C_{\eta_1}\left(f^{-1}(\mathcal{B}),r\right)\right)\right)\subseteq f^{-1}\left(C_{\eta_2}(\mathcal{B},r)\right)$.

(3) Since f is an LFPT continuous function, then

$$\begin{aligned}
f^{-1}\left(\mathcal{I}_{\eta_2}(C,r)\right) &= f^{-1}\left(\bigcup\left\{\mathcal{D}\in 2^{\mathcal{Y}}:\overline{\eta}_2(\mathcal{D},C)\ll r'\right\}\right)\\
&= \bigcup\left\{f^{-1}(\mathcal{D})\in 2^{\mathcal{X}}:\overline{\eta}_2(\mathcal{D},C)\ll r'\right\}\\
&\subseteq \bigcup\left\{f^{-1}(\mathcal{D})\in 2^{\mathcal{X}}:\overline{\eta}_1(f^{-1}(\mathcal{D}),f^{-1}(C))\ll r'\right\}\\
&= \bigcup\left\{\mathcal{E}\in 2^{\mathcal{X}}:\overline{\eta}_1(\mathcal{E},f^{-1}(C))\ll r'\right\}\\
&= \mathcal{I}_{\eta_1}(f^{-1}(C),r).
\end{aligned}$$

(4) Suppose $\mathcal{B}\in 2^{\mathcal{Y}}$ such that $\mathcal{I}_{\eta_2}(\mathcal{B},r)=\mathcal{B}$. From (3) we have $f^{-1}(\mathcal{B})=f^{-1}(\mathcal{I}_{\eta_2}(\mathcal{B},r))\subseteq\mathcal{I}_{\eta_1}(f^{-1}(\mathcal{B}),r)$. But $\mathcal{I}_{\eta_1}(f^{-1}(\mathcal{B}),r)\subseteq f^{-1}(\mathcal{B})$. Then $\mathcal{I}_{\eta_1}(f^{-1}(\mathcal{B}),r)=f^{-1}(\mathcal{B})$. Therefore, f is L-fuzzifying continuous function. □

5. Maps between LFPT spaces

Extensions of standard mathematical notions abound, and the value of any resulting theory should be judged by the strength of its link with initial structures. This section gives the maps between L-fuzzifying pretopogenous structures and initial fuzzifying structures.

Definition 5.1. Let f be a function from a set $\mathcal{X}$ to a set $\mathcal{Y}$ and η be an LFPT order on $\mathcal{Y}$. Define a function $\eta_1:2^{\mathcal{X}}\times 2^{\mathcal{X}}\longrightarrow L$ by

$$\eta_1(\mathcal{A},\mathcal{B})=\eta(f(\mathcal{A}),(f(\mathcal{B}^c))^c),\ \forall\mathcal{A},\mathcal{B}\in 2^{\mathcal{X}}.$$

We will call η_1 is the inverse image of η by the mapping f and is denoted by $f^{-1}(\eta)$.

Proposition 5.2. *Let $f:\mathcal{X}\longrightarrow\mathcal{Y}$ be a surjective function and η be an LFPT order on $\mathcal{Y}$. Then the following hold:*
(1) $f^{-1}(\eta)$ is an LFPT order on $\mathcal{X}$;
(2) $(f^{-1}(\eta))^s=f^{-1}(\eta^s)$ and if η is symmetrical, then so is $f^{-1}(\eta)$;
*(3) If η is of type **I**, then so is $f^{-1}(\eta)$;*
*(4) If η is of type **D**, then so is $f^{-1}(\eta)$.*

Proof. (1) Since f is surjective, then:

(PT1) $f^{-1}(\eta)(\phi,\phi)=\eta(f(\phi),(f(\phi)^c)^c)=\eta(\phi,(f(\mathcal{X}))^c)=\eta(\phi,\mathcal{Y}-f(\mathcal{X}))=\eta(\phi,\phi)=\top$. Similarly, $f^{-1}(\eta)(\mathcal{X},\mathcal{X})=\top$.

(PT2) Suppose $\overline{f^{-1}(\eta)}(\mathcal{A},\mathcal{B})\ll\top$. Then $\overline{\eta}(f(\mathcal{A}),(f(\mathcal{B}^c))^c)\ll\top$. Since η is an LFPT order on $\mathcal{Y}$, then $f(\mathcal{A})\subseteq(f(\mathcal{B}^c))^c$ which implies $f(\mathcal{B}^c)\subseteq(f(\mathcal{A}))^c$. Then $f(\mathcal{B}^c)\subseteq\mathcal{Y}-f(\mathcal{A})\subseteq f(\mathcal{X}-\mathcal{A})$. So, $f(\mathcal{B}^c)\subseteq f(\mathcal{A}^c)$ which implies $\mathcal{A}\subseteq\mathcal{B}$. Hence $f^{-1}(\eta)$ is an LFPT order on $\mathcal{X}$.

(2) $(f^{-1}(\eta))^s(\mathcal{A},\mathcal{B})=f^{-1}(\eta)(\mathcal{B}^c,\mathcal{A}^c)=\eta(f(\mathcal{B}^c),(f(\mathcal{A}))^c)=\eta^s(f(\mathcal{A}),(f(\mathcal{B}^c))^c)=f^{-1}(\eta^s)(\mathcal{A},\mathcal{B})$.
Also, $f^{-1}(\eta)(\mathcal{A},\mathcal{B})=\eta(f(\mathcal{A}),(f(\mathcal{B}^c))^c)=\eta^s(f(\mathcal{A}),(f(\mathcal{B}^c))^c)=f^{-1}(\eta^s)(\mathcal{A},\mathcal{B})=(f^{-1}(\eta))^s(\mathcal{A},\mathcal{B})$.
Hence $f^{-1}(\eta)$ is symmetrical.

(3) Suppose η is of type **I**, $\mathcal{A}\subseteq\mathcal{A}_1$ and $\mathcal{B}_1\subseteq\mathcal{B}$. Then $f(\mathcal{A})\subseteq f(\mathcal{A}_1)$ and $(f(\mathcal{B}_1^c))^c\subseteq(f(\mathcal{B}^c))^c$.
So, $f^{-1}(\eta)(\mathcal{A}_1,\mathcal{B}_1)=\eta(f(\mathcal{A}_1),(f(\mathcal{B}_1^c))^c)\le\eta(f(\mathcal{A}),(f(\mathcal{B}^c))^c)=f^{-1}(\eta)(\mathcal{A},\mathcal{B})$. Therefore, $f^{-1}(\eta)$ is of type **I**.

(4) Suppose η is of type **D** and f is a surjective function. Then:

(PT4)
$$\begin{aligned}
f^{-1}(\eta)(\mathcal{A}_1\cup\mathcal{A}_2,\mathcal{B}) &= \eta(f(\mathcal{A}_1\cup\mathcal{A}_2),(f(\mathcal{B}^c))^c)\\
&= \eta(f(\mathcal{A}_1)\cup f(\mathcal{A}_2),(f(\mathcal{B}^c))^c)\\
&= \eta(f(\mathcal{A}_1),(f(\mathcal{B}^c))^c)\wedge\eta(f(\mathcal{A}_2),(f(\mathcal{B}^c))^c)\\
&= f^{-1}(\eta)(\mathcal{A}_1,\mathcal{B})\wedge f^{-1}(\eta)(\mathcal{A}_2,\mathcal{B}).
\end{aligned}$$

(PT5)
$$\begin{aligned}
f^{-1}(\eta)(\mathcal{A},\mathcal{B}_1\cap\mathcal{B}_2) &= \eta(f(\mathcal{A}),(f(\mathcal{B}_1\cap\mathcal{B}_2)^c)^c)\\
&= \eta(f(\mathcal{A}),(f(\mathcal{B}_1^c\cup\mathcal{B}_2^c))^c)\\
&= \eta(f(\mathcal{A}),(f(\mathcal{B}_1^c)\cup f(\mathcal{B}_2^c))^c)\\
&= \eta(f(\mathcal{A}),(f(\mathcal{B}_1^c))^c\cap(f(\mathcal{B}_2^c))^c)
\end{aligned}$$

$$= \eta(f(\mathcal{A}), (f(\mathcal{B}_1^c))^c) \wedge \eta(f(\mathcal{A}), (f(\mathcal{B}_2^c))^c)$$
$$= f^{-1}(\eta)(\mathcal{A}, \mathcal{B}_1) \wedge f^{-1}(\eta)(\mathcal{A}, \mathcal{B}_2).$$

Therefore, $f^{-1}(\eta)$ is of type **D**. □

Definition 5.3. Let $f_i : \mathcal{X} \longrightarrow (\mathcal{Y}_i, \eta_i)$, $i \in I$ be surjective functions, where η_i is an LFPT orders on $\mathcal{Y}_i$. Then the initial L-fuzzifying structure $\mathfrak{E}$ on $\mathcal{X}$ is the coarsest one for which f_i are LFPT maps. That is $\mathfrak{E}(\mathcal{A}, \mathcal{B}) = f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B})$, for all $i \in I$.

Theorem 5.4. *Let $\mathfrak{E}$ be an initial L-fuzzifying structure on $\mathcal{X}$. Then:*

(1) If η_i are symmetrical, then so is $\mathfrak{E}$;
*(2) If η_i are of type **I**, then so is $\mathfrak{E}$;*
*(3) If η_i are of type **D**, then so is $\mathfrak{E}$.*

Proof. (1) Suppose η_i are symmetrical. Then we have

$$\mathfrak{E}(\mathcal{A}, \mathcal{B}) = f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B})$$
$$= \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}_i^c))^c)$$
$$= \eta_i^s(f_i(\mathcal{A}), (f_i(\mathcal{B}_i^c))^c)$$
$$= f_i^{-1}(\eta_i^s)(\mathcal{A}, \mathcal{B})$$
$$= \mathfrak{E}^s(\mathcal{A}, \mathcal{B}).$$ Hence $\mathfrak{E}$ is symmetrical.

(2) Suppose η_i are of type **I**, $\mathcal{A} \subseteq \mathcal{A}_1$ and $\mathcal{B}_1 \subseteq \mathcal{B}$. Then $f_i(\mathcal{A}) \subseteq f_i(\mathcal{A}_1)$ and $(f_i(\mathcal{B}_1^c))^c \subseteq (f_i(\mathcal{B}^c))^c$. So,

$$\mathfrak{E}(\mathcal{A}_1, \mathcal{B}_1) = f_i^{-1}(\eta_i)(\mathcal{A}_1, \mathcal{B}_1)$$
$$= \eta_i(f_i(\mathcal{A}_1), (f_i(\mathcal{B}_1^c))^c)$$
$$\leq \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}^c))^c)$$
$$= f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B})$$
$$= \mathfrak{E}(\mathcal{A}, \mathcal{B}).$$

Then $\mathfrak{E}$ is of type **I**.

(3) Suppose η_i are of type **D** and f_i are surjective. Then:

(PT4) $$\mathfrak{E}(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}) = f_i^{-1}(\eta_i)(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B})$$
$$= \eta_i(f_i(\mathcal{A}_1 \cup \mathcal{A}_2), (f_i(\mathcal{B}^c))^c)$$
$$= \eta_i(f_i(\mathcal{A}_1) \cup f_i(\mathcal{A}_2), (f(\mathcal{B}^c))^c)$$
$$= \eta_i(f_i(\mathcal{A}_1), (f_i(\mathcal{B}^c))^c) \wedge \eta_i(f_i(\mathcal{A}_2), (f_i(\mathcal{B}^c))^c)$$
$$= f_i^{-1}(\eta_i)(\mathcal{A}_1, \mathcal{B}) \wedge f_i^{-1}(\eta_i)(\mathcal{A}_2, \mathcal{B})$$
$$= \mathfrak{E}(\mathcal{A}_1, \mathcal{B}) \wedge \mathfrak{E}(\mathcal{A}_2, \mathcal{B}).$$

(PT5) $$\mathfrak{E}(\mathcal{A}, \mathcal{B}_1 \cap \mathcal{B}_2) = f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B}_1 \cap \mathcal{B}_2)$$
$$= \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}_1 \cap \mathcal{B}_2)^c)^c)$$
$$= \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}_1^c \cup \mathcal{B}_2^c))^c)$$
$$= \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}_1^c) \cup f_i(\mathcal{B}_2^c))^c)$$
$$= \eta_i(f_i(A), (f_i(\mathcal{B}_1^c))^c \cap (f_i(\mathcal{B}_2^c))^c)$$
$$= \eta_i(fi(A), (f_i(\mathcal{B}_1^c))^c) \wedge \eta_i(f_i(\mathcal{A}), (f_i(\mathcal{B}_2^c))^c)$$
$$= f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B}_1) \wedge f_i^{-1}(\eta_i)(\mathcal{A}, \mathcal{B}_2)$$
$$= \mathfrak{E}(\mathcal{A}, \mathcal{B}_1) \wedge \mathfrak{E}(\mathcal{A}, \mathcal{B}_2).$$

Hence $\mathfrak{E}$ is of type **D**. □

6. The relationships among different structures

It is interesting to discuss the links between L-fuzzifying preproximity, L-preuniformity and L- fuzzifying pretopogenous on $\mathcal{X}$.

Proposition 6.1. *(1) Let δ be an L-fuzzifying preproximity on $\mathcal{X}$ in [1], then $\eta_\delta(\mathcal{A},\mathcal{B}) = \overline{\delta}(\mathcal{A},\mathcal{B}^c)$ (where $\overline{\delta}$ is negation of δ), defines an LFPT order on $\mathcal{X}$. When δ is symmetrical, then so is η_δ. When δ is of type **I** (resp. **D**), then so is η_δ.*

*(2) Let η be an LFPT on $\mathcal{X}$, then $\delta_\eta(\mathcal{A},\mathcal{B}) = \overline{\eta}(\mathcal{A},\mathcal{B}^c)$ defines an L-fuzzifying preproximity on $\mathcal{X}$. When η is symmetrical, then so is δ_η. When η is of type **I** (resp. **D**), then so is δ_η.*

Proof. (1) Suppose δ be an L-fuzzifying preproximity on $\mathcal{X}$.

(PT1) $\eta_\delta(\mathcal{X},\mathcal{X}) = \overline{\delta}(\mathcal{X},\phi) = \overline{\bot} = \top$, $\eta_\delta(\phi,\phi) = \overline{\delta}(\phi,\mathcal{X}) = \overline{\bot} = \top$.

(PT2) If $\overline{\eta_\delta}(\mathcal{A},\mathcal{B}) = \overline{\overline{\delta}}(\mathcal{A},\mathcal{B}^c) \ll \top$, then $\delta(\mathcal{A},\mathcal{B}^c) \ll \top$. So, $\mathcal{A} \subseteq \mathcal{B}$. Hence η_δ is an LFPT order on $\mathcal{X}$.

(PT3) If $\mathcal{A} \subseteq \mathcal{B}, \mathcal{A}_1 \subseteq \mathcal{B}_1$, then $\overline{\delta}(\mathcal{B},\mathcal{A}_1^c) \leq \overline{\delta}(\mathcal{A},\mathcal{B}_1^c)$. So, $\eta_\delta(\mathcal{B},\mathcal{A}_1) \leq \eta_\delta(\mathcal{A},\mathcal{B}_1)$. Then η_δ is of type **I**.

(PT4)
$$\begin{aligned}\eta_\delta(\mathcal{A}\cup\mathcal{B},\mathcal{C}) &= \overline{\delta}(\mathcal{A}\cup\mathcal{B},\mathcal{C}^c)\\ &= \overline{\delta}(\mathcal{A},\mathcal{C}^c)\wedge\overline{\delta}(\mathcal{B},\mathcal{C}^c)\\ &= \eta_\delta(\mathcal{A},\mathcal{C})\wedge\eta_\delta(\mathcal{B},\mathcal{C}).\end{aligned}$$

(PT5)
$$\begin{aligned}\eta_\delta(\mathcal{A},\mathcal{B}\cap\mathcal{C}) &= \overline{\delta}(\mathcal{A},(\mathcal{B}\cap\mathcal{C})^c)\\ &= \overline{\delta}(\mathcal{A},\mathcal{B}^c\cup\mathcal{C}^c)\\ &= \overline{\delta}(\mathcal{A},\mathcal{B}^c)\wedge\overline{\delta}(\mathcal{A},\mathcal{C}^c)\\ &= \eta_\delta(\mathcal{A},\mathcal{B})\wedge\eta_\delta(\mathcal{A},\mathcal{C}).\end{aligned}$$
Thus η_δ is of type **D**.

Suppose δ is symmetrical. Then $\overline{\delta}(\mathcal{A},\mathcal{B}^c) = \overline{\delta}(\mathcal{B}^c,\mathcal{A})$. So, $\eta_\delta(\mathcal{A},\mathcal{B}) = \eta_\delta(\mathcal{B}^c,\mathcal{A}^c) = \eta_\delta^s(\mathcal{A},\mathcal{B})$. Hence η_δ is symmetrical.

(2) The proof is similar to (1). □

Lemma 6.2. *(1) $\delta_{\eta_\delta} = \delta$. (2) $\eta_{\delta_\eta} = \eta$.*

Proof. (1) $\delta_{\eta_\delta}(\mathcal{A},\mathcal{B}) = \overline{\eta_\delta}(\mathcal{A},\mathcal{B}^c) = \delta(\mathcal{A},\mathcal{B})$.

(2) $\eta_{\delta_\eta}(\mathcal{A},\mathcal{B}) = \overline{\delta_\eta}(\mathcal{A},\mathcal{B}^c) = \eta(\mathcal{A},\mathcal{B})$. □

Definition 6.3. Let $(\mathcal{X},\mathfrak{U})$ be an L-preuniform space in [2]. Define the function $\eta_\mathfrak{U} : 2^\mathcal{X}\times 2^\mathcal{X} \longrightarrow L$ as follows:

$$\eta_\mathfrak{U}(\mathcal{A},\mathcal{B}) = \sup\{\mathfrak{U}(\mathcal{U})\,|\mathcal{U}[\mathcal{A}]\subseteq\mathcal{B}\}$$

Theorem 6.4. *Let $(\mathcal{X},\mathfrak{U})$ be an L-preuniform space in* [2]. If $\top \not\ll \top$, then $(\mathcal{X},\eta_\mathfrak{U})$ is an LFPT space.

Proof. It suffices to check (PT1) and (PT2) for $\eta_\mathfrak{U}$.

(PT1) Since for each $\mathcal{U}\in 2^{\mathcal{X}\times\mathcal{X}}$, $\mathcal{U}[\phi] = \phi$ and from (PU1), $\mathfrak{U}(\mathcal{U}) \neq \top$. So, $\eta_\mathfrak{U}(\phi,\phi) = \top$. Similarly, $\eta_\mathfrak{U}(\mathcal{X},\mathcal{X}) = \top$.

(PT2) Assume that $\overline{\eta_\mathfrak{U}}(\mathcal{A},\mathcal{B}) \ll \top$ and $\top \not\ll \top$, then $\overline{\eta_\mathfrak{U}}(\mathcal{A},\mathcal{B}) \neq \top$. So, $\overline{\eta_\mathfrak{U}}(\mathcal{A},\mathcal{B}) < \top$. Thus there exist $\mathcal{U}\in 2^{\mathcal{X}\times\mathcal{X}}$ such that $\mathfrak{U}(\mathcal{U}) > \bot$. Hence $\mathcal{A}\subseteq\mathcal{U}[\mathcal{A}]\subseteq\mathcal{B}$. Therefore $(\mathcal{X},\eta_\mathfrak{U})$ is an LFPT space.

(PT3) Suppose that $\mathcal{A}\subseteq\mathcal{A}_1$ and $\mathcal{B}_1\subseteq\mathcal{B}$. Then we have $\mathcal{U}[\mathcal{A}]\subseteq\mathcal{U}[\mathcal{A}_1]$. So, we obtain $\sup\{\mathfrak{U}(\mathcal{U})\,|\mathcal{U}[\mathcal{A}_1]\subseteq\mathcal{B}_1\} \leq \sup\{\mathfrak{U}(\mathcal{U})\,|\mathcal{U}[\mathcal{A}]\subseteq\mathcal{B}\}$. Hence $\eta_\mathfrak{U}(\mathcal{A}_1,\mathcal{B}_2) \leq \eta_\mathfrak{U}(\mathcal{A},\mathcal{B})$. Therefore $\eta_\mathfrak{U}$ is of type **I**. □

7. Conclusion

We primarily generalized the concept of pretopogenous structure in this paper by employing the way below relation. It is also possible to obtain the relationship between L-fuzzifying pretopogenous and L-fuzzifying topology. Furthermore, the relationships between various structures are introduced, including: L-fuzzifying topology, L-fuzzifying preuniform, L-fuzzifying preproximity, and L-fuzzifying topogenous. The representation given in Section 3 will enable us to give interpretations of compactness and connectedness which seems appropriate for applications. Furthermore, we believe that this approach would be interesting to extend to other structures such as proximity, topogenous, syntopogenous, homotopy, and so on. All of these issues will be investigated further in future research projects. Another point of contention is evolutionary biology, because key concepts in this field are intrinsically topological [28]. Classical population genetics and quantitative genetics models rely on a Euclidean vector space as a natural framework for studying the evolution of phenotypic adaptation and the process of speciation. It would be interesting to create a mathematical framework that includes graphs, recombination sets, and Euclidean vector spaces as special cases. When phenotypes are organized based on genetic accessibility, the resulting space lacks a metric and is formalized by an unknown structure. Future research will look into whether and how L-fuzzifying pretopogenous spaces can help with this. We anticipate that the properties of this space will result in patterns of phenotypic evolution such as punctuation, irreversibility, or modularity. Future research could be inspired by [28] to investigate the applicability of L-fuzzifying pretopogenous spaces to combinatorial search spaces, fitness landscapes, evolutionary trajectories, and artificial chemistry.

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