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On basic Horn hypergeometric functions $\mathbf{H}_3$ and $\mathbf{H}_4$

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Abstract

The purpose of this work is to demonstrate several interesting contiguous function relations and q-differential formulas for basic Horn hypergeometric functions $\mathbf{H}_3$ and $\mathbf{H}_4$. Some properties of our main results are also constructed.

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1 Introduction

Quantum calculus (or, q-calculus) is the study of calculus without limits. Many extensions of q-calculus have been developed and applied as one of the most active areas of research in mathematics and physics. These new extensions have proved to be very useful in various fields such as physics, engineering, statistics, actuarial sciences, economics, survival analysis, life checking out and telecommunications, and many others (see for example [17, 23, 24, 31, 33]). The applications have largely stimulated our present study. One of the most important branches of q-calculus is q-special functions. Jackson [19, 20], Andrews [4, 5], Gupta [16], Agarwal [2, 3], Ismail and Libis [18], Jain [21, 22], Jain and Vertna [23], Mishra [24], Sahai and Verma [25–27], Srivastava [30], Srivastava and Jain [31], Swarttouw [33], Verma and Sahai [34] introduced and discussed some interesting properties for various families of the basic Appell series, basic hypergeometric series, and q-Lauricella series by applying certain operators of q-calculus and its applications. Acikgoz et al. [1] and Araci et al. [6, 7] introduced a class of q-Euler, q-Frobenius–Euler, and q-Bernoulli polynomials based on q-exponential functions. Duran et al. [13, 14] introduced q-Bernoulli, q-Euler, and q-Genocchi polynomials and obtained the q-analogues of familiar earlier formulas and identities. In [8], Bagdasaryan et al. constructed Apostol q-Bernoulli, Apostol q-Genocchi, and Apostol q-Euler polynomials. Bansal and Choi [9], Bansal and Kumar [10], and Bansal et al. [11, 12] introduced and investigated the Pathway fractional integral formulas, fractional integral operators and integral transform of incomplete H-functions, incomplete $\aleph$-functions, incomplete I-functions, and S-generalized Gauss hypergeometric function. In [28, 29], Shehata investigated and discussed the generating functions for (p,q)-Bessel and (p,q)-Humbert functions.

Throughout this paper, we observe the following notations $0 < |q| < 1$, $q \in \mathbb{C} - \{1\}$. Let $\mathbb{N}$ and $\mathbb{C}$ be the sets of natural numbers and complex numbers, respectively.

The q-number (basic or quantum number) $[\beta]_q$ is defined by [5, 15, 17]

$$[\beta]_q = \frac{1 - q^\beta}{1 - q}, \quad \beta \in \mathbb{C}, 0 < |q| < 1, q \in \mathbb{C} - \{1\}. \tag{1.1}$$

The q-number and q-factorial are given by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{N}, 0 < |q| < 1, q \in \mathbb{C} - \{1\}; \\ 1, & n = 0, \end{cases} \tag{1.2}$$

and

$$[n]_q! = \begin{cases} \prod_{r=1}^{n} [r]_q = [n]_q [n-1]_q \dots [2]_q [1]_q = \frac{(q;q)_n}{(1-q)^n}, \\ \qquad n \in \mathbb{N}, 0 < |q| < 1, q \in \mathbb{C} - \{1\}; \\ 1, \quad n = 0, 0 < |q| < 1, q \in \mathbb{C} - \{1\}, \end{cases} \tag{1.3}$$

where $(\beta;q)_n$ is the q-shifted factorial (q-Pochhammer symbol) which is definoted as follows: for $n \in \mathbb{N}$, $\beta \in \mathbb{C} \setminus \{1, q^{-1}, q^{-2}, \dots, q^{1-n}\}$, $0 < |q| < 1$, $q \in \mathbb{C} - \{1\}$;

$$\begin{aligned} (\beta;q)_n &= \prod_{r=0}^{n-1} (1 - \beta q^r) = (1-\beta)(1-\beta q)\dots(1-\beta q^{n-1}), \\ (\beta;q)_0 &= 1, \quad \beta \in \mathbb{C}, 0 < |q| < 1, q \in \mathbb{C} - \{1\}. \end{aligned} \tag{1.4}$$

Note that taking limit as tends to (1.1) in the above relations gives the shifted factorial $(\beta)_n$ (see [32])

$$\lim_{q \longrightarrow 1^-} \frac{(q^\beta;q)_n}{(1-q)^n} = (\beta)_n.$$

We recall some notations and defifinitions from q-calculus for $\beta \in \mathbb{C}$, $n \in \mathbb{N}$, $0 < |q| < 1$, $q \in \mathbb{C} - \{1\}$, which are essential in the sequel (see [15]):

$$\begin{aligned} (\beta q;q)_n &= \frac{1 - \beta q^n}{1 - \beta} (\beta;q)_n \\ &= (1 - \beta q^{n-1})(\beta;q)_{n-1}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} (\beta q^{-1};q)_n &= \frac{1 - \beta q^{-1}}{1 - \beta q^{n-1}} (\beta;q)_n \\ &= (1 - \beta q^{-1})(\beta;q)_{n-1}, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} (\beta;q)_{n+k} &= (\beta;q)_n (\beta q^n;q)_k \\ &= (\beta;q)_k (\beta q^k;q)_n. \end{aligned} \tag{1.7}$$

Definition 1.1 Based on q-Pochhammer' symbol (1.4), we define the basic Horn hypergeometric functions $\mathbf{H}_3$ and $\mathbf{H}_4$ as follows:

$$\mathbf{H}_3(a,b;c;q,x,y) = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n, \quad c \neq 1, q^{-1}, q^{-2}, \dots, \tag{1.8}$$

and

$$\begin{aligned}&\mathbf{H}_4(a,b;c,d;q,x,y)\\ &\quad = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n, \quad c,d \neq 1, q^{-1}, q^{-2}, \dots .\end{aligned} \tag{1.9}$$

Remark 1.1 If $q \to 1$, the basic Horn hypergeometric functions reduce to the Horn hypergeometric functions defined in [[32], p. 56, Eq. (27), p. 57, Eq. (28)].

To simplify the notation, we write $\mathbf{H}_3$ for the series $\mathbf{H}_3(a,b;c;q,x,y)$, $\mathbf{H}_3(aq^{\pm 1})$ for the series $\mathbf{H}_3(aq^{\pm 1},b;c;q,x,y)$, $\mathbf{H}_4$ for the series $\mathbf{H}_4$, …, and $\mathbf{H}_4(cq^{\pm 1})$ stands for the series $\mathbf{H}_4(a;b,cq^{\pm 1};q,x,y)$.

For a wide variety of other investigations involving basic Horn hypergeometric functions, see, for instance, [2, 3, 5, 30]. Motivated by the previous works in q-analysis (see [25, 26, 34]), in this paper we introduce a class of new extended forms of the basic Horn hypergeometric function. Our study can be detailed as follows: In Sect. 2, we introduce and study some contiguous functions relations and q-differential formulas for our considered basic Horn hypergeometric functions $\mathbf{H}_3$ by permuting parameters. We discuss some family relations between basic Horn hypergeometric functions $\mathbf{H}_4$ in Sect. 3. Finally, we discuss our main results and related results involving contiguous relations for $\mathbf{H}_3$ and $\mathbf{H}_4$ in Sect. 4.

2 Contiguous functions for $\mathrm{H}_3(a,b;c;q,x,y)$

Here we derive several properties as well as the contiguous function relations for $\mathbf{H}_3$ with $c \neq 1, q^{-1}, q^{-2}, \dots$.

Theorem 2.1 *For $c \neq 1$, the contiguous relations of $\mathbf{H}_3$ hold true for the numerator parameter a:*

$$\begin{aligned}\mathbf{H}_3(aq) = \mathbf{H}_3 &+ \frac{ax(1-aq)}{1-c}\mathbf{H}_3\bigl(aq^2,b;cq;q,x,y\bigr)\\ &+ \frac{axq(1-aq)}{1-c}\mathbf{H}_3\bigl(aq^2,b;cq;q,xq,y\bigr)\\ &+ \frac{ay(1-b)}{1-c}\mathbf{H}_3\bigl(aq,bq;cq;q,xq^2,y\bigr),\end{aligned} \tag{2.1}$$

$$\begin{aligned}\mathbf{H}_3(aq) = \mathbf{H}_3 &+ \frac{ay(1-b)}{1-c}\mathbf{H}_3(aq,bq;cq;q,x,y)\\ &+ \frac{ax(1-aq)}{1-c}\mathbf{H}_3\bigl(aq^2,b;cq;q,x,yq\bigr)\\ &+ \frac{axq(1-aq)}{1-c}\mathbf{H}_3\bigl(aq^2,b;cq;q,xq,yq\bigr),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathbf{H}_3\big(aq^{-1}\big) &= \mathbf{H}_3 - \frac{ax(1-a)}{q(1-c)}\mathbf{H}_3(aq,b;cq;q,x,y)\\ &\quad - \frac{ax(1-a)}{1-c}\mathbf{H}_3(aq,b;cq;q,xq,y) - \frac{ay(1-b)}{q(1-c)}\mathbf{H}_3\big(a,bq;cq;q,xq^2,y\big),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathbf{H}_3\big(aq^{-1}\big) &= \mathbf{H}_3 - \frac{ay(1-b)}{q(1-c)}\mathbf{H}_3(a,bq;cq;q,x,y)\\ &\quad - \frac{ax(1-a)}{q(1-c)}\mathbf{H}_3(aq,b;cq;q,x,yq)\\ &\quad - \frac{ax(1-a)}{1-c}\mathbf{H}_3(aq,b;cq;q,xq,yq).\end{aligned} \tag{2.4}$$

Proof Replacing a by aq in (1.8), we get

$$\mathbf{H}_3(aq) - \mathbf{H}_3 = \sum_{m,n=0}^{\infty} \frac{(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}\big[(aq;q)_{2m+n} - (a;q)_{2m+n}\big]x^m y^n.$$

The relations

$$(a;q)_{2m+n} = (1-a)(aq;q)_{2m+n-1}$$

and

$$(aq;q)_{2m+n} = \big(1-aq^{2m+n}\big)(aq;q)_{2m+n-1},$$

imply

$$\begin{aligned}&\mathbf{H}_3(aq) - \mathbf{H}_3\\ &\quad= \sum_{m,n=0}^{\infty} \frac{a(1-q^{2m+n})(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n\\ &\quad= \sum_{m,n=0}^{\infty} \frac{a(1-q^m+q^m(1-q^m)+q^{2m}(1-q^n))(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n\\ &\quad= \sum_{m=1,n=0}^{\infty} \frac{a(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n}x^m y^n + \sum_{m=1,n=0}^{\infty} \frac{aq^m(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n}x^m y^n\\ &\qquad+ \sum_{m=0,n=1}^{\infty} \frac{aq^{2m}(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_{n-1}}x^m y^n\\ &\quad= \frac{ax(1-aq)}{1-c}\sum_{m,n=0}^{\infty} \frac{(aq^2;q)_{2m+n}(b;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n\\ &\qquad+ \frac{axq(1-aq)}{1-c}\sum_{m,n=0}^{\infty} \frac{q^m(aq^2;q)_{2m+n}(b;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n\\ &\qquad+ \frac{ay(1-b)}{1-c}\sum_{m,n=0}^{\infty} \frac{q^{2m}(aq;q)_{2m+n}(bq;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n\end{aligned}$$

$$= \frac{ax(1-aq)}{1-c}\mathbf{H}_3\big(aq^2,b;cq;q,x,y\big) + \frac{axq(1-aq)}{1-c}\mathbf{H}_3\big(aq^2,b;cq;q,xq,y\big) + \frac{ay(1-b)}{1-c}\mathbf{H}_3\big(aq,bq;cq;q,xq^2,y\big), \quad c \neq 1.$$

By using the relation $1-q^{2m+n} = 1-q^n + q^n(1-q^m) + q^{m+n}(1-q^m)$, we get (2.2). Performing the replacement $a \to aq^{-1}$ in the contiguous relations (2.1) and (2.2), we obtain (2.3) and (2.4). □

Theorem 2.2 *For $c \neq 1$, $\mathbf{H}_3$ satisfies the derivative equations*

$$\theta_{x,q}\mathbf{H}_3 = x\frac{(1-a)(1-aq)}{(1-c)(1-q)}\mathbf{H}_3\big(aq^2,b;cq;q,x,y\big) \tag{2.5}$$

and

$$\theta_{y,q}\mathbf{H}_3 = y\frac{(1-a)(1-b)}{(1-c)(1-q)}\mathbf{H}_3(aq,bq;cq;q,x,y). \tag{2.6}$$

Proof From (1.8), we consider the operators $\theta_{x,q} = x\frac{\partial}{\partial x} = xD_{x,q}$ and $\theta_{y,q} = y\frac{\partial}{\partial y} = yD_{y,q}$ to get

$$\begin{aligned}
\theta_{x,q}\mathbf{H}_3 &= \sum_{m,n=0}^{\infty}\left[\frac{1-q^m}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n \\
&= \sum_{m=1,n=0}^{\infty}\left[\frac{1}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n}x^m y^n \\
&= \sum_{m,n=0}^{\infty}\left[\frac{1}{1-q}\right]\frac{(a;q)_{2m+n+2}(b;q)_n}{(c;q)_{m+n+1}(q;q)_m(q;q)_n}x^{m+1} y^n \\
&= x\frac{(1-a)(1-aq)}{(1-c)(1-q)}\mathbf{H}_3\big(aq^2,b;cq;q,x,y\big), \quad c \neq 1,
\end{aligned}$$

and

$$\begin{aligned}
\theta_{y,q}\mathbf{H}_3 &= \sum_{m,n=0}^{\infty}\left[\frac{1-q^n}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n \\
&= \sum_{m=0,n=1}^{\infty}\left[\frac{1}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_{n-1}}x^m y^n \\
&= \sum_{m,n=0}^{\infty}\left[\frac{1}{1-q}\right]\frac{(a;q)_{2m+n+1}(b;q)_{n+1}}{(c;q)_{m+n+1}(q;q)_m(q;q)_n}x^m y^{n+1} \\
&= y\frac{(1-a)(1-b)}{(1-c)(1-q)}\mathbf{H}_3(aq,bq;cq;q,x,y), \quad c \neq 1.
\end{aligned}$$

□

Theorem 2.3 $\mathbf{H}_3$ *satisfies the difference equations*

$$\left[a\theta_{x,q} + \frac{1-a}{1-q}\right]\mathbf{H}_3 + a\theta_{x,q}\mathbf{H}_3(xq) + a\theta_{y,q}\mathbf{H}_3\big(xq^2\big) = \frac{1-a}{1-q}\mathbf{H}_3(aq), \tag{2.7}$$

$$\left[a\theta_{y,q} + \frac{1-a}{1-q}\right]\mathbf{H}_3 + a\theta_{x,q}\mathbf{H}_3(yq) + a\theta_{x,q}\mathbf{H}_3(xq,yq) = \frac{1-a}{1-q}\mathbf{H}_3(aq), \tag{2.8}$$

$$\left[aq^{-1}\theta_{x,q} + \frac{1-aq^{-1}}{1-q}\right]\mathbf{H}_3\left(aq^{-1}\right) + aq^{-1}\theta_{y,q}\mathbf{H}_3\left(aq^{-1},xq\right) + aq^{-1}\theta_{y,q}\mathbf{H}_3\left(aq^{-1},xq,yq\right)$$
$$= \frac{1-aq^{-1}}{1-q}\mathbf{H}_3, \tag{2.9}$$

and

$$\left[aq^{-1}\theta_{y,q} + \frac{1-aq^{-1}}{1-q}\right]\mathbf{H}_3\left(aq^{-1}\right) + aq^{-1}\theta_{x,q}\mathbf{H}_3\left(aq^{-1},yq\right)$$
$$+ aq^{-1}\theta_{x,q}\mathbf{H}_3\left(aq^{-1},xq,yq\right)$$
$$= \frac{1-aq^{-1}}{1-q}\mathbf{H}_3. \tag{2.10}$$

Proof With the help of the above differential operators and using (2.5) and (2.6) for $\mathbf{H}_3$, we get the results (2.7)–(2.10). □

Theorem 2.4 *For $c \neq 1$, the contiguous function relations of $\mathbf{H}_3$ with the numerator parameter b give*

$$\mathbf{H}_3(bq) = \mathbf{H}_3 + \frac{by(1-a)}{1-c}\mathbf{H}_3(aq,bq;cq;q,x,y) \tag{2.11}$$

and

$$\mathbf{H}_3\left(bq^{-1}\right) = \mathbf{H}_3 - \frac{by(1-a)}{q(1-c)}\mathbf{H}_3(aq,b;cq;q,x,y). \tag{2.12}$$

Proof If we replace b by bq in (1.8), we get

$$\mathbf{H}_3(bq) - \mathbf{H}_3 = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}}{(c;q)_{m+n}(q;q)_m(q;q)_n}\left[(bq;q)_n - (b;q)_n\right]x^m y^n.$$

Using the relations

$$(b;q)_n = (1-b)(bq;q)_{n-1}$$

and

$$(bq;q)_n = \left(1-bq^n\right)(bq;q)_{n-1},$$

we have

$$\mathbf{H}_3(bq) - \mathbf{H}_3 = \sum_{m,n=0}^{\infty} \frac{b(1-q^n)(a;q)_{2m+n}(bq;q)_{n-1}}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n$$
$$= \sum_{m=0,n=1}^{\infty} \frac{b(a;q)_{2m+n}(bq;q)_{n-1}}{(c;q)_{m+n}(q;q)_m(q;q)_{n-1}}x^m y^n$$

$$= \frac{by(1-a)}{1-c} \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n}(bq;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n$$

$$= \frac{by(1-a)}{1-c} \mathbf{H}_3(aq, bq; cq; q, x, y), \quad c \neq 1.$$

Replacing $b \to bq^{-1}$ in relation (2.11), we obtain (2.12). □

Theorem 2.5 *The difference equations hold true for* $\mathbf{H}_3$:

$$\left[b\theta_{y,q} + \frac{1-b}{1-q}\right]\mathbf{H}_3 = \frac{1-b}{1-q}\mathbf{H}_3(bq) \tag{2.13}$$

and

$$\left[bq^{-1}\theta_{y,q} + \frac{1-bq^{-1}}{1-q}\right]\mathbf{H}_3\left(bq^{-1}\right) = \frac{1-bq^{-1}}{1-q}\mathbf{H}_3. \tag{2.14}$$

Proof From (2.5) and (2.6), we get (2.13) and (2.14). □

Theorem 2.6 *For* $c \neq 1$, $b \neq q$, *the formulas hold true for* $\mathbf{H}_3$:

$$\begin{aligned}
&\mathbf{H}_3\left(aq, bq^{-1}; c; q, x, y\right) \\
&\quad = \mathbf{H}_3 + \frac{ay}{1-c}\mathbf{H}_3(aq, b; c, dq; q, x, y) \\
&\qquad + \frac{ax(1-aq)}{1-bq^{-1}}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, x, yq\right) \\
&\qquad + \frac{axq(1-aq)}{1-bq^{-1}}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, xq, yq\right) \\
&\qquad - \frac{abq^{-1}x(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, x, yq\right) \\
&\qquad - \frac{abx(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, xq, yq\right) \\
&\qquad - \frac{bq^{-1}y}{1-c}\mathbf{H}_3(aq, b; cq; q, x, y)
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\mathbf{H}_3\left(aq, bq^{-1}; c; q, x, y\right) \\
&\quad = \mathbf{H}_3 + \frac{ax(1-aq)}{(1-c)(1-bq^{-1})}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, x, y\right) \\
&\qquad + \frac{axq(1-aq)}{1-bq^{-1}}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, xq, y\right) \\
&\qquad + \frac{ay}{1-bq^{-1}}\mathbf{H}_3\left(aq^2, bq; cq; q, xq^2, y\right) \\
&\qquad - \frac{abxq(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_3\left(aq^2, bq^{-1}; cq; q, x, yq\right)
\end{aligned} \tag{2.16}$$

$$- \frac{abx(1-aq)}{(1-bq^{-1})(1-c)} \mathbf{H}_3\big(aq^2, bq^{-1}; cq; q, xq, yq\big)$$
$$- \frac{bq^{-1}(1-aq)y}{1-c} \mathbf{H}_3\big(aq^2, b; cq; q, x, y\big).$$

Proof Using (1.8), (1.5), (1.6), (1.7) and the relation $(bq^{-1};q)_n = (1-bq^{-1})(b;q)_{n-1}$ implies

$$\begin{aligned}
&\mathbf{H}_3\big(aq, bq^{-1}; c; q, x, y\big) - \mathbf{H}_3 \\
&= \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_m(q;q)_n} \big[abq^{2m+n-1} - bq^{-1} - aq^{2m+n} + a - abq^{n-1} + bq^{n-1}\big] x^m y^n \\
&= a \sum_{m=0,n=1}^{\infty} \frac{(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_m(q;q)_{n-1}} x^m y^n + a \sum_{m=1,n=0}^{\infty} \frac{q^n(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n} x^m y^n \\
&\quad + a \sum_{m=1,n=0}^{\infty} \frac{q^{m+n}(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n} x^m y^n \\
&\quad - ab \sum_{m=1,n=0}^{\infty} \frac{q^{n-1}(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n} x^m y^n \\
&\quad - ab \sum_{m=1,n=0}^{\infty} \frac{q^{m+n-1}(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_{m-1}(q;q)_n} x^m y^n \\
&\quad - bq^{-1} \sum_{m=0,n=1}^{\infty} \frac{(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_{m+n}(q;q)_m(q;q)_{n-1}} x^m y^n \\
&= \frac{a}{1-c} \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n}(b;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^m y^{n+1} \\
&\quad + \frac{a(1-aq)}{(1-bq^{-1})(1-c)} \sum_{m,n=0}^{\infty} \frac{q^n(aq^2;q)_{2m+n}(bq^{-1};q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^{m+1} y^n \\
&\quad + \frac{a(1-aq)}{(1-bq^{-1})(1-c)} \sum_{m,n=0}^{\infty} \frac{q^{m+n+1}(aq^2;q)_{2m+n}(bq^{-1};q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^{m+1} y^n \\
&\quad - \frac{ab}{(1-bq^{-1})(1-c)} \sum_{m,n=0}^{\infty} \frac{q^{n-1}(aq^2;q)_{2m+n}(bq^{-1};q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^{m+1} y^n \\
&\quad - \frac{ab(1-aq)}{(1-bq^{-1})(1-c)} \sum_{m,n=0}^{\infty} \frac{q^{m+n}(aq^2;q)_{2m+n}(bq^{-1};q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^{m+1} y^n \\
&\quad - \frac{bq^{-1}}{1-c} \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n}(b;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n} x^m y^{n+1} \\
&= \frac{ay}{1-c} \mathbf{H}_3(aq, b; cq; q, x, y) + \frac{ax(1-aq)}{1-bq^{-1}} \mathbf{H}_3\big(aq^2, bq^{-1}; cq; q, x, yq\big) \\
&\quad + \frac{axq(1-aq)}{1-bq^{-1}} \mathbf{H}_3\big(aq^2, bq^{-1}; cq; q, xq, yq\big) \\
&\quad - \frac{abq^{-1}x(1-aq)}{(1-bq^{-1})(1-c)} \mathbf{H}_3\big(aq^2, bq^{-1}; cq; q, x, yq\big)
\end{aligned}$$

$$- \frac{abx(1-aq)}{(1-bq^{-1})(1-c)} \mathbf{H}_3\bigl(aq^2, bq^{-1}; cq; q, xq, yq\bigr)$$
$$- \frac{bq^{-1}y}{1-c} \mathbf{H}_3(aq, b; cq; q, x, y), \quad c \neq 1, b \neq q.$$

Using the relation $a(1-q^{2m+n}) - abq^{n-1}(1-q^{2m}) - bq^{-1}(1-q^n) = a(1-q^m) + aq^m(1-q^m) + aq^{2m}(1-q^n) - abq^{n-1}(1-q^m) - abq^{m+n-1}(1-q^m) - bq^{-1}(1-q^n)$, we obtain (2.16). □

Theorem 2.7 *Each of the following properties for* $\mathbf{H}_3$ *in* (1.8) *holds true*:

$$\begin{aligned} &b(1-a)\mathbf{H}_3(aq) - a(1-b)\mathbf{H}_3(bq) \\ &\quad = (b-a)\mathbf{H}_3 + \frac{abx(1-a)(1-aq)}{1-c} \mathbf{H}_3\bigl(aq^2, b; cq; q, x, yq\bigr) \\ &\qquad + \frac{abxq(1-a)(1-aq)}{1-c} \mathbf{H}_3\bigl(aq^2, b; cq; q, xq, yq\bigr), \quad c \neq 1. \end{aligned} \tag{2.17}$$

Proof From (1.8), we have

$$\begin{aligned} &b(1-a)\mathbf{H}_3(aq) - a(1-b)\mathbf{H}_3(bq) \\ &\quad = \sum_{m,n=0}^{\infty} \frac{b(1-a)(aq;q)_{2m+n}(b;q)_n - a(1-b)(a;q)_{2m+n}(bq;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n. \end{aligned}$$

By using the equation

$$(1-a)(aq;q)_{2m+n} = \bigl(1-aq^{2m+n}\bigr)(a;q)_{2m+n}$$

and

$$(1-b)(bq;q)_n = \bigl(1-bq^n\bigr)(b;q)_n,$$

we get

$$\begin{aligned} &b(1-a)\mathbf{H}_3(aq) - a(1-b)\mathbf{H}_3(bq) \\ &\quad = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n[b(1-aq^{2m+n}) - a(1-bq^n)]}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \\ &\quad = (b-a) \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \\ &\qquad + ab \sum_{m,n=0}^{\infty} \frac{q^n(1-q^m)(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(1-q^m)(q;q)_{m-1}(q;q)_n} x^m y^n \\ &\qquad + ab \sum_{m,n=0}^{\infty} \frac{q^{n+m}(1-q^m)(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(1-q^m)(q;q)_{m-1}(q;q)_n} x^m y^n \\ &\quad = (b-a)\mathbf{H}_3 + ab \sum_{m,n=0}^{\infty} \frac{q^n(a;q)_{2m+n+2}(b;q)_n}{(c;q)_{m+n+1}(q;q)_m(q;q)_n} x^{m+1} y^n \end{aligned}$$

$$+ ab\sum_{m,n=0}^{\infty}\frac{q^{n+m+1}(a;q)_{2m+n+2}(b;q)_n}{(c;q)_{m+n+1}(q;q)_m(q;q)_n}x^{m+1}y^n$$

$$= (b-a)\mathbf{H}_3 + \frac{abx(1-a)(1-aq)}{1-c}\mathbf{H}_3\big(aq^2,b;cq;q,x,yq\big)$$

$$+ \frac{abxq(1-a)(1-aq)}{1-c}\mathbf{H}_3\big(aq^2,b;cq;q,xq,yq\big),\quad c\neq 1.\qquad \square$$

Theorem 2.8 *The contiguous function relations of* $\mathbf{H}_3$ *with denominator parameter* c *are valid*:

$$\begin{aligned}\mathbf{H}_3\big(cq^{-1}\big) &= \mathbf{H}_3 + \frac{cx(1-a)(1-aq)}{(q-c)(1-c)}\mathbf{H}_3\big(aq^2,b;cq;q,x,y\big)\\ &\quad + \frac{cy(1-a)(1-b)}{(q-c)(1-c)}\mathbf{H}_3(aq,bq;cq;q,xq,y),\quad c\neq 1,q,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\mathbf{H}_3\big(cq^{-1}\big) &= \mathbf{H}_3 + \frac{cy(1-a)(1-b)}{(q-c)(1-c)}\mathbf{H}_3(aq,bq;cq;q,x,y)\\ &\quad + \frac{cx(1-a)(1-aq)(1-b)}{(q-c)(1-c)}\mathbf{H}_3\big(aq^2,b;cq;q,x,yq\big),\quad c\neq 1,q,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathbf{H}_3(cq) &= \mathbf{H}_3 - \frac{cx(1-a)(1-aq)}{(1-cq)(1-c)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big)\\ &\quad - \frac{cy(1-a)(1-b)}{(1-cq)(1-c)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq,y\big),\quad c\neq 1,q^{-1},\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\mathbf{H}_3(cq) &= \mathbf{H}_3 - \frac{cy(1-a)(1-b)}{(1-cq)(1-c)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big)\\ &\quad - \frac{cx(1-a)(1-aq)}{(1-cq)(1-c)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big),\quad c\neq 1,q^{-1}.\end{aligned} \tag{2.21}$$

Proof By the definition of basic Horn function, we get

$$\begin{aligned}\mathbf{H}_3\big(cq^{-1}\big) - \mathbf{H}_3 &= \sum_{m,n=0}^{\infty}\frac{(a;q)_{2m+n}(b;q)_n}{(q;q)_m(q;q)_n}\left[\frac{1}{(cq^{-1};q)_{m+n}} - \frac{1}{(c;q)_{m+n}}\right]x^my^n\\ &= \sum_{m,n=0}^{\infty}\frac{(a;q)_{2m+n}(b;q)_n}{(q;q)_m(q;q)_n}\frac{(c;q)_{m+n}-(cq^{-1};q)_{m+n}}{(cq^{-1};q)_{m+n}(c;q)_{m+n}}x^my^n.\end{aligned}$$

Using

$$\big(cq^{-1};q\big)_{m+n} = \big(1-cq^{-1}\big)(c;q)_{m+n-1}$$

and

$$(c;q)_{m+n} = \big(1-cq^{m+n-1}\big)(c;q)_{m+n-1},$$

we can rewrite the above equation as follows:

$$\begin{aligned}
&\mathbf{H}_3\bigl(cq^{-1}\bigr) - \mathbf{H}_3 \\
&= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(q;q)_m(q;q)_n} \frac{[1-q^{m+n}]cq^{-1}(c;q)_{m+n-1}}{[1-cq^{-1}](c;q)_{m+n-1}(c;q)_{m+n}} x^m y^n \\
&= c \sum_{m,n=0}^{\infty} \frac{1-q^m}{q-c} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \\
&\quad + c \sum_{m,n=0}^{\infty} q^m \frac{1-q^n}{q-c} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \\
&= \frac{c}{q-c} \sum_{m,n=0}^{\infty} \frac{(1-a)(1-aq)(aq^2;q)_{2m+n}(b;q)_n}{(1-c)(cq;q)_{m+n}(q;q)_m(q;q)_n} x^{m+1} y^n \\
&\quad + \frac{c}{q-c} \sum_{m,n=0}^{\infty} q^m \frac{(1-a)(a;q)_{2m+n}(1-b)(b;q)_n}{(1-c)(c;q)_{m+n+1}(q;q)_m(q;q)_n} x^m y^{n+1} \\
&= \frac{cx(1-a)(1-aq)}{(q-c)(1-c)} \mathbf{H}_3\bigl(aq^2,b;cq;q,x,y\bigr) \\
&\quad + \frac{cy(1-a)(1-b)}{(q-c)(1-c)} \mathbf{H}_3(aq,bq;cq;q,xq,y), \quad c \neq 1,q,
\end{aligned}$$

which is the desired result. The proof of Eq. (2.19) can run parallel to Eq. (2.18), so details are omitted here.

If we replace $c \to cq$ in relation (2.18), we obtain

$$\begin{aligned}
\mathbf{H}_3(cq) = \mathbf{H}_3 &- \frac{cx(1-a)(1-aq)}{(1-cq)(1-c)} \mathbf{H}_3\bigl(aq^2,b;cq^2;q,x,y\bigr) \\
&- \frac{cy(1-a)(1-b)}{(1-cq)(1-c)} \mathbf{H}_3\bigl(aq,bq;cq^2;q,xq,y\bigr), \quad c \neq 1,q^{-1}.
\end{aligned}$$

The proof of Eq. (2.21) would run parallel to Eq. (2.20), so we may skip the involved details. □

Theorem 2.9 *The derivative formulas for* $\mathbf{H}_3$ *are satisfied*:

$$\mathbf{H}_3\bigl(cq^{-1}\bigr) = \frac{c}{c-q}\mathbf{H}_3(a,b;c;q,xq,yq) - \frac{q}{c-q}\mathbf{H}_3, \quad c \neq q, \tag{2.22}$$

$$\mathbf{H}_3(cq) + (c-1)\mathbf{H}_3 = c\mathbf{H}_3(a,b;cq;q,xq,yq), \tag{2.23}$$

$$\biggl[cq^{-1}\theta_{x,q} + \frac{1-cq^{-1}}{1-q}\biggr]\mathbf{H}_3 + cq^{-1}\theta_{y,q}\mathbf{H}_3(xq) = \frac{1-cq^{-1}}{1-q}\mathbf{H}_3\bigl(cq^{-1}\bigr), \tag{2.24}$$

and

$$\biggl[cq^{-1}\theta_{y,q} + \frac{1-cq^{-1}}{1-q}\biggr]\mathbf{H}_3 + cq^{-1}\theta_{x,q}\mathbf{H}_3(yq) = \frac{1-cq^{-1}}{1-q}\mathbf{H}_3\bigl(cq^{-1}\bigr). \tag{2.25}$$

Proof Using the definition $\mathbf{H}_3$ in (1.8) with the relation $\frac{1}{(cq^{-1};q)_{m+n}} = \frac{1}{(c;q)_{m+n}}[\frac{c}{c-q}q^{m+n} - \frac{q}{c-q}]$, we get

$$\begin{aligned}\mathbf{H}_3(cq^{-1}) &= \sum_{m,n=0}^{\infty}\left[\frac{c}{c-q}q^{m+n} - \frac{q}{c-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}x^m y^n \\ &= \frac{c}{c-q}\mathbf{H}_3(a,b;c;q,xq,yq) - \frac{q}{c-q}\mathbf{H}_3, \quad c \neq q.\end{aligned}$$

Replacing $c = cq$ in (2.22) implies the contiguous relation

$$\mathbf{H}_3(cq) + (c-1)\mathbf{H}_3 = c\mathbf{H}_3(a,b;cq;q,xq,yq).$$

By using Eqs. (2.5) and (2.6), we obtain the required results (2.23) and (2.24). □

Theorem 2.10 *For $c \neq 1$ and $cq \neq 1$, the following formulas are valid*:

$$\begin{aligned}&\mathbf{H}_3(aq,b;cq;q,x,y) - \mathbf{H}_3 \\ &\quad= \frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,x,y) \\ &\qquad+ \frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,x,y) \\ &\qquad+ \frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,xq,y) \\ &\qquad+ \frac{cy(1-b)}{(1-c)(1-cq)}\mathbf{H}_3(aq,bq;cq^2;q,xq,y) \\ &\qquad+ \frac{ay(1-b)}{(1-c)(1-cq)}\mathbf{H}_3(aq,bq;cq^2;q,xq^2,y) \\ &\qquad- \frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,bq;cq^2;q,xq,yq),\end{aligned} \tag{2.26}$$

$$\begin{aligned}&\mathbf{H}_3(aq,b;cq;q,x,y) - \mathbf{H}_3 \\ &\quad= \frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,x,y) \\ &\qquad+ \frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,xq,y) \\ &\qquad+ \frac{cy(1-b)}{(1-c)(1-cq)}\mathbf{H}_3(aq,bq;cq^2;q,xq^2,y) \\ &\qquad+ \frac{ay(1-b)}{(1-c)(1-cq)}\mathbf{H}_3(aq,bq;cq^2;q,xq^2,y) \\ &\qquad+ \frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,x,yq) \\ &\qquad- \frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3(aq^2,b;cq^2;q,xq,yq),\end{aligned} \tag{2.27}$$

$$\begin{aligned}
&\mathbf{H}_3(aq,b;cq;q,x,y)-\mathbf{H}_3\\
&\quad=\frac{ax}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big)\\
&\qquad+\frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,xq,yq\big)\\
&\qquad+\frac{cy(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big)\\
&\qquad+\frac{ay(1-b)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq,y\big)\\
&\qquad-\frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,bq;cq^2;q,xq,yq\big),
\end{aligned}\tag{2.28}$$

and

$$\begin{aligned}
&\mathbf{H}_3(aq,b;cq;q,x,y)-\mathbf{H}_3\\
&\quad=\frac{ay(1-b)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big)\\
&\qquad+\frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,xq,yq\big)\\
&\qquad+\frac{cy(1-b)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big)\\
&\qquad+\frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big)\\
&\qquad-\frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,bq;cq^2;q,xq,yq\big).
\end{aligned}\tag{2.29}$$

Proof Replacing a and c by aq and cq in (1.8), we get

$$\begin{aligned}
&\mathbf{H}_3(aq,b;cq;q,x,y)-\mathbf{H}_3\\
&\quad=\sum_{m,n=0}^{\infty}\biggl[\frac{(aq;q)_{2m+n}(b;q)_n}{(cq;q)_{m+n}(q;q)_m(q;q)_n}-\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n}\biggr]x^my^n\\
&\quad=\sum_{m,n=0}^{\infty}\frac{(aq;q)_{2m+n-1}(cq;q)_{m+n-1}(b;q)_n}{(q;q)_m(q;q)_n}\\
&\qquad\times\biggl[\frac{(1-aq^{2m+n})(1-c)-(1-a)(1-cq^{m+n})}{(1-cq^{m+n})(cq;q)_{m+n-1}(c;q)_{m+n}}\biggr]x^my^n\\
&\quad=\frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big)+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,xq,y\big)\\
&\qquad+\frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big)+\frac{cy(1-b)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq,y\big)
\end{aligned}$$

$$+ \frac{ay(1-b)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq^2,y\big)$$
$$- \frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,bq;cq^2;q,xq,yq\big), \quad c \neq 1, q^{-1}.$$

By using the relations

$$\begin{aligned}
&a\big(1-q^{2m+n}\big) + c\big(1-q^{m+n}\big) - acq^{m+n}\big(1-q^m\big) \\
&\quad = a\big(1-q^m\big) + aq^m\big(1-q^m\big) + aq^{2m}\big(1-q^n\big) + c\big(1-q^n\big) \\
&\qquad + cq^n\big(1-q^m\big) - acq^{m+n}\big(1-q^m\big), \\
&\quad = a\big(1-q^n\big) + aq^n\big(1-q^m\big) + aq^{m+n}\big(1-q^m\big) + c\big(1-q^m\big) \\
&\qquad + cq^m\big(1-q^n\big) - acq^{m+n}\big(1-q^m\big), \\
&\quad = a\big(1-q^n\big) + aq^n\big(1-q^m\big) + aq^{m+n}\big(1-q^m\big) + c\big(1-q^n\big) \\
&\qquad + cq^n\big(1-q^m\big) - acq^{m+n}\big(1-q^m\big).
\end{aligned}$$

The proof of Eqs. (2.27)–(2.29) would run parallel to Eq. (2.26) by using the above relations, so we omit the involved details. □

Theorem 2.11 *For* $\mathbf{H}_3$ *defined by* (1.8), *each of the formulas holds*:

$$\begin{aligned}
&\mathbf{H}_3(a,bq;cq;q,x,y) \\
&\quad = \mathbf{H}_3 + \frac{by(1-a)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big) \\
&\qquad + \frac{cx(1-a)(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big) \\
&\qquad + \frac{cy(1-a)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq,y\big) \\
&\qquad + \frac{cbx(1-a)(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big)
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
&\mathbf{H}_3(a,bq;cq;q,x,y) \\
&\quad = \mathbf{H}_3 + \frac{(b+c)y(1-a)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big) \\
&\qquad + \frac{c(b+1)x(1-a)(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big)
\end{aligned} \tag{2.31}$$

for $c \neq 1$ *and* $cq \neq 1$.

Proof From (1.8), we have

$$
\begin{aligned}
&\mathbf{H}_3(a,bq;cq;q,x,y)-\mathbf{H}_3\\
&=\sum_{m,n=0}^{\infty}\frac{(a;q)_{2m+n}}{(q;q)_m(q;q)_n}\left[\frac{(bq;q)_n}{(cq;q)_{m+n}}-\frac{(b;q)_n}{(c;q)_{m+n}}\right]x^my^n\\
&=\sum_{m,n=0}^{\infty}\frac{(a;q)_{2m+n}(cq;q)_{m+n-1}(bq;q)_{n-1}}{(q;q)_m(q;q)_n}\left[\frac{(1-bq^n)(1-c)-(1-b)(1-cq^{m+n})}{(1-cq^{m+n})(cq;q)_{m+n-1}(c;q)_{m+n}}\right]x^my^n\\
&=\sum_{m,n=0}^{\infty}\frac{(a;q)_{2m+n}(bq;q)_{n-1}}{(q;q)_m(q;q)_n}\left[\frac{b(1-q^n)+c(1-q^{m+n})+bcq^n(1-q^m)}{(1-c)(cq;q)_{m+n}}\right]x^my^n\\
&=\frac{by(1-a)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,x,y\big)+\frac{cx(1-a)(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,y\big)\\
&\quad+\frac{cy(1-a)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq,bq;cq^2;q,xq,y\big)\\
&\quad+\frac{cbx(1-a)(1-aq)}{(1-c)(1-cq)}\mathbf{H}_3\big(aq^2,b;cq^2;q,x,yq\big),\qquad c\neq 1,cq\neq 1.
\end{aligned}
$$

Hence, we obtain (2.30), one can derive the result (2.31) by a similar way. □

3 Contiguous functions for H_4

Relying on a similar procedure as the one used in the previous section, we obtain the following list of results for basic Horn function $\mathbf{H}_4(a,b;c,d;q,x,y)$ with $c,d\neq 1,q^{-1},q^{-2},\ldots$.

Theorem 3.1 *For $c\neq 1$ and $d\neq 1$, the contiguous function relations of $\mathbf{H}_4$ with numerator parameter a hold true*:

$$
\begin{aligned}
\mathbf{H}_4(aq)&=\mathbf{H}_4+\frac{ax(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,x,y\big)\\
&\quad+\frac{axq(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,xq,y\big)\\
&\quad+\frac{ay(1-b)}{1-d}\mathbf{H}_4\big(aq,bq;c,dq;q,xq^2,y\big),
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\mathbf{H}_4(aq)&=\mathbf{H}_4+\frac{ay(1-b)}{1-d}\mathbf{H}_4(aq,bq;c,dq;q,x,y)\\
&\quad+\frac{ax(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,x,yq\big)\\
&\quad+\frac{axq(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,xq,yq\big),
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\mathbf{H}_4\big(aq^{-1}\big)&=\mathbf{H}_4-\frac{aq^{-1}x(1-a)}{1-c}\mathbf{H}_4(aq,b;cq,d;q,x,y)\\
&\quad-\frac{ax(1-a)}{1-c}\mathbf{H}_4(aq,b;cq,d;q,xq,y)\\
&\quad-\frac{aq^{-1}y(1-b)}{1-d}\mathbf{H}_4\big(a,bq;c,dq;q,xq^2,y\big),
\end{aligned}
\tag{3.3}
$$

and

$$\begin{aligned}\mathbf{H}_4\big(aq^{-1}\big) &= \mathbf{H}_4 - \frac{ay(1-b)}{1-d}\mathbf{H}_4(a,bq;c,dq;q,x,y)\\ &\quad - \frac{aq^{-1}x(1-a)}{1-c}\mathbf{H}_4(aq,b;cq,d;q,x,yq)\\ &\quad - \frac{ax(1-a)}{1-c}\mathbf{H}_4(aq,b;cq,d;q,xq,yq).\end{aligned} \tag{3.4}$$

Proof In (1.9), replacing a by aq, we get

$$\mathbf{H}_4(aq) - \mathbf{H}_4 = \sum_{m,n=0}^{\infty} \frac{(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}\big[(aq;q)_{2m+n} - (a;q)_{2m+n}\big]x^m y^n.$$

Using the relations

$$(a;q)_{2m+n} = (1-a)(aq;q)_{2m+n-1}$$

and

$$(aq;q)_{2m+n} = \big(1-aq^{2m+n}\big)(aq;q)_{2m+n-1},$$

we get

$$\begin{aligned}\mathbf{H}_4(aq) - \mathbf{H}_4 &= \sum_{m,n=0}^{\infty} \frac{a(1-q^{2m+n})(aq;q)_{2m+n-1}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n\\ &= \sum_{m,n=0}^{\infty} \frac{a(aq;q)_{2m+n+1}(b;q)_n}{(c;q)_{m+1}(d;q)_n(q;q)_m(q;q)_n}x^{m+1}y^n\\ &\quad + \sum_{m,n=0}^{\infty} \frac{aq^{m+1}(aq;q)_{2m+n+1}(b;q)_n}{(c;q)_{m+1}(d;q)_n(q;q)_m(q;q)_n}x^{m+1}y^n\\ &\quad + \sum_{m,n=0}^{\infty} \frac{aq^{2m}(aq;q)_{2m+n}(b;q)_{n+1}}{(c;q)_m(d;q)_{n+1}(q;q)_m(q;q)_n}x^m y^{n+1}\\ &= \frac{ax(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,x,y\big)\\ &\quad + \frac{axq(1-aq)}{1-c}\mathbf{H}_4\big(aq^2,b;cq,d;q,xq,y\big)\\ &\quad + \frac{ay(1-b)}{1-d}\mathbf{H}_4\big(aq,bq;c,dq;q,xq^2,y\big), \quad c\neq 1, d\neq 1.\end{aligned}$$

The proof of Eq. (3.2) would run parallel to Eq. (3.2), so details are omitted here. Performing the replacement $a \to aq^{-1}$ in the contiguous relations (3.1) and (3.2), we get (3.3) and (3.4). □

Theorem 3.2 *The q-derivatives of* $\mathbf{H}_4$ *defined in* (1.9) *are valid*:

$$D_{x,q}^r \mathbf{H}_4 = \frac{(a;q)_{2r}}{(c;q)_r(1-q)^r}\mathbf{H}_4\big(aq^{2r},b;cq^r,d;q,x,y\big), \quad c\neq 1,q^{-1},q^{-2},\ldots \tag{3.5}$$

and

$$D^r_{y,q}\mathbf{H}_4 = \frac{(a;q)_r(b;q)_r}{(d;q)_r(1-q)^r}\mathbf{H}_4\big(aq^r, bq^r; c, dq^r; q, x, y\big), \quad d \neq 1, q^{-1}, q^{-2}, \ldots. \tag{3.6}$$

Proof In (1.9), we apply the operators $\frac{\partial}{\partial x} = D_{x,q}$ and $\frac{\partial}{\partial y} = D_{y,q}$ to get

$$\begin{aligned} D_{x,q}\mathbf{H}_4 &= \sum_{m,n=0}^{\infty}\left[\frac{1-q^m}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^{m-1}y^n \\ &= \frac{(1-a)(1-aq)}{(1-q)(1-c)}\sum_{m,n=0}^{\infty}\frac{(aq^2;q)_{2m+n}(b;q)_n}{(cq;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n \\ &= \frac{(1-a)(1-aq)}{(1-c)(1-q)}\mathbf{H}_4\big(aq^2, b; cq, d; q, x, y\big), \quad c \neq 1 \end{aligned}$$

and

$$\begin{aligned} D_{y,q}\mathbf{H}_4 &= \sum_{m,n=0}^{\infty}\left[\frac{1-q^n}{1-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^{n-1} \\ &= \frac{(1-a)(1-b)}{(1-q)(1-c)}\sum_{m,n=0}^{\infty}\frac{(aq;q)_{2m+n}(bq;q)_n}{(c;q)_m(dq;q)_n(q;q)_m(q;q)_n}x^m y^n \\ &= \frac{(1-a)(1-b)}{(1-d)(1-q)}\mathbf{H}_4(aq, bq; cq, d; q, x, y), \quad d \neq 1. \end{aligned}$$

Iterating this technique n times on $\mathbf{H}_4$, we obtain (3.5) and (3.6). □

Theorem 3.3 *For* $\mathbf{H}_4$ *defined in* (1.9), *we have*

$$\left[a\theta_{x,q} + \frac{1-a}{1-q}\right]\mathbf{H}_4 + a\theta_{x,q}\mathbf{H}_4(xq) + a\theta_{y,q}\mathbf{H}_4\big(xq^2\big) = \frac{1-a}{1-q}\mathbf{H}_4(aq), \tag{3.7}$$

$$\left[a\theta_{y,q} + \frac{1-a}{1-q}\right]\mathbf{H}_4 + a\theta_{x,q}\mathbf{H}_4(yq) + a\theta_{x,q}\mathbf{H}_4(xq, yq) = \frac{1-a}{1-q}\mathbf{H}_4(aq), \tag{3.8}$$

$$\begin{aligned} &\left[aq^{-1}\theta_{x,q} + \frac{1-aq^{-1}}{1-q}\right]\mathbf{H}_4\big(aq^{-1}\big) + aq^{-1}\theta_{y,q}\mathbf{H}_4\big(aq^{-1}, xq\big) + aq^{-1}\theta_{y,q}\mathbf{H}_4\big(aq^{-1}, xq, yq\big) \\ &\quad = \frac{1-aq^{-1}}{1-q}\mathbf{H}_4, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\left[aq^{-1}\theta_{y,q} + \frac{1-aq^{-1}}{1-q}\right]\mathbf{H}_4\big(aq^{-1}\big) + aq^{-1}\theta_{x,q}\mathbf{H}_4\big(aq^{-1}, yq\big) + aq^{-1}\theta_{x,q}\mathbf{H}_4\big(aq^{-1}, xq, yq\big) \\ &\quad = \frac{1-aq^{-1}}{1-q}\mathbf{H}_4. \end{aligned} \tag{3.10}$$

Proof By using these q-derivatives of $\mathbf{H}_4$ in (3.5) and (3.6), we get the recursion formulas (3.7) and (3.8).

Using (3.5), (3.7), and (3.8) for $\mathbf{H}_4$, we obtain (3.9) and (3.10). □

Theorem 3.4 *For $d \neq 1$, the contiguous function relations of $\mathbf{H}_4$ with the numerator parameter b hold true*:

$$\mathbf{H}_4(bq) = \mathbf{H}_4 + \frac{by(1-a)}{1-d}\mathbf{H}_4(aq, bq; c, dq; q, x, y) \tag{3.11}$$

and

$$\mathbf{H}_4\bigl(bq^{-1}\bigr) = \mathbf{H}_4 - \frac{by(1-a)}{q(1-d)}\mathbf{H}_4(aq, b; c, dq; q, x, y). \tag{3.12}$$

Proof In (1.9), we replace b by bq to obtain

$$\mathbf{H}_4(bq) - \mathbf{H}_4 = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}\bigl[(bq;q)_n - (b;q)_n\bigr]x^m y^n.$$

By means of the relations

$$(b;q)_n = (1-b)(bq;q)_{n-1}$$

and

$$(bq;q)_n = \bigl(1-bq^n\bigr)(bq;q)_{n-1},$$

we have

$$\begin{aligned}
\mathbf{H}_4(bq) - \mathbf{H}_4 &= \sum_{m,n=0}^{\infty} \frac{b(1-q^n)(a;q)_{2m+n}(bq;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n \\
&= \sum_{m,n=0}^{\infty} \frac{b(a;q)_{2m+n+1}(bq;q)_n}{(c;q)_{m+1}(d;q)_n(q;q)_m(q;q)_n}x^m y^{n+1} \\
&= \frac{by(1-a)}{1-d}\mathbf{H}_4(aq, bq; c, dq; q, x, y), \quad d \neq 1.
\end{aligned}$$

Replacing $b \to bq^{-1}$ in relation (3.12), we get

$$\mathbf{H}_4\bigl(a, bq^{-1}; c, d; q, x, y\bigr) = \mathbf{H}_4 - \frac{by(1-a)}{q(1-d)}\mathbf{H}_4(aq, b; c, dq; q, x, y), \quad d \neq 1. \qquad \square$$

Theorem 3.5 *The q-differential formulas are valid*:

$$\left[b\theta_{y,q} + \frac{1-b}{1-q}\right]\mathbf{H}_4 = \frac{1-b}{1-q}\mathbf{H}_4(bq) \tag{3.13}$$

and

$$\left[bq^{-1}\theta_{y,q} + \frac{1-bq^{-1}}{1-q}\right]\mathbf{H}_4\bigl(bq^{-1}\bigr) = \frac{1-bq^{-1}}{1-q}\mathbf{H}_4. \tag{3.14}$$

Proof From (3.5) and (3.6), we obtain (3.13) and (3.14). □

Theorem 3.6 *For $c \neq 1$, $d \neq 1$, and $b \neq q$, the contiguous function relations hold true for the numerator parameters a and b of $\mathbf{H}_4$:*

$$\begin{aligned}
&\mathbf{H}_4\bigl(aq, bq^{-1}; c, d; q, x, y\bigr)\\
&\quad= \mathbf{H}_4 + \frac{ay}{1-d}\mathbf{H}_4(aq, b; c, dq; q, x, y)\\
&\qquad+ \frac{ax(1-aq)}{1-bq^{-1}}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, x, yq\bigr)\\
&\qquad+ \frac{axq(1-aq)}{1-bq^{-1}}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, xq, yq\bigr)\\
&\qquad- \frac{abq^{-1}x(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, x, yq\bigr)\\
&\qquad- \frac{abx(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, xq, yq\bigr)\\
&\qquad- \frac{bq^{-1}y}{1-d}\mathbf{H}_4(aq, b; c, dq; q, x, y)
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\mathbf{H}_4\bigl(aq, bq^{-1}; c, d; q, x, y\bigr)\\
&\quad= \mathbf{H}_4 + \frac{ax(1-aq)}{(1-c)(1-bq^{-1})}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, x, y\bigr)\\
&\qquad+ \frac{axq(1-aq)}{1-bq^{-1}}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, xq, y\bigr)\\
&\qquad+ \frac{ay}{1-d}\mathbf{H}_4\bigl(aq, bq; c, dq; q, xq^2, y\bigr)\\
&\qquad- \frac{abxq(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, x, yq\bigr)\\
&\qquad- \frac{abx(1-aq)}{(1-bq^{-1})(1-c)}\mathbf{H}_4\bigl(aq^2, bq^{-1}; cq; q, xq, yq\bigr)\\
&\qquad- \frac{bq^{-1}y}{1-d}\mathbf{H}_4(aq, b; c, dq; q, x, y).
\end{aligned} \tag{3.16}$$

Proof Using the definition of $\mathbf{H}_4$ with the relation $(bq^{-1};q)_n = (1-bq^{-1})(b;q)_{n-1}$, we get

$$\begin{aligned}
&\mathbf{H}_4\bigl(aq, bq^{-1}; c, d; q, x, y\bigr) - \mathbf{H}_4\\
&\quad= \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}\bigl[\bigl(1-aq^{2m+n}\bigr)\bigl(1-bq^{-1}\bigr) - (1-a)\bigl(1-bq^{n-1}\bigr)\bigr]x^m y^n\\
&\quad= \sum_{m,n=0}^{\infty} \frac{a(1-q^n)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n\\
&\qquad+ \sum_{m,n=0}^{\infty} \frac{aq^n(1-q^m)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n\\
&\qquad+ \sum_{m,n=0}^{\infty} \frac{aq^{m+n}(1-q^m)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^m y^n
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{m,n=0}^{\infty} \frac{abq^{n-1}(1-q^m)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n \\
&- \sum_{m,n=0}^{\infty} \frac{abq^{m+n-1}(1-q^m)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n \\
&- \sum_{m,n=0}^{\infty} \frac{bq^{-1}(1-q^n)(aq;q)_{2m+n-1}(b;q)_{n-1}}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n \\
= &\frac{ay}{1-d} \mathbf{H}_4(aq,b;c,dq;q,x,y) \\
&+ \frac{ax(1-aq)}{1-bq^{-1}} \mathbf{H}_4\big(aq^2,bq^{-1};cq,d;q,x,yq\big) \\
&+ \frac{axq(1-aq)}{1-bq^{-1}} \mathbf{H}_4\big(aq^2,bq^{-1};cq,d;q,xq,yq\big) \\
&- \frac{abq^{-1}x(1-aq)}{(1-bq^{-1})(1-c)} \mathbf{H}_4\big(aq^2,bq^{-1};cq,d;q,x,yq\big) \\
&- \frac{abx(1-aq)}{(1-bq^{-1})(1-c)} \mathbf{H}_4\big(aq^2,bq^{-1};cq,d;q,xq,yq\big) \\
&- \frac{bq^{-1}y}{1-d} \mathbf{H}_4(aq,b;c,dq;q,x,y), \quad c \neq 1, d \neq 1, b \neq q.
\end{aligned}
$$

Proceeding as above, one can get the proof of Eq. (3.16) that would run parallel to Eq. (3.15), so details are omitted here. □

Theorem 3.7 *For $c \neq 1$, the recursion formulas hold true for the numerator parameter b of $\mathbf{H}_4$:*

$$
\begin{aligned}
&b(1-a)\mathbf{H}_4(aq) - a(1-b)\mathbf{H}_4(bq) \\
&\quad = (b-a)\mathbf{H}_4 + \frac{abx(1-a)(1-aq)}{1-c} \mathbf{H}_4\big(aq^2,b;cq,d;q,x,yq\big) \\
&\quad + \frac{abxq(1-a)(1-aq)}{1-c} \mathbf{H}_4\big(aq^2,b;cq,d;q,xq,yq\big).
\end{aligned} \tag{3.17}
$$

Proof From (1.9) we have

$$
\begin{aligned}
&b(1-a)\mathbf{H}_4(aq) - a(1-b)\mathbf{H}_4(bq) \\
&\quad = \sum_{m,n=0}^{\infty} \frac{b(1-a)(aq;q)_{2m+n}(b;q)_n - a(1-b)(a;q)_{2m+n}(bq;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n.
\end{aligned}
$$

By using the equation

$$
(1-a)(aq;q)_{2m+n} = \big(1-aq^{2m+n}\big)(a;q)_{2m+n}
$$

and

$$
(1-b)(bq;q)_n = \big(1-bq^n\big)(b;q)_n,
$$

we get

$$\begin{aligned}
&b(1-a)\mathbf{H}_4(aq) - a(1-b)\mathbf{H}_4(bq) \\
&= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n[b(1-aq^{2m+n}) - a(1-bq^n)]}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n \\
&= (b-a) \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n} x^m y^n \\
&\quad + ab \sum_{m,n=0}^{\infty} \frac{q^n(1-q^m)(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(1-q^m)(q;q)_{m-1}(q;q)_n} x^m y^n \\
&\quad + ab \sum_{m,n=0}^{\infty} \frac{q^{n+m}(1-q^m)(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(1-q^m)(q;q)_{m-1}(q;q)_n} x^m y^n \\
&= (b-a)\mathbf{H}_4 + \frac{abx(1-a)(1-aq)}{1-c} \mathbf{H}_4\bigl(aq^2, b; cq; q, x, yq\bigr) \\
&\quad + \frac{abxq(1-a)(1-aq)}{1-c} \mathbf{H}_4\bigl(aq^2, b; cq; q, xq, yq\bigr), \quad c \neq 1.
\end{aligned}$$

□

Theorem 3.8 *The contiguous function relations of* $\mathbf{H}_4$ *with denominator parameters c and d hold true*:

$$\mathbf{H}_4\bigl(cq^{-1}\bigr) = \mathbf{H}_4 + \frac{cx(1-a)(1-aq)}{(q-c)(1-c)} \mathbf{H}_4\bigl(aq^2, b; cq, d; q, x, y\bigr), \quad c \neq 1, q \tag{3.18}$$

and

$$\mathbf{H}_4\bigl(dq^{-1}\bigr) = \mathbf{H}_4 + \frac{dy(1-a)(1-b)}{(q-d)(1-d)} \mathbf{H}_4(aq, bq; c, dq; q, x, y), \quad d \neq 1, q. \tag{3.19}$$

Proof By the definition of $\mathbf{H}_4$, we have

$$\mathbf{H}_4\bigl(a, b; cq^{-1}; q, x, y\bigr) - \mathbf{H}_4 = \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(d;q)_n(q;q)_m(q;q)_n} \frac{(c;q)_m(d;q)_n - (cq^{-1};q)_m}{(cq^{-1};q)_m(c;q)_m} x^m y^n.$$

Using

$$\bigl(cq^{-1};q\bigr)_m = \bigl(1-cq^{-1}\bigr)(c;q)_{m-1}$$

and

$$(c;q)_m = \bigl(1-cq^{m-1}\bigr)(c;q)_{m-1},$$

we get

$$\begin{aligned}
&\mathbf{H}_4\bigl(a, b; cq^{-1}; q, x, y\bigr) - \mathbf{H}_4 \\
&= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(q;q)_m(q;q)_n} \frac{[1-q^m]cq^{-1}(c;q)_{m-1}}{[1-cq^{-1}](c;q)_{m-1}(c;q)_m(d;q)_n} x^m y^n
\end{aligned}$$

$$= c\sum_{m,n=0}^{\infty}\frac{1-q^m}{q-c}\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(1-q^m)(q;q)_{m-1}(q;q)_n}x^my^n$$
$$= \frac{cx(1-a)(1-aq)}{(q-c)(1-c)}\mathbf{H}_4\big(aq^2,b;cq;q,x,y\big), \quad c\neq 1,q,$$

which is the desired result. The recursion formula (3.19) can be proved in a similar manner. □

Theorem 3.9 *The contiguous function relations hold true for the denominator parameter c of $\mathbf{H}_4$*:

$$\begin{aligned}&\mathbf{H}_4(a,b;cq,d;q,x,y)\\&\quad= \mathbf{H}_4-\frac{cx(1-a)(1-aq)}{(1-cq)(1-c)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,y\big), \quad c\neq 1, cq\neq 1,\\&\mathbf{H}_4(a,b;c,dq;q,x,y)\\&\quad= \mathbf{H}_4-\frac{dy(1-a)(1-b)}{(1-dq)(1-d)}\mathbf{H}_4\big(aq,bq;c,dq^2;q,x,y\big), \quad d\neq 1, dq\neq 1.\end{aligned} \tag{3.20}$$

Proof Replacing $c\to cq$ in (3.18) and $d\to dq$ in (3.19), we obtain (3.20). □

Theorem 3.10 *The following results of $\mathbf{H}_4$ hold well*:

$$\mathbf{H}_4\big(cq^{-1}\big) = \frac{c}{c-q}\mathbf{H}_4(a,b;c,d;q,xq,y)-\frac{q}{c-q}\mathbf{H}_4, \quad c\neq q, \tag{3.21}$$

$$\mathbf{H}_4\big(dq^{-1}\big) = \frac{d}{d-q}\mathbf{H}_4(a,b;c,d;q,x,yq)-\frac{q}{d-q}\mathbf{H}_4, \quad d\neq q, \tag{3.22}$$

$$\mathbf{H}_4(cq)+(c-1)\mathbf{H}_4 = c\mathbf{H}_4(a,b,cq,xq,y), \tag{3.23}$$

$$\mathbf{H}_4(dq)+(d-1)\mathbf{H}_4 = d\mathbf{H}_4(a,b,c,dq,x,yq), \tag{3.24}$$

$$\left[cq^{-1}\theta_{x,q}+\frac{1-cq^{-1}}{1-q}\right]\mathbf{H}_4 = \frac{1-cq^{-1}}{1-q}\mathbf{H}_4\big(cq^{-1}\big), \tag{3.25}$$

and

$$\left[dq^{-1}\theta_{y,q}+\frac{1-dq^{-1}}{1-q}\right]\mathbf{H}_4 = \frac{1-dq^{-1}}{1-q}\mathbf{H}_4\big(dq^{-1}\big). \tag{3.26}$$

Proof We replace $\frac{1}{(cq^{-1};q)_m} = \frac{1}{(c;q)_m}[\frac{c}{c-q}q^m-\frac{q}{c-q}]$ in relation (1.9) to get

$$\begin{aligned}\mathbf{H}_4\big(cq^{-1}\big) &= \sum_{m,n=0}^{\infty}\left[\frac{c}{c-q}q^m-\frac{q}{c-q}\right]\frac{(a;q)_{2m+n}(b;q)_n}{(c;q)_m(d;q)_n(q;q)_m(q;q)_n}x^my^n\\&= \frac{c}{c-q}\mathbf{H}_4(a,b;c,d;q,xq,y)-\frac{q}{c-q}\mathbf{H}_4; \quad c\neq q.\end{aligned}$$

Replacing $c = cq$ in (3.21), we get

$$\mathbf{H}_4(cq)+(c-1)\mathbf{H}_4 = c\mathbf{H}_4(a,b,cq,xq,y).$$

The results (3.25), (3.26), (3.22), and (3.24) are along the same lines as those of Eqs. (2.22)–(2.25). □

Theorem 3.11 *For $c, d, cq \neq 1$, the results of $\mathbf{H}_4$ hold*:

$$\begin{aligned}
&\mathbf{H}_4(aq,b;cq,d;q,x,y) - \mathbf{H}_4\\
&\quad= \frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,x,y\bigr)\\
&\qquad+ \frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,xq,y\bigr)\\
&\qquad+ \frac{ay(1-b)}{(1-c)(1-d)}\mathbf{H}_4\bigl(aq,bq;cq,dq;q,xq^2,y\bigr)\\
&\qquad+ \frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,x,y\bigr)\\
&\qquad- \frac{acy(1-b)}{(1-c)(1-d)}\mathbf{H}_4(aq,bq;cq,dq;q,xq,y)\\
&\qquad- \frac{acxq(1-aq}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,xq,yq\bigr)
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\mathbf{H}_4(aq,b;cq,d;q,x,y) - \mathbf{H}_4\\
&\quad= \frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,x,y\bigr)\\
&\qquad+ \frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,xq,y\bigr)\\
&\qquad+ \frac{ay(1-b)}{(1-c)(1-d)}\mathbf{H}_4\bigl(aq,bq;cq,dq;q,xq^2,y\bigr)\\
&\qquad+ \frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,x,y\bigr)\\
&\qquad- \frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,bq;cq^2,d;q,xq,y\bigr)\\
&\qquad- \frac{acy(1-b)}{(1-c)(1-d)}\mathbf{H}_4\bigl(aq^2,bq;cq,dq;q,xq^2,y\bigr).
\end{aligned} \tag{3.28}$$

Proof In (1.9), replacing a and c by aq and cq, we have

$$\begin{aligned}
&\mathbf{H}_4(aq,b;cq,d;q,x,y) - \mathbf{H}_4\\
&\quad= \sum_{m,n=0}^{\infty} \frac{(b;q)_n}{(q;q)_m(q;q)_n}\biggl[\frac{(aq;q)_{2m+n}(c;q)_m(d;q)_n - (a;q)_{2m+n}(cq;q)_m}{(cq;q)_m(c;q)_m(d;q)_n}\biggr]x^m y^n\\
&\quad= \sum_{m,n=0}^{\infty} \frac{(aq;q)_{2m+n-1}(cq;q)_{m-1}(b;q)_n}{(q;q)_m(q;q)_n}\biggl[\frac{(1-aq^{2m+n})(1-c) - (1-a)(1-cq^m)}{(1-cq^m)(cq;q)_{m-1}(c;q)_m(d;q)_n}\biggr]x^m y^n\\
&\quad= \frac{ax(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\bigl(aq^2,b;cq^2,d;q,x,y\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,xq,y\big)\\
&+\frac{ay(1-b)}{(1-c)(1-d)}\mathbf{H}_4\big(aq,bq;cq,dq;q,xq^2,y\big)\\
&+\frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,y\big)\\
&-\frac{acxq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,bq;cq^2,d;q,xq,y\big)\\
&-\frac{acy(1-b)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2,bq;cq,dq;q,xq,yq\big),\quad c,d,cq\neq 1.
\end{aligned}
$$

By using the relation

$$
\begin{aligned}
&a\big(1-q^{2m+n}\big)+c\big(1-q^m\big)-acq^m\big(1-q^{m+n}\big)\\
&\quad=a\big(1-q^m\big)+aq^m\big(1-q^m\big)\\
&\qquad+aq^{2m}\big(1-q^n\big)+c\big(1-q^m\big)-acq^m\big(1-q^m\big)-acq^{2m}\big(1-q^n\big).
\end{aligned}
$$

The proof of Eq. (3.27) is similar as that of Eq. (3.28). □

Theorem 3.12 *For $c,d,cq\neq 1$, the following contiguous function relations hold*:

$$
\begin{aligned}
&\mathbf{H}_4(aq,b;cq,d;q,x,y)-\mathbf{H}_4\\
&\quad=\frac{ay(1-b)}{(1-c)(1-d)}\mathbf{H}_4(aq,bq;cq,dq;q,x,y)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,yq\big)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,xq,yq\big)\\
&\qquad+\frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,y\big)\\
&\qquad-\frac{acy(1-b)}{(1-c)(1-d)}\mathbf{H}_4(aq,bq;cq,dq;q,xq,y)\\
&\qquad-\frac{acxq(1-aq}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,xq,yq\big)
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&\mathbf{H}_4(aq,b;cq,d;q,x,y)-\mathbf{H}_4\\
&\quad=\frac{ay(1-b)}{(1-c)(1-d)}\mathbf{H}_4(aq,bq;cq,dq;q,x,y)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,yq\big)\\
&\qquad+\frac{axq(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,xq,yq\big)\\
&\qquad+\frac{cx(1-aq)}{(1-c)(1-cq)}\mathbf{H}_4\big(aq^2,b;cq^2,d;q,x,y\big)
\end{aligned}
\tag{3.30}
$$

$$- \frac{acxq(1-aq)}{(1-c)(1-cq)} \mathbf{H}_4\big(aq^2, bq; cq^2, d; q, xq, y\big)$$
$$- \frac{acy(1-b)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, bq; cq, dq; q, xq^2, y\big).$$

Proof By virtue of our calculation in Theorem 3.11, and with the help of the relations

$$\begin{aligned} &a\big(1-q^{2m+n}\big) + c\big(1-q^m\big) - acq^m\big(1-q^{m+n}\big) \\ &\quad = a\big(1-q^n\big) + aq^n\big(1-q^m\big) \\ &\qquad + aq^{m+n}\big(1-q^m\big) + c\big(1-q^m\big) - acq^m\big(1-q^n\big) - acq^{m+n}\big(1-q^m\big) \end{aligned}$$

and

$$\begin{aligned} &a\big(1-q^{2m+n}\big) + c\big(1-q^m\big) - acq^m\big(1-q^{m+n}\big) \\ &\quad = a\big(1-q^n\big) + aq^n\big(1-q^m\big) \\ &\qquad + aq^{m+n}\big(1-q^m\big) + c\big(1-q^m\big) - acq^m\big(1-q^m\big) - acq^{2m}\big(1-q^n\big). \end{aligned}$$

To simplify the above relationships, we get (3.29) and (3.30). □

Theorem 3.13 *For $c, d, dq \neq 1$, the contiguous relations hold true for the parameters of* $\mathbf{H}_4$:

$$\begin{aligned} &\mathbf{H}_4(aq, b; c, dq; q, x, y) - \mathbf{H}_4 \\ &\quad = \frac{ax(1-aq)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, b; cq, dq; q, x, y\big) \\ &\qquad + \frac{axq(1-aq)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, b; cq, dq; q, xq, y\big) \\ &\qquad + \frac{ax(1-b)}{(1-d)(1-dq)} \mathbf{H}_4\big(aq, b; c, dq^2; q, xq^2, y\big) \\ &\qquad + \frac{d(1-b)}{(1-d)(1-dq)} \mathbf{H}_4(aq, bq; c, dq; q, x, y) \\ &\qquad - \frac{adx(1-aq)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, b; cq, dq; q, x, yq\big) \\ &\qquad - \frac{adyq(1-b}{(1-d)(1-dq)} \mathbf{H}_4\big(aq, bq; c, dq^2; q, xq, yq\big), \end{aligned} \tag{3.31}$$

$$\begin{aligned} &\mathbf{H}_4(aq, b; c, dq; q, x, y) - \mathbf{H}_4 \\ &\quad = \frac{ax(1-aq)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, b; cq, dq; q, x, y\big) \\ &\qquad + \frac{axq(1-aq)}{(1-c)(1-d)} \mathbf{H}_4\big(aq^2, b; cq, dq; q, xq, y\big) \\ &\qquad + \frac{ax(1-b)}{(1-d)(1-dq)} \mathbf{H}_4\big(aq, b; c, dq^2; q, xq^2, y\big) \\ &\qquad + \frac{d(1-b)}{(1-d)(1-dq)} \mathbf{H}_4(aq, bq; c, dq; q, x, y) \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
&- \frac{adyq(1-b)}{(1-d)(1-dq)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, yq\big) \\
&- \frac{adx(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, x, yq^2\big),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{H}_4(aq, b; c, dq; q, x, y) - \mathbf{H}_4 \\
&\quad = \frac{ay(1-b)}{(1-d)(1-dq)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, y\big) \\
&\qquad + \frac{axq(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, x, yq\big) \\
&\qquad + \frac{dy(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, xq, yq\big) \\
&\qquad + \frac{dy(1-b)}{(1-c)(1-d)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, y\big) \\
&\qquad - \frac{adx(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq, d; q, x, yq\big) \\
&\qquad - \frac{ady(1-b)}{(1-d)(1-dq)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, xq, yq\big),
\end{aligned}
\tag{3.33}
$$

and

$$
\begin{aligned}
&\mathbf{H}_4(aq, b; c, dq; q, x, y) - \mathbf{H}_4 \\
&\quad = \frac{ay(1-b)}{(1-d)(1-dq)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, y\big) \\
&\qquad + \frac{axq(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, x, yq\big) \\
&\qquad + \frac{dy(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, xq, yq\big) \\
&\qquad + \frac{dy(1-b)}{(1-c)(1-d)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, y\big) \\
&\qquad - \frac{adyq(1-b)}{(1-d)(1-dq)}\mathbf{H}_4\big(aq, bq; c, dq^2; q, x, yq\big) \\
&\qquad - \frac{adx(1-aq)}{(1-c)(1-d)}\mathbf{H}_4\big(aq^2, b; cq, dq; q, x, yq^2\big).
\end{aligned}
\tag{3.34}
$$

Proof Using the relations

$$
\begin{aligned}
&a\big(1-q^{2m+n}\big) + d\big(1-q^n\big) - adq^n\big(1-q^{m+n}\big) \\
&\quad = a\big(1-q^m\big) + aq^m\big(1-q^m\big) \\
&\qquad + aq^{2m}\big(1-q^n\big) + d\big(1-q^n\big) - adq^n\big(1-q^m\big) - adq^{m+n}\big(1-q^n\big), \\
&a\big(1-q^{2m+n}\big) + d\big(1-q^n\big) - adq^n\big(1-q^{m+n}\big) \\
&\quad = a\big(1-q^m\big) + aq^m\big(1-q^m\big) \\
&\qquad + aq^{2m}\big(1-q^n\big) + d\big(1-q^n\big) - adq^n\big(1-q^n\big) - adq^{2n}\big(1-q^m\big), \\
&a\big(1-q^{2m+n}\big) + d\big(1-q^n\big) - adq^n\big(1-q^{m+n}\big)
\end{aligned}
$$

$$= a(1-q^n) + aq^n(1-q^m) + aq^{m+n}(1-q^m) + d(1-q^n) - adq^n(1-q^m) - adq^{m+n}(1-q^n),$$

and

$$\begin{aligned} & a(1-q^{2m+n}) + d(1-q^n) - adq^n(1-q^{m+n}) \\ &= a(1-q^n) + aq^n(1-q^m) \\ &\quad + aq^{m+n}(1-q^m) + d(1-q^n) - adq^n(1-q^n) - adq^{2n}(1-q^m). \end{aligned}$$

Simplifying the above relations, we obtain (3.31)–(3.34). □

Theorem 3.14 *The formulas hold true of* $\mathbf{H}_4$:

$$\begin{aligned} & \mathbf{H}_4(a, bq; cq, d; q, x, y) \\ &= \mathbf{H}_4 + \frac{by(1-a)}{(1-d)(1-cq)} \mathbf{H}_4(aq, bq; cq^2, dq; q, x, y) \\ &\quad - \frac{cx(1-a)(1-aq)(1-b)}{(1-c)(1-cq)} \mathbf{H}_4(aq^2, b; cq^2, d; q, x, y), \quad c, d, cq \neq 1 \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} & \mathbf{H}_4(a, bq; c, dq; q, x, y) \\ &= \mathbf{H}_4 + \frac{(b-d)y(1-a)}{(1-d)(1-dq)} \mathbf{H}_4(aq, bq; c, dq^2; q, x, y), \quad d, dq \neq 1. \end{aligned} \tag{3.36}$$

Proof By using the definition of $\mathbf{H}_4$, we get

$$\begin{aligned} & \mathbf{H}_4(a, bq; cq; q, x, y) - \mathbf{H}_4 \\ &= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(b;q)_n}{(d;q)_n(q;q)_m(q;q)_n} \left[\frac{(bq;q)_n(c;q)_m - (b;q)_n(cq;q)_m}{(cq;q)_m(c;q)_m} \right] x^m y^n \\ &= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(cq;q)_{m-1}(bq;q)_{n-1}}{(d;q)_n(q;q)_m(q;q)_n} \left[\frac{(1-bq^n)(1-c) - (1-b)(1-cq^m)}{(1-cq^m)(cq;q)_{m-1}(c;q)_m} \right] x^m y^n \\ &= \sum_{m,n=0}^{\infty} \frac{(a;q)_{2m+n}(bq;q)_{n-1}}{(d;q)_n(q;q)_m(q;q)_n} \left[\frac{b(1-c)(1-q^n) + c(b-1)(1-q^m)}{(1-c)(cq;q)_m} \right] x^m y^n \\ &= \frac{by(1-a)}{(1-d)(1-cq)} \mathbf{H}_4(aq, bq; cq^2, dq; q, x, y) \\ &\quad + \frac{c(b-1)x(1-a)(1-aq)}{(1-c)(1-cq)} \mathbf{H}_4(aq^2, b; cq^2, d; q, x, y), \quad c, d, cq \neq 1. \end{aligned}$$

We prove relation (3.36) in a similar way as relation (3.35). □

4 Conclusion

We conclude by the remark that the results established in this paper are general forms and one can deduce several contiguous function relations and q-differential relations of basic Horns hypergeometric functions $\mathbf{H}_3$ and $\mathbf{H}_4$ as different cases of our main findings. Also, other types of these extensions are recommended for a parallel study of this work. More work will be carried out in the coming future results in other fields of interest for fractional quantum calculus.

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