Hindawi
Journal of Function Spaces
Volume 2020, Article ID 4650761, 10 pages
https://doi.org/10.1155/2020/4650761

Research Article

$\mathscr{L}$-Simulation Functions over b-Metric-Like Spaces and Fractional Hybrid Differential Equations

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Received 12 May 2020; Accepted 1 December 2020; Published 12 December 2020

Academic Editor: Gen Qi Xu

In this paper, we establish some fixed point results for α_{qs^p}-admissible mappings embedded in $\mathscr{L}$-simulation functions in the context of b-metric-like spaces. As an application, we discuss the existence of a unique solution for fractional hybrid differential equation with multipoint boundary conditions via Caputo fractional derivative of order $1 < \alpha \leq 2$. Some examples and corollaries are also considered to illustrate the obtained results.

1. Introduction

Fixed point theory has received much attention due to its applications in pure mathematics and applied sciences. Generalization of this theory depends on generalizing the metric type space or the contractive type mapping. The concept of metric spaces has been extended in various directions by reducing or modifying the metric axioms. Since, losing or weakening some of the metric axioms causes loss of some topological properties, hence bringing obstacles in proving some fixed point theorems. These obstacles force researchers to develop new techniques in the development of fixed point theory in order to resolve more real concrete applications.

In 1989, Bakhtin [1] (and also Czerwik [2]) introduced the concept of $b-$metric spaces and presented a generalization of Banach contraction principle. Amini-Harandi [3] introduced the notion of metric-like spaces which play an important role in topology and logical programming. In 2013, Alghamdi et al. [4] generalized the notions of $b-$metric and metric-like spaces by introducing a new space called $b-$metric-like space and proved some related fixed point results. Recently, many results of fixed point of mappings under certain contractive conditions in such spaces have been obtained (see [5–8]).

Zoto et al. [9] introduced the concept of $\alpha_{qs^p}-$admissible mappings and provided some fixed point theorems for these mappings under some new conditions of contractivity in the setting of $b-$metric-like spaces. Recently, Cho [10] proposed the notion of $\theta\mathscr{L}-$contractions and confine his fixed point results for such contractions to generalized metric spaces. Aydi et al. [11] proved that those results are also valid in partial metric spaces.

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of arbitrary order. Indeed, it is found to be more realistic in describing and modeling several natural phenomena than the classical one. In recent years, many researchers have focused on joining fixed point theory with fractional calculus, see for example [12–15].

The study of differential equations with fractional order has attracted many authors because of its intensive development of fractional calculus itself and its applications in various fields of science and engineering, see [16–19].

On the other hand, hybrid differential equations have attracted much attention after the pioneering works appeared in [20, 21] which discussed main aspects about first-order hybrid differential equations with perturbations of 1st and 2nd types, respectively.

Fractional hybrid differential equations (in short, FHDEs) have been studied using Riemann-Liouville and

Caputo fractional derivatives of order $\alpha > 0$ in many literatures, see [22–27].

In [23], Derbazi et al. applied Dhage hybrid fixed point theorem [28] to provide sufficient conditions that guarantee the existence only of solutions for a class of FHDEs with three-point boundary conditions due to Caputo fractional derivative of order $1 < \alpha \leq 2$ in Banach algebra spaces.

Inspired by the above works, we investigate the existence of a unique fixed point for α_{qs^p}-admissible mapping via $\mathscr{L}$-simulation function $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ and control function $\theta : (0,\infty) \to (1,\infty)$ in more general setting (b-metric-like space) than partial metric, $b-$metric and metric-like spaces. Also, as an application, we provide appropriate conditions that guarantee the existence of a unique solution to the following FHDE.

$$\begin{gathered} {}^c\mathscr{D}^{\alpha}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h(t,x(t),I^q x(t)), \quad t \in J = [0,T], \\ \zeta_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + \zeta_2\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1, \\ \sum_{i=3}^{m}\zeta_i\, {}^c\mathscr{D}^{\beta}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta_i} + \zeta_{m+1}\, {}^c\mathscr{D}^{\beta}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2, \end{gathered} \tag{1}$$

where ${}^c\mathscr{D}^{\alpha}$ and ${}^c\mathscr{D}^{\beta}$ denote the Caputo fractional derivatives of orders α and β, respectively, $0 < \alpha \leq 2$, $q > 0$, $0 < \beta \leq 1$, $0 < \eta_i < T$, ζ_i, $i = 1,2,3,\cdots,m+1$, $m \in \mathbb{N}$ are real constants such that

$$\begin{gathered} \zeta_1 + \zeta_2 \neq 0, \quad \sum_{i=3}^{m}\zeta_i\eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta} \neq 0, \\ g \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\}) \quad \text{and} \quad f, h \in C(J \times \mathbb{R}, \mathbb{R}). \end{gathered} \tag{2}$$

2. Basic Concepts

In order to fix the framework needed to state our main results, we recall the following notions.

Definition 1 [1]. Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is a $b-$metric if for all $x, y, z \in X$, the following conditions are satisfied.

(b_1) $d(x,y) = 0 \Leftrightarrow x = y$

(b_2) $d(x,y) = d(y,x)$

(b_3) $d(x,y) \leq s[d(x,z) + d(z,y)]$

The pair (X,d) is called a $b-$metric space, and s is the coefficient of it.

Note that, every metric space is a $b-$metric space with coefficient $s = 1$.

Definition 2 [3]. A metric-like space on a nonempty set X is a function $\sigma : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$:

(σ_1) $\sigma(x,y) = 0 \Rightarrow x = y$

(σ_2) $\sigma(x,y) = \sigma(y,x)$

(σ_3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$

Then, the pair (X,σ) is called a metric-like space.

It should be noted that σ satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$.

Definition 3 [4]. A function $\sigma_b : X \times X \to [0,\infty)$ on a nonempty set X is called $b-$metric-like if for any $x, y, z \in X$, the following conditions hold true.

(σ_{b1}) $\sigma_b(x,y) = 0 \Rightarrow x = y$

(σ_{b2}) $\sigma_b(x,y) = \sigma_b(y,x)$

(σ_{b3}) $\sigma_b(x,y) \leq s[\sigma_b(x,z) + \sigma_b(z,y)]$

The pair (X,σ_b) is called a $b-$metric-like space.

Remark 4. The class of $b-$metric-like spaces is considerably larger than both $b-$metric spaces and metric-like spaces. Since, every $b-$metric is a $b-$metric-like with same coefficient and zero self-distance. Also, every metric-like is a $b-$metric-like with $s = 1$. However, the converse implications do not hold (see for example, [1, 4]).

Example 5. Let $X = R$ and $\sigma_b : X \times X \to [0,\infty)$ be defined as

$$\sigma_b(x,y) = (|x|+|y|)^p, \forall x, y \in X \text{ and } p \in \mathbb{N}, \tag{3}$$

then, (X,σ_b) is a $b-$metric-like space with parameter $s = 2^{p-1}$.

Example 6. Let $X \subseteq \mathbb{R}$ and $C_b(X) = \{f : X \to R : \sup_{x\in X} |f(x)| < \infty\}$. The function $\sigma_b : X \times X \to [0,\infty)$, defined by

$$\sigma_b(f,g) = \sqrt[n]{\sup_{x\in X}(|f(x)|,|g(x)|)^n}, \forall f, g \in C_b(X), n \in \mathbb{N}, \tag{4}$$

is a $b-$metric-like with constant $s = \sqrt[n]{2^{n-1}}$, and so, (X,σ_b) is a $b-$metric-like space.

Definition 7. Let (X,σ_b) be a $b-$metric-like space and $\{x_n\}$ be a sequence in X, and $x \in X$. Then,

(1) The set

$$B(x,r) = \{y \in X : |\sigma_b(x,y) - \sigma_b(x,x)| < r\}, \tag{5}$$

is called an open ball with center x and radius r. Also, the family

$$\{B(x,r), \forall x \in X, r > 0\}, \tag{6}$$

forms a base of the topology τ_{σ_b} generated by σ_b on X.

(2) $\{x_n\}$ is said to converge to x w.r.t. τ_{σ_b} if

$$\lim_{n\to\infty}\sigma_b(x_n,x) = \sigma_b(x,x). \tag{7}$$

(3) $\{x_n\}$ is said to be σ_b-Cauchy if

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m), \tag{8}$$

exists and is finite.

(4) (X,σ_b) is said to be complete if for every Cauchy sequence $\{x_n\}$ in X, there exists $x\in X$ such that

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m)=\lim_{n\to\infty}\sigma_b(x_n,x)=\sigma_b(x,x). \tag{9}$$

Lemma 8. *Let (X,σ_b) be a $b-$metric-like space with parameter $s\geq 1$ and $\{x_n\}$ be a convergent sequence in X such that*

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m)=\lim_{n\to\infty}\sigma_b(x_n,x)=\sigma_b(x,x)=0,\quad x\in X. \tag{10}$$

Then, every subsequence $\{x_{n_k}\}$ with $n_k\geq k\in\mathbb{N}$ converges to the same limit $x\in X$.

Proof. Since $x_n\to x$ and $\lim_{n\to\infty}\sigma_b(x_n,x)=0$, then for a given $\varepsilon>0$

$$\exists n_0\in\mathbb{N}: n>n_0\Rightarrow\sigma_b(x_n,x)<\varepsilon. \tag{11}$$

From (σ_{b3}) and (10), we have

$$\sigma_b\left(x_{n_k},x\right)<s\left[\sigma_b\left(x_{n_k},x_k\right)+\sigma_b(x_k,x)\right]\to 0 \text{ as } n_k\geq k\to\infty. \tag{12}$$

Therefore,

$$\lim_{k\to\infty}\sigma_b\left(x_{n_k},x\right)=0=\sigma_b(x,x). \tag{13}$$

Definition 9 [10]. Let $\mathscr{L}$ be the set of all $\mathscr{L}-$simulation functions $\xi:[1,\infty)\times[1,\infty)\to\mathbb{R}$ that fulfil:

(ξ_1) $\xi(1,1)=1$

(ξ_2) $\xi(u,v)<v/u, \forall u,v>1$

(ξ_3) For any two sequences $\{u_n\},\{v_n\}\in(1,\infty)$ with $u_n<v_n$, $\forall\, n\in\mathbb{N}$

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}v_n>1\Rightarrow\limsup_{n\to\infty}\xi(u_n,v_n)<1. \tag{14}$$

In [10], authors used the function $\theta:(0,\infty)\to(1,\infty)$ defined by Jleli and Samet in [29] to propose the following result.

Definition 10 [29]. Let Θ be the set of all functions $\theta:(0,\infty)\to(1,\infty)$ that fulfil:

(θ_1) θ is non-decreasing

(θ_2) For any sequence $\{u_n\}\in(0,\infty)$

$$\lim_{n\to\infty}\theta(u_n)=1\Rightarrow\lim_{n\to\infty}u_n=0. \tag{15}$$

Theorem 11 [10]. *Let (X,d) be a complete generalized metric space and $T:X\to X$ satisfy*

$$\xi(\theta(d(Tx,Ty)),\theta(d(x,y)))\geq 1,\forall x,y\in X \text{ with } d(Tx,Ty)>0,\xi\in\mathscr{L},\theta\in\Theta. \tag{16}$$

Then, T has a unique fixed point, and for every initial point $x_0\in X$, the Picard sequence $\{T^nx_0\}$ converges to that fixed point.

Definition 12 [9]. Let (X,σ_b) be a $b-$metric-like space with parameter $s\geq 1$, $\alpha:X\times X\to[0,\infty)$ be a function, and $q\geq 1$ and $p\geq 2$ be arbitrary constants. A mapping $T:X\to X$ is $\alpha_{qs^p}-$admissible if

$$\alpha(x,Tx)\geq qs^p\Rightarrow\alpha\left(Tx,T^2x\right)\geq qs^p,\forall x\in X. \tag{17}$$

In addition, T is said to be triangular $\alpha_{qs^p}-$admissible if it is $\alpha_{qs^p}-$admissible and

$$\alpha(x,y) \text{ and } \alpha(y,Ty)\geq qs^p\Rightarrow\alpha(x,Ty)\geq qs^p,\forall x,y\in X. \tag{18}$$

Definition 13 [18, 19]. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x:[0,\infty)\to\mathbb{R}$ is given by

$$I^\alpha x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds. \tag{19}$$

The Caputo fractional derivative of order α of x is given by

$${}^c\mathscr{D}^\alpha x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds, \tag{20}$$

where $n=[\alpha]+1$ and Γ denote the gamma function, provided that the right side is point-wise defined on $[0,\infty)$.

Lemma 14 [23]. *Let $0<\beta\leq\alpha$ and $x\in C^n([0,T])$. Then, for all $t\in[0,T]$, we have:*

(1) $I^\alpha I^\beta x(t)=I^{\alpha+\beta}x(t)$ and ${}^c\mathscr{D}^\beta I^\alpha x(t)=I^{\alpha-\beta}f(t)$

(2) $I^{\alpha c}\mathscr{D}^\alpha x(t)=x(t)+\sum_{i=0}^{n-1}c_it^i$, for some $c_0,\cdots,c_{n-1}\in\mathbb{R}$

(3) $I^\alpha:C([0,T])\to C([0,T])$

3. A Set of Fixed Point Results

Our first main result is the following theorem.

Theorem 15. *Let (X,σ_b) be a complete $b-$metric-like space with parameter $s\geq 1$. Suppose that $T:X\to X$ is a triangular*

α_{qs^p} *− admissible mapping and satisfy*

$$\xi(\alpha(x,y)\theta(\sigma_b(Tx,Ty)),\theta(\sigma_b(x,y)))\geq 1,\forall x,y\in X,\xi\in\mathscr{L},\theta\in\Theta. \tag{21}$$

Consider that the following properties hold true

(a) If $\{x_n\}$ is a sequence in X such that $x_n\to x\in X$ as $n\to\infty$ and $\alpha(x_n,x_{n+1})\geq qs^p$, then $\alpha(x_n,x)\geq qs^p,\forall n\in\mathbb{N}$

(b) For all $x,y\in Fix(T)$, we have $\alpha(x,y)\geq qs^p$, where $Fix(T)$ denotes the set of fixed points of T

Moreover, if there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq qs^p$, then T has a unique fixed point.

Proof. Starting with that point $x_0\in X:\alpha(x_0,Tx_0)\geq qs^p$. We define a sequence $\{x_n\}\subset X$ by

$$x_{n+1}=Tx_n,\forall n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}. \tag{22}$$

Regarding that T is an α_{qs^p} − admissible, then by induction, we get

$$\alpha(x_n,x_{n+1})\geq qs^p,\forall n\in\mathbb{N}_0. \tag{23}$$

If $\sigma_b(x_n,x_{n+1})=0$ for some n, then $x_n=x_{n+1}$, that is, x_n is a fixed point of T, and the proof is completed. So, we assume that

$$\sigma_b(x_n,x_{n+1})>0,\ \forall\ n. \tag{24}$$

From (23) and (24), we apply (21) at $x=x_{n-1}$ and $y=x_n$ to get

$$\begin{aligned}1&\leq\xi(\alpha(x_{n-1},x_n)\theta(\sigma_b(Tx_{n-1},Tx_n)),\theta(\sigma_b(x_{n-1},x_n)))\\&<\frac{\theta(\sigma_b(x_{n-1},x_n))}{\alpha(x_{n-1},x_n)\theta(\sigma_b(x_n,x_{n+1}))},\Rightarrow\theta(\sigma_b(x_n,x_{n+1}))\\&\leq\alpha(x_{n-1},x_n)\theta(\sigma_b(x_n,x_{n+1}))<\theta(\sigma_b(x_{n-1},x_n)).\end{aligned} \tag{25}$$

Hence, the sequence $\{\theta(\sigma_b(x_n,x_{n+1}))\}$ is monotone decreasing and bounded below by 1. Therefore, there exists $m\geq 1$ such that

$$\lim_{n\to\infty}\theta(\sigma_b(x_n,x_{n+1}))=m. \tag{26}$$

To prove that $m=1$, suppose the contrary that $m>1$ and obtain a contradiction. From (25), (26), and (ξ_3), we have

$$1\leq\limsup_{n\to\infty}\xi(\alpha(x_{n-1},x_n)\theta(\sigma_b(Tx_{n-1},Tx_n)),\theta(\sigma_b(x_{n-1},x_n)))<1, \tag{27}$$

that is all we need. Thus, $m=1$ and

$$\lim_{n\to\infty}\theta(\sigma_b(x_n,x_{n+1}))=1. \tag{28}$$

Also, (θ_2) implies

$$\lim_{n\to\infty}\sigma_b(x_n,x_{n+1})=0. \tag{29}$$

Now, we show that

$$\lim_{n,m\to\infty}\sigma_b(x_n,x_m)=0. \tag{30}$$

Consider the sequence

$$R_k=\sup\{\theta(\sigma_b(x_n,x_m)):\ m\geq n\geq k\},\forall k=1,2,3,\cdots. \tag{31}$$

It is easy to verify that

$$\begin{gathered}\lim_{k\to\infty}R_k=1\ \Rightarrow\ \lim_{n,m\to\infty}\theta(\sigma_b(x_n,x_m))=1,\\1\leq\cdots\leq R_{k+1}\leq R_k\leq\cdots\leq R_1.\end{gathered} \tag{32}$$

Hence, the sequence $\{R_k\}$ is decreasing and bounded below by 1. Consequently, there exists $r\geq 1$ such that

$$\lim_{k\to\infty}R_k=r. \tag{33}$$

Assume that $r>1$, then from (31), we conclude that

$$\begin{gathered}\forall k=1,2,3,\cdots\left(\frac{1}{k}>0\right),\exists m_k\geq n_k\geq k\\:R_k-\frac{1}{k}<\theta\left(\sigma_b\left(x_{n_k},x_{m_k}\right)\right)<R_k.\end{gathered} \tag{34}$$

Taking limit as $k\to\infty$, together with (33), implies

$$\lim_{k\to\infty}\theta\left(\sigma_b\left(x_{n_k},x_{m_k}\right)\right)=r. \tag{35}$$

Also, we have

$$\begin{aligned}&\theta\left(\sigma_b\left(x_{n_k},x_{m_k}\right)-\sigma_b\left(x_{n_k},x_{n_k-1}\right)-\sigma_b\left(x_{m_k-1},x_{m_k}\right)\right)\\&\quad\leq\theta\left(\sigma_b\left(x_{n_k-1},x_{m_k-1}\right)\right)<R_k(\text{or }R_{k-1}).\end{aligned} \tag{36}$$

Again, taking limit as $k\to\infty$, together with (33), (35) implies

$$\lim_{k\to\infty}\theta\left(\sigma_b\left(x_{n_k-1},x_{m_k-1}\right)\right)=r. \tag{37}$$

According to (23) and the fact that T is a triangular α_{qs^p} − admissible, we derive

$$\alpha\left(x_{n_k-1},x_{m_k-1}\right)\geq 1. \tag{38}$$

On account of the above observations, we apply condition (21) and then (ξ_3) to obtain

$$\begin{aligned}1 &\le \xi\left(\alpha\left(x_{n_k-1}, x_{m_k-1}\right)\theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right), \theta\left(\sigma_b\left(x_{n_k-1}, x_{m_k-1}\right)\right)\right) \\ &< \frac{\theta\left(\sigma_b\left(x_{n_k-1}, x_{m_k-1}\right)\right)}{\alpha\left(x_{n_k-1}, x_{m_k-1}\right)\theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right)}, \Rightarrow \theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right) \\ &\le \alpha\left(x_{n_k-1}, x_{m_k-1}\right)\theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right) < \theta\left(\sigma_b\left(x_{n_k-1}, x_{m_k-1}\right)\right).\end{aligned} \tag{39}$$

Therefore, we have

$$\lim_{k\to\infty} \alpha\left(x_{n_k-1}, x_{m_k-1}\right)\theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right) = r. \tag{40}$$

And hence

$$1 \le \limsup_{k\to\infty} \xi\left(\alpha\left(x_{n_k-1}, x_{m_k-1}\right)\theta\left(\sigma_b\left(x_{n_k}, x_{m_k}\right)\right), \theta\left(\sigma_b\left(x_{n_k-1}, x_{m_k-1}\right)\right)\right) < 1, \tag{41}$$

which is a contradiction, then $r = 1$ and

$$\lim_{n,m\to\infty} \theta(\sigma_b(x_n, x_m)) = 1. \tag{42}$$

Thus, (30) holds true, and the sequence $\{x_n\}$ is $\sigma_b -$ Cauchy. By the completeness of (X, σ_b), there exists $x \in X$ such that

$$\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) = \lim_{n,m\to\infty} \sigma_b(x_n, x_m) = 0. \tag{43}$$

Now, consider the subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that

$$\sigma_b\left(x_{n_k}, x\right) > 0 \text{ and } \sigma_b\left(x_{n_k+1}, Tx\right) > 0, \forall n_k \ge k \in \mathbb{N}. \tag{44}$$

Lemma 8, together with (43), imply that

$$\lim_{k\to\infty} \sigma_b\left(x_{n_k}, x\right) = 0. \tag{45}$$

Apply (21), we obtain

$$\begin{aligned}1 &\le \xi\left(\alpha\left(x_{n_k}, x\right)\theta\left(\sigma_b\left(x_{n_k+1}, Tx\right)\right), \theta\left(\sigma_b\left(x_{n_k}, x\right)\right)\right) \\ &< \frac{\theta\left(\sigma_b\left(x_{n_k}, x\right)\right)}{\alpha\left(x_{n_k}, x\right)\theta\left(\sigma_b\left(x_{n_k+1}, Tx\right)\right)}.\end{aligned}$$

$$\begin{aligned}&\Rightarrow \theta\left(\sigma_b\left(x_{n_k+1}, Tx\right)\right) \le \alpha\left(x_{n_k}, x\right)\theta\left(\sigma_b\left(x_{n_k+1}, Tx\right)\right) \\ &\quad < \theta\left(\sigma_b\left(x_{n_k}, x\right)\right), 0 \le \sigma_b\left(x_{n_k+1}, Tx\right) \le \sigma_b\left(x_{n_k}, x\right) \\ &\quad \to 0 \text{ as } k \to \infty.\end{aligned} \tag{46}$$

Hence,

$$\lim_{k\to\infty} \sigma_b\left(x_{n_k+1}, Tx\right) = 0. \tag{47}$$

From (45), (47), and (σ_3), we conclude that

$$\begin{aligned}&\sigma_b(x, Tx) \le s\left[\sigma_b\left(x, x_{n_k}\right) + \sigma_b\left(x_{n_k}, Tx\right)\right] \to 0 \text{ as } k \to \infty, \\ &\Rightarrow \sigma_b(x, Tx) = 0 \Rightarrow x = Tx.\end{aligned} \tag{48}$$

To see that this fixed point in unique, suppose that $y \in X$ is another fixed point of T and apply (21) to get the opposite.

$$\begin{aligned}1 &\le \xi(\alpha(x, y)\theta(\sigma_b(Tx, Ty)), \theta(\sigma_b(x, y))) \\ &< \frac{\theta(\sigma_b(x, y))}{\alpha(x, y)\theta(\sigma_b(Tx, Ty))}, \\ &\Rightarrow \theta(\sigma_b(x, y)) \le \alpha(x, y)\theta(\sigma_b(x, y)) < \theta(\sigma_b(x, y)).\end{aligned} \tag{49}$$

This is impossible, so $x = y$, and the fixed point is unique.

Let $\mathfrak{R}$ denote the class of $\beta : [0,\infty) \to [0, 1)$ which satisfies the condition

$$\lim_{n\to\infty} t_n > 0 \Rightarrow \lim_{n\to\infty} \beta(t_n) < 1. \tag{50}$$

Remark 16. Since $b -$ metric-like space is a proper extension of partial metric, metric-like, and $b -$ metric spaces. Then, we can derive our main results in the setting of these spaces.

Corollary 17. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be a mapping such that*

$$\sigma(Tx, Ty) \le \beta(\sigma(x, y))\sigma(x, y), \forall x, y \in X, \beta \in \mathfrak{R}. \tag{51}$$

Then, T has a unique fixed point.

Proof. For all $x, y \in X$ with $x \ne y$ and $Tx \ne Ty$, condition (51) can be written as

$$e^{\sigma(Tx,Ty)} \le \left(e^{\sigma(x,y)}\right)^{\beta\left(\ln\, e^{\sigma(x,y)}\right)}. \tag{52}$$

Therefore, Corollary 17 follows from Theorem 15 by taking

$$q = s = 1, \;; \alpha(x, y) = 1, \quad \theta(t) = e^t \quad \text{and} \quad \xi(u, v) = \frac{v^{\beta(\ln\, v)}}{u}, \tag{53}$$

for all $x, y \in X$, $t \in (0,\infty)$, and $u, v \in (1,\infty)$.

4. Fractional Hybrid Differential Equations

Here, we place our considered problem (1) in the space

$$X = C(J, \mathbb{R}) = \{x : J \to \mathbb{R}\, ; x \text{ is continuous on } J\}, \tag{54}$$

with a mapping $\sigma_b : X \times X \to \mathbb{R}^+ \cup \{0\}$ defined on it as:

$$\sigma_b(x, y) = \sup_{t\in J}(|x(t)|+|y(t)|)^p, \forall x, y \in X, t \in J, p > 1. \quad (55)$$

It is evident that (X, σ_b) is a complete $b-$metric-like space with coefficient $s = 2^{p-1}$.

For convenience, we define the following functions $\rho_i : J \to \mathbb{R}$, $i = 1,2,3,4$:

$$\rho_1(t) = \left[\frac{\zeta_2 \Gamma(2-\beta)T}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}\right)} - \frac{\Gamma(2-\beta)t}{\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}}\right],$$

$$\rho_2(t) = \left[\frac{\zeta_2 \zeta_{m+1} \Gamma(2-\beta)T}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}\right)} - \frac{\zeta_{m+1}\Gamma(2-\beta)t}{\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}}\right],$$

$$\rho_3(t) = -\frac{\zeta_2}{\zeta_1+\zeta_2},$$

$$\rho_4(t) = \left[-\frac{\zeta_1 \Gamma(2-\beta)T\lambda_2}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}\right)} + \frac{\lambda_1}{\zeta_1+\zeta_2} + \frac{\Gamma(2-\beta)t\lambda_2}{\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}}\right]. \quad (56)$$

Lemma 18. *Let $h \in C(J)$, then the integral representation of the boundary value problem*

$$^c\mathscr{D}^\alpha\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h(t), \forall t \in J, \quad (57)$$

$$\zeta_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + \zeta_2\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1,$$

$$\sum_{i=3}^m \zeta_i\, {}^c\mathscr{D}^\beta\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta_i} + \zeta_{m+1}\, {}^c\mathscr{D}^\beta\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2, \quad (58)$$

where $1 < \alpha \le 2$, $0 < \beta \le 1$, $0 < \eta_i < T$ and ζ_i, $i = 1, 2, 3, \cdots, m+1$, $m \in \mathbb{N}$ are real constants such that

$$\zeta_1 + \zeta_2 \neq 0, \quad \sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta} \neq 0, \quad (59)$$

is given by

$$x(t) = g(t, x(t))\left[\int_0^T G(t,s)h(s)ds + \rho_4(t)\right] + f(t, x(t)), \quad (60)$$

where $G(t, s)$ is the Green function and is given by

$$\begin{aligned} G(t,s) = {} & \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=i}^m \zeta_j\left(\eta_j - s\right)^{\alpha-\beta-1} \\ & + \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1} + \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}, if \\ & \cdot (s \leqq t, \eta_{i-1} \leqq s \leqq \eta_i), = \frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=i}^m \zeta_j\left(\eta_j - s\right)^{\alpha-\beta-1} \\ & + \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1} + \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}, if \\ & \cdot (s \ge t, \eta_{i-1} \leqq s \leqq \eta_i). \end{aligned} \quad (61)$$

Proof. Applying the operator I^α on both sides of (57) and using Lemma 14, we have

$$\frac{x(t)-f(t,x(t))}{g(t,x(t))} = I^\alpha h(t) - c_0 - c_1 t. \quad (62)$$

Using the boundary conditions (58), we get

$$-\zeta_1 c_0 + \zeta_2(I^\alpha - c_0 - c_1 T) = \lambda_1, \quad (63)$$

$$\sum_{i=3}^m \zeta_i\left[I^{\alpha-\beta}h(\eta_i) - c_1\frac{\eta_i^{1-\beta}}{\Gamma(2-\beta)}\right] + \zeta_{m+1}\left[I^{\alpha-\beta}h(T) - c_1\frac{T^{1-\beta}}{\Gamma(2-\beta)}\right] = \lambda_2. \quad (64)$$

Note that,

$$^c\mathscr{D}^\beta c_0 = 0 \text{ and} {}^c\mathscr{D}^\beta t = \frac{t^{1-\beta}}{\Gamma(2-\beta)}. \quad (65)$$

Solving (63) and (64) for c_0, c_1 yields

$$c_0 = -\frac{\zeta_2 T\Gamma(2-\beta)\left[\sum_{i=3}^m \zeta_i I^{\alpha-\beta}h(\eta_i) + \zeta_{m+1}I^{\alpha-\beta}h(T) - \lambda_2\right]}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}\right)} + \frac{\zeta_2}{\zeta_1+\zeta_2}I^\alpha h(T) - \frac{\lambda_1}{\zeta_1+\zeta_2},$$

$$c_1 = \frac{\Gamma(2-\beta)\left[\sum_{i=3}^m \zeta_i I^{\alpha-\beta}h(\eta_i) + \zeta_{m+1}I^{\alpha-\beta}h(T) - \lambda_2\right]}{\sum_{i=3}^m \zeta_i \eta_i^{1-\beta} + \zeta_{m+1}T^{1-\beta}}. \quad (66)$$

Substituting the values of c_0 and c_1 into (62), we get

$$\begin{aligned}\frac{x(t)-f(t,x(t))}{g(t,x(t))} &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &+ \left[\frac{\zeta_2\Gamma(2-\beta)T}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}\right)}\right. \\ &\left.- \frac{\Gamma(2-\beta)t}{\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}}\right]\cdot\sum_{i=3}^m\zeta_i\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)ds \\ &+ \left[\frac{\zeta_2\zeta_{m+1}\Gamma(2-\beta)T}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}\right)}\right. \\ &\left.- \frac{\zeta_{m+1}\Gamma(2-\beta)t}{\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}}\right]\cdot\int_0^T\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)ds \\ &- \frac{\zeta_2}{\zeta_1+\zeta_2}\cdot\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &+ \left[-\frac{\zeta_1\Gamma(2-\beta)T\lambda_2}{(\zeta_1+\zeta_2)\left(\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}\right)}\right. \\ &\left.+ \frac{\lambda_1}{\zeta_1+\zeta_2} + \frac{\Gamma(2-\beta)t\lambda_2}{\sum_{i=3}^m\zeta_i\eta_i^{1-\beta}+\zeta_{m+1}T^{1-\beta}}\right].\end{aligned} \tag{67}$$

Provided the functions $\rho_i(t)$, $i=1,2,3,4$ are defined as in Eq. (56), we obtain

$$\begin{aligned}\frac{x(t)-f(t,x(t))}{g(t,x(t))} &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{i=3}^m\zeta_i\int_0^{\eta_i} \\ &\cdot(\eta_i-s)^{\alpha-\beta-1}h(s)ds + \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}\int_0^T \\ &\cdot(T-s)^{\alpha-\beta-1}h(s)ds + \frac{\rho_3(t)}{\Gamma(\alpha)}\int_0^T \\ &\cdot(T-s)^{\alpha-1}h(s)ds + \rho_4(t).\end{aligned} \tag{68}$$

One can easily verify that

$$\begin{aligned}\sum_{i=3}^m\zeta_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-\beta-1}h(s)ds &= \int_0^{\eta_3}\sum_{j=3}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}h(s)ds \\ &+ \sum_{i=4}^m\int_{\eta_{i-1}}^{\eta_i}\sum_{j=i}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}h(s)ds.\end{aligned} \tag{69}$$

Thus, for $0\le t\le\eta_3$

$$\begin{aligned}\frac{x(t)-f(t,x(t))}{g(t,x(t))} &= \int_0^t\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=3}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right. \\ &\left.+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\int_t^{\eta_3}\left[\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=3}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}\right. \\ &\left.\cdot(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds+\sum_{i=4}^m\int_{\eta_{i-1}}^{\eta_i} \\ &\cdot\left[\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=i}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}\right. \\ &\left.\cdot(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds+\int_{\eta_m}^T \\ &\cdot\left[\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\rho_4(t).\end{aligned} \tag{70}$$

For $\eta_{k-1}\le t\le\eta_k$, we have

$$\begin{aligned}\frac{x(t)-f(t,x(t))}{g(t,x(t))} &= \int_0^{\eta_3}\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=3}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right. \\ &\left.+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\sum_{i=4}^{k-1}\int_{\eta_{i-1}}^{\eta_i}\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=i}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right. \\ &\left.+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\int_{\eta_{k-1}}^t\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=k}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right. \\ &\left.+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\int_t^{\eta_{k-1}}\left[\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=k}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}\right. \\ &\left.\cdot(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds+\sum_{i=k+1}^m\int_{\eta_{i-1}}^{\eta_i} \\ &\cdot\left[\frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=1}^m\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}+\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}\right. \\ &\left.\cdot(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds+\int_{\eta_m}^T \\ &\cdot\left[\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}+\frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds \\ &+\rho_4(t).\end{aligned} \tag{71}$$

For $\eta_m \le t \le T$, we get

$$\begin{aligned}\frac{x(t)-f(t,x(t))}{g(t,x(t))} &= \int_0^{\eta_3}\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=3}^{m}\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right.\\ &\left.+ \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1} + \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds\\ &+ \sum_{i=4}^{m}\int_{\eta_{i-1}}^{\eta_i}\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\rho_1(t)}{\Gamma(\alpha-\beta)}\sum_{j=i}^{m}\zeta_j\left(\eta_j-s\right)^{\alpha-\beta-1}\right.\\ &\left.+ \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1} + \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds\\ &+ \int_{\eta_m}^{t}\left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1}\right.\\ &\left.+ \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds + \int_t^T\\ &\cdot\left[\frac{\rho_2(t)}{\Gamma(\alpha-\beta)}(T-s)^{\alpha-\beta-1} + \frac{\rho_3(t)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\right]h(s)ds\\ &+ \rho_4(t).\end{aligned} \tag{72}$$

Joining the previous three cases together, we get (60).

Define the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$Tx(t) = g(t,x(t))\left[\int_0^T G(t,s)h(s,x(s),I^q x(s))ds + \rho_4(t)\right] + f(t,x(t)). \tag{73}$$

In view of Lemma 18, fixed points of T are solutions of the FHDE (1). Now, we assume the following conditions which allow us to establish the existence and uniqueness results for the solution of the multipoint boundary value problem (1) by applying Theorem 15.

(C_1) The functions $g : J\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ and $f, h : J\times\mathbb{R} \to \mathbb{R}$ are continuous

(C_2) There exist two functions $\varphi_1, \varphi_2 : J \to \mathbb{R}$ with bounds $\|\varphi_1\|$ and $\|\varphi_2\|$, respectively, such that

$$\begin{aligned}&|f(t,x)| + |f(t,y)| \le |\varphi_1(t)|(|x|+|y|),\\ &|g(t,x)| + |g(t,y)| \le |\varphi_2(t)|(|x|+|y|), \forall (t,x,y) \in \left(J,\mathbb{R}^2\right).\end{aligned} \tag{74}$$

(C_3) There exist a functions $p \in L^\infty(J,\mathbb{R})$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$|h(t,x,y)| \le |p(t)|\psi(|x|+|y|), \forall (t,x) \in (J,\mathbb{R}). \tag{75}$$

(C_4) Let ℓ_1 and ℓ_2 be equal

$$\ell_1 = 8^{p-1}\|\varphi_2\|^p\left(\int_0^T |G(t,s)p(s)|ds\right)^p, \tag{76}$$

$\ell_2 = 1 + (T/\Gamma(q+1))$, $\ell_3 = 8^{p-1}\|\varphi_2\|^p\|\rho_4\|^p + 4^{p-1}\|\varphi_1\|^p$, then,

$$\ell_1\psi(\ell_2 \ln t)^p + \ell_3 < 1, \forall t > 1. \tag{77}$$

Theorem 19. *If $(C_1) - (C_4)$ hold true, then the problem (1) has one solution in X.*

Proof. First, we show that T is $\theta\mathfrak{L}$ – contraction on X.

$$\begin{aligned}\sigma_b(Tx,Ty) &= \sup_{t\in J}(|Tx(t)| + |Ty(t)|)^p \le \sup_{t\in J}(|g(t,x(t))|\\ &\cdot\left[\psi\left(\left(1+\frac{T}{\Gamma(q+1)}\right)\sigma_b(x,y)\right)\int_0^T\right.\\ &\cdot|G(t,s)p(s)|ds + |\rho_4(t)|\Big] + |f(t,x(t))|\\ &+ |g(t,y(t))|\left[\psi\left(\left(1+\frac{T}{\Gamma(q+1)}\right)\sigma_b(x,y)\right)\int_0^T |G\right.\\ &\cdot(t,s)p(s)|ds + |\rho_4(t)|\Big] + |f(t,y(t))|)^p \le \sup_{t\in J}\\ &\cdot\left((|\varphi_2(t)|(|x(t)| + |y(t)|))\left[\psi\left(\left(1+\frac{T}{\Gamma(q+1)}\right)\sigma_b\right.\right.\right.\\ &\cdot(x,y)\Big)\int_0^T|G(t,s)p(s)|ds + |\rho_4(t)|\Big] + |\varphi_1(t)|(|x(t)|\\ &+ |y(t)|)\Big)^p \le 2^{p-1}\Big(2^{p-1}\|\varphi_2\|^p\sigma_b(x,y)\\ &\cdot\left[\psi\left(\left(1+\frac{T}{\Gamma(q+1)}\right)\sigma_b(x,y)\right)^p\left(\int_0^T|G(t,s)p(s)|ds\right)^p\right.\\ &+ \|\rho_4\|^p\Big] + \left(\|\varphi_1\|^p\sigma_b(x,y)\right) \le 2^{p-1}\Big(2^{p-1}\|\varphi_2\|^p\\ &\cdot\left[\psi\left(\left(1+\frac{T}{\Gamma(q+1)}\right)\sigma_b(x,y)\right)^p\left(\int_0^T|G(t,s)p(s)|ds\right)^p\right.\\ &+ \|\rho_4\|^p\Big] + \|\varphi_1\|^p)\sigma_b(x,y).\end{aligned} \tag{78}$$

Now, we define the functions $\phi : [1,\infty) \to [0,1)$, $\theta \in \Theta$, and $\xi \in \mathfrak{L}$as

$$\phi(t) = \ell_1\psi(\ell_2 \ln t)^p + \ell_3, \tag{79}$$

$\theta(t) = e^t$,
$\xi(s,t) = t^{\phi(t)}/s$.
Consequently, we obtain

$$\begin{aligned}\alpha(x,y)e^{\sigma_b(Tx,Ty)} &\le \left(e^{\sigma_b(x,y)}\right)^{\phi\left(e^{\sigma_b(x,y)}\right)}1 \le \xi(\alpha(x,y)\theta(\sigma_b(Tx,Ty)),\theta\\ &\cdot(\sigma_b(x,y))),\end{aligned} \tag{80}$$

where $\alpha(x,y) = 2^{p-1}$, $\forall x, y \in X$. Also, other hypothesis of Theorem 15 is satisfied. Hence, T has a unique fixed point, and then, FHDE (1) has on solution in X. This completes the proof.

Now, we present an example to support our results.

Example 20. Let us consider the following FHDE with five-point boundary conditions.

$$
\begin{aligned}
{}^{c}\mathscr{D}^{3/2}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] &= h\left(t,x(t),I^{5/2}x(t)\right), \quad t\in J=[0,1],\\
&\cdot\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0}\\
&+\frac{1}{2}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1}\\
&=\frac{1}{2},\sum_{i=3}^{5}\zeta_i{}^{c}\mathscr{D}^{1/2}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta_i}\\
&+\frac{1^{c}}{4}\mathscr{D}^{1/2}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1}=1,
\end{aligned}
\tag{81}
$$

where $\nu>0$. To show the existence of a unique solution of (81), we apply Theorem 19 with

$$
\begin{aligned}
\alpha&=\frac{3}{2},\beta=\frac{1}{2},q=\frac{5}{2},\zeta_1=1,\zeta_2=\frac{1}{2},\zeta_3=\frac{5}{8},\zeta_4=\frac{5}{12},\zeta_5\\
&=\frac{5}{16},\zeta_6=\frac{1}{4},\eta_3=\frac{4}{25},\eta_4=\frac{9}{25},\eta_5=\frac{16}{25},\\
f(t,x)&=\frac{1}{4}\left(\frac{1}{2}\left(x+\sqrt{x^2+1}\right)-e^{\nu t}\right),\\
g(t,x)&=\frac{1}{4}\frac{\ln\left(1+x^2\right)}{x},\\
h(t,x,y)&=\cos\left(|x|+|y|\right),\forall t\in[0,1]\text{ and } x,y\in\mathbb{R}.
\end{aligned}
\tag{82}
$$

By computation, we can show that the hypotheses $(C_1)-(C_4)$ are satisfied with

$$
\|p\|=1,\ \|\varphi_1\|=\|\varphi_2\|=\frac{1}{4}. \tag{83}
$$

Therefore, we conclude that problem (81) has one solution.

5. Concluding Remarks and Observations

Our results extend the results of [10, 11] and many others. Indeed, we deal with a class of α_{qs^p}-admissible $\theta\mathscr{L}$-contractions in a larger structure such as a b-metric-like space. Also, the problem we used as an application is different from that one appeared in [23] in the sense that we discussed not only existence but also uniqueness for our considered problem under multipoint boundary conditions via Caputo fractional derivatives.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors of this paper declare that they have no conflicts of interest.

Acknowledgments

The authors thank the anonymous referees for their constructive comments and suggestions given for improving the presentation of the present paper.

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