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Timelike sweeping surfaces and singularities

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We consider a timelike sweeping surface with rotation minimizing frames in Minkowski 3-Space $\mathbb{E}_1^3$. We introduce a new geometric "invariant", which demonstrates the geometric properties and local singularities of the surface. Subsequently, we give the sufficient and necessary conditions for this surface to be a developable ruled surface. Finally, the singularities of these ruled surfaces are investigated.

Keywords: Spine curve; rotation minimizing frame; local singularities.

Mathematics Subject Classification 2020: 53A04, 53A05, 53A17

1. Introduction

Singularity theory of curves is an active area of research in different branches of mathematics and physics. In view of differential geometry, curves and surfaces are represented by functions with one variable and two variables, respectively. In recent years, singularity theory of curves and surfaces has become an important tool for various interesting fields such as medical imaging and computer vision (see, e.g. [1–5]).

As we know, the envelope of a family of sphere whose centers follow a space curve is called a canal surface. It is traced by a one-parameter family of spheres represented by the radius function and center curves: When the radius function is constant, the canal surface is called a sweeping surface. There are several different names of sweeping surfaces that we are familiar with, such as tubular surface,

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pipe surface, and string surface. Sweeping surfaces have important applications in (CAD/CAM) such as shape reconstruction, and providing a way of constructing the trajectory of the robots (see, [6–10]).

In 1975, Bishop [11] introduced a new frame called the alternative frame or Bishop frame, which could provide the desired means to slide along a space curve. It has become a useful tool for animations, motion planning, computer vision, and related applications where the Serret–Frenet frame may prove unsuitable. For example, it may be used to compute the shape of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also produce a way to control virtual cameras in computer animations [12–14]. Corresponding to the Bishop frame in Euclidean 3-space, there exists a Minkowski version moving frame that is named a Minkowski Bishop frame as applied to Minkowski geometry. If we study a space curve, it is more convenient for us to apply the Minkowski version of the Bishop moving frame as an important tool than the Serret–Frenet moving frame in Lorentzian space. There exist a lot of papers dealing with Minkowski Bishop frame, for example [15–17].

In this paper, we present the notion of timelike sweeping surfaces with rotation minimizing frames in Minkowski 3-Space $\mathbb{E}_1^3$. By applying singularity theory we classify the generic properties, and present new invariant connected to the singularity of this timelike sweeping surface. Then, the main generic singularity of this sweeping surface includes the well-known cuspidal edge, swallowtail, and these are characterized by this new invariant. Finally, to explain our results, we illustrated some examples.

2. Preliminaries

We introduce in this section some basic notations on Minkowski space [18, 19]. Let $\mathbb{E}_1^3$ denote the three-dimensional Minkowski space, i.e. $\mathbb{R}^3$ equipped with the metric

$$\langle d\mathbf{r}, d\mathbf{r}\rangle = dr_1^2 + dr_2^2 - dr_3^2,$$

where (r_1, r_2, r_3) denotes the canonical coordinates in $\mathbb{R}^3$. We say that a nonzero vector $\mathbf{r} \in \mathbb{E}_1^3$ is spacelike, null, or timelike if $\langle \mathbf{r}, \mathbf{r}\rangle$ is positive, zero, or negative, respectively. In addition, with a norm $\|\mathbf{r}\| = \sqrt{|\langle \mathbf{r}, \mathbf{r}\rangle|}$, then the vector $\mathbf{r}$ is a spacelike unit if $\langle \mathbf{r}, \mathbf{r}\rangle = 1$ and a timelike unit if $\langle \mathbf{r}, \mathbf{r}\rangle = -1$. Therefore, we say that a smooth curve $\beta : I \to \mathbb{E}_1^3$ is spacelike, timelike, or lightlike, if its velocity β' is spacelike, timelike, or lightlike, respectively. In a similar form, a surface is spacelike, timelike, or lightlike, if its normal vector is timelike, spacelike, or lightlike, respectively. Given two vectors $\mathbf{r}$, $\mathbf{p} \in \mathbb{E}_1^3$, the inner product is the real number $\langle \mathbf{r}, \mathbf{p}\rangle = r_1p_1 + r_2p_2 - r_3p_3$ and the vector product is defined by

$$\mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{f}_1 & \mathbf{f}_2 & -\mathbf{f}_3 \\ r_1 & r_2 & r_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = ((r_2p_3 - r_3p_2), (r_3p_1 - r_1p_3), -(r_1p_2 - x_2p_1)),$$

where $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ is the canonical basis of $\mathbb{E}^3_1$. For a fixed point $\mathbf{p} \in \mathbb{E}^3_1$, and a positive number $c > 0$, the hyperbolic and Lorentzian (de Sitter space) spheres, respectively, are defined by

$$\mathbb{H}^2_+(\mathbf{p}, \mathbf{c}) = \{\mathbf{a} \in \mathbb{E}^3_1 \,|\, \langle \mathbf{a} - \mathbf{p}, \mathbf{a} - \mathbf{p}\rangle = -c^2\}, \tag{1}$$

and

$$\mathbb{S}^2_1(\mathbf{p}, c) = \{\mathbf{a} \in \mathbb{E}^3_1 \,|\, \langle \mathbf{a} - \mathbf{p}, \mathbf{a} - \mathbf{p}\rangle = c^2\}. \tag{2}$$

We define

$$\mathbb{LC}^*_p = \{\mathbf{a} \in \mathbb{E}^3_1 \,|\, \langle \mathbf{a} - \mathbf{p}, \mathbf{a} - \mathbf{p}\rangle = 0\} - \{\mathbf{p}\}, \tag{3}$$

and we call it the (open) lightcone at the vertex $\mathbf{p}$. When $\mathbf{p} = \mathbf{0}$ and $c = 1$, we simply denote $\mathbb{LC}^*_0$, $\mathbb{H}^2_+$, and $\mathbb{S}^2_1$, respectively.

Let $\beta = \beta(s)$ be a unit speed spacelike curve with spacelike principal normal $\zeta_2(s)$ represented by its arc-length s; by $\kappa(s)$ and $\tau(s)$ are the natural curvature and torsion of $\beta(s)$, respectively. Let $\{\zeta_1(s),\, \zeta_2(s),\, \zeta_3(s)\}$ be the Serret–Frenet frame associated with $\beta(s)$. The Serret–Frenet vector fields satisfy the relations:

$$\begin{aligned} &\langle \zeta_1, \zeta_1\rangle = \langle \zeta_2, \zeta_2\rangle = 1, \quad \langle \zeta_3, \zeta_3\rangle = -1, \\ &\zeta_1 \times \zeta_2 = -\zeta_3, \quad \zeta_1 \times \zeta_3 = -\zeta_2, \quad \zeta_2 \times \zeta_3 = \zeta_1. \end{aligned} \tag{4}$$

Then the Serret–Frenet formulae read as follows:

$$\begin{pmatrix} \zeta_1' \\ \zeta_2' \\ \zeta_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \psi \times \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}, \tag{5}$$

where $\psi(s) = -\tau\zeta_1 + \kappa\zeta_3$ is Darboux vector along the curve β. In what follows, dash is differentiation with respect to s.

Definition 2.1. A pseudo orthogonal moving frame $\{\xi_1,\, \xi_3,\, \xi_3\}$, along a non-null space curve $\alpha(s)$, is called rotation minimizing frame or Bishop frame (RMF) with respect to ξ_1 if its angular velocity ω satisfies $\langle \omega, \xi_1\rangle = 0$ or, equivalently, the derivatives of ξ_2 and ξ_3 are both parallel to ξ_1. An analogous characterization holds when ξ_2 or ξ_3 is chosen as the reference direction.

According to Definition 2.1, we can see that the Serret–Frenet moving frame is RMF with respect to ξ_2, but not with respect to ζ_1 and ζ_3. Although the Serret–Frenet moving frame is not RMF with respect to ζ_1, one can easily obtain such a RMF from it. New normal plane vectors $(\mathbf{N}_1,\mathbf{N}_2)$ are determined through a rotation of (ζ_2,ζ_3) according to

$$\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\theta & \sinh\theta\\ 0 & \sinh\theta & \cosh\theta\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}, \tag{6}$$

with a certain spacelike angle $\theta(s)\geq 0$. Here, we will call the set $\{\mathbf{T}_1, \mathbf{N}_1, \mathbf{N}_2\}$ as RMF or Bishop moving frame. The RMF vector satisfies the relations

$$\begin{aligned}&\langle\mathbf{T}_1,\mathbf{T}_1\rangle=\langle\mathbf{N}_1,\mathbf{N}_1\rangle=1,\quad \langle\mathbf{N}_2,\mathbf{N}_2\rangle=-1,\\ &\mathbf{T}_1\times\mathbf{N}_1=-\mathbf{N}_2,\quad \mathbf{N}_2\times\mathbf{N}_1=-\mathbf{T}_1,\quad \mathbf{T}_1\times\mathbf{N}_2=-\mathbf{N}_1.\end{aligned} \tag{7}$$

The angle θ specifies the difference between these two frames; it can be computed from the integral formula

$$\theta(s)=-\int_{s_0}^{s}\tau ds+\theta_0, \tag{8}$$

where s_0 is the initial value of the arc length. Generally, we put $s_0=0$, so $\theta_0=\theta(0)$. Therefore, we have the alternative frame equations

$$\begin{pmatrix}\mathbf{T}_1'\\ \mathbf{N}_1'\\ \mathbf{N}_2'\end{pmatrix}=\begin{pmatrix}0 & \kappa_1 & -\kappa_2\\ -\kappa_1 & 0 & 0\\ -\kappa_2 & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}, \tag{9}$$

where $\widetilde{\omega}(s)=-\kappa_2\mathbf{N}_1+\kappa_1\mathbf{N}_2$ is the Bishop Darboux vector. Here, the functions $\kappa_1(s)=\kappa\cosh\vartheta$, and $\kappa_2(s)=\kappa\sinh\vartheta$ are called the Bishop curvature functions. One can show that

$$\left.\begin{aligned}&\kappa_1^2-\kappa_2^2=\kappa^2,\quad\text{and}\quad \vartheta=\tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right);\quad \kappa_1\neq 0,\\ &\theta(s)=-\int_{s_0}^{s}\tau ds+\theta_0,\quad \theta_0=\theta(0).\end{aligned}\right\} \tag{10}$$

Consequently, the Serret–Frenet frame and the RMF are identical if and only if $\beta(s)$ is a planar, i.e. $\tau=0$. Now we define a Bishop spherical Darboux image $\mathbf{e}(s): I\to\mathbb{H}_+^2$, by

$$\mathbf{e}(s):=\frac{\omega(s)}{\|\omega(s)\|}=\frac{-\kappa_1}{\sqrt{\kappa_1^2-\kappa_2^2}}\left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1-\mathbf{N}_2\right). \tag{11}$$

Therefore, we consider a new geometric invariant $\rho(s)=\kappa_1\kappa_2'-\kappa_2\kappa_1'$.

Definition 2.2. A sweeping surface along $\beta(s)$ is a surface defined by

$$M:\mathbf{R}(s,u)=\beta(s)+F(s)\mathbf{x}(u)=\beta(s)+x_1(u)\mathbf{N}_1(s)+x_2(u)\mathbf{N}_2(s), \tag{12}$$

where $\beta(s)$ is called the (at least C^1-continuous) spine curve, $0 \leq s \leq T$, s is its arc length parameter. $\mathbf{x}(u) = (0, x_2(u), x_3(u))^T$ is called planar profile (cross-section) curve and represented by $\mathbf{x}(u) = 0$, $u \in I \subseteq \mathbb{R}$ and the symbol "T" means transposition. The semi-orthogonal matrix $\{\mathbf{T}_1(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\} = F(s)$ specifies the RMF along $\beta(s)$.

3. Timelike Sweeping Surface and Its Singularities

In this section, we present timelike sweeping surfaces in Minkowski 3-space $\mathbb{E}^3_1$. Consider the planar profile curve given by $\mathbf{x}(u) = (0, \sinh u, \cosh u)$. By using Eq. (12), it follows that

$$M : \mathbf{R}(s, u) = \beta(s) + \sinh u\mathbf{N}_1 + \cosh u\mathbf{N}_2. \tag{13}$$

By the formulae expressed in Eq. (9), we can calculate

$$\left.\begin{aligned} \mathbf{R}_u(s, u) &= \cosh u\mathbf{N}_1 + \sinh u\mathbf{N}_2,\\ \mathbf{R}_s(s, u) &= (1 - \kappa_1 \sinh u - \kappa_2 \cosh u)\mathbf{T}_1. \end{aligned}\right\} \tag{14}$$

The unit normal vector is

$$\mathbf{N}(s, u) = \frac{\mathbf{R}_s \times \mathbf{R}_u}{\|\mathbf{R}_s \times \mathbf{R}_u\|} = \cosh u\mathbf{N}_1 + \sinh u\mathbf{N}_2. \tag{15}$$

Note that $\|\mathbf{N}(s, u)\|^2 = 1$ means that M is a timelike surface. The vital aim of this work is the following theorem.

Theorem 3.1. *Let $\beta : I \to \mathbb{E}^3_1$ be a unit speed spacelike curve with a timelike principal normal, and $\kappa_1^2 - \kappa_2^2 \neq 0$. Then, for any fixed $\mathbf{x} \in \mathbb{H}^2_+(0, 1)$, one has the following*:

(A) (1) $\mathbf{e} = \mathbf{e}(s)$ *is locally diffeomorphic to a line $\{\mathbf{0}\} \times \mathbb{R}$ at s_0 iff $\rho(s_0) \neq 0$;*
  (2) $\mathbf{e} = \mathbf{e}(s)$ *is locally diffeomorphic to the cusp $C \times \mathbb{R}$ at s_0 iff $\rho(s_0) = 0$, and $\rho'(s_0) \neq 0$.*

(B) (1) *M is locally diffeomorphic to a timelike plane at (s_0, u_0) iff $a_1\kappa_1(s_0) + a_2\kappa_2(s_0) \neq 0$, where a_1 and a_2 are real numbers such that $a_1^2 - a_2^2 = -1$.*
  (2) *M is locally diffeomorphic to Cuspidal edge CE at (s_0, u_0) iff $\mathbf{x} = \pm\mathbf{e}(s_0)$, and $\rho(s_0) \neq 0$.*
  (3) *M is locally diffeomorphic to Swallowtail SW at (s_0, u_0) iff $\mathbf{x} = \pm\mathbf{e}(s_0)$, $\rho(s_0) = 0$, and $\rho'(s_0) = 0$.*

The proof will appear later.

Here, $C \times \mathbb{R} = \{(x_1, x_2)|x_1^2 = x_2^3\} \times \mathbb{R}$, $CE = \{(x_1, x_2, x_3)|\mathrm{x}_1{=}\mathrm{u},\ \mathrm{x}_2{=}\mathrm{v}^2,\ \mathrm{x}_3{=}\mathrm{v}^3\}$, and SW$= \{(x_1, x_2, x_3)|x_1 = u,\ x_2 = 3v^2 + uv^2,\ x_3 = 4v^3 + 2uv\}$. The pictures of $C \times \mathbb{R}$, CE, and SW will be seen in Figs. 1–3.

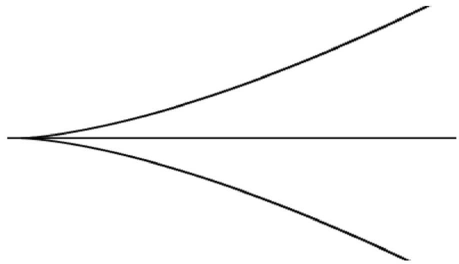

Fig. 1. CxR.

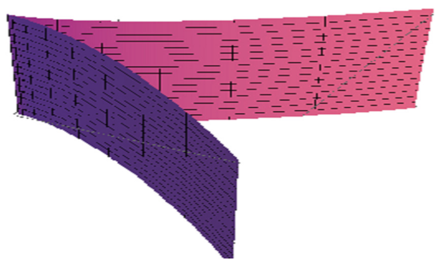

Fig. 2. CE.

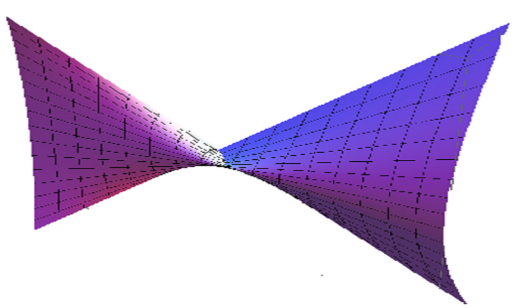

Fig. 3. SW.

3.1. *Hyperbolic Bishop height functions*

Now, we will give two different types of Hyperbolic Bishop height functions which will be important to study the singularities of M as follows: $H : I \times \mathbb{H}^2_+ \to \mathbb{R}$, by $H(s, \mathbf{x}) = \langle \beta(\mathbf{s}), \mathbf{x} \rangle$. We call it Hyperbolic Bishop height function. We use the notation $h_{\mathbf{x}}(s) = H(s, \mathbf{x})$ for any fixed $\mathbf{x} \in \mathrm{H}^2_+(0, 1)$. We can also define $\widetilde{H} : I \times \mathrm{H}^2_+(0, 1) \times \mathbb{R} \to \mathbb{R}$, by $\widetilde{H}(s, \mathbf{x}, w) = \langle \beta, \mathbf{x} \rangle - w$. We call it extended Hyperbolic Bishop height function of $\beta(\mathbf{s})$. We also use the notation $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{H}(s, \mathbf{x})$. From now on, we shall often omit to indicate the parameter s. Then, we state the following fundamental proposition.

Proposition 3.1. *Let $\beta : I \to \mathbb{E}_1^3$ be a unit speed spacelike curve with a timelike principal normal, with $\kappa_1^2 - \kappa_2^2 \neq 0$. Then the following holds*:

(A) (1) $h'_{\mathbf{x}}(s) = 0$ *iff* $\mathbf{x} = a_1\mathbf{N}_1 + a_2\mathbf{N}_2$, *and* $-a_1^2 + a_2^2 = -1$.

(2) $h'_{\mathbf{x}}(s) = h''_x(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$.

(3) $h'_{\mathbf{x}}(s) = h''_x(s) = h'''_{\mathbf{x}}(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, *and* $\rho(s) = 0$.

(4) $h'_{\mathbf{x}}(s) = h''_x(s) = h'''_{\mathbf{x}}(s) = h^{(4)}_{\mathbf{x}}(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, *an* $\rho(s) = \rho'(s) = 0$.

(5) $h'_{\mathbf{x}}(s) = h''_x(s) = h'''_{\mathbf{x}}(s) = h^{(4)}_{\mathbf{x}}(s) = h^{(5)}_{\mathbf{x}}(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, *and* $\rho(s) = \rho'(s) = \rho''(s) = 0$.

(B) (1) $\widetilde{h}_{\mathbf{x}}(s) = 0$ *iff there exist* $\langle\beta, \mathbf{x}\rangle = w$.

(2) $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}'_x(s) = 0$ *iff there exist* $a_1,\ a_2 \in \mathbb{R}$ *such that* $\mathbf{x} = \sinh u\mathbf{N}_1 + \cosh u\mathbf{N}_2$, *and* $\langle\beta, \mathbf{x}\rangle = w$.

(3) $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}'_{\mathbf{x}}(s) = \widetilde{h}''_x(s) = \widetilde{h}''_x(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, $\langle\beta, \mathbf{x}\rangle = w$, *and* $\rho(s) = 0$.

(4) $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}'_{\mathbf{x}}(s) = \widetilde{h}''_x(s) = \widetilde{h}''_x(s) = \widetilde{h}'''_x(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, $\langle\beta, \mathbf{x}\rangle = w$, *and* $\rho(s) = \rho'(s) = 0$.

(5) $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}'_{\mathbf{x}}(s) = \widetilde{h}''_x(s) = \widetilde{h}''_x(s) = \widetilde{h}'''_x(s) = \widetilde{h}^{(4)}_x(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, $\langle\beta, \mathbf{x}\rangle = w$ *and* $\rho(s) = \rho'(s) = \rho''(s) = 0$.

(6) $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}'_{\mathbf{x}}(s) = \widetilde{h}''_x(s) = \widetilde{h}''_x(s) = \widetilde{h}'''_x(s) = \widetilde{h}^{(4)}_x(s) = \widetilde{h}^{(4)}_x(s) = 0$ *iff* $\mathbf{x} = \pm\mathbf{e}(s)$, $\langle\beta, \mathbf{x}\rangle = w$, *and* $\rho(s) = \rho'(s) = \rho''(s) = \rho'''(s) = 0$.

Proof. According to Eq. (9) we have that $\|\mathbf{T}_1'\|^2 \neq 0$ iff $\kappa_1^2 - \kappa_2^2 \neq 0$.

(A) (1) Since $h'_{\mathbf{x}}(s) = \langle\mathbf{T}_1, \mathbf{x}\rangle = 0$, and $\{\mathbf{T}_1,\ \mathbf{N}_1,\ \mathbf{N}_2\}$ is RMF along $\beta(s)$, then there exist $a_1,\ a_2 \in \mathbb{R}$ such that $\mathbf{x} = a_1\mathbf{N}_1 + a_2\mathbf{N}_2$. By the condition that $\mathbf{x} \in \mathbb{H}_+^2$, we get $a_1^2 - a_2^2 = -1$. The converse of the theorem also holds true.

(2) Since $h''_{\mathbf{x}}(s) = \langle\mathbf{T}_1', \mathbf{x}\rangle = \langle\kappa_1\mathbf{N}_1 - \kappa_2\mathbf{N}_2, \mathbf{x}\rangle = 0$, we have that $a_1\kappa_1 + a_2\kappa_2 = 0$. It follows from the fact $a_1^2 - a_2^2 = -1$ that $a_1 = \mp\kappa_2/\sqrt{\kappa_1^2 - \kappa_2^2}$, and $a_2 = \pm\kappa_1/\sqrt{\kappa_1^2 - \kappa_2^2}$. Therefore, we have that

$$\mathbf{x} = \left(= \pm\frac{\kappa_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 - \mathbf{N}_2\right)\right)(s) = \pm\mathbf{e}(s).$$

(3) Since $h'''_{\mathbf{x}}(s) = \langle\mathbf{T}_1'', \mathbf{x}\rangle = \langle\kappa_1'\mathbf{N}_1 - \kappa_2'\mathbf{N}_2, \mathbf{x}\rangle = 0$, by the conditions of (2), we have that

$$\pm\frac{\kappa_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\frac{\kappa_1'\kappa_2 - \kappa_2'\kappa_1}{\kappa_1}\right)(s) = 0.$$

Therefore, $h'_{\mathbf{x}}(s) = h''_x(s) = h'''_{\mathbf{x}}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, and $\rho(s) = 0$.

(4) Since $h_{\mathbf{x}}^{(4)}(s) = \langle \mathbf{T}_1''', \mathbf{x} \rangle = 0$, we have

$$\langle 3(\kappa_2'\kappa_2 - \kappa_1'\kappa_1)\mathbf{T}_1 + (\kappa_1(\kappa_2^2 - \kappa_1^2) + \kappa_1'')\mathbf{N}_1 \\ - (\kappa_2(\kappa_2^2 - \kappa_1^2) + \kappa_2'')\mathbf{N}_2, \mathbf{x} \rangle = 0,$$

by the conditions of (3), we have that

$$\pm \frac{\kappa_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\frac{(\kappa_1'\kappa_2 - \kappa_2'\kappa_1)'}{\kappa_1} \right)(s) = \pm \left(\frac{\kappa_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \frac{\rho'}{\kappa_1} \right)(s) = 0.$$

Hence, $h_{\mathbf{x}}'(s) = h_x''(s) = h_{\mathbf{x}}'''(s) = h_{\mathbf{x}}^{(4)}(s) = 0$ if and only if $\mathbf{x} = \pm \mathbf{e}(s)$, and $\rho(s) = \rho'(s) = 0$.

(5) Since $h_{\mathbf{x}}^{(5)}(s) = \langle \mathbf{T}_1^{(4)}, \mathbf{x} \rangle 0$, we have

$$\left. \begin{aligned} &\langle ((\kappa_1^2 - \kappa_2^2)^2 + 4(\kappa_2\kappa_2'' - \kappa_1\kappa_1''))\mathbf{T}_1 \\ &\quad + (\kappa_1''' + 5\kappa_1(\kappa_2'\kappa_2 - \kappa_1'\kappa_1) + \kappa_1'\kappa_2^2 - \kappa_2'\kappa_1^2)\mathbf{N}_1 \\ &\quad + (\kappa_2''' + 5\kappa_2(\kappa_1'\kappa_1 - \kappa_2'\kappa_2) + \kappa_2'\kappa_1^2 - \kappa_1'\kappa_2^2)\mathbf{N}_2, \mathbf{x} \rangle = 0, \end{aligned} \right\}$$

or

$$\pm \frac{1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\frac{\kappa_2\kappa_1''' - \kappa_1\kappa_2''' - (\kappa_1'\kappa_2 - \kappa_2'\kappa_1)(\kappa_1^2 - \kappa_2^2)}{\kappa_1} \right) = 0.$$

Hence, $h_{\mathbf{x}}'(s) = h_x''(s) = h_{\mathbf{x}}'''(s) = h_{\mathbf{x}}^{(4)}(s) = h_{\mathbf{x}}^{(5)}(s) = 0$ if and only if $\mathbf{x} = \pm \mathbf{e}(s)$, and $\rho(s) = \rho'(s) = \rho''(s) = 0$.

(B) The proof of (B) follows from the proof of (A); so, we omit it. □

Proposition 3.2. *Under the same assumption of Proposition* 3.1, *we have* $\rho(s) = 0$ *if and only if*

$$\mathbf{e}(s) = \frac{-\kappa_1}{\sqrt{\kappa_1^2 - \kappa_2^2}} \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 - \mathbf{N}_2 \right)$$

is a constant vector.

Proof. Assume that $\kappa_1^2 - \kappa_2^2 \neq 0$. By simple computations, we find

$$\mathbf{e}'(s) = \frac{\rho(s)}{(\sqrt{\kappa_1^2 - \kappa_2^2})^3}(\kappa_1\mathbf{N}_1 - \kappa_2\mathbf{N}_2).$$

Thus, $\mathbf{e}'(s) = \mathbf{0}$ if and only if $\rho(s) = \kappa_1\kappa_2' - \kappa_2\kappa_1' = 0$. □

Proposition 3.3. *Under the same assumption of Proposition* 3.2, *we state the following*:

(a) *β is a B-slant helix iff κ_2/κ_1 is constant.*
(b) *$\mathbf{N}_1$ is a part of a circle on $\mathbb{H}_+^2$ whose center is the timelike constant vector $\mathbf{e}_0$.*

Proof. (a) Suppose that $\rho(s) = \kappa_2\kappa_1' - \kappa_1\kappa_2' = 0$. Hence, we can write

$$\left(\frac{\kappa_2}{\kappa_1}\right)' = \frac{\kappa_1\kappa_2' - \kappa_2\kappa_1'}{\kappa_2^2} = \frac{\rho(s)}{\kappa_2^2} = 0.$$

This means that $\frac{\kappa_2}{\kappa_1}$=constant, that is, β is a B-slant helix.

(b) Let $\kappa_1^2 - \kappa_2^2 \neq 0$. Since

$$\begin{aligned}\langle \mathbf{e}, \mathbf{N}_1 \rangle &= \frac{-\kappa_1}{\sqrt{\sqrt{\kappa_1^2 - \kappa_2^2}}} \left\langle \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 - \mathbf{N}_2\right), \mathbf{N}_1 \right\rangle \\ &= \frac{\frac{\kappa_2}{\kappa_1}}{\sqrt{1 - \left(\frac{\kappa_2}{\kappa_1}\right)^2}} = \text{const.}\end{aligned}$$

This means that $\mathbf{N}_1$ is a part of circle on $\mathbb{H}_+^2$ whose center is the constant timelike vector $\mathbf{e}_0(s)$. □

3.2. *Unfolding of functions by one-variable*

Let us review some properties on the singularity theory (see [20, 21]). Let $F\colon(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a smooth function, and $f(s) = F_{x_0}(s, \mathbf{x}_0)$. Then F is called an r-parameter unfolding of $f(s)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^p(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{k+1}(s_0) \neq 0$. We also say that f has $A_{\geq k}$-singularity ($k \geq 1$) at s_0. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 be $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0))(s_0) = \Sigma_{j=0}^{k-1} L_{ji}(s - s_0)^j$ (without the constant term), for $i = 1, \ldots, r$. Then $F(s)$ is called a p-versal unfolding if the $k \times r$ matrix of coefficients (L_{ji}) has rank k ($k \leq r$).

Now, we remind the definitions of some relevant sets about the unfolding related with the above notations. The discriminant set of F is the set

$$\mathfrak{D}_F = \left\{\mathbf{x}{\in}\mathbb{R}^r |\text{ there exists } s \text{ with } F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = 0 \text{ at } (s, \mathbf{x})\right\}. \quad (16)$$

The bifurcation set of F is the set

$$\mathfrak{B}_F = \left\{\mathbf{x}{\in}\mathbb{R}^r |\text{ there exists } s \text{ with } \frac{\partial F}{\partial s}(s, \mathbf{x}) = \frac{\partial^2 F}{\partial s^2}(s, \mathbf{x}) = 0 \text{ at } (s, \mathbf{x})\right\}. \quad (17)$$

Then similar to [1–3], we give the following theorem.

Theorem 3.2. *Let $F\colon(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$, which has the A_k singularity at s_0.*

Suppose that F is a p-versal unfolding.

(a) *If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{\mathbf{0}\}{\times}\mathbb{R}^{r-1}$, and $\mathfrak{B}_F = \varnothing$;*
(b) *If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\mathbf{C}{\times}\mathbb{R}^{r-2}$, and $\mathfrak{B}_F$ is locally diffeomorphic to $\{\mathbf{0}\}{\times}\mathbb{R}^{r-1}$;*

(c) *If $k=3$, then $\mathfrak{D}_F$ is locally diffeomorphic to* $\mathbf{SW}\times\mathbb{R}^{r-3}$, *and $\mathfrak{B}_F$ is locally diffeomorphic to* $\mathbf{C}\times\mathbb{R}^{r-2}$.

Then, we have the following fundamental proposition.

Proposition 3.4. *Let $\beta : I \to \mathbb{E}^3_1$ be a unit speed spacelike curve with a timelike principal normal, and $\kappa_1^2-\kappa_2^2 \neq 0$. (1) If $h_{\mathbf{x}}(s)=H(s,\mathbf{x})$ has an A_k-singularity ($k=2,3$) at $s_0\in\mathbb{R}$, then H is a $p-$versal unfolding of $h_{\mathbf{x}_0}(s_0)$. (2) If $\widetilde{h}_{\mathbf{x}}(s)=\widetilde{H}(s,\mathbf{x},w)$ has an A_k-singularity ($k=2,3$) at $s_0\in\mathbb{R}$, then $\widetilde{H}$ is a $p-$versal unfolding of $\widetilde{h}_{\mathbf{x}_0}(s_0)$.*

Proof. (1) Let $\mathbf{x}=(x_0,x_1,x_2)\in\mathbb{H}^2_+$, $-x_0^2+x_1^2+x_2^2=-1$, x_0, x_1, and x_2 cannot be all zero. Without loss of generality, suppose that $x_2\neq 0$. Then by $x_2=\sqrt{x_0^2-x_1^2-1}$, Therefore,

$$H(s,\mathbf{x})=-x_0\beta_0(s)+x_1\beta_1(s)+\sqrt{x_0^2-x_1^2-1}\,\beta_2(s). \tag{18}$$

Thus, we have that

$$\left.\begin{aligned}&\frac{\partial H}{\partial x_0}=-\beta_0(s)+\frac{x_0\beta_2(s)}{\sqrt{1+x_0^2-x_1^2}}, \quad \frac{\partial H}{\partial x_1}=\beta_1(s)-\frac{x_1\beta_2(s)}{\sqrt{1+x_1^2-x_2^2}},\\ &\frac{\partial^2 H}{\partial s\partial x_0}=-\beta_0'(s)+\frac{x_0\beta_2'(s)}{\sqrt{1+x_0^2-x_1^2}}, \quad \frac{\partial^2 H}{\partial s\partial x_1}=\beta_1'(s)-\frac{x_1\beta_2'(s)}{\sqrt{1+x_1^2-x_2^2}}.\end{aligned}\right\}$$

Therefore, the 2-jets of $\frac{\partial H}{\partial x_i}$ at s_0 $(i=0,1)$ are as follows: Let $\mathbf{x}_0=(x_{00},x_{10},x_{20})\in\mathbb{H}^2_+$, and assume $x_{20}\neq 0$, then

$$\left.\begin{aligned}j^1\left(\frac{\partial H}{\partial x_0}(s,\mathbf{x}_0)\right)&=\left(-\beta_0'(s)+\frac{x_{00}\beta_2'(s)}{x_{20}}\right)(s-s_0),\\ j^1\left(\frac{\partial H}{\partial x_1}(s,\mathbf{x}_0)\right)&=\left(\beta_1'(s)-\frac{x_{10}\beta_2'(s)}{x_{20}}\right)(s-s_0),\end{aligned}\right\} \tag{19}$$

and

$$\left.\begin{aligned}j^2\left(\frac{\partial H}{\partial x_0}(s,\mathbf{x}_0)\right)&=\left(-\beta_0'(s)+\frac{x_{00}\beta_2(s)}{x_{20}}\right)(s-s_0)\\ &\quad+\frac{1}{2}\left(-\beta_0''+\frac{x_{00}\beta_2''(s)}{x_{20}}\right)(s-s_0)^2,\\ j^2\left(\frac{\partial H}{\partial x_1}(s,\mathbf{x}_0)\right)&=\left(\beta_1'(s)-\frac{x_{10}\beta_2'(s)}{x_{20}}\right)(s-s_0)\\ &\quad+\frac{1}{2}\left(\beta_1''(s)-\frac{x_{10}\beta_2''(s)}{x_{20}}\right)(s-s_0)^2.\end{aligned}\right\} \tag{20}$$

(i) If $h_{\mathbf{x}_0}(s_0)$ has the A_2-singularity at s_0, then $h'_{\mathbf{x}_0}(s_0) = 0$. So the $(2-1) \times 2$ matrix of coefficients (L_{ji}) is

$$A = \left(-\beta_0'(s) + \frac{x_{00}\beta_2'(s)}{x_{20}} \beta_1'(s) - \frac{x_{10}\beta_2'(s)}{x_{20}} \right). \tag{21}$$

If $\operatorname{rank}(A) = 0$, then we obtain

$$\beta_0'(s) = \frac{x_{00}\beta_2'(s)}{x_{20}}, \quad \beta_1'(s) = \frac{x_{10}\beta_2'(s)}{x_{20}}. \tag{22}$$

Since $\|\beta'(s_0)\| = \|\mathbf{T}_1(s_0)\| = 1$, we have $\beta_2'(s_0) \neq 0$, so that we have the contradiction as follows:

$$\begin{aligned} 0 &= \langle (\beta_0'(s_0), \beta_1'(s_0), \beta_2'(s_0)), (x_{00}, x_{10}, x_{20}) \rangle \qquad (23) \\ &= -\beta_0'(s_0)x_{00} + \beta_1'(s_0)x_{10} + \beta_2'(s_0)x_{20} \\ &= -\frac{x_{00}^2\beta_2'(s_0)}{x_{20}} + \frac{x_{10}^2\beta_2'(s_0)}{x_{20}} + \beta_2'(s_0)x_{20} \qquad (24) \\ &= \frac{\beta_2'(s_0)}{x_{20}}(-x_{00}^2 + x_{10}^2 + x_{20}^2) \\ &= \frac{\beta_2'(s_0)}{x_{20}} \neq 0. \end{aligned}$$

Therefore, $\operatorname{rank}(A) = 1$, and H is the (p) versal unfolding of $h_{\mathbf{x}_0}$ at s_0.

(ii) If $h_{\mathbf{x}_0}(s_0)$ has the A_3-singularity at $s_0 \in \mathbb{R}$, then $h'_{\mathbf{x}_0}(s_0) = h''_{\mathbf{x}_0}(s_0) = 0$, and by Proposition 3.2,

$$\mathbf{e}(s_0) = \frac{\kappa_1}{\sqrt{\kappa_2^2 - \kappa_1^2}} \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 + \mathbf{N}_2 \right), \tag{25}$$

where $\kappa_1^2 - \kappa_2^2 > 0$, $\rho'(s_0) = 0$, and $\rho''(s_0) \neq 0$. It is obvious that the $(3-1) \times 2$ matrix of the coefficients (L_{ji}) is

$$B = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} -\beta_0'(s) + \dfrac{x_{00}\beta_2(s)}{x_{20}} & \beta_1'(s) - \dfrac{x_{10}\beta_2'(s)}{x_{20}} \\ -\beta_0'' + \dfrac{x_{00}\beta_2''(s)}{x_{20}} & \beta_1''(s) - \dfrac{x_{10}\beta_2''(s)}{x_{20}} \end{pmatrix}. \tag{26}$$

For the purpose, we also need the 2×2 matrix B to be non-singular, which always holds true. Actually, the determinant of this matrix at s_0 is

$$\det(B) = \frac{1}{x_{20}} \begin{vmatrix} -\beta_0' & \beta_1' & \beta_2' \\ -\beta_0'' & \beta_1'' & \beta_2'' \\ x_{00} & x_{10} & x_{20} \end{vmatrix} \tag{27}$$

$$= \frac{1}{x_{20}}\langle \beta' \times \beta'', \mathbf{e_0}\rangle$$

$$= \mp\frac{\kappa_1}{x_{20}\sqrt{\kappa_2^2-\kappa_1^2}}\left\langle \beta' \times \beta'', \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1+\mathbf{N}_2\right)\right\rangle. \tag{28}$$

Since $\beta' = \mathbf{T}_1$, we have $\beta'' = \kappa_1\mathbf{N}_1 + \kappa_2\mathbf{N}_2$. By making use of these relations in the above equality, we have that

$$\det(B) = \mp\frac{\sqrt{\kappa_2^2-\kappa_1^2}}{x_{20}} \neq 0. \tag{29}$$

This means that $\operatorname{rank}(B) = 2$.

(2) As in (1), we have

$$\widetilde{H}(s, \mathbf{x}, x_2) = -x_0\beta_0(s) + x_1\beta_1(s) + \sqrt{x_0^2 - x_1^2 - 1}\beta_2(s) - x_2. \tag{30}$$

We require the 2×3 matrix

$$G = \begin{pmatrix} -\beta_0'(s) + \dfrac{x_{00}\beta_2(s)}{x_{20}} & \beta_1'(s) - \dfrac{x_{10}\beta_2'(s)}{x_{20}} & -1 \\ -\beta_0'' + \dfrac{x_{00}\beta_2''(s)}{x_{20}} & \beta_1''(s) - \dfrac{x_{10}\beta_2''(s)}{x_{20}} & 0 \end{pmatrix},$$

to have the maximal rank. By case (i) in Eq. (20), the second row of G does not vanish, so $\operatorname{rank}(G) = 2$. □

Proof of Theorem 3.1. (1) By Proposition 3.1, the bifurcation set of $H(s, \mathbf{x})$ is

$$\mathfrak{B}_H = \left\{\frac{\kappa_1}{\sqrt{\kappa_2^2-\kappa_1^2}}\left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1+\mathbf{N}_2\right)|s \in \mathbb{R}|s \in \mathbb{R}\right\}. \tag{31}$$

The assertion (1) of Theorem 3.1 follows from Propositions 3.1 and 3.4, and Theorem 3.2. The discriminant set of $\widetilde{H}(s, \mathbf{x})$ is given as follows:

$$\mathfrak{D}_{\widetilde{H}} = \{\mathbf{x}_0 = \beta + \sinh u\mathbf{N}_1 + \cosh u\mathbf{N}_2 | s \in \mathbb{R}\}. \tag{32}$$

The assertion (1) of Theorem 3.1 follows from Propositions 3.1 and 3.4, and Theorem 3.2. □

Example 3.1. Given the spacelike circular helix: $\beta(s) = (\sqrt{2}s, \cosh s, \sinh s)$, $-1 \leq s \leq 1$. It is easy to show that

$$\left.\begin{aligned} &\zeta_1(s) = (\sqrt{2}, \sinh s, \cosh s),\\ &\zeta_2(s) = (0, \cosh s, \sinh s),\\ &\zeta_3(s) = (-1, -\sqrt{2}\sinh s, -\sqrt{2}\cosh s),\\ &\kappa(s) = 1, \text{ and } \tau(s) = -\sqrt{2}. \end{aligned}\right\}$$

If $\theta_0 = 0$, we have $\theta(s) = -\sqrt{2}s$. Using the Eq. (2.9), we obtain

$$\kappa_1(s) = \cosh\sqrt{2}s, \quad \text{and} \quad \kappa_2(s) = -\sinh\sqrt{2}s.$$

Hence, the geometric invariant is

$$\rho(s) = -1.$$

The transformation matrix can be expressed as

$$\begin{pmatrix} \mathbf{T}_1 \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\frac{s}{2} & -\sinh\frac{s}{2} \\ 0 & -\sinh\frac{s}{2} & \cosh\frac{s}{2} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}.$$

Subsequently, we have

$$\mathbf{N}_1 = \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} = \begin{pmatrix} \cosh\frac{s}{2}\sinh s - \frac{1}{2}\sinh\frac{s}{2}\cosh s \\ \frac{\sqrt{3}}{2}\sinh\frac{s}{2} \\ \cosh\frac{s}{2}\cosh s - \frac{1}{2}\sinh\frac{s}{2}\sinh s \end{pmatrix},$$

$$\mathbf{N}_2 = \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cosh\frac{s}{2}\cosh s - \sinh\frac{s}{2}\sinh s \\ -\frac{\sqrt{3}}{2}\cosh\frac{s}{2} \\ \frac{1}{2}\cosh\frac{s}{2}\sinh s - \sinh\frac{s}{2}\cosh s \end{pmatrix}.$$

Therefore, the timelike sweeping surface is (Fig. 4)

$$M : \mathbf{R}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + \sinh u \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} + \cosh u \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix}.$$

The Bishop spherical Darboux image is (Fig. 5)

$$\mathbf{e}(s) = -\sinh\frac{s}{2} \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \end{pmatrix} + \cosh\frac{s}{2} \begin{pmatrix} N_{21} \\ N_{22} \\ N_{23} \end{pmatrix}.$$

3.3. *Singularities of developable surfaces*

Developable surfaces are special cases of ruled surfaces. Developable surfaces are widely used in the airplane wings, manufacture of automobile body parts and ship

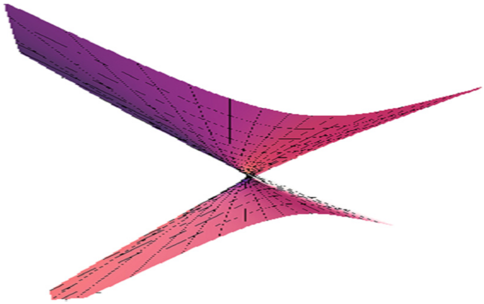

Fig. 4. Timelike sweeping surface with spacelike helix singularity curve.

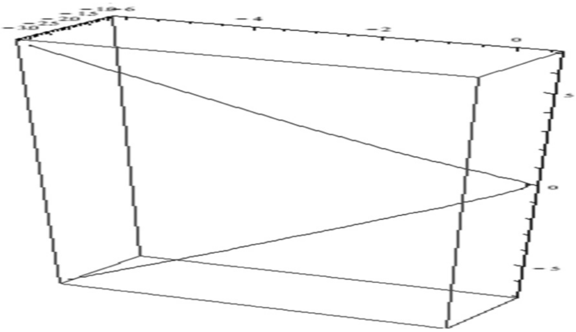

Fig. 5. Bishop spherical Darboux image which has a cusp point.

hulls [1, 7]. Therefore, we investigate the case that the profile curve $x(u)$ degenerates into a line. Then, we have the following spacelike developable surface:

$$M : \mathbf{Q}(s,u) = \beta(s) + u\mathbf{N}_2(s), \quad u \in \mathbb{R}, \tag{33}$$

and the timelike developable surface

$$M^{\perp} : \mathbf{Q}^{\perp}(s,u) = \beta(s) + u\mathbf{N}_1(s), \quad u \in \mathbb{R}. \tag{34}$$

Obviously, $\mathbf{P}(s,0) = \alpha(s)$ (resp. $\mathbf{P}^{\perp}(s,0) = \alpha(s)$), $0 \leq s \leq L$, that is, the surface M (respectively, $M^{\perp}$) interpolates the curve $\alpha(s)$. We can also calculate that

$$M : \mathbf{Q}_s \times \mathbf{Q}_u = (1 - u\kappa_2)\mathbf{N}_1(s),$$

and

$$M^{\perp} : \mathbf{Q}_s^{\perp} \times \mathbf{Q}_u^{\perp} = (1 + u\kappa_1)\mathbf{N}_1(s).$$

Then we have M (resp. $M^{\perp}$) is non-singular at (s_0, u_0) iff $1 - u_0\kappa_2(s_0) \neq 0$ (respectively, $(1 + u_0\kappa_1(s_0) \neq 0)$. Based on [22, Theorem 3.3], we can classify the singularities of developable surface M by using κ_2.

Theorem 3.3. *Let M be the spacelike developable expressed by Eq.* (33)*. Then we have the following*:

(1) *M is locally diffeomorphic to Cuspidal edge at (s_0, u_0) iff $\kappa_2(s_0) = 0$, and $\kappa_2'(s_0) \neq 0$;*
(2) *M is locally diffeomorphic to Swallowtail at (s_0, u_0) iff $\kappa_2(s_0) \neq 0$, and $\frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} \neq 0$.*

Proof. If there exists a parameter s_0 such that $\kappa_2(s_0) = 0$, and $u_0' = \frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} \neq 0$ $(\kappa_2'(s_0) \neq 0)$, then M is locally diffeomorphic to Cuspidal edge at (s_0, u_0). So, assertion (1) holds. Also, if there exists a parameter s_0 such that $u_0 = \frac{1}{\kappa_2(s_0)} \neq 0$, $u_0' = \frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} = 0$, and $(\frac{1}{\kappa_2(s_0)})'' \neq 0$, then M is locally diffeomorphic to Swallowtail at (s_0, u_0), assertion (2) holds. □

Example 3.2. Based on Example 3.2, we have the following:

(1) If $s_0 = 0$, then $\kappa_2(s_0) = 0$, $\kappa_2'(s_0) \neq 0$. The timelike developable surface

$$M : \mathbf{Q}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + u\begin{pmatrix} \frac{1}{2}\cosh\frac{s}{2}\cosh s - \sinh\frac{s}{2}\sinh s \\ -\frac{\sqrt{3}}{2}\cosh\frac{s}{2} \\ \frac{1}{2}\cosh\frac{s}{2}\sinh s - \sinh\frac{s}{2}\cosh s \end{pmatrix},$$

is locally diffeomorphic to the cuspidal edge, $u \in \mathbb{R}$, see Fig. 6.

(2) If $s_0 = 0$, then $\kappa_1(s_0) \neq 0$, $\kappa_1'(s_0) = 0$. The spacelike developable surface

$$M^{\perp} : \mathbf{Q}^{\perp}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + u\begin{pmatrix} \cosh\frac{s}{2}\sinh s - \frac{1}{2}\sinh\frac{s}{2}\cosh s \\ \frac{\sqrt{3}}{2}\sinh\frac{s}{2} \\ \cosh\frac{s}{2}\cosh s - \frac{1}{2}\sinh\frac{s}{2}\sinh s \end{pmatrix},$$

is locally diffeomorphic to swallowtail, $u \in \mathbb{R}$, see Fig. 7.

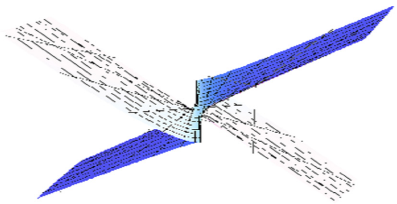

Fig. 6. Timelike developale surface.

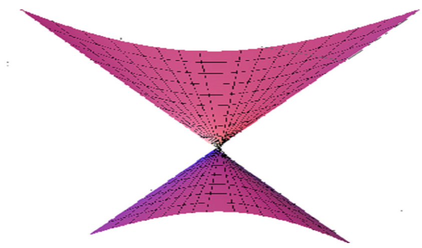

Fig. 7. Spacelike developale surface.

4. Conclusion

This paper studies the properties of timelike sweeping surface by setting up an orthonormal RMF to each point of the spine curve. Then, some general results of the singularity theory are used for families of function germs, and the main result is proved. Also, conditions for a sweeping surface to be developable ruled surface are derived. The similar problem addressed in this paper may be considered for 3-surfaces in 4-space.

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