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Multidimensional fixed point results for two hybrid pairs in partially ordered metric space

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***Abstract*- In this paper, we introduce the notion of mixed weakly monotone property for two hybrid pairs (S, g) and (T, g) each of them consists of multi-valued mapping S or $T: X^n \to CB(X)$ and single valued mapping $g: X \to X$ defined on partially ordered metric space and then we prove coincidence and common fixed point theorems for two hybrid pairs under different contractive conditions. These theorems extend and generalize very recent results that can be found in [12] and many others.**

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***Keywords*: Metric space, partially ordered set, fixed point of $n-$ order, multi-valued mapping, mixed weakly monotone property, $w-$compatible mappings.**

I. Introduction

In recent years there has been a growing interest in studying the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. This trend was initiated by Ran and Reurings in [16] where they extended the famous Banach contraction principle in partially ordered sets with some applications to matrix equations.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [15] who extended the Banach contraction principle from single valued to multi-valued mapping. Later many authors developed the existence of fixed points for various multi-valued contractions under different conditions. For details, we refer the reader to [1, 3, 6, 7, 10, 18, 19, 21] and the references therein. Fixed point theory of such mappings has applications in control theory, convex optimization, differential inclusion and economics.

Gnana-Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point and proved some coupled fixed point results under certain conditions in a complete metric space endowed with a partial order. They applied their results to study the existence of a unique solution for a periodic boundary value problem associated with a first order ordinary differential equation. Later, Lakshmikantham and C'iric' [13] established the existence of coupled coincidence and coupled common fixed point results to generalize the results in [5].

Beg and Butt [4] have followed the technique of Bhaskar and Lakshmikantham and proved some coupled fixed point results for multi-valued mappings in partially ordered metric spaces. For this purpose, they introduced a generalized mixed monotone property for a set valued mapping.

II. Preliminaries

Throughout this paper, n will be a positive integer, m and p will be non-negative integers and $i, j \in \Lambda_n = \{1,2,\ldots,n\}$. Furthermore, X will denote a non-empty set and X^n will denote the product space $X^n = \overbrace{X \times \ldots \times X}^{n-times}$. Unless otherwise stated, "for all m and i" will mean "for all $m \geq 0$ and $i \in \Lambda_n$", respectively.

Definition 2.1 *A metric on* X *is a mapping* $d : X \times X \to R$ *satisfying, for all* $x, y, z \in X$ *:*

(d1) $d(x,y) = 0 \iff x = y$;

(d2) $d(x,y) \geq 0$;

(d3) $d(x,y) = d(y,x)$;

(d4) $d(x,z) \leq d(x,y) + d(y,z)$.

The last requirement is called the triangle inequality. If d is a metric on X , we say that (X,d) is a metric space.

Definition 2.2 *[8] A triple* $(X,d,\leq)$ *is called an ordered metric space iff*

i. (X,d) is a metric space and,

ii. $(X,\leq)$ is a partially ordered set.

Definition 2.3 *[5] Let* $(X,d,\leq)$ *be an ordered metric space.* X *is said to have the sequential monotone property if it verifies the following properties:*

i. If $\{x_n\}$ is an increasing sequence with $x_n \to x$ then $x_n \leq x$ for all $n \in N$,

ii. If $\{y_n\}$ is a decreasing sequence with $y_n \to y$ then $y_n \geq y$ for all $n \in N$.

Lakshmikantham and C′iric′ [13] introduced the following concepts for two single valued mappings $F : X \times X \to X$ and $g : X \to X$ defined on partially ordered set $(X,\leq)$.

Definition 2.4 *[13] An element* $(x,y) \in X \times X$ *is said to be*

i. coupled coincidence point of the mappings F and g if $gx = F(x,y)$ and $gy = F(y,x)$;

ii. coupled common fixed point of the mappings F and g if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.

Definition 2.5 *[13] The mapping* F *has the mixed g-monotone property if* $F(x,y)$ *is* $g-$*monotone non-decreasing in its first argument and* $g-$*monotone non-increasing in its second argument, that is, for any* $x, y \in X$ *,*

$$x_1, x_2 \in X,\ g(x_1) \leq g(x_2) \text{ implies } F(x_1,y) \leq F(x_2,y)$$

and

$$y_1, y_2 \in X,\ g(y_1) \leq g(y_2) \text{ implies } F(x,y_1) \geq F(x,y_2).$$

If g is the identity mapping, we obtain the Bhaskar and lakshmikantham's notion of a mixed monotone property of the mapping F.

Definition 2.6 *[2] The mapping* F *and* g *are called* $w-$*compatible if* $g(F(x,y)) = F(gx,gy)$*, whenever* $gx = F(x,y)$ *and* $gy = F(y,x)$.

Then Roldán et al [20] extended the previous notions by defining coincidence point between two mappings in any number of variables. Fix a partition $\{A,B\}$ of Λ_n, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$ and $\sigma_1,\ldots,\sigma_n,\tau$ be mappings from Λ_n into itself. We will denote

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}. \tag{2.1}$$

If $(X,\leq)$ is a partially ordered space, $x, y \in X$, we will use the following notation

$$x \leq_i y \iff \begin{cases} x \leq y, & i \in A, \\ x \geq y, & i \in B. \end{cases}$$

Inspired by the above notation, we endow the product space X^n with the following order:

For $(x^1,\ldots,x^n)$ and $(u^1,\ldots,u^n) \in X^n$, we say

$$(x^1,\ldots,x^n) \leq (u^1,\ldots,u^n) \iff x^i \leq_i u^i.$$

Also we say that $(x^1,\ldots,x^n)$ and $(u^1,\ldots,u^n)$ are comparable if $(x^1,\ldots,x^n) \leq (u^1,\ldots,u^n)$ or $(x^1,\ldots,x^n) \geq (u^1,\ldots,u^n)$.

Definition 2.7 *[20] Let $(X,\leq)$ be a partially ordered set and $F: X^n \to X$, $g: X \to X$ be mappings. We say that F has the mixed g-monotone property if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, that is, for all $x^1, x^2, \ldots, x^n, y, z \in X$ and all i.*

$$gy \leq gz \Rightarrow F(x^1,\ldots,x^{i-1},y,x^{i+1},\ldots,x^n) \leq_i F(x^1,\ldots,x^{i-1},z,x^{i+1},\ldots,x^n).$$

Definition 2.8 *[20] A point $(x^1, x^2, \ldots, x^n) \in X^n$ is called*

i. $n-$ fixed point of the mapping F if $x^i = F(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), \forall i;$

ii. $n-$coincidence point of the mappings F and g if $gx^i = F(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), \forall i;$

iii. $n-$common fixed point of the mappings F and g if $x^i = gx^i = F(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), \forall i.$

Theorem 2.1 *[20] Let $(X,d,\leq)$ be a complete ordered metric space. Let $\phi = (\sigma_1,\sigma_2,\ldots,\sigma_n,\tau)$ be a $(n+1)$-tuple of mappings from Λ_n into itself such that $\tau \in \Omega_{A,B}$ is a permutation and verifying that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F: X^n \to X$ and $g: X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$ and g commutes with F. Assume that there exists $k \in [0,1)$ verifying*

$$d(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n)) \leq k \max_{1<i<n} d(gx_i, gy_i) \tag{2.2}$$

for which $gx_i \leq_i gy_i$ for all i. Suppose either F is continuous or X has the sequential monotone property. If there exist $x_0^1,\ldots.x_0^n \in X$ verifying

$$gx_0^{\tau(i)} \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)},\ldots,x_0^{\sigma_i(n)}), \forall i.$$

Then F and g have, at least, one n-coincidence point.

Let (X,d) be a metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, let

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where

$$d(x,A) = \inf_{a\in A} d(x,a).$$

H is said to be a Hausdorff metric induced by d. Let $F: X \times X \to 2^X$ (the power set of X) be a set valued mapping i.e, $X \times X \ni (x,y) \mapsto F(x,y)$ is a subset of X.

Definition 2.9 *[4] Let $(X,\leq)$ be a partially ordered set and $F: X \times X \to CB(X)$ be a set valued mapping. F is said to be a mixed monotone mapping if F is order-preserving in x and order-reversing in y i.e., $x_1 \leq x_2$, $y_2 \leq y_1$, $x_i, y_i \in X (i=1,2)$ imply for all $u_1 \in F(x_1,y_1)$ there exists $u_2 \in F(x_2,y_2)$ such that $u_1 \leq u_2$ and for all $v_1 \in F(y_1,x_1)$ there exists $v_2 \in F(y_2,x_2)$ such that $v_2 \leq v_1$.*

Definition 2.10 *[4] A point $(x,y) \in X \times X$ is said to be a coupled fixed point of the set valued mapping F if $x \in F(x,y)$ and $y \in F(y,x)$.*

Definition 2.11 *[1] Let $F: X \times X \to CB(X)$ and $g: X \to X$ be hybrid pair of mappings. An element $(x,y) \in X \times X$ is called*

i. coupled coincidence point of a hybrid pair F,g if $gx \in F(x,y)$ and $gy \in F(y,x)$;

ii. coupled common fixed point of a hybrid pair F,g if $x = gx \in F(x,y)$ and $y = gy \in F(y,x)$.

Lemma 2.1 *[15] Let $A,B \in CB(X)$ and $\alpha > 1$. Then, for every $a \in A$ there exist $b \in B$ such that*

$$d(a,b) \leq \alpha H(A,B).$$

Lemma 2.2 *[15] Let* $A, B \in CB(X)$ *and* $\alpha > 0$. *Then, for every* $a \in A$ *there exist* $b \in B$ *such that*

$$d(a,b) \le H(A,B) + \alpha.$$

In our recent paper [18], we give the following definitions for hybrid pair of set valued mapping with $n-$ variable and single valued mapping.

Definition 2.12 *[18] Let* $(X, \le)$ *be a partially ordered set and* $F : X^n \to CB(X)$, $g : X \to X$ *be mappings. We say that* F *has the mixed* g *-monotone property if* $F(x^1, \dots, x^n)$ *is* g *-monotone non-decreasing in arguments of* A *and* g *-monotone non-increasing in arguments of* B, *that is, for any* $x^1, \dots, x^{i-1}, x^i, x^{i+1}, \dots, x^n, y \, and \, z \in X$ *we have*

$$gy \le gz \Rightarrow F(x^1, \dots, x^{i-1}, y, x^{i+1}, \dots, x^n) \le_i F(x^1, \dots, x^{i-1}, z, x^{i+1}, \dots, x^n),$$

that is for any element $u \in F(x^1, \dots, x^{i-1}, y, x^{i+1}, \dots, x^n)$ and $v \in F(x^1, \dots, x^{i-1}, z, x^{i+1}, \dots, x^n)$ we have $u \le_i v$.

Definition 2.13 *[18] A point* $(x^1, x^2, \dots, x^n) \in X^n$ *is called*

i. $n-$ fixed point of the mappings F if $x^i \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), \forall i;$

ii. $n-$coincidence point of the mappings F and g if $gx^i \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), \forall i;$

iii. $n-$common fixed point of the mappings F and g if $x^i = gx^i \in F(x^{\sigma_i(1)}, \dots, x^{\sigma_i(n)}), \forall i.$

In 2012, Gordii et al. [9] introduced the concept of the mixed weakly increasing property of mappings and proved some coupled fixed point results.

Definition 2.14 *[9] Let* $(X, \le)$ *be a partially ordered set and* $S, T : X \times X \to X$ *be mappings. We say that a pair* S, T *has the mixed weakly monotone property on* X *if for any* $x, y \in X$

$$x \le S(x,y), \quad y \ge S(y,x),$$
$$\Rightarrow S(x,y) \le T(S(x,y), S(y,x)) \quad , S(y,x) \ge T(S(y,x), S(x,y))$$

and

$$x \le T(x,y) \quad , \quad y \ge T(y,x),$$
$$\Rightarrow T(x,y) \le S(T(x,y), T(y,x)) \quad , T(y,x) \ge S(T(y,x), T(x,y)).$$

In [12], the authors obtained tripled coincidence and common fixed point results for two hybrid pairs consisting of multi-valued and single valued mappings (S, g) and (T, g) under two different contractive conditions.

Theorem 2.2 ***[12]*** *Let* (X, d) *be a metric space,* $S, T: X \times X \times X \to CB(X)$ *and* $g: X \to X$ *be mappings such that*

$$H\big(S(x,y,z), T(u,v,w)\big) \le a_1\, d(gx, gu) + a_2\, d(gy, gv) + a_3\, d(gz, gw) + a_4\, d(S(x,y,z), gx) + a_5\, d(T(u,v,w), gu) + a_6 d\,(S(x,y,z), gu) + a_7\, d(T(u,v,w), gx), \quad (2.3)$$

For all $x, y, z, u, v, w \in X$, *where* $a_i, \forall i = 1,2, \dots, 7$, *are non-negative real numbers such that* $\sum_{i=1}^{7} a_i \le h < 1$. *If* $S(X^3) \cup T(X^3) \subseteq g(X)$ *and* $g(X)$ *is complete subset of* X, *then* (S, g) *and* (T, g) *have tripled coincidence point . Furthermore, (S,g) and (T,g) have tripled common fixed point if one of the following conditions holds for some* $x, y, z \in \gamma(S, g) \cap \gamma(T, g)$ *and* $u, v, w \in X$. *:*

(a) (S,g) and (T,g) are w- compatible, $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = u$ *and* $\lim_{n\to\infty} g^n z = w$ *and* g *is continuous at u, v, w ;*

(b) if $g^2 x = gx, g^2 y = y, g^2 z = z$ *and* g *is S, T- idempotent;*

(c) g is continuous at x, y, z *and* $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$ *and* $\lim_{n\to\infty} g^n w = z$. .

Inspired by the results of Roldán et al [20], Gordii et al. [9] and the previous result of Kutbi et al. [12], in this paper we establish n- coincidence and n- common fixed point theorems for two hybrid pairs each of them consists of multi-valued mapping with n- variable and single valued mapping under different contractive conditions by using the notion of mixed weakly monotone property. Our results improve and extend all above results.

III. MAIN RESULT

First of all, we give the following definitions.

Definition 3.1 *Let $(X,\leq)$ be a partially ordered set and $S,T:X^n\to CB(X)$ be mappings. We say that a pair (S,T) has the mixed weakly $g-$monotone property on X if for any $x^1,\ldots,x^n\in X$*

$$\{gx^i\}\leq_i S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),\ \forall\, i$$

$$\Rightarrow S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})\ \leq_i T(S(x^{\sigma_{\sigma_i(1)}(1)},\ldots,x^{\sigma_{\sigma_i(1)}(n)}),\ldots,S(x^{\sigma_{\sigma_i(n)}(1)},\ \ldots,x^{\sigma_{\sigma_i(n)}(n)}))$$

and

$$\{gx^i\}\leq_i T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),\ \forall\, i$$

$$\Rightarrow T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})\ \leq_i S(T(x^{\sigma_{\sigma_i(1)}(1)},\ldots,x^{\sigma_{\sigma_i(1)}(n)}),\ldots,T(x^{\sigma_{\sigma_i(n)}(1)},\ \ldots,x^{\sigma_{\sigma_i(n)}(n)})).$$

Definition 3.2 *The mappings $S:X^n\to CB(X)$ and $g:X\to X$ are called $w-$compatible if $g(S(x^1,\ldots,x^n))\subseteq S(gx^1,\ldots,gx^n)$, whenever $(x^1,\ldots,x^n)\in\gamma(S,g)$, where* $\gamma(S,g)$ is the set of all n-coincidence points of *S* and g.

Definition 3.3 *The mapping g is called $S-$idempotent at some point $(x^1,\ldots,x^n)\in X^3$ if $g^2x^i\in S(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}),\ \forall\, i$.*

Now, we state our main results.

Theorem 3.1 *Let $(X,d,\leq)$ be an ordered metric space and $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ be $n-$tuple of mappings from Λ_n into itself and for which $\{\sigma_i(j),\forall\, i \text{ and } \text{fixed } j\}=\Lambda_n$, i.e., $\{\sigma_1(j),\ldots,\sigma_n(j)\}=\Lambda_n$, for fixed j. Let $S,T:X^n\to CB(X)$ and $g:X\to X$ be mappings such that S and T have the mixed weakly $g-$monotone property on X, $S(X^n)\cup T(X^n)\subseteq g(X)$, $T(or\ S)(x_1,\ldots,x_n)\ \subseteq\ T(or\ S)(gx_1,\ldots,gx_n)$, $\forall$ $(x_1,\ldots,x_n)\in X$ and $g(X)$ is complete subspace of X. Assume that there exist non-negative real numbers $a_i\in[0,1)$, $1\leq i\leq n+4$ such that $\sum_{i=1}^{n+4}a_i\leq h<1$ and*

$$\begin{aligned}H(S(x^{\sigma_i(1)},\ \ldots,x^{\sigma_i(n)}),T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}))&\leq\sum_{j=1}^{n}a_jd(gx^{\sigma_i(j)},gu^{\sigma_i(j)})\\&+a_{n+1}d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),gx^i)+a_{n+2}d(T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gu^i)\\&+a_{n+3}d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),gu^i)+a_{n+4}d(T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gx^i),\end{aligned}\tag{3.1}$$

for only comparable elements $(x^1,\ldots,x^n)$ and $(u^1,\ldots,u^n)$ in X^n. Also assume that X has the sequential monotone property. If there exist $x_0^i\in X$, $i\in\Lambda_n$ with $\{gx_0^i\}\leq_i S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$ or $\{gx_0^i\}\leq_i T(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$, then (S,g) and (T,g) have $n-$coincidence point in X. Furthermore, (S,g) and (T,g) have $n-$common fixed point if one of the following conditions holds for some $(x^1,\ldots,x^n)\in\gamma(S,g)\cap\gamma(T,g)$ and $(u^1,\ldots,u^n)\in X^n$:

(a) (S,g) and (T,g) are $w-$compatible, $\lim_{m\to\infty}g^mx^i=u^i$, $\forall\, i$ and g is continuous at u^i, $\forall i$;

(b) if $g^2x^i=gx^i,\forall i$ and g is S,T idempotent;

(c) g is continuous at x^i and $\lim_{m\to\infty}g^mu^i=x^i$, $\forall\, i$.

Proof. Consider $\{gx_0^i\}\leq_i S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$. Using the mixed weakly $g-$monotonicity for S and T yields

$$S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})\leq_i T(S(x_0^{\sigma_{\sigma_i(1)}(1)},\ldots,x_0^{\sigma_{\sigma_i(1)}(n)}),\ldots,S(x_0^{\sigma_{\sigma_i(n)}(1)},\ldots,x_0^{\sigma_{\sigma_i(n)}(n)})).$$

Then for $gx_1^i\in S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$, we have

$$\begin{aligned} S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)}) &\leq_i T(gx_1^{\sigma_i(1)},\ldots,gx_1^{\sigma_i(n)}) \supseteq T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)}) \\ \Rightarrow \{gx_1^i\} &\leq_i T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)}), \end{aligned} \tag{3.2}$$

and then,

$$T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)}) \leq_i S(T(x_1^{\sigma_{\sigma_i(1)}(1)},\ldots,x_1^{\sigma_{\sigma_i(1)}(n)}),\ldots,T(x_1^{\sigma_{\sigma_i(n)}(1)},\ldots,x_1^{\sigma_{\sigma_i(n)}(n)})).$$

That is, for $gx_2^i \in T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})$, we have

$$\begin{aligned} T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)}) &\leq_i S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)}) \\ \Rightarrow \{gx_2^i\} &\leq_i S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)}). \end{aligned} \tag{3.3}$$

For $gx_1^i \in \{gx_1^i\}$ and $gx_2^i \in T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})$, by using (3.2) we get

$$gx_1^i \leq_i gx_2^i. \tag{3.4}$$

Also, by (3.3), for $gx_2^i \in \{gx_2^i\}$ and $gx_3^i \in S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})$ we have

$$gx_2^i \leq_i gx_3^i. \tag{3.5}$$

Continuing in this way, we can construct n sequences $\{gx_m^i\}$ in X for which

$$gx_{2m+1}^i \in S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}),\ gx_{2m+2}^i \in T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)}) \tag{3.6}$$

and

$$gx_m^i \leq_i gx_{m+1}^i, \forall\ m \text{ and } i. \tag{3.7}$$

If $a_i = 0$ for all $i \in \Lambda_{n+4}$ then from (3.1)

$$\begin{aligned} d(gx_1^i,T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})) &\leq H(S(x_0^{\sigma_i(1)},\ \ldots,x_0^{\sigma_i(n)}),T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})) = 0, \\ d(gx_2^i,S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})) &\leq H(T(x_1^{\sigma_i(1)},\ \ldots,x_1^{\sigma_i(n)}),S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})) = 0. \end{aligned}$$

Imply that

$$gx_1^i \in \overline{T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})} = T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})$$

and

$$gx_2^i \in \overline{S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})} = S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})$$

for all $i \in \Lambda_n$. Hence $(x_1^1,\ldots,x_1^n)$ and $(x_2^1,\ldots,x_2^n)$ are $n-$coincidence points of pairs (T,g) and (S,g), respectively. So we assume that $a_i > 0$ for some i in $\{1,\ldots,n+4\}$ which gives that $0 < h < 1$.

Now we apply Lemma 2.2 and then contraction condition (3.1) to can say that for $gx_{2m+1}^i \in S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)})$ there exist $gx_{2m+2}^i \in T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)})$ such that for all i and m,

$$
\begin{aligned}
d(gx^i_{2m+1}, gx^i_{2m+2}) &\le H(S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}), T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)})) + \frac{h^{2m+1}}{2n} \\
&\le \sum_{j=1}^{n} a_j d(gx_{2m}^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)}) + a_{n+1} d(S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}), gx^i_{2m}) \\
&+ a_{n+2} d(T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)}), gx^i_{2m+1}) + a_{n+3} d(S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}), gx^i_{2m+1}) \\
&+ a_{n+4} d(T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)}), gx^i_{2m}) + \frac{h^{2m+1}}{2n} \\
&\le \sum_{j=1}^{n} a_j d(gx_{2m}^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)}) \\
&+ a_{n+1} d(gx^i_{2m+1}, gx^i_{2m}) + a_{n+2} d(gx^i_{2m+2}, gx^i_{2m+1}) \\
&+ a_{n+3} d(gx^i_{2m+1}, gx^i_{2m+1}) + a_{n+4} d(gx^i_{2m+2}, gx^i_{2m}) + \frac{h^{2m+1}}{2n}
\end{aligned} \tag{3.8}
$$

$$
(1 - a_{n+2} - a_{n+4}) \quad d(gx^i_{2m+1}, gx^i_{2m+2}) \le \sum_{j=1}^{n} a_j d(gx_{2m}^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)}) + (a_{n+1} + a_{n+4}) d(gx^i_{2m+1}, gx^i_{2m}) + \frac{h^{2m+1}}{2n}
$$

Now we interchange the role of S and T in (3.1) to get

$$
d(gx^i_{2m+2}, gx^i_{2m+1}) \le H(T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)}), S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)})) + \frac{h^{2m+1}}{2n}
$$

$$
(1 - a_{n+1} - a_{n+3}) d(gx^i_{2m+1}, gx^i_{2m+2}) \le \sum_{j=1}^{n} a_j d(gx_{2m}^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)}) + (a_{n+2} + a_{n+3}) d(gx^i_{2m+1}, gx^i_{2m}) + \frac{h^{2m+1}}{2n} \tag{3.9}
$$

Adding over all i in (3.8) and (3.9), this gives

$$
\begin{aligned}
(1 - a_{n+2} - a_{n+4}) \quad & \sum_i d(gx^i_{2m+1}, gx^i_{2m+2}) \le \sum_i \left(\sum_j a_j d(gx_{2m}^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)})\right) \\
& + (a_{n+1} + a_{n+4}) \sum_i d(gx^i_{2m+1}, gx^i_{2m}) + \frac{h^{2m+1}}{2}
\end{aligned} \tag{3.10}
$$

$$
(1 - a_{n+2} - a_{n+4}) \quad \delta_{2m+1} \le \left(\sum_i a_i + a_{n+1} + a_{n+4}\right)\delta_{2m} + \frac{h^{2m+1}}{2}
$$

and

$$
(1 - a_{n+1} - a_{n+3}) \quad \delta_{2m+1} \le \left(\sum_i a_i + a_{n+2} + a_{n+3}\right)\delta_{2m} + \frac{h^{2m+1}}{2} \tag{3.11}
$$

Again, from (3.10) and (3.11)

$$
\begin{aligned}
&(2 - a_{n+1} - a_{n+2} - a_{n+3} - a_{n+4})\delta_{2m+1} \le \left(2\sum_i a_i + a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}\right)\delta_{2m} + h^{2m+1} \\
&\delta_{2m+1} \le \frac{2\sum_i a_i + a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}}{2 - a_{n+1} - a_{n+2} - a_{n+3} - a_{n+4}}\delta_{2m} + \frac{h^{2m+1}}{2 - a_{n+1} - a_{n+2} - a_{n+3} - a_{n+4}}.
\end{aligned} \tag{3.12}
$$

Since,

$$
\begin{aligned}
&\sum_i a_i + a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} \le h < 1 \Rightarrow \\
&2\sum_i a_i + a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} \le 2h - (a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}) \\
&\le 2h - h(a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}) = h(2 - a_{n+1} - a_{n+2} - a_{n+3} - a_{n+4}) \Rightarrow \\
&\frac{2\sum_i a_i + a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4}}{2 - a_{n+1} - a_{n+2} - a_{n+3} - a_{n+4}} \le h,
\end{aligned}
$$

and $a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}<1$ then $2-a_{n+1}-a_{n+2}-a_{n+3}-a_{n+4}>1$, that is $\frac{1}{2-a_{n+1}-a_{n+2}-a_{n+3}-a_{n+4}}<1$. Thus from (3.12), we obtain

$$\delta_{2m+1}\leq h\delta_{2m}+h^{2m+1}. \tag{3.13}$$

By similar way as above, applying Lemma 2.2 and then contraction condition (3.1) yield that for $gx^i_{2m+2}\in T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1})$ there exist $gx^i_{2m+3}\in S(x^{\sigma_i(1)}_{2m+2},\ldots,x^{\sigma_i(n)}_{2m+2})$ such that for all i and all m

$$\begin{aligned} d(gx^i_{2m+2},gx^i_{2m+3}) &\leq H(T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1}),S(x^{\sigma_i(1)}_{2m+2},\ldots,x^{\sigma_i(n)}_{2m+2}))+\frac{h^{2m+2}}{2n}\\ &\leq\sum_{j=1}^{n}a_j d(gx^{\sigma_i(j)}_{2m+1},gx^{\sigma_i(j)}_{2m+2})+a_{n+1}d(gx^i_{2m+2},gx^i_{2m+1})\\ &\quad+a_{n+2}d(gx^i_{2m+3},gx^i_{2m+2})+a_{n+3}d(gx^i_{2m+2},gx^i_{2m+2})\\ &\quad+a_{n+4}d(gx^i_{2m+3},gx^i_{2m+1})+\frac{h^{2m+2}}{2n}\end{aligned} \tag{3.14}$$

$$(1-a_{n+2}-a_{n+4})\quad d(gx^i_{2m+2},gx^i_{2m+3})\leq\sum_{j=1}^{n}a_j d(gx^{\sigma_i(j)}_{2m+1},gx^{\sigma_i(j)}_{2m+2})+(a_{n+1}+a_{n+4})d(gx^i_{2m+2},gx^i_{2m+1})+\frac{h^{2m+2}}{2n}.$$

Also, we have

$$\begin{aligned} d(gx^i_{2m+3},gx^i_{2m+2}) &\leq H(S(x^{\sigma_i(1)}_{2m+2},\ldots,x^{\sigma_i(n)}_{2m+2}),T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1}))+\frac{h^{2m+2}}{2n}\\ &\leq\sum_{j=1}^{n}a_j d(gx^{\sigma_i(j)}_{2m+2},gx^{\sigma_i(j)}_{2m+1})+a_{n+1}d(gx^i_{2m+3},gx^i_{2m+2})\\ &\quad+a_{n+2}d(gx^i_{2m+2},gx^i_{2m+1})+a_{n+3}d(gx^i_{2m+3},gx^i_{2m+1})\\ &\quad+a_{n+4}d(gx^i_{2m+2},gx^i_{2m+2})+\frac{h^{2m+2}}{2n}\end{aligned} \tag{3.15}$$

$$(1-a_{n+1}-a_{n+3})\quad d(gx^i_{2m+2},gx^i_{2m+3})\leq\sum_{j=1}^{n}a_j d(gx^{\sigma_i(j)}_{2m+1},gx^{\sigma_i(j)}_{2m+2})+(a_{n+2}+a_{n+3})d(gx^i_{2m+1},gx^i_{2m+2})+\frac{h^{2m+2}}{2n}.$$

Adding over all i in (3.14) and (3.15), this gives

$$\begin{aligned}(1-a_{n+2}-a_{n+4})\quad\sum_i d(gx^i_{2m+2},gx^i_{2m+3}) &\leq\sum_j a_j\sum_i d(gx^{\sigma_i(j)}_{2m+1},gx^{\sigma_i(j)}_{2m+2})\\ &\quad+(a_{n+1}+a_{n+4})\sum_i d(gx^i_{2m+1},gx^i_{2m+2})+\frac{h^{2m+2}}{2}\\ (1-a_{n+2}-a_{n+4})\quad\delta_{2m+2} &\leq(\sum_i a_i+a_{n+1}+a_{n+4})\delta_{2m+1}+\frac{h^{2m+2}}{2}\end{aligned} \tag{3.16}$$

and

$$(1-a_{n+1}-a_{n+3})\quad\delta_{2m+2}\leq(\sum_i a_i+a_{n+2}+a_{n+3})\delta_{2m+1}+\frac{h^{2m+2}}{2}. \tag{3.17}$$

Again, from (3.16) and (3.17)

$$\begin{aligned}\delta_{2m+2} &\leq\frac{2\sum_i a_i+a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}}{2-a_{n+1}-a_{n+2}-a_{n+3}-a_{n+4}}\delta_{2m+1}+\frac{h^{2m+2}}{2-a_{n+1}-a_{n+2}-a_{n+3}-a_{n+4}}\ .\\ &\leq h\delta_{2m+1}+h^{2m+2}.\end{aligned} \tag{3.18}$$

From (3.13) and (3.18)

$$\delta_{m+1}\leq h\delta_m+h^{m+1}. \tag{3.19}$$

Continuing this process, we obtain

$$\begin{aligned}\delta_{m+1} &\le h\delta_m + h^{m+1} \\ &\le h(h\delta_{m-1} + h^{m-1}) + h^{m+1} \\ &\le h^2(h\delta_{m-2} + h^{m-2}) + 2h^{m+1} \\ &\vdots \\ &\le h^{m+1}\delta_0 + (m+1)h^{m+1}.\end{aligned} \tag{3.20}$$

By repeatedly use of triangle inequality, for every $m, p \in N$ with $p > m$, we obtain

$$\begin{aligned}\sum_i d(gx_m^i, gx_p^i) &\le \sum_i [d(gx_m^i, gx_{m+1}^i) + \ldots + d(gx_{p-1}^i, gx_p^i)] \\ &\le h^m\delta_0 + (m)h^m + \ldots + h^{p-1}\delta_0 + (p-1)h^{p-1} \\ &\le \sum_{l=m}^{p-1} [h^l\delta_0 + lh^l].\end{aligned}$$

Since $h < 1$, we conclude that $\{gx_m^i\}$ are Cauchy sequences in $g(X)$ which is complete then there exist $x^i \in X$ such that

$$gx_m^i \to gx^i \text{ as } m \to \infty. \tag{3.21}$$

Using (3.7) and (3.21) and having in mind the properties on X then $gx_n^i \le_i gx^i$. From (3.1) we get

$$\begin{aligned}d(gx^i, T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) &\le d(gx^i, gx_{2m+1}^i) + H(S(x_{2m}^{\sigma_i(1)}, \ldots, x_{2m}^{\sigma_i(n)}), T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) \\ &\le d(gx^i, gx_{2m+1}^i) + \sum_{j=1}^n a_j d(gx_{2m}^{\sigma_i(j)}, gx^{\sigma_i(j)}) \\ &\quad + a_{n+1} d(S(x_{2m}^{\sigma_i(1)}, \ldots, x_{2m}^{\sigma_i(n)}), gx_{2m}^i) + a_{n+2} d(T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i) \\ &\quad + a_{n+3} d(S(x_{2m}^{\sigma_i(1)}, \ldots, x_{2m}^{\sigma_i(n)}), gx^i) + a_{n+4} d(T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx_{2m}^i) \\ &\le d(gx^i, gx_{2m+1}^i) + \sum_{j=1}^n a_j d(gx_{2m}^{\sigma_i(j)}, gx^{\sigma_i(j)}) \\ &\quad + a_{n+1} d(gx_{2m+1}^i, gx_{2m}^i) + a_{n+2} d(T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i) \\ &\quad + a_{n+3} d(gx_{2m+1}^i, gx^i) + a_{n+4} d(T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx_{2m}^i).\end{aligned} \tag{3.22}$$

On taking limit as $m \to \infty$, we get

$$d(gx^i, T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) \le (a_{n+2} + a_{n+4}) d(T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx_{2m}^i)$$

which yields to $d(gx^i, T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})) = 0$ and $gx^i \in T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), \forall\, i$. Similarly we have

$$\begin{aligned}d(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i) &\le H(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), T(x_{2m+1}^{\sigma_i(1)}, \ldots, x_{2m+1}^{\sigma_i(n)})) + d(gx_{2m+2}^i, gx^i) \\ &\le d(gx_{2m+2}^i, gx^i) + \sum_{j=1}^n a_j d(gx^{\sigma_i(j)}, gx_{2m+1}^{\sigma_i(j)}) \\ &\quad + a_{n+1} d(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i) + a_{n+2} d(gx_{2m+2}, gx_{2m+1}^i) \\ &\quad + a_{n+3} d(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx_{2m+1}^i) + a_{n+4} d(gx_{2m+2}, gx^i).\end{aligned} \tag{3.23}$$

As $m \to \infty$, we have

$$d(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i) \le (a_{n+1} + a_{n+3}) d(S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}), gx^i).$$

Thus, $(x^1, \ldots, x^n)$ is an $n-$ coincidence point for the pairs (S, g) and (T, g).

Suppose that (a) holds, by induction we can prove that $(g^k x^1, \ldots, g^k x^n) \in \gamma(S, g) \cap \gamma(T, g)$ for all $k \ge 0$. Since S and T are $w-$compatible and $(x^1, \ldots, x^n) \in \gamma(S, g) \cap \gamma(T, g)$ then we have for all i,

$$gx^i \in S(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)}),\ gx^i \in T(x^{\sigma_i(1)}, \ldots, x^{\sigma_i(n)})$$

and

$$g(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}))\subseteq S(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}),$$
$$g(T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}))\subseteq T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}).$$

So

$$g^2x^i\in S(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}),\ \ g^2x^i\in T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}).$$

That is $(gx^1,\ldots,gx^n)\in\gamma(S,g)\cap\gamma(T,g)$. If $(g^kx^1,\ldots,g^kx^n)\in\gamma(S,g)\cap\gamma(T,g)$ for some $k\geq 0$ then we get for all i,

$$g^{k+1}x^i\in S(g^kx^{\sigma_i(1)},\ldots,g^kx^{\sigma_i(n)}),\ \ g^{k+1}x^i\in T(g^kx^{\sigma_i(1)},\ldots,g^kx^{\sigma_i(n)}) \tag{3.24}$$

and

$$\begin{aligned} g(S(g^kx^{\sigma_i(1)},\ldots,g^kx^{\sigma_i(n)})) &\subseteq S(g^{k+1}x^{\sigma_i(1)},\ldots,g^{k+1}x^{\sigma_i(n)}),\\ g(T(g^kx^{\sigma_i(1)},\ldots,g^kx^{\sigma_i(n)})) &\subseteq T(g^{k+1}x^{\sigma_i(1)},\ldots,g^{k+1}x^{\sigma_i(n)}).\end{aligned}$$

So

$$g^{k+2}x^i\in S(g^{k+1}x^{\sigma_i(1)},\ldots,g^{k+1}x^{\sigma_i(n)}),\ \ g^{k+2}x^i\in T(g^{k+1}x^{\sigma_i(1)},\ldots,g^{k+1}x^{\sigma_i(n)}).$$

That is $(g^mx^1,\ldots,g^mx^n)\in\gamma(S,g)\cap\gamma(T,g),\ \forall\ m$. Also, as $g^mx^i\to u^i$ and g is continuous at u^i, $\forall\ i$, these give

$$u^i=\lim_{m\to\infty}g^{m+1}x^i=\lim_{g^mx^i\to u^i}g(g^mx^i)=gu^i.$$

Now we prove that $\{g^mx^i\}$ are monotone sequence (decreasing or increasing) for all i. Since $gx^i\in S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})$ then $\{gx^i\}\leq_i S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})$, which implies

$$S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})\leq_i T(S(x^{\sigma_{\sigma_i(1)}(1)},\ldots,x^{\sigma_{\sigma_i(1)}(n)}),\ldots,S(x^{\sigma_{\sigma_i(n)}(1)},\ldots,x^{\sigma_{\sigma_i(n)}(n)})). \tag{3.25}$$

Then by (3.24) and (3.25)

$$\begin{aligned} S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}) &\leq_i T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)})\\ \Rightarrow\{gx^i\} &\leq_i T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)})\\ \Rightarrow gx^i &\leq_i g^2x^i.\end{aligned} \tag{3.26}$$

Again by monotonicity for S and T we have

$$T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)})\leq_i S(T(gx^{\sigma_{\sigma_i(1)}(1)},\ldots,gx^{\sigma_{\sigma_i(1)}(n)}),\ldots,T(gx^{\sigma_{\sigma_i(n)}(1)},\ldots,gx^{\sigma_{\sigma_i(n)}(n)})). \tag{3.27}$$

From (3.24) and (3.27)

$$\begin{aligned} T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}) &\leq_i S(g^2x^{\sigma_i(1)},\ldots,g^2x^{\sigma_i(n)})\\ \Rightarrow\{g^2x^i\} &\leq_i S(g^2x^{\sigma_i(1)},\ldots,g^2x^{\sigma_i(n)})\\ \Rightarrow g^2x^i &\leq_i g^3x^i.\end{aligned} \tag{3.28}$$

Continuing in this way we can get

$$g^mx^i\circ_i g^{m+1}x^i\ \forall\ m\ and\ i. \tag{3.29}$$

Therefore, for each i, $\{g^mx^i\}$ is monotone sequence and converges to u^i then $g^mx^i\leq_i gu^i$.

Now we apply (3.1) with $x=u$ and $u=g^{m-1}x$

$$\begin{aligned} d(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gu^i) &\leq H(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),T(g^{m-1}x^{\sigma_i(1)},\ldots,g^{m-1}x^{\sigma_i(n)}))+d(g^mx^i,gu^i))\\ &\leq\sum_{j=1}^n a_jd(gu^{\sigma_i(j)},g^mx^{\sigma_i(j)})\\ &+a_{n+1}d(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gu^i)+a_{n+2}d(g^mx^i,g^{m-1}x^i)\\ &+a_{n+3}d(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),g^{m-1}x^i)+a_{n+4}d(g^mx^i,gu^i)\to 0\ as\ m\to\infty.\end{aligned} \tag{3.30}$$

Hence $gu^i\in S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)})$. Consequently,

$$u^i = gu^i \in S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}).$$

Similarly

$$u^i = gu^i \in T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}).$$

Suppose that (b) holds. Since g is S,T idempotent for some $(x^1,\ldots,x^n) \in \gamma(S,g) \cap \gamma(T,g)$ and $g^2x^i = gx^i$, we have

$$gx^i = g^2x^i \in S(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)})$$

and

$$gx^i = g^2x^i \in T(gx^{\sigma_i(1)},\ldots,gx^{\sigma_i(n)}).$$

Finally suppose that (c) holds. Since g is continuous at x^i for some $(x^1,\ldots,x^n) \in \gamma(S,g) \cap \gamma(T,g)$ and $\lim_{n\to\infty} g^m u^i = x^i, \forall\, i$ then

$$x^i = \lim_{n\to\infty} g^{m+1}u^i = \lim_{g^m u^i \to x^i} g(g^m u^i) = gx^i.$$

Hence

$$x^i = gx^i \in S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})$$

and

$$x^i = gx^i \in T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}).$$

Corollary 3.1 *Let $(X,d,\leq)$ be an ordered metric space, $S,T : X^3 \to CB(X)$ and $g : X \to X$ be mappings such that S and T have the mixed weakly $g-$monotone property on X, $S(X^3) \cup T(X^3) \subseteq g(X)$, $T(or\ S)(x,y,z) \subseteq T(or\ S)(gx,gy,gz)$, for any $x,y,z \in X$ and $g(X)$ is complete subspace of X. Assume that there exist non-negative real numbers $a_i, i = 1,2,\ldots,7$ such that $\sum_{i=1}^{7} a_i \leq h < 1$ and*

$$\begin{aligned} H(S(x,y,z),T(u,v,w)) \quad &\leq a_1 d(gx,gu) + a_2 d(gy,gv) + a_3 d(gz,gw) \\ &+ a_4 d(S(x,y,z),gx) + a_5 d(T(u,v,w),gu) \\ &+ a_6 d(S(x,y,z),gu) + a_7 d(T(u,v,w),gx), \end{aligned} \tag{3.31}$$

for all $x,y,z,u,v,w \in X$, where $gx \leq gu$, $gy \geq gv$, $gz \leq gw$. If there exist $x_0,y_0,z_0 \in X$ with

[$\{gx_0\} \leq S(x_0,y_0,z_0)$, $\{gy_0\} \geq S(y_0,z_0,x_0)$ and $\{gz_0\} \leq S(z_0,x_0,y_0)$] or

[$\{gx_0\} \leq T(x_0,y_0,z_0)$, $\{gy_0\} \geq T(y_0,z_0,x_0)$ and $\{gz_0\} \leq T(z_0,x_0,y_0)$].

Then (S,g) and (T,g) have tripled coincidence point, that is there exist $(x,y,z) \in X^3$ such that $gx \in F(x,y,z)$, $gy \in F(y,z,x)$ and $gz \in F(z,x,y)$.

Moreover (S,g) and (T,g) have tripled common fixed point (i.e., $\exists (x,y,z) \in X^3$ such that $x = gx \in F(x,y,z)$, $y = gy \in F(y,z,x)$ and $z = gz \in F(z,x,y)$) if one of the following conditions holds for some $(x,y,z) \in \gamma(S,g) \cap \gamma(T,g)$ and $u,v,w \in X$:

a) (S,g) and (T,g) are $w-$compatible, $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and $\lim_{n\to\infty} g^n z = w$ and g is continuous at u,v,w;

b) if $g^2x = gx$, $g^2y = gy$ and $g^2z = gz$ and g is S,T idempotent;

c) g is continuous at x,y,z and $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$, $\lim_{n\to\infty} g^n w = z$.

Proof. Consider $n = 3$, A is the set of odd numbers in Λ_3 and B is the set of even numbers. Define $(\sigma_1,\sigma_2,\sigma_3) : \Lambda_3 \to \Lambda_3$ by

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\ and\ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Hence, Corollary 3.1 follows from Theorem 3.1 and the definition of tripled coincidence and common fixed point follows directly from Definition 2.13.

Remark 3.1 *Corollary 3.1 is the ordered version of Theorem 7 of Kutbi et al. [12]. Note that Theorem 2.1 is not useful here because under previous choice for* σ_2 *and* σ_3 *we have* σ_i *is not in* $\Omega_{A,B}$ *or in* $\Omega'_{A,B}$.

Theorem 3.2 *Let* $(X,d,\leq)$ *be an ordered metric space and* $(\sigma_1,\ldots,\sigma_n)$ *be* $n-$*tuple of mappings from* Λ_n *into itself. Let* $S,T: X^n \to CB(X)$ *and* $g: X \to X$ *be mappings such that* S *and* T *have the mixed weakly* $g-$*monotone property on* X, $S(X^n)\cup T(X^n)\subseteq g(X)$, $T(or\ S)(x_1,\ldots,x_n) \subseteq T(or\ S)(gx_1,\ldots,gx_n)$, $\forall$ $(x_1,\ldots,x_n)\in X$ *and* $g(X)$ *is complete subspace of* X. *Assume that there exists a non-negative real number* $h<1$ *such that*

$$\begin{aligned} H(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),\quad & T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)})) \leq h\max\{d(gx^{\sigma_i(1)},gu^{\sigma_i(1)}),\ldots,d(gx^{\sigma_i(n)},gu^{\sigma_i(n)}),\\ & d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),gx^i), d(T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gu^i),\\ & \frac{d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}),gu^i)+d(T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}),gx^i)}{2}\} \end{aligned} \tag{3.32}$$

for $(x^1,\ldots,x^n)$ and $(u^1,\ldots,u^n)$ in X^n, where $gx^i \leq_i gu^i$. Also assume that X has the sequential monotone property. If there exist $x_0^i \in X, i\in\Lambda_n$ with $\{gx_0^i\}\leq_i S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$ or $\{gx_0^i\}\leq_i T(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$, then (S,g) and (T,g) have $n-$coincidence point in X. Furthermore, (S,g) and (T,g) have $n-$common fixed point if one of the conditions (a), (b) or (c) of Theorem 3.1 holds.

Proof. As in Theorem 3.1, we begin with $\{gx_0^i\}\leq_i S(x_0^{\sigma_i(1)},\ldots,x_0^{\sigma_i(n)})$ and use the mixed weakly $g-$monotone property for S and T to construct sequences $\{gx_m^i\}$ for all i while

$$gx_{2m+1}^i \in S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}),\ \ gx_{2m+2}^i \in T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)}) \tag{3.33}$$

and

$$gx_m^i \leq_i gx_{m+1}^i, \forall\ m\ and\ i. \tag{3.34}$$

If $h=0$ and (3.32), then

$$\begin{aligned} d(gx_1^i,T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})) &\leq H(S(x_0^{\sigma_i(1)},\ \ldots,x_0^{\sigma_i(n)}),T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)}))=0,\\ d(gx_2^i,S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)})) &\leq H(T(x_1^{\sigma_i(1)},\ \ldots,x_1^{\sigma_i(n)}),S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)}))=0. \end{aligned}$$

Imply that

$$gx_1^i \in T(x_1^{\sigma_i(1)},\ldots,x_1^{\sigma_i(n)})\ \ and\ \ gx_2^i \in S(x_2^{\sigma_i(1)},\ldots,x_2^{\sigma_i(n)}),\ \forall\ i.$$

Hence $(x_1^1,\ldots,x_1^n)$ and $(x_2^1,\ldots,x_2^n)$ are $n-$coincidence points of pairs (T,g) and (S,g), respectively. So we assume that $h>0$ and $k=\dfrac{1}{\sqrt{h}}>1.$

Now we apply Lemma 2.1 and then contraction condition (3.32) to can say that for $gx_{2m+1}^i \in S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)})$ there exist $gx_{2m+2}^i \in T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)})$ such that for all i and m

$$\begin{aligned} d(gx^i_{2m+1}, gx^i_{2m+2}) &\le kH(S(x^{\sigma_i(1)}_{2m},\ldots,x^{\sigma_i(n)}_{2m}), T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1})) \le \sqrt{h} \\ &\max\{d(gx^{\sigma_i(1)}_{2m}, gx^{\sigma_i(1)}_{2m+1}),\ldots,d(gx^{\sigma_i(n)}_{2m}, gx^{\sigma_i(n)}_{2m+1}), \\ &d(S(x^{\sigma_i(1)}_{2m},\ldots,x^{\sigma_i(n)}_{2m}), gx^i_{2m}), d(T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1}), gx^i_{2m+1}), \\ &\frac{d(S(x^{\sigma_i(1)}_{2m},\ldots,x^{\sigma_i(n)}_{2m}), gx^i_{2m+1}) + d(T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1}), gx^i_{2m})}{2}\} \\ &\le \sqrt{h}\max\{d(gx^{\sigma_i(1)}_{2m}, gx^{\sigma_i(1)}_{2m+1}),\ldots,d(gx^{\sigma_i(n)}_{2m}, gx^{\sigma_i(n)}_{2m+1}), \\ &d(gx^i_{2m+1}, gx^i_{2m}), d(gx^i_{2m+2}, gx^i_{2m+1}), \frac{d(gx^i_{2m+1}, gx^i_{2m+1}) + d(gx^i_{2m+2}, gx^i_{2m})}{2}\} \\ &\le \sqrt{h}\max\{d(gx^1_{2m}, gx^1_{2m+1}),\ldots,d(gx^n_{2m}, gx^n_{2m+1}), d(gx^i_{2m+2}, gx^i_{2m+1})\}. \end{aligned} \tag{3.35}$$

If $\max\{d(gx^1_{2m}, gx^1_{2m+1}),\ldots,d(gx^n_{2m}, gx^n_{2m+1}), d(gx^i_{2m+2}, gx^i_{2m+1})\} = d(gx^i_{2m+2}, gx^i_{2m+1})$, then (3.35) $\Rightarrow$ $d(gx^i_{2m+1}, gx^i_{2m+2}) = 0$, i.e., $gx^i_{2m+1} \in T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1})$, $\forall\, i$. Therefore,

$$d(gx^i_{2m+1}, gx^i_{2m+2}) \le \sqrt{h}\max\{d(gx^1_{2m}, gx^1_{2m+1}),\ldots,d(gx^n_{2m}, gx^n_{2m+1})\}. \tag{3.36}$$

Also for $gx^i_{2m+2} \in T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1})$ there exist $gx^i_{2m+3} \in S(x^{\sigma_i(1)}_{2m+2},\ldots,x^{\sigma_i(n)}_{2m+2})$ such that for all i and m

$$\begin{aligned} d(gx^i_{2m+2}, gx^i_{2m+3}) &\le kH(T(x^{\sigma_i(1)}_{2m+1},\ldots,x^{\sigma_i(n)}_{2m+1}), S(x^{\sigma_i(1)}_{2m+2},\ldots,x^{\sigma_i(n)}_{2m+2})) \\ &\le \sqrt{h}\max\{d(gx^{\sigma_i(1)}_{2m+1}, gx^{\sigma_i(1)}_{2m+2}),\ldots,d(gx^{\sigma_i(n)}_{2m+1}, gx^{\sigma_i(n)}_{2m+2}), \\ &d(gx^i_{2m+2}, gx^i_{2m+1}), d(gx^i_{2m+3}, gx^i_{2m+2}), \frac{d(gx^i_{2m+2}, gx^i_{2m+2}) + d(gx^i_{2m+3}, gx^i_{2m+1})}{2}\} \\ &\le \sqrt{h}\max\{d(gx^1_{2m+1}, gx^1_{2m+2}),\ldots,d(gx^n_{2m+1}, gx^n_{2m+2})\}. \end{aligned} \tag{3.37}$$

Using (3.36) and (3.37) we get for all i

$$d(gx^i_m, gx^i_{m+1}) \le \sqrt{h}\max\{d(gx^1_{m-1}, gx^1_m),\ldots,d(gx^n_{m-1}, gx^n_m)\} = \sqrt{h}\delta_{m-1}. \tag{3.38}$$

Consider $\delta_m = \max\{d(gx^1_m, gx^1_{m+1}),\ldots,d(gx^n_m, gx^n_{m+1})\} = d(gx^i_m, gx^i_{m+1})$ for some $i \in \Lambda_n$. Thus from (3.38)

$$\begin{aligned} \delta_m &\le \sqrt{h}\delta_{m-1} \\ &\le (\sqrt{h})^2\delta_{m-2} \\ &\vdots \\ &\le (\sqrt{h})^m\delta_0. \end{aligned} \tag{3.39}$$

For $m, p \in N$ with $p > m$, we have

$$\begin{aligned} d(gx^i_m, gx^i_p) &\le [d(gx^i_m, gx^i_{m+1}) + \ldots + d(gx^i_{p-1}, gx^i_p)] \\ &\le [(\sqrt{h})^m + \ldots + (\sqrt{h})^{p-1}]\delta_0 \\ &\le \sum_{i=m}^{p-1}(\sqrt{h})^i\delta_0. \end{aligned}$$

Since $h < 1$, we conclude that $\{gx^i_m\}$ are Cauchy sequences in $g(X)$ which is complete then there exist $x^i \in X$ such that

$$gx^i_m \to gx^i \ as\ m \to \infty. \tag{3.40}$$

Using the sequential property on X, (3.32), (3.33), (3.34) and (3.40), all will give $gx^i_m \le_i gx^i$ and

$$\begin{aligned} d(gx^i, \quad T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})) &\le d(gx^i, gx^i_{2m+1}) + H(S(x_{2m}^{\sigma_i(1)},\ldots,x_{2m}^{\sigma_i(n)}), T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})) \\ &\le d(gx^i, gx^i_{2m+1}) + h\max\{d(gx_{2m}^{\sigma_i(1)}, gx^{\sigma_i(1)}),\ldots,d(gx_{2m}^{\sigma_i(n)}, gx^{\sigma_i(n)}), d(gx^i_{2m+1}, gx^i_{2m}), \\ &\quad d(T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), gx^i), \frac{d(gx^i_{2m+1}, gx^i) + d(T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), gx^i_{2m})}{2}\}. \end{aligned} \tag{3.41}$$

By taking limit as $m \to \infty$, we get

$$d(gx^i, T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)})) = 0.$$

Hence, $gx^i \in T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), \forall\, i$. Also we have

$$\begin{aligned} d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), gx^i) &\le H(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), T(x_{2m+1}^{\sigma_i(1)},\ldots,x_{2m+1}^{\sigma_i(n)})) + d(gx^i_{2m+2}, gx^i) \\ &\le h\max\{d(gx^1, gx^1_{2m+1}),\ldots,d(gx^n, gx^n_{2m+1}), \\ &\quad d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), gx^i), d(gx^i_{2m+2}, gx^i_{2m+1}), \\ &\quad \frac{d(S(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), gx^i_{2m+1}) + d(gx_{2m+2}, gx^i)}{2}\}. \end{aligned} \tag{3.42}$$

At $n \to \infty$, we obtain $gx^i \in T(x^{\sigma_i(1)},\ldots,x^{\sigma_i(n)}), \forall\, i.$ Thus $(x^1,\ldots,x^n)$ is a $n-$ coincidence point for the pairs (S,g) and (T,g).

Suppose that (a) holds, by Theorem 3.1 we have, $u^i = gu^i, \forall\, i$ and $(g^m x^1,\ldots,g^m x^n) \in \gamma(S,g) \cap \gamma(T,g), \forall\, m$. Hence,

$$g^{m+1}x^i \in S(g^m x^{\sigma_i(1)},\ldots,g^m x^{\sigma_i(n)}),\ \ g^{m+1}x^i \in T(g^m x^{\sigma_i(1)},\ldots,g^m x^{\sigma_i(n)}). \tag{3.43}$$

Also we can get

$$g^m x^i \le_i g^{m+1}x^i, \forall\, m \text{ and } i. \tag{3.44}$$

Therefore, $\{g^k x^i\}$ is monotone sequence and converges to u^i then $g^k x^i \le_i gu^i$.

Now we apply (3.32) with $x = u$ and $u = g^{m-1}x$

$$\begin{aligned} d(S(u^{\sigma_i(1)}, \quad \ldots,u^{\sigma_i(n)}), gu^i) &\le H(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}), T(g^{m-1}x^{\sigma_i(1)},\ldots,g^{m-1}x^{\sigma_i(n)})) + d(g^m x^i, gu^i)) \\ &\le h\max\{d(gu^{\sigma_i(1)}, g^m x^{\sigma_i(1)}),\ldots,d(gu^{\sigma_i(n)}, g^m x^{\sigma_i(n)}), d(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}), gu^i), \\ &\quad d(g^m x^i, g^{m-1}x^i), \frac{d(S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}), g^{m-1}x^i) + d(g^m x^i, gu^i)}{2}\} + d(g^m x^i, gu^i) \\ &\to 0 \ as\ n \to \infty. \end{aligned} \tag{3.45}$$

Hence $gu^i \in S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)})$. Consequently,

$$u^i = gu^i \in S(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}).$$

Similarly

$$u^i = gu^i \in T(u^{\sigma_i(1)},\ldots,u^{\sigma_i(n)}).$$

If (b) or (c), we can prove the existence of $n-$ common fixed point as in Theorem 3.1.

Corollary 3.2 *Assume same hypothesis of Corollary 3.1 but replace Condition (3.31) with another one. For* $0 < h < 1$,

$$H(S(x,y,z), T(u,v,w)) \le h\max\{\ d(gx,gu), d(gy,gv), d(gz,gw), d(S(x,y,z),gx), d(T(u,v,w),gu), \frac{d(S(x,y,z),gu) + d(T(u,v,w),gx)}{2}\}.$$

Then we have the same results.

Remark 3.2 Corollary 3.2 is the ordered version of Theorem 8 of Kutbi et al. [12].

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