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Research Article

New Rough Approximations Based on E-Neighborhoods

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This paper puts forward some rough approximations which are motivated from topology. Given a subset $R \subseteq U \times U$, we can use 8 types of E-neighborhoods to construct approximations of an arbitrary $X \subseteq U$ on the one hand. On the other hand, we can also construct approximations relying on a topology which is induced by an E-neighborhood. Properties of these approximations and relationships between them are studied. For convenience of use, we also give some useful and easy-to-understand examples and make a comparison between our approximations and those in the published literature.

1. Introduction and Preliminaries

The problem of imperfect knowledge became a crucial issue for computer scientists, especially in the area of information system and artificial intelligence [1, 2]. There are various approaches to manipulate and understand imperfect knowledge, among which is rough set theory. Rough set was proposed by Pawlak [3, 4] in 1982 which has been generalized in many ways [5–13]. What we are concerned about are those methods whose ideas are motivated from topology, for example, methods constructing the lower and upper approximations by using different kinds of neighborhoods, such as right and left neighborhoods [10, 14], minimal right neighborhoods [15], and intersection and union neighborhoods (see [16, 17]). In fact, a combination of rough set theory and topological theory became the main goal of many studies (see [18–25]).

The present paper is a continuation to these works where we initiate new types of neighborhoods (namely, E_j-neighborhoods) to give lower and upper approximations of an arbitrary set X directly or indirectly. In Section 2, we apply N_j-neighborhoods to establish the concepts of E_j-neighborhoods and discuss the main properties. With the help of examples, we show the relationships between them as well as with N_j-neighborhoods and P_j-adhesion neighborhoods. In Section 3, we formulate and study the concepts of E_j-lower and E_j-upper approximations, E_j-boundary region, E_j-positive and E_j-negative regions, and E_j-accuracy measure of a subset and make comparisons between them with respect to different types of j. In Section 4, we study the previous concepts from a topological view and explore their main properties. In Section 5, we give some conclusions and make a plan for future works.

Now, we recall some basic properties and results of rough set theory, particularly those related to some types of neighborhood systems.

Definition 1 (see [3, 5, 16])

(1) A subset $R \subseteq U \times U$ (also called a binary relation on U) is said to be an equivalence relation if it is reflexive (i.e., $(v, v) \in R$ for each $v \in U$), symmetric (i.e., $(u, v) \in R$ if $(v, u) \in R$), and transitive (i.e., $(u, w) \in R$ whenever $(u, v) \in R$ and $(v, w) \in R$). It is said to be a preorder (or quasi-order) if it is reflexive and transitive. It is said to be a partial order if it is an antisymmetric (i.e., $u = v$ whenever $(u, v) \in R$ and

$(v,u) \in R$) preorder. It is said to be a diagonal if $R = \{(v,v)\colon v \in U\}$. It said to be serial if for every $v \in U$, $w \in U$ such that $(v,w) \in R$ (also written as vRw).

(2) (See [4, 26] for a special case). For an equivalence relation R on U and a subset $X \subseteq U$, the two related sets $\underline{R}(X) = \bigcup\{A \in U/R\colon A \subseteq X\}$ and $\underline{R}(X) = \bigcup\{A \in U/R\colon A \subseteq X \neq \varnothing\}$ are called the Pawlak lower approximation and upper approximation of X, respectively (where $U/R = \{[x]\colon x \in U\}$ is the set of all equivalence classes).

Proposition 1 (see [3, 4] for a special case). *The lower approximations and the upper approximations have the following properties ($\{E,F\} \cup \{E_i\}_{i\in I} \subseteq 2^U$, the power set of U):*

(1) $\underline{R}(\varnothing) = \overline{R}(\varnothing) = \varnothing$ and $\underline{R}(U) = \overline{R}(U) = U$.

(2) $\underline{R}(E) \subseteq E \subseteq \overline{R}(E)$ and $\underline{R}(E) = [\overline{R}(E^c)]^c$.

(3) $\underline{R}(\cup_{i\in I}E_i) = \cup_{i\in I}\underline{R}(E_i)$ and $\underline{R}(\cup_{i\in I}E_i) = \cup_{i\in I}\underline{R}(E_i)$. Particularly, $\underline{R}(E) \subseteq \underline{R}(F)$ and $\underline{R}(E) \subseteq \underline{R}(F)$ if $E \subseteq F$.

(4) $\underline{R}[\underline{R}(E)] = \underline{R}(E)$ and $\overline{R}[\overline{R}(E)] = \overline{R}(E)$.

Definition 2. A subset $\mathcal{J} \subseteq 2^U$ is called a topology on U (and $(U,\mathcal{J})$ is called a topological space) if it is closed under arbitrary union and finite intersection. A topology satisfying that every open set is also closed is called a clopen topology. We will use A^o to denote the interior of A (i.e., the union of all open sets that are contained in A) and A^- to denote the closure of A (i.e., the intersection of all closed sets containing A) in this paper.

Let $\mathcal{J}_R = \{A \in 2^U\colon [x] \subseteq A\,(\forall x \in A)\}$. Then, $(U,\mathcal{J}_R)$ is a topological space, $\underline{R}(X) = X^o$, $\overline{R}(X) = X^-$, and $[x]$ is the smallest open neighborhood of x ($\forall x \in U$). This inspires many people to define the lower approximation and the upper approximation by neighborhoods; actually, our approach in this paper is also motivated from topology. In the following, we write $J = \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$.

Definition 3 (see [10, 14, 16]). Let $R \subseteq U^2$ and $j \in J$.

(1) The 8 kinds of j-neighborhoods are defined as follows: $N_r(v) = \{w \in U\colon vRw\}$, $N_l(v) = \{w \in U\colon wRv\}$, $N_{\langle r\rangle}(v) = \cap\{N_r\}(w)\colon v \in N_r(w)$ (if $v \in N_r(w)$ for some $w \in U$ or $\varnothing$, otherwise), $N_{\langle l\rangle}(v) = \cap\{N_l\}(w)\colon v \in N_l(w)$ (if $v \in N_l(w)$ for some $w \in U$ or $\varnothing$, otherwise), $N_i(v) = N_r(v) \cap N_l(v)$, $N_u(v) = N_r(v) \cup N_l(v)$, $N_{\langle i\rangle}(v) = N_{\langle r\rangle}(v) \cap N_{\langle l\rangle}(v)$, and $N_{\langle u\rangle}(v) = N_{\langle r\rangle}(v) \cup N_{\langle l\rangle}(v)$

(2) The triple (U,R,λ_j) is called a j-neighborhood space (in brief, j-NS), where λ_j is a mapping from U to 2^U which associated each $v \in U$ with a j-neighborhood

Definition 4 (see [6]). Let $R \subseteq U^2$. The j-adhesion neighborhoods are defined as follows:

(1) $P_r(v) = \{w \in U\colon N_r(w) = N_r(v)\}$

(2) $P_l(v) = \{w \in U\colon N_l(w) = N_l(v)\}$

(3) $P_{\langle r\rangle}(v) = \{w \in U\colon N_{\langle r\rangle}(w) = N_{\langle r\rangle}(v)\}$

(4) $P_{\langle l\rangle}(v) = \{w \in U\colon N_{\langle l\rangle}(w) = N_{\langle l\rangle}(v)\}$

(5) $P_i(x) = P_r(v) \cap P_l(v)$

(6) $P_u(x) = P_r(v) \cup P_l(v)$

(7) $P_{\langle i\rangle}(v) = P_{\langle r\rangle}(v) \cap P_{\langle l\rangle}(v)$

(8) $P_{\langle u\rangle}(v) = P_{\langle r\rangle}(v) \cup P_{\langle l\rangle}(v)$

2. E-Neighborhoods

In this section, we introduce the notions of E-neighborhoods using j-neighborhoods and study their properties.

Definition 5. Let $R \subseteq U^2$. The E-neighborhoods are defined as follows:

(1) $E_r(x) = \{y \in U\colon N_r(y) \cap N_r(x) \neq \varnothing\}$

(2) $E_l(x) = \{y \in U\colon N_l(y) \cap N_l(x) \neq \varnothing\}$

(3) $E_i(x) = E_r(x) \cap E_l(x)$

(4) $E_u(x) = E_r(x) \cup E_l(x)$

(5) $E_{\langle r\rangle}(x) = \{y \in U\colon N_{\langle r\rangle}(y) \cap N_{\langle r\rangle}(x) \neq \varnothing\}$

(6) $E_{\langle l\rangle}(x) = \{y \in U\colon N_{\langle l\rangle}(y) \cap N_{\langle l\rangle}(x) \neq \varnothing\}$

(7) $E_{\langle i\rangle}(x) = E_{\langle r\rangle}(x) \cap E_{\langle l\rangle}(x)$

(8) $E_{\langle u\rangle}(x) = E_{\langle r\rangle}(x) \cup E_{\langle l\rangle}(x)$

We give the following example to illustrate how we calculate different types of neighborhoods. Also, we will benefit from this example to clarify some obtained results.

Example 1. Let $U = \{v,w,x,y\}$ and $R = \{(v,v), (y,y), (v,x), (v,y), (y,w), (w,y)\}$. Then, the j-neighborhoods and E-neighborhoods of a point are as in Table 1.

Theorem 1. *E-neighborhoods have the following properties ($R \subseteq U^2, v \in U$):*

(1) $E_i(v) \subseteq E_r(v) \cap E_l(v) \subseteq E_r(v) \cup E_l(v) \subseteq E_u(v)$

(2) $E_{\langle i\rangle}(v) \subseteq E_{\langle r\rangle}(v) \cap E_{\langle l\rangle}(v) \subseteq E_{\langle r\rangle}(v) \cup E_{\langle l\rangle}(v) \subseteq E_{\langle u\rangle}(v)$

(3) $v \in E_j(x)$ iff $x \in E_j(v)\,(j \in J)$

(4) If R is reflexive, then $E_{\langle j\rangle}(v) \subseteq E_j(v)$ and $P_j(v) \cup N_j(v) \subseteq E_j(v)$ $(j \in J)$

(5) If R is symmetric, then $E_r(v) = E_l(v) = E_i(v) = E_u(v)$ and $E_{\langle r\rangle}(x) = E_{\langle l\rangle}(v) = E_{\langle i\rangle}(v) = E_{\langle u\rangle}(v)$

(6) If R is transitive, then $E_j(v) \subseteq E_{\langle j\rangle}(v)$ $(j \in \{r, lu, i\})$

(7) If R is serial, then $P_j(v) \subseteq E_j(v)$ $(j \in J)$

(8) If R is symmetric and transitive, then $E_j(v) \subseteq N_j(v)$ and $E_j(v) \subseteq E_j(w)$ (if $v \in E_j(w)$) for each $j \in J$

(9) If R is preorder, then $E_j(v) \subseteq E_{\langle j\rangle}(v)$ $(j \in \{r, l, u, i\})$

(10) If R is a equivalence relation, then for each $j \in J$, all $E_j(v)$ are identical, $E_j(v) = N_j(v) = P_j(v)$, and $v \in E_j(w)$ iff $E_j(v) = E_j(w)$

Table 1: j-neighborhoods and E-neighborhoods.

	v	w	x	y
N_r	$\{v,x,y\}$	$\{y\}$	$\varnothing$	$\{w,y\}$
N_l	$\{v\}$	$\{y\}$	$\{v\}$	$\{v,w,y\}$
N_i	$\{v\}$	$\{y\}$	$\varnothing$	$\{w,y\}$
N_u	$\{v,x,y\}$	$\{y\}$	$\{v\}$	$\{v,w,y\}$
$N_{\langle r\rangle}$	$\{v,x,y\}$	$\{w,y\}$	$\{v,x,y\}$	$\{y\}$
$N_{\langle l\rangle}$	$\{v\}$	$\{v,w,y\}$	$\varnothing$	$\{y\}$
$N_{\langle i\rangle}$	$\{v\}$	$\{w,y\}$	$\varnothing$	$\{y\}$
$N_{\langle u\rangle}$	$\{v,x,y\}$	$\{v,w,y\}$	$\{v,x,y\}$	$\{y\}$
P_r	$\{v\}$	$\{w\}$	$\{x\}$	$\{y\}$
P_l	$\{v,x\}$	$\{w\}$	$\{v,x\}$	$\{y\}$
P_i	$\{v\}$	$\{w\}$	$\{x\}$	$\{y\}$
P_u	$\{v,x\}$	$\{w\}$	$\{v,x\}$	$\{y\}$
$P_{\langle r\rangle}$	$\{v,x\}$	$\{w\}$	$\{v,x\}$	$\{y\}$
$P_{\langle l\rangle}$	$\{v\}$	$\{w\}$	$\{x\}$	$\{y\}$
$P_{\langle i\rangle}$	$\{v\}$	$\{w\}$	$\{x\}$	$\{y\}$
$P_{\langle u\rangle}$	$\{v,x\}$	$\{w\}$	$\{v,x\}$	$\{y\}$
E_r	$\{v,w,y\}$	$\{v,w,y\}$	$\varnothing$	$\{v,w,y\}$
E_l	$\{v,x,y\}$	$\{w,y\}$	$\{v,x,y\}$	U
E_i	$\{v,y\}$	$\{w,y\}$	$\varnothing$	$\{v,w,y\}$
E_u	U	$\{v,w,y\}$	$\{v,x,y\}$	U
$E_{\langle r\rangle}$	U	U	U	U
$E_{\langle l\rangle}$	$\{v,w\}$	$\{v,w,y\}$	$\varnothing$	$\{w,y\}$
$E_{\langle i\rangle}$	$\{v,w\}$	$\{v,w,y\}$	$\varnothing$	$\{w,y\}$
$E_{\langle u\rangle}$	U	U	U	U

Proof. (3) Obviously, $v \in E_j(x) \Leftrightarrow N_j(v) \cap N_j(x) \neq \varnothing \Leftrightarrow x \in E_j(v)$ for each $j \in \{r, l, \langle r\rangle, \langle l\rangle\}$. Then, $v \in E_j(x) \Leftrightarrow x \in E_j(v)$ for each $j \in \{i, u, \langle i\rangle, \langle u\rangle\}$.

(4) Step 1: since R is reflexive, $\cap_{v \in N_r(x)} N_r(x) \subseteq N_r(v)$ and $\cap_{v \in N_l(x)} N_l(x) \subseteq N_l(v)$. This implies that $E_{\langle r\rangle}(v) \subseteq E_r(v)$ and $E_{\langle l\rangle}(v) \subseteq E_l(v)$. Consequently, $E_{\langle i\rangle}(v) \subseteq E_i(v)$ and $E_{\langle u\rangle}(v) \subseteq E_u(v)$.

Step 2: let $x \in N_j(v)$. By reflexivity of R, we have $\{x\} \subseteq N_j(x) \cap N_j(v)$. Therefore, $x \in E_j(v)$; thus, $N_j(v) \subseteq E_j(v)$. Also, let $x \in P_j(v)$. Then, $N_j(x) = N_j(v)$. Since R is reflexive, $N_j(x) \cap N_j(v) \neq \varnothing$. Therefore, $x \in E_j(v)$; thus, $P_j(v) \subseteq E_j(v)$. Hence, we obtain the desired result.

(5) Since R is symmetric, $N_r(v) = N_l(v)$. Therefore, $N_r(v) \cap N_r(z) \neq \varnothing \Leftrightarrow N_l(v) \cap N_l(z) \neq \varnothing$; thus, $E_r(v) = E_l(v)$. Consequently, $E_r(v) = E_l(x) = E_i(x) = E_u(x)$. Similarly, $E_{\langle r\rangle}(x) = E_{\langle l\rangle}(v) = E_{\langle i\rangle}(v) = E_{\langle u\rangle}(v)$.

(6) Let $a \in N_r(v)$. Then, vRa. For each $N_r(x)$ containing v, we have xRv. Since R is transitive, xRa. Therefore, $a \in N_r(x)$; thus, $a \in \cap_{v \in N_r(x)} N_r(x) = N_{\langle r\rangle}(v)$. Hence, $N_r(v) \subseteq N_{\langle r\rangle}(v)$; consequently, $E_r(v) \subseteq E_{\langle r\rangle}(v)$. Similarly, $E_l(v) \subseteq E_{\langle l\rangle}(v)$. This implies that $E_i(v) \subseteq E_{\langle i\rangle}(v)$ and $E_u(v) \subseteq E_{\langle u\rangle}(v)$.

(7) It follows from (4) and the fact $v \in E_j(x) \Leftrightarrow N_j(v) \cap N_j(x) \neq \varnothing \Leftrightarrow x \in E_j(v)$ $(j \in \{r, l, \langle r\rangle, \langle l\rangle\})$.

(8) Step 1: we prove the case $j = r$. Let $x \in E_r(v)$. Then, $N_r(x) \cap N_r(v) \neq \varnothing$, i.e., there exists $z \in N_r(x) \cap N_r(v)$, xRz, and vRz. Since R is symmetric and transitive, vRx. Therefore, $x \in N_r(v)$; thus, $E_r(v) \subseteq N_r(v)$.

Step 2: since R is symmetric, $E_r(v) = E_l(v) = E_i(v) = E_u(v)$ by (5). We only prove the case $j = l$. Let $v \in E_l(w)$. Then, $N_l(v) \cap N_l(w) \neq \varnothing$; therefore, there is $a \in U$ such that aRv and aRw. Now, let $x \in E_l(v)$. Then, $N_l(x) \cap N_l(v) \neq \varnothing$, i.e., there exists a $b \in U$ such that bRx and bRv. Note that aRb; this means that aRx; consequently, $a \in N_l(x)$; also, $a \in N_l(w)$; thus, $N_l(x) \cap N_l(x) \neq \varnothing$. Hence, $x \in E_l(w)$, as required.

(10) Step 1: by (4) and (8), $E_j(v) = N_j(v)$ and $P_j(v) \subseteq E_j(v)$. It remains to prove $E_j(v) \subseteq P_j(v)$. Since R is symmetric, $E_r(v) = E_l(v) = E_i(v) = E_u(v)$ by (5). Let $x \in E_l(v)$. Then, $N_l(x) \cap N_l(v) \neq \varnothing$. Since R is equivalence, $N_l(x) = N_l(v)$. Therefore, $x \in P_l(v)$; consequently, $E_l(v) \subseteq P_l(v)$. Thus, $P_j(v) = E_j(v)$ is proved.

Step 2: we only prove the case $j = l$. Let $v \in E_l(w)$. Then, $N_l(v) \cap N_l(w) \neq \varnothing$. Since R is equivalence, $N_l(v) = N_l(w) \neq \varnothing$. Therefore, $E_j(v) = E_j(w)$. Conversely, $E_j(v) = E_j(w)$. Since R is reflexive, $v \in N_l(v)$. Hence, $v \in E_j(w)$. □

3. Rough Approximations Using E-Neighborhoods Directly

Let (U, R, λ_j) be j-NS and $\mathscr{R}_j = \{E_j(v): v \in U\}$ $(j \in J)$. We devote this section to formulate the following concepts using E-neighborhoods directly: $\mathscr{R}_j$-lower approximation, $\mathscr{R}_j$-upper approximation, $\mathscr{R}_j$-boundary region, $\mathscr{R}_j$-positive region, $\mathscr{R}_j$-negative region, and $\mathscr{R}_j$-accuracy measure of a subset X. We will also illustrate the relationships between them and reveal the main properties with the help of examples.

Definition 6. Let $X \subseteq U$ and $j \in J$. Then, $\mathscr{R}_j^-(X) = \{x \in U: E_j(x) \subseteq X\}$ and $\mathscr{R}_j^+(X) = \{x \in U: E_j(x) \subseteq X \neq \varnothing\}$ are called $\mathscr{R}_j$-lower approximation and $\mathscr{R}_j$-upper approximation, respectively; $\mathscr{B}_j(X) = \mathscr{R}_j^+(X) - \mathscr{R}_j^-(X)$, $\mathrm{POS}_j(X) = \mathscr{R}_j^-(X)$, and $\mathrm{NEG}_j(X) = U - \mathscr{R}_j^+(X)$ are called $\mathscr{R}_j$-boundary, $\mathscr{R}_j$-positive, and $\mathscr{R}_j$-negative regions of X, respectively; $\mathscr{M}_j(X) = (|\mathscr{R}_j^-(X) \cap X| / |\mathscr{R}_j^+(X) \cap X|)$ is called the $\mathscr{R}_j$-accuracy measure of X (if $0 < |\mathscr{R}_j^+(X) \cup X| < \aleph_0$), where $|X|$ denotes the cardinality of a set X and $\aleph_0$ denotes the cardinality of the set of natural numbers.

Theorem 2. *$\mathscr{R}_j$-approximations have the following properties ($R \subseteq U^2$, $\{E, F\} \cup \{E_i\}_{i \in I} \subseteq 2^U$, $j \in J$):*

(1) $\mathscr{R}_j^+(\varnothing) = \varnothing \subseteq \mathscr{R}_j^-(\varnothing)$.

(2) $\mathscr{R}_j^+(U) \subseteq U = \mathscr{R}_j^-(U)$.

(3) $\mathscr{R}_j^-(\cap_{i\in I}E_i) = \cap_{i\in I}\mathscr{R}_j^-(E_i)$ *and* $\mathscr{R}_j^+(\cap_{i\in I}E_i) = \cap_{i\in I}\mathscr{R}_j^+(E_i)$. *Particularly,* $\mathscr{R}_j^-(E)\subseteq\mathscr{R}_j^-(F)$ *and* $\mathscr{R}_j^+(E)\subseteq\mathscr{R}_j^+(F)$ *if* $E\subseteq F$.

(4) $\mathscr{R}_j^-(E) = [\mathscr{R}_j^+(E^c)]^c$ *and* $\mathscr{R}_j^+(E) = [\mathscr{R}_j^-(E^c)]^c$.

(5) *In general,* $\mathscr{R}_j^-[\mathscr{R}_j^-(E)] \neq \mathscr{R}_j^-(E)$ *and* $\mathscr{R}_j^+[\mathscr{R}_j^+(E)] \neq \mathscr{R}_j^+(E)$.

Proof. We only prove (3) and (4).

(3) Step 1: obviously, $\mathscr{R}_j^-(\cap_{i\in I}E_i)\subseteq(\cap_{i\in I}\mathscr{R}_j^-)(E_i)$. Conversely, for each $x\in\cap_{i\in I}\mathscr{R}_j^-(E_i)$ (i.e., $x\in\mathscr{R}_j^-(E_i)$ for each $i\in I$), we have $E_j(x)\subseteq E_i$ ($\forall i\in I$), and thus, $E_j(x)\subseteq\cap_{i\in I}E_i$. Hence, $x\in\mathscr{R}_j^-(\cap_{i\in I}E_i)$, as required.
Step 2: obviously, $\cup_{i\in I}\mathscr{R}_j^+(E_i)\subseteq\mathscr{R}_j^+(\cup_{i\in I}E_i)$. Conversely, for each $x\in\mathscr{R}_j^+(\cup_{i\in I}E_i)$, we have $E_j(x)\cap(\cup_{i\in I}E_i)\neq\varnothing$, i.e., $E_j(x)\cap E_{i_0}\neq\varnothing$ for some $i_0\in I$. Thus, $x\in\mathscr{R}_j^+(E_{i_0})\subseteq\cup_{i\in I}\mathscr{R}_j^+(E_i)$, as required.
(4) $x\in\mathscr{R}_j^-(E^c)\Leftrightarrow E_j(x)\subseteq E^c\Leftrightarrow E_j\ (x)\cap E=\varnothing\Leftrightarrow x\notin\mathscr{R}_j^+(E)\Leftrightarrow x\in[\mathscr{R}_j^+(E)]^c$. The second equality immediately comes by putting $E=F^c$ in the first. □

Example 2. For j-NS (U,R,λ_j) in Example 1, the $\mathscr{R}_j$-lower approximations and $\mathscr{R}_j$-upper approximations ($j\in\{r,l,i,u\}$), the $\mathscr{R}_i$-accuracy measure, the lower approximations X_- and upper approximations X_+ in the sense of [6] for $P_{<r>}$, and the lower approximations X^- and upper approximations X^+ in the sense of [10] for N_r are given in Table 2 (where $Y=\{v,w,y\}$ and $Z=\{v,x,y\}$). From Table 2, we can see the approximations in this paper and those in [6] (resp., [10]) are incomparable.

4. Rough Approximations Induced by E-Topologies

We will construct rough approximations using E-neighborhoods indirectly in this section. Let (U,R,λ_j) be j-NS ($j\in J$). We first employ E_j-neighborhoods to generate a topology $\mathscr{J}_{R,j}$ (called an E_j-topology) and then call the interior $\mathscr{J}_{R,j}^-(X)$ and the closure $\mathscr{J}_{R,j}^+(X)$ of a subset $X\subseteq U$ the $\mathscr{J}_{R,j}$-lower approximation and $\mathscr{J}_{R,j}$-upper approximation of X, respectively. These kinds of approximations are compared with those in Section 3.

Theorem 3

(1) $\mathscr{J}_{R,j}=\{A\in 2^U: E_j(x)\subseteq A\ (\forall x\in A)\}$ *is a topology on* U *satisfying* $U-A\in\mathscr{J}_{R,j}$ *whenever* $A\in\mathscr{J}_{R,j}$ $(j\in J)$

(2) *Both* $\mathscr{J}_{R,N,j}=\{A\in 2^U: N_j(x)\subseteq A\ (\forall x\in A)\}$ *and* $\mathscr{J}_{R,P,j}=\{A\in 2^U: P_j(x)\subseteq A\ (\forall x\in A)\}$ *are topologies on* U $(j\in J)$

(3) $\mathscr{J}_{R,u}\subseteq\mathscr{J}_{R,r}\cap\mathscr{J}_{R,l}\subseteq\mathscr{J}_{R,r}\cup\mathscr{J}_{R,l}\subseteq\mathscr{J}_{R,i}$

(4) $\mathscr{J}_{R,\langle u\rangle}\subseteq\mathscr{J}_{R,\langle r\rangle}\cap\mathscr{J}_{R,\langle l\rangle}\subseteq\mathscr{J}_{R,\langle r\rangle}\cup\mathscr{J}_{R,\langle l\rangle}\subseteq\mathscr{J}_{R,\langle i\rangle}$

(5) *If R is reflexive, then* $\mathscr{J}_{R,P,j}\subseteq\mathscr{J}_{R,j}$ $(j\in J)$

(6) *If R is serial, then* $\mathscr{J}_{R,P,j}\subseteq\mathscr{J}_{R,j}$ $(j\in\{r,l,u,i\})$

(7) *If R is a equivalence relation, then* $\mathscr{J}_{R,j}$ *is constant for all* $j\in J$

Proof. We only prove (1). Obviously, $\mathscr{J}_{R,j}$ is a topology on U. For each $A\in\mathscr{J}_{R,j}$ and each $x\in U-A$, we need to prove $E_j(x)\subseteq U-A$. Without loss of generality, we assume $E_j(x)\neq\varnothing$. Suppose $v\in E_j(x)\cap A$. Then, $x\in E_j(v)$ (by Theorem 1 (3)) and $E_j(v)\subseteq A$ (as $A\in\mathscr{J}_{R,j}$), and thus, $x\in A$. This is a contradiction. Therefore, $E_j(x)\cap A=\varnothing$, i.e., $E_j(x)\subseteq U-A$. □

Definition 7. Let $X\subseteq U$ and $j\in J$. Then, $\mathscr{J}_{R,j}^-(X)=X^o$ (the interior of X in $(U,\mathscr{J}_{R,j})$) and $\mathscr{J}_{R,j}^+(X)=X^-$ (the closure of X in $(U,\mathscr{J}_{R,j})$) are called the $\mathscr{J}_{R,j}$-lower approximation and $\mathscr{J}_{R,j}$-upper approximation, respectively; $\mathscr{J}_{R,j}^+(X)-\mathscr{J}_{R,j}^-(X)$, $\mathscr{J}_{R,j}^+(X)$, and $U-\mathscr{J}_{R,j}^+(X)$ are called $\mathscr{J}_{R,j}$-boundary, $\mathscr{J}_{R,j}$-positive, and $\mathscr{J}_{R,j}$-negative regions of X, respectively; $\mathscr{M}_{J,j}(X)=(|\mathscr{J}_{R,j}^-(X)|/|\mathscr{J}_{R,j}^+(X)|)$ is called the $\mathscr{J}_{R,j}$-accuracy measure of X (if $0<|\mathscr{J}_{R,j}^+(X)|<\aleph_0$).

The relation between $\mathscr{M}_{J,j}(X)$ and $\mathscr{M}_j(X)$ $(j\in J)$ is given by the following.

Theorem 4. $\mathscr{M}_{J,j}(X)\leq\mathscr{M}_j(X)$ *for each* $j\in J$ *(if* $0<|\mathscr{J}_{R,j}^+(X)|<\aleph_0$*).*

Proof

(i) Step 1: for each $z\in\mathscr{J}_{R,j}^-(X)$, we have $z\in\mathscr{R}_j^-(X)$ (because $\mathscr{J}_{R,j}^-(X)=X^o$ and $E_j(z)\subseteq X^o\subseteq X$), and thus, $\mathscr{J}_{R,j}^-(X)\subseteq\mathscr{R}_j^-(X)\cap X$, which implies

$$\left|\mathscr{J}_{R,j}^-(X)\right|\leq\left|\mathscr{R}_j^-(X)\cap X\right|. \tag{1}$$

(ii) Step 2: let $z\in\mathscr{R}_j^+(X)\cup X$. If $z\in X$, then $z\in\mathscr{J}_{R,j}^-(X)$. If $z\notin X$, then $z\in\mathscr{R}_j^+(X)$, and thus, $E_j(z)\cap X\neq\varnothing$. This means there exists $v\in U-\{z\}$ such that $v\in E_j(z)$ and $v\in X$. Consequently, for any $V\in\mathscr{J}_{R,j}$ containing z, we have $v\in V$. Therefore, $V\cap X\neq\varnothing$, and thus, $z\in\mathscr{J}_{R,j}^+(X)$ (because $\mathscr{J}_{R,j}^+(X)=X^-$). It follows that $\mathscr{R}_j^+(X)\cup X\subseteq\mathscr{J}_{R,j}^+(X)$, and thus,

$$\frac{1}{\left|\mathscr{J}_{R,j}^+(X)\right|}\leq\frac{1}{\left|\mathscr{R}_j^+(X)\cup X\right|}. \tag{2}$$

From (1) and (2), we can see that $\mathscr{M}_{J,j}(X)=(|\mathscr{J}_{R,j}^-(X)|/|\mathscr{J}_{R,j}^+(X)|)\leq(|\mathscr{R}_j^-(X)\cap X|/|\mathscr{R}_j^+(X)\cup X|)=\mathscr{M}_j(X)$ $(j\in J)$. □

Example 3

(1) Now, we exemplify an application of rough approximations introduced in this paper. Let $X=\{x_1,x_2,\ldots,x_{50}\}$ be a group of people who have just reached Xian Yang Airport (but are not allowed to outbound station) from two countries (by two planes involving 200 people denoted by a set

Table 2: $\mathscr{R}_j$-lower approximations and $\mathscr{R}_j$-upper approximations ($j \in \{r, l, i, u\}$).

X	$\mathscr{R}_u^-(X)$	$\mathscr{R}_u^+(X)$	$\mathscr{R}_r^-(X)$	$\mathscr{R}_r^+(X)$	$\mathscr{R}_l^-(X)$	$\mathscr{R}_l^+(X)$	$\mathscr{R}_i^-(X)$	$\mathscr{R}_i^+(X)$	$\mathscr{M}_i(X)$	X_-	X_+	X^-	X^+
$\{v\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	Z	$\{x\}$	$\{v, y\}$	0	$\varnothing$	$\{v, x\}$	$\{x\}$	$\{v\}$
$\{w\}$	$\varnothing$	Y	$\{x\}$	Y	$\varnothing$	$\{w, y\}$	$\{x\}$	$\{w, y\}$	0	$\{w\}$	$\{w\}$	$\{x\}$	$\{y\}$
$\{x\}$	$\varnothing$	Z	$\{x\}$	$\varnothing$	$\varnothing$	Z	$\{x\}$	$\varnothing$	1	$\varnothing$	$\{v, x\}$	$\{x\}$	$\{v\}$
$\{y\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{x\}$	Y	0	$\{y\}$	$\{y\}$	$\{w, x\}$	Y
$\{v, w\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{v, x\}$	Y	1/3	$\{w\}$	$\{v, w, x\}$	$\{x\}$	$\{v, y\}$
$\{v, x\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	Y	$\{x\}$	$\{v, y\}$	1/3	$\{v, x\}$	$\{v, x\}$	$\{x\}$	$\{v\}$
$\{v, y\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{v, x\}$	Y	1/3	$\{y\}$	Z	$\{w, x\}$	Y
$\{w, x\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{x\}$	$\{w, y\}$	1	$\{w\}$	$\{v, w, x\}$	$\{x\}$	$\{v, y\}$
$\{w, y\}$	$\varnothing$	U	$\{x\}$	Y	$\{w\}$	U	$\{w, x\}$	Y	1/3	$\{w, y\}$	$\{w, y\}$	$\{w, x, y\}$	Y
$\{x, y\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{x\}$	Y	1/4	$\{y\}$	Z	$\{w, x\}$	Y
$\{v, w, x\}$	$\varnothing$	U	$\{x\}$	Y	$\varnothing$	U	$\{x\}$	Y	1/4	$\{v, w, x\}$	$\{v, w, x\}$	$\{x\}$	$\{v, y\}$
Y	Y	U	U	Y	$\{w\}$	U	U	Y	1	$\{w, y\}$	U	$\{w, x, y\}$	Y
Z	Z	U	$\{x\}$	Y	$\{v, x\}$	U	$\{v, x\}$	Y	1	Z	Z	$\{v, w, x\}$	Y
$\{w, x, y\}$	$\varnothing$	U	$\{x\}$	Y	$\{w\}$	U	$\{w, x\}$	Y	2/3	$\{w, y\}$	U	$\{w, x, y\}$	Y
U	U	U	U	Y	U	U	U	Y	1	U	U	U	Y
$\varnothing$	$\varnothing$	$\varnothing$	$\{x\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{x\}$	$\varnothing$					

Table 3: $\mathscr{J}_{R,j}$-lower approximations and $\mathscr{J}_{R,j}$-upper approximations ($j \in \{r, l, i, u\}$).

X	$\mathscr{J}_{R,u}^-(X)$	$\mathscr{J}_{R,u}^+(X)$	$\mathscr{J}_{R,r}^-(X)$	$\mathscr{J}_{R,r}^+(X)$	$\mathscr{J}_{R,l}^-(X)$	$\mathscr{J}_{R,l}^+(X)$	$\mathscr{J}_{R,i}^-(X)$	$\mathscr{J}_{R,i}^+(X)$	$\mathscr{M}_{J,i}(X)$	$\mathscr{M}_i(X)$
$\{v\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	0
$\{w\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	0
$\{x\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1
$\{y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	0
$\{v, w\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	1/3
$\{v, x\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1/3
$\{v, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	1/3
$\{w, x\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1
$\{w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	$\varnothing$	U	$\varnothing$	$\{v, w, y\}$	0	1/3
$\{x, y\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1/4
$\{v, w, x\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1/4
$\{v, w, y\}$	$\varnothing$	U	$\{v, w, y\}$	$\{v, w, y\}$	$\varnothing$	U	$\{v, w, y\}$	$\{v, w, y\}$	1	1
$\{v, x, y\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	1
$\{w, x, y\}$	$\varnothing$	U	$\{x\}$	U	$\varnothing$	U	$\{x\}$	U	1/4	2/3
U	U	U	U	U	U	U		U	1	1
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$		

$U = \{x_1, x_2, \ldots, x_{200}\}$) and will attend a meeting holding in Xi'an. Assume that x_1 and x_2 are actually infected of asymptomatic infection of new coronavirus, $E_j(x) \in 2^U$ consists of x and all $y \in U$ who contacted x after x had contacted one of x_1 and x_2 ($x \in U - \{x_1, x_2\}$), and $E_j(x_1) = \{x_1\}$ and $E_j(x_2) = \{x_2\}$. To insure the safety of this meeting, $\mathscr{R}_j^-(X) = \{x \in U | E_j(x) \subseteq X\} = \{x_1, x_2\} \cup \{x \in X|$ each person contacting x directly after x contacts one of x_1 and x_2 is in $X\}$ can be looked to be the set of all people who must run a nucleic acid test, and $\mathscr{R}_j^+(X) = X \cup \{x \in U |E_j(x) \cap X \neq \varnothing\} = X \cup \{x \in U - X|$ some person contacting x directly after x contacts one of x_1 and x_2 is in $X\}$ can be looked to be the set of all people who should run a nucleic acid test.

(2) For j-NS (U, R, λ_j) in Example 1, the $\mathscr{J}_{R,j}$-lower approximations and $\mathscr{J}_{R,j}$-upper approximations ($j \in \{r, l, i, u\}$) and the $\mathscr{M}_{J,j}$-accuracy measure are given in Table 3.

5. Concluding Remarks

Motivated by topology, this article has initiated two new rough approximations by introducing a new class of neighborhood systems (called E_j-neighborhoods) using j-neighborhoods. We have probed the main features and formulated the concepts of E_j-lower and E_j-upper approximations and E_j-accuracy measure which are had contacted one of induced from different types of j and compared them. We complete this work by studying these concepts from a topological view and comparing them. In all comparisons, we obtain higher accurate approximations in the case of $j = i$.

In the upcoming works, we will study new types of neighborhoods in rough set theory and use them to define a topological structure. Also, we will investigate the E_j-neighborhoods and approximations on the fuzzy rough set motivated from fuzzy control problems and fractional-order nonlinear systems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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