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On coupled coincidence and common fixed point theorems for different types of mappings satisfying rational type contractions in $b-$metric spaces

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Abstract

In this paper, we study the existence and uniqueness of coupled coincidence and common fixed points for single-valued and fuzzy mappings under contraction condition of rational type in $b-$metric spaces and obtain the corresponding results for hybrid pair of single-valued and multi-valued mappings. We consider $b-$metric space with a partial order and prove the existence of coupled coincidence and common fixed points for two single-valued mappings.

Keywords: Fuzzy, multi-valued and single-valued mappings, coupled coincidence and common fixed points, $b-$metric and ordered $b-$metric spaces, rational type contractions.
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1. Introduction and Preliminaries

It is well known that the Banach's contraction principle for single-valued contractions in metric spaces was extended in various ways. One of these extensions was given for the so called $b-$metric spaces by Czerwik [15]. For more results on b-metric spaces and discussion on the topological structure introduced on it, we refer to [3, 6, 7, 19].

We shall recall some well notions and definitions of the $b-$metric spaces.

Definition 1.1. [31] Let X be a set and $s \geq 1$ be a given real number. A functional $d : X \times X \to R_+$ is said to be a $b-$metric if the following axioms are satisfied, for all $x, y, z \in X$:

(1) $d(x, y) = 0 \Leftrightarrow x = y$,

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(2) $d(x,y) = d(y,x)$,
(3) $d(x,z) \leq s\Big[d(x,y) + d(y,z)\Big]$.

A pair (X,d) is called a $b-$metric space.

Lemma 1.2. [31] Let (X,d) be a $b-$metric space. Then the sequence $\{x_n\}_{n\in N} \subset X$ is called:

(1) Convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = x$.
(2) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.3. Let X be a nonempty set. Then $(X, \preceq, d)$ is called an ordered $b-$metric space if and only if:

(1) (X,d) is a $b-$ metric space,
(2) $(X, \preceq)$ is a partially ordered set.

Notice that we can endow the product space $X \times X$ with the partial order $\preceq_P$ given by

$$(x,y) \preceq_p (u,v) \Leftrightarrow x \preceq u,\ \ y \succeq v.$$

Definition 1.4. Let $(X, \preceq)$ be a partially ordered set. An elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$.

Definition 1.5. Let $(X, \preceq)$ be a partially ordered set. A mapping f on X is said to be monotone non-decreasing with respect to the order $\preceq$ if, for all $x, y \in X$, $x \preceq y$ implies $fx \preceq fy$.

In [16], Dass and Gupta proved the following fixed point theorem for contractions of rational type.

Theorem 1.6. Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)} + \beta d(x,y), \ \ \texttt{for all}\ \ x,y \in X.$$

Then T has a unique fixed point.

Then, Cabrera et al.[8] gave a version of Theorem 1.6 in the context of partially ordered metric spaces. Recently, Oprea and Petruşel [31] obtained coupled fixed point theorems for monotone rational contractions in ordered $b-$metric spaces, see also [9, 11].

The concept of coupled fixed point problem was considered, for the first time, by Opoitsev in [29, 30] but a very fruitful approach in this field was proposed by Guo, Lakshmikantham [17] and Bhaskar, Lakshmikantham [5]. Then Lakshmikantham and Ćirić [24] extended these results by defining the mixed g-monotone property and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings $F : X \times X \to X$ and $g : X \to X$ in a partially ordered metric space.

Definition 1.7. [24] Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in its first argument and monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \ \ g(x_1) \preceq g(x_2) \ \text{ implies } \ F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \ \ g(y_1) \preceq g(y_2) \ \text{ implies } \ F(x, y_1) \succeq F(x, y_2).$$

If g is the identity mapping, we obtain the Bhaskar and Lakshmikantham's notion of a mixed monotone property of the mapping F.

Definition 1.8. [24]An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), F(y, x) = g(y).$$

Also, if g is the identity mapping, then (x, y) is called a coupled fixed point of the mapping F.

Later on, many results related to this kind of problem appeared (see, for example [21, 26, 28, 35]).

The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [34] and Markin [27]. Later many authors developed the existence of fixed points for various multi-valued contractive mappings under different conditions. For details, we refer the reader to [1, 12, 13, 20, 22, 23, 25, 32, 33] and the references therein. The theory of multi-valued mappings has applications in control theory, convex optimization, differential inclusion and economics.

Let (X, d) be a $b-$metric space. We denote by $CB(X)$ the family of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$ and $x \in X$, we denote

$$D(x, A) = \inf_{a \in A} d(x, a).$$

Let H be the Hausdorff metric in $CB(X)$ induced by the metric d on X, that is

$$H(A, B) = \max\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\}.$$

It is clear that for any $A, B \in CB(X)$ and $a \in A$, we have

$$D(a, B) \leq H(A, B).$$

Definition 1.9. An element $x \in X$ is said to be a fixed point of a set valued mapping $T : X \to CB(X)$ if and only if $x \in Tx$.

In 1969, Nadler [34] extended the famous Banach contraction principle from single-valued to multi-valued mapping and prove the following theorem.

Theorem 1.10. [34] Let (X, d) be a complete metric space and T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0, 1]$ such that

$$H\big(Tx, Ty\big) \leq cd(x, y),$$

for all $x, y \in X$. Then T has a fixed point.

The concept of coupled fixed point for multi-valued mapping $F : X \times X \to CB(X)$ was introduced by Beg and Butt [4] who followed the technique of Bhaskar and Lakshmikantham to define the mixed monotone property for F and give sufficient conditions for the existence of its coupled fixed point (not necessarily unique) in an ordered space $(X, \leq, d)$.

Definition 1.11. [1] Let X be a nonempty set, $F : X \times X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $(x, y) \in X \times X$ is called:

(1) coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$,
(2) coupled coincidence point of a hybrid pair F, g if $g(x) \in F(x, y)$ and $g(y) \in F(y, x)$,
(3) coupled common fixed point of a hybrid pair F, g if $x = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.

We denote the set of coupled coincidence point of mappings F and g by $\curlyvee(F, g)$. Note that if $(x, y) \in \curlyvee(F, g)$, then (y, x) is also in $\curlyvee(F, g)$.

Definition 1.12. [1] Let $F : X \times X \to 2^X$ be a multi-valued mapping and g be a single-valued mapping on X. The hybrid pair (F, g) is called $w-$compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \in \curlyvee(F, g)$.

It is well known that the fuzzy set concept plays an important role in many scientific and engineering applications. The fuzziness appears when we need to perform calculations with imprecision variables. The concept of fuzzy sets was introduced initially by Zadeh [36] in 1965. Then, Heilpern [18] introduced the concept of fuzzy contraction mappings and developed Banach contraction principle for fuzzy mappings in complete metric linear space. Subsequently several other authors have studied existence of fixed points of fuzzy mappings satisfying some different contractive type conditions (see for example [2, 10, 14] and references within). Very recently, Zhu et all. [37] introduced the concepts of coupled coincidence and coupled common fixed points of single-valued and fuzzy mappings and established some coupled coincidence and coupled common fixed point theorems for this hybrid pair.

Let (X, d) be a $b-$metric space with constant $s \geq 1$ and $I = [0, 1]$. A fuzzy set in X is a function which associates with each element in X a real number in the interval I. That is, A is a fuzzy set in X if for any $x \in X$,

$$\begin{aligned} A :&X \to I, \\ &x \mapsto Ax \in I, \end{aligned}$$

the function value Ax, is called the grade of membership of x in A. For $x \in X$ and $r \in (0, 1]$, the fuzzy point x_r of X is the fuzzy set of X given by

$$x_r(z) = \begin{cases} r, & z = x, \\ 0, & z \neq x. \end{cases}$$

The $r-$level set of A, denoted by $[A]_r$, is defined by

$$\begin{aligned} [A]_r &= \{x : Ax \geq r\}, \text{ if } r \in (0, 1], \\ [A]_0 &= \overline{\{x : Ax > 0\}}, \end{aligned}$$

where $\overline{B}$ denotes the closure of the non-fuzzy set B.
A fuzzy set A is said to be an approximate quantity if and only if $[A]_r$ is a nonempty convex and compact in X for each $r \in I$ and $\sup_{x \in X} Ax = 1$. We denote by $W(X)$ the family of all approximate quantities in X.

Suppose that $A, B \in W(X)$, then A is said to be more accurate than B, denoted by $A \subset B$, if and only if $Ax \leq Bx$ for each $x \in X$.

Let I^X be the collection of all fuzzy subsets in X and $W(X)$ be a sub-collection of all approximate quantities. For $A, B \in W(X)$, $r \in I$, define

$$\begin{aligned} p_r(A, B) &= \inf_{x \in [A]_r, y \in [B]_r} d(x, y), \\ D_r(A, B) &= H([A]_r, [B]_r) \\ &= \max\{\sup_{x \in [A]_r} p_r(x, B), \sup_{y \in [B]_r} p_r(A, y)\}. \end{aligned}$$

Let X be an arbitrary set, Y be a metric linear space. A mapping T is called fuzzy mapping if T is a mapping from X into I^Y, that is

$$\begin{aligned} T :&X \to I^Y, \\ &x \mapsto Tx \in I^Y, (Tx \text{ is a fuzzy set on a metric linear space } Y), \\ &Tx : Y \to I, \\ &\quad y \mapsto Tx(y) \in I, (Tx(y) \text{ is the grade of membership of y in } Tx, \text{ for all } y \in Y). \end{aligned}$$

Therefore, a fuzzy mapping T is a fuzzy set on $X \times Y$,

$$\begin{aligned} T :&X \to I^Y, \\ &(x, y) \mapsto Tx(y) \in I. \end{aligned}$$

Lemma 1.13. Let (X, d) be a $b-$metric space with constant $s \geq 1$, $x, y \in X$ and $A, B \in W(X)$:

(1) If $p_r(x, A) = 0$ then $x \in [A]_r$ and $x_r \subset A$.
(2) If $x_r \subset A$, then $p_r(x, B) \leq D_r(A, B)$.
(3) $p_r(x, A) \leq s\Big[d(x, y) + p_r(y, A)\Big]$.

Proof. For (1). Let $p_r(x, A) = \inf_{a\in[A]_r} d(x, a) = d(x, [A]_r) = 0 \Rightarrow x \in [A]_r$. By the definition of the $r-$level set of A and the fuzzy point x_r we have $x_r(x) = r \leq A(x)$ and $x_r(z) = 0 \leq r \leq A(x)$ for $z \neq x$. Then $x_r(u) \leq A(u)$ for all $u \in X \Rightarrow x_r \subset A$.
For (2). Let $x_r \subset A$ then $A(x) \geq x_r(x) = r$, i. e., $x \in [A]_r$. Hence

$$\begin{aligned} p_r(x, B) &\leq \sup_{x\in[A]_r} p_r(x, B) \\ &\leq \max\{\sup_{x\in[A]_r} p_r(x, B), \sup_{b\in[B]_r} p_r(A, b)\} = D_r(A, B). \end{aligned}$$

For (3).

$$\begin{aligned} p_r(x, A) &= \inf_{a\in[A]_r} d(x, a) \\ &\leq \inf_{a\in[A]_r} [sd(x, y) + sd(y, a)] \\ &\leq \inf_{a\in[A]_r} sd(x, y) + \inf_{a\in[A]_r} sd(y, a) \\ &\leq sd(x, y) + sp_r(y, A). \end{aligned}$$

□

Definition 1.14. [37] Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ a single-valued mapping. An element $(x, y) \in X \times X$ is said to be

(1) fuzzy coupled fixed point of F if there exists $r \in (0, 1]$ such that $x \in [F(x, y)]_r$ and $y \in [F(y, x)]_r$,
(2) fuzzy coupled coincidence point of F and g if there exists $r \in (0, 1]$ such that $gx \in [F(x, y)]_r$ and $gy \in [F(y, x)]_r$,
(3) fuzzy coupled common fixed point of F and g if there exists $r \in (0, 1]$ such that $x = gx \in [F(x, y)]_r$ and $y = gy \in [F(y, x)]_r$.

We denote by $\curlyvee_r(F, g) = \{(x, y) \in X\times X \mid gx \in [F(x, y)]_r \text{ and } gy \in [F(y, x)]_r\}$ the set of fuzzy coupled coincidence points of F and g. Note that if $(x, y) \in \curlyvee_r(F, g)$, then (y, x) is also in $\curlyvee_r(F, g)$.

Definition 1.15. [37] Let $F : X \times X \to I^X$ be a fuzzy mapping and $g : X \to X$ be a single-valued mapping. The hybrid pair $\{F, g\}$ is said to be $r - w-$compatible if there exists $r \in (0, 1]$ such that $g[F(x, y)]_r \subseteq [F(gx, gy)]_r$, whenever $(x, y) \in \curlyvee_r(F, g)$.

The following lemma is important in proving our main results

Lemma 1.16. [34] Let $A, B \in CB(X)$ and $\eta > 0$ then for each $a \in A$ there exists $b \in B$ such that

$$d(A, B) \leq H(A, B) + \eta.$$

Since every compact subset of a $b-$metric space is bounded and closed, then we can apply Lemma 1.16 for $A, B \in W(X)$.

Starting with the papers of Zhu et all [37] and Orea and Petruşel [31], in the frame work of complete $b-$metric spaces, in section two we present some existence and uniqueness theorems for coupled coincidence and coupled common fixed point for hybrid pair of single-valued and fuzzy mappings under some rational type contractions. As a consequence of section two we obtain the corresponding results for hybrid pair of single-valued and multi-valued mappings. Section three is devoted to establish some coupled coincidence and common fixed point results in partially ordered $b-$metric spaces for mappings having mixed monotone property and satisfying certain rational contractive condition.

2. Coupled coincidence and common fixed point theorems for single-valued and fuzzy mappings

Theorem 2.1. Let (X, d) be a $b-$metric space with constant $s \geq 1$, $F : X \times X \to W(X)$ be a fuzzy mapping and $g : X \to X$ be a single-valued mapping on X. Suppose that there exist non-negative real numbers $\alpha, \beta, \gamma, \delta \in [0, 1)$ and $r \in (0, 1]$ with $(\alpha + \gamma + \delta s) + (\beta + \gamma + \delta s)s < 1$ such that for all $x, y, u, v \in X$

$$\begin{aligned} H\Big([F(x,y)]_r, [F(u,v)]_r\Big) \leq & \frac{\alpha d(gu, [F(u,v)]_r)\Big[1 + d(gx, [F(x,y)]_r)\Big]}{1 + d(gx, gu)} \\ & + \beta d(gx, gu) + \gamma\Big[d(gx, [F(x,y)]_r) + d(gu, [F(u,v)]_r)\Big] \\ & + \delta\Big[d(gx, [F(u,v)]_r) + d(gu, [F(x,y)]_r)\Big]. \end{aligned} \tag{2.1}$$

Suppose also that $[F(x, y)]_r \subseteq g(X)$ for all $x, y \in X$ and $g(X)$ is complete subspace of X.
Then F and g have at least one fuzzy coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $[F(x_0, y_0)]_r$ and $[F(y_0, x_0)]_r$ are nonempty and well defined. Since $[F(x, y)]_r \subseteq g(X)$, there exist $x_1, y_1 \in X$ such that $gx_1 \in [F(x_0, y_0)]_r$ and $gy_1 \in [F(y_0, x_0)]_r$. By Lemma 1.16, we obtain that for this point $gx_1 \in [F(x_0, y_0)]_r$ and $\beta + \gamma + \delta s > 0$ there exists $gx_2 \in [F(x_1, y_1)]_r$ such that

$$d(gx_1, gx_2) \leq H\Big([F(x_0, y_0)]_r, [F(x_1, y_1)]_r\Big) + (\beta + \gamma + \delta s).$$

Therefore, by Eq. (2.1), we have

$$\begin{aligned} d(gx_1, gx_2) &\leq H\Big([F(x_0,y_0)]_r, [F(x_1,y_1)]_r\Big) + (\beta + \gamma + \delta s) \\ &\leq \frac{\alpha d(gx_1, [F(x_1,y_1)]_r)\Big[1 + d(gx_0, [F(x_0,y_0)]_r)\Big]}{1 + d(gx_0, gx_1)} \\ &+ \beta d(gx_0, gx_1) + \gamma\Big[d(gx_0, [F(x_0,y_0)]_r) + d(gx_1, [F(x_1,y_1)]_r)\Big] \\ &+ \delta\Big[d(gx_0, [F(x_1,y_1)]_r) + d(gx_1, [F(x_0,y_0)]_r)\Big] + (\beta + \gamma + \delta s) \\ &\leq \frac{\alpha d(gx_1, gx_2)\Big[1 + d(gx_0, gx_1)\Big]}{1 + d(gx_0, gx_1)} + \beta d(gx_0, gx_1) + \gamma\Big[d(gx_0, gx_1) + d(gx_1, gx_2)\Big] \\ &+ \delta\Big[d(gx_0, gx_2) + d(gx_1, gx_1)\Big] + (\beta + \gamma + \delta s) \\ &\leq \alpha d(gx_1, gx_2) + \beta d(gx_0, gx_1) + \gamma\Big[d(gx_0, gx_1) + d(gx_1, gx_2]_r)\Big] \\ &+ \delta\Big[sd(gx_0, gx_1) + sd(gx_1, gx_2)\Big] + (\beta + \gamma + \delta s) \\ &\leq (\alpha + \gamma + \delta s)d(gx_1, gx_2) + (\beta + \gamma + \delta s)d(gx_0, gx_1) + (\beta + \gamma + \delta s), \end{aligned}$$

which implies that

$$d(gx_1, gx_2) \leq \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)} d(gx_0, gx_1) + \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)}.$$

Let $k = \frac{\beta+\gamma+\delta s}{1-(\alpha+\gamma+\delta s)}$. Since $(\alpha + \gamma + \delta s) + (\beta + \gamma + \delta s)s < 1$, then we obtain that $k < \frac{1}{s}$. Thus,

$$d(gx_1, gx_2) \leq kd(gx_0, gx_1) + k. \tag{2.2}$$

Similarly, we can find $gy_2 \in [F(y_1, x_1)]_r$ such that

$$\begin{aligned}
d(gy_1, gy_2) &\leq H\Big([F(y_0, x_0)]_r, [F(y_1, x_1)]_r\Big) + (\beta + \gamma + \delta s) \\
&\leq \frac{\alpha d(gy_1, [F(y_1, x_1)]_r)\Big[1 + d(gy_0, [F(y_0, x_0)]_r)\Big]}{1 + d(gy_0, gy_1)} \\
&+ \beta d(gy_0, gy_1) + \gamma\Big[d(gy_0, [F(y_0, x_0)]_r) + d(gy_1, [F(y_1, x_1)]_r)\Big] \\
&+ \delta\Big[d(gy_0, [F(y_1, x_1)]_r) + d(gy_1, [F(y_0, x_0)]_r)\Big] + (\beta + \gamma + \delta s) \\
&\leq \alpha d(gy_1, gy_2) + \beta d(gy_0, gy_1) + \gamma\Big[d(gy_0, gy_1) + d(gy_1, gy_2]_r)\Big] \\
&+ \delta\Big[sd(gy_0, gy_1) + sd(gy_1, gy_2)\Big] + (\beta + \gamma + \delta s) \\
&\leq (\alpha + \gamma + \delta s)d(gy_1, gy_2) + (\beta + \gamma + \delta s)d(gy_0, gy_1) + (\beta + \gamma + \delta s),
\end{aligned}$$

which gives

$$d(gy_1, gy_2) \leq \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)} d(gy_0, gy_1) + \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)},$$

or

$$d(gy_1, gy_2) \leq kd(gy_0, gy_1) + k. \tag{2.3}$$

Again for the point $gx_2 \in [F(x_1, y_1)]_r$ and $\frac{(\beta+\gamma+\delta s)^2}{1-(\alpha+\gamma+\delta s)} > 0$, we can find $gx_3 \in [F(x_2, y_2)]_r$ such that

$$\begin{aligned}
d(gx_2, gx_3) &\leq H\Big([F(x_1, y_1)]_r, [F(x_2, y_2)]_r\Big) + \frac{(\beta + \gamma + \delta s)^2}{1 - (\alpha + \gamma + \delta s)} \\
&\leq \frac{\alpha d(gx_2, [F(x_2, y_2)]_r)\Big[1 + d(gx_1, [F(x_1, y_1)]_r)\Big]}{1 + d(gx_1, gx_2)} \\
&+ \beta d(gx_1, gx_2) + \gamma\Big[d(gx_1, [F(x_1, y_1)]_r) + d(gx_2, [F(x_2, y_2)]_r)\Big] \\
&+ \delta\Big[d(gx_1, [F(x_2, y_2)]_r) + d(gx_2, [F(x_1, y_1)]_r)\Big] + \frac{(\beta + \gamma + \delta s)^2}{1 - (\alpha + \gamma + \delta s)} \\
&\leq \alpha d(gx_2, gx_3) + \beta d(gx_1, gx_2) + \gamma\Big[d(gx_1, gx_2) + d(gx_2, gx_3)\Big] \\
&+ \delta d(gx_1, gx_3) + \frac{(\beta + \gamma + \delta s)^2}{1 - (\alpha + \gamma + \delta s)} \\
&\leq (\alpha + \gamma + \delta s)d(gx_2, gx_3) + (\beta + \gamma + \delta s)d(gx_1, gx_2) + \frac{(\beta + \gamma + \delta s)^2}{1 - (\alpha + \gamma + \delta s)}.
\end{aligned}$$

This implies that

$$d(gx_2, gx_3) \leq \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)} d(gx_1, gx_2) + \Big(\frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)}\Big)^2.$$

So

$$d(gx_2, gx_3) \leq kd(gx_1, gx_2) + k^2. \tag{2.4}$$

Also for the point $gy_2 \in [F(y_1, gx_1)]_r$, there is another point $gy_3 \in [F(y_2, gx_2)]_r$ with

$$d(gy_2, gy_3) \leq kd(gy_1, gy_2) + k^2. \tag{2.5}$$

Continuing this process, one obtains two sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that

$$gx_{n+1} \in [F(x_n, y_n)]_r \quad , \quad gy_{n+1} \in [F(y_n, gx_n)]_r \tag{2.6}$$

and

$$\begin{aligned} d(gx_n, gx_{n+1}) &\le kd(gx_{n-1}, gx_n) + k^n, \\ d(gy_n, gy_{n+1}) &\le kd(gy_{n-1}, gy_n) + k^n. \end{aligned} \tag{2.7}$$

Next, we will show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. By using Eq. (2.7) we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\le kd(gx_{n-1}, gx_n) + k^n \\ &\le k\Big(kd(gx_{n-2}, gx_{n-1}) + k^{n-1}\Big) + k^n \\ &\le k^2 d(gx_{n-2}, gx_{n-1}) + 2k^n \\ &\vdots \\ &\le k^n d(gx_0, gx_1) + nk^n,\ k < 1. \end{aligned} \tag{2.8}$$

Then, for each $p \in N$, we obtain by repeated application of triangle inequality that

$$\begin{aligned} d(gx_n, gx_{n+p}) &\le sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots + s^p d(gx_{n+p-1}, gx_{n+p}) \\ &\le [sk^n + s^2 k^{n+1} + \cdots + s^p k^{n+p-1}] d(gx_0, gx_1) + \\ &[nsk^n + (n+1)s^2 k^{n+1} + \cdots + (n+p-1)s^p k^{n+p-1}] \\ &\le sk^n \frac{1-(sk)^p}{1-sk} d(gx_0, gx_1) + [nk^{n-1} + (n+1)k^{n-1} + \cdots + (n+p-1)k^{n-1}] \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that $\{gx_n\}$ is Cauchy sequence. Similarly, we can show that $\{gy_n\}$ is also Cauchy sequence. Since $g(X)$ is complete, there exist $x, y \in X$ such that

$$gx_n \to gx \text{ and } gy_n \to gy. \tag{2.9}$$

By Lemma 1.13

$$\begin{aligned} d(gx, [F(x,y)]_r) &\le sd(gx, gx_{n+1}) + sd\Big(gx_{n+1}, [F(x,y)]_r\Big) \\ d(gx, [F(x,y)]_r) - sd(gx, gx_{n+1}) &\le sd\Big(gx_{n+1}, [F(x,y)]_r\Big) \\ &\le sH\Big([F(x,y)]_r, [F(x_n,y_n)]_r\Big) \\ &\le s\Bigg(\frac{\alpha d(gx_n, [F(x_n,y_n)]_r)\Big[1 + d(gx, [F(x,y)]_r)\Big]}{1 + d(gx, gx_n)} \\ &+ \beta d(gx, gx_n) + \gamma\Big[d(gx, [F(x,y)]_r) + d(gx_n, [F(x_n,y_n)]_r)\Big] \\ &+ \delta\Big[d(gx, [F(x_n,y_n)]_r) + d(gx_n, [F(x,y)]_r)\Big]\Bigg) \\ &\le s\Bigg(\frac{\alpha d(gx_n, gx_{n+1})\Big[1 + d(gx, [F(x,y)]_r)\Big]}{1 + d(gx, gx_n)} + \beta d(gx, gx_n) \\ &+ \gamma\Big[d(gx, [F(x,y)]_r) + d(gx_n, gx_{n+1})\Big] + \delta\Big[d(gx, gx_{n+1}) + d(gx_n, [F(x,y)]_r)\Big]\Bigg). \end{aligned}$$

Since, from Eq. (2.8), $d(gx_n, [F(x_n,y_n)]_r) \le d(gx_n, gx_{n+1}) \to 0$ as $n \to \infty$ and from Eq. (2.9) and Lemma 1.2, $d(gx, gx_n) \to 0$ as n tends to infinity. Also, we have, $d(gx_n, [F(x,y)]_r) \le sd(gx_n, gx) + sd(gx, [F(x,y)]_r)$. Then we can take limit as $n \to \infty$ to get

$$d(gx, [F(x,y)]_r) \le s\Big(\gamma d(gx, [F(x,y)]_r) + s\delta d(gx, [F(x,y)]_r)\Big).$$

That is

$$\Big[1 - s(\gamma + s\delta)\Big]d(gx, [F(x, y)]_r) \leq 0.$$

This implies that $d(gx, [F(x, y)]_r) = 0$. That is $p_r(gx, F(x, y)) = 0$, then by Lemma 1.13 we get $gx \in [F(x, y)]_r$. A similar argument can be derived to show that $gy \in [F(y, x)]_r$. This completes the proof and (x, y) is a fuzzy coupled coincidence point of the mappings F and g.

The existence and uniqueness of fuzzy coupled common fixed point for F and g is discussed in the following theorem.

Theorem 2.2. In addition to hypotheses of Theorem 2.1, F and g have fuzzy coupled common fixed point if one of the following conditions holds:

(a) F and g are $r - w-$compatible, $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in \curlyvee_r(F, g)$, $u, v \in X$ and g is continuous at u and v.

(b) g is continuous at x, y for some $(x, y) \in \curlyvee_r(F, g)$ and there exist $u, v \in X$ with $\lim_{n\to\infty} g^n u = x$, $\lim_{n\to\infty} g^n v = y$.

(c) There exists $(x, y) \in X \times X$ such that $gx = g^2x$, $gy = g^2y$ and $(gx, gy) \in \curlyvee_r(F, g)$.

(d) For any two fuzzy coupled coincidence points for F and g (say, (x, y) and (u, v)) the set $g\Big(\curlyvee_r(F, g)\Big)$ is singleton under the following condition

$$d(gx, gu) \leq H([F(x, y)]_r, [F(u, v)]_r)$$

and

$$d(gy, gv) \leq H([F(y, x)]_r, [F(v, u)]_r).$$

Furthermore, in addition to what mentioned in (d) assume one of the following holds

(i) $g\Big(\curlyvee_r(F, g)\Big) \subseteq \curlyvee_r(F, g)$,

(ii) F and g are $r - w-$compatible.

Proof. Theorem 2.1 ensure the existence of at least one fuzzy coupled coincidence point for F and g, i.e., $\curlyvee_r(F, g) \neq \emptyset$.

Suppose that (a) holds. Then for some $(x, y) \in \curlyvee_r(F, g)$, there exist $u, v \in X$ with $\lim_{n\to\infty} g^n x = u$, $\lim_{n\to\infty} g^n y = v$ and g is continuous at u and v. Hence

$$u = \lim_{n\to\infty} g^{n+1}x = \lim_{g^n x\to u} g(g^n x) = gu \tag{2.10}$$

and

$$v = \lim_{n\to\infty} g^{n+1}y = \lim_{g^n y\to v} g(g^n y) = gv. \tag{2.11}$$

As F and g are $r - w-$compatible, we have

$$g[F(x, y)]_r \subseteq [F(gx, gy)]_r \tag{2.12}$$

and

$$g[F(y, x)]_r \subseteq [F(gy, gx)]_r. \tag{2.13}$$

Note that if $(x, y) \in \curlyvee_r(F, g)$, then (y, x) is also in $\curlyvee_r(F, g)$. From (2.12), (2.13), $gx \in [F(x, y)]_r$ and $gy \in [F(y, x)]_r$, we can get that

$$g(gx) \in [F(gx, gy)]_r \tag{2.14}$$

and

$$g(gy) \in [F(gy, gx)]_r. \tag{2.15}$$

Thus,

$$(gx, gy) \in \curlyvee_r(F, g).$$

Again by the $r-w-$compatibility of F and g, we obtain that

$$g[F(gx, gy)]_r \subseteq [F(g(gx), g(gy))]_r \tag{2.16}$$

and

$$g[F(gy, gx)]_r \subseteq [F(g(gy), g(gx))]_r \tag{2.17}$$

Eqs. (2.14) and (2.16) imply to

$$g(g^2x) \in [F(g^2x, g^2y)]_r.$$

Also, (2.15) and (2.17) yield to

$$g(g^2y) \in [F(g^2y, g^2x)]_r.$$

So,

$$(g^2x, g^2y) \in \curlyvee_r(F, g).$$

Continuing this process, we can get that

$$g^n x \in [F(g^{n-1}x, g^{n-1}y)]_r, \quad g^n y \in [F(g^{n-1}y, g^{n-1}x)]_r \tag{2.18}$$

and

$$(g^n x, g^n y) \in \curlyvee_r(F, g) \ \texttt{for all} \ n \geq 1. \tag{2.19}$$

Then by (2.1), (2.18) and Lemma 1.13

$$\begin{aligned}
d(gu,[F(u, v)]_r) - sd(gu, g^n x) &\leq sd\Big(g^n x, [F(u, v)]_r\Big)\\
&\leq sH\Big([F(u, v)]_r, [F(g^{n-1}x, g^{n-1}y)]_r\Big)\\
&\leq s\Big(\frac{\alpha d(g^n x, [F(g^{n-1}x, g^{n-1}y)]_r)\big[1 + d(gu, [F(u, v)]_r)\big]}{1 + d(gu, g^n x)}\\
&+ \beta d(gu, g^n x) + \gamma\Big[d(gu, [F(u, v)]_r) + d(g^n x, [F(g^{n-1}x, g^{n-1}y)]_r)\Big]\\
&+ \delta\Big[d(gu, [F(g^{n-1}x, g^{n-1}y)]_r) + d(g^n x, [F(u, v)]_r)\Big]\Big)\\
&\leq s\Big(\frac{\alpha d(g^n x, g^n x)\big[1 + d(gu, [F(u, v)]_r)\big]}{1 + d(gu, g^n x)} + \beta d(gu, g^n x)\\
&+ \gamma\Big[d(gu, [F(u, v)]_r) + d(g^n x, g^n x)\Big] + \delta d(gu, g^n x) + \delta s\Big[d(g^n x, gu) + d(gu, [F(u, v)]_r)\Big]\Big)\\
&\leq s(\beta + \delta + \delta s)d(gu, g^n x) + (\gamma + \delta s)d(gu, [F(u, v)]_r).
\end{aligned}$$

Letting $n \to \infty$ above to get

$$[1 - (\gamma + \delta s)]d(gu, [F(u, v)]_r) \leq 0.$$

Therefore, $u = gu \in [F(u, v)]_r$. Similarly, $v = gv \in [F(u, v)]_r$. Hence (u, v) is a fuzzy coupled common fixed point of F and g.

Suppose that (b) holds. Then g is continuous at (x, y), $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in \curlyvee_r(F, g)$ and $u, v \in X$. It follows from the continuity of g at x, y that

$$x = \lim_{n\to\infty} g^{n+1}u = \lim_{g^n u \to x} g(g^n u) = gx \in [F(x, y)]_r$$

and

$$y = \lim_{n\to\infty} g^{n+1}v = \lim_{g^n v\to y} g(g^n v) = gy \in [F(y,x)]_r.$$

Hence (x, y) is a fuzzy coupled common fixed point of F and g.

Suppose that (c) holds. Then there exists $(x, y) \in X \times X$ such that $gx = g^2x$, $gy = g^2y$ and $(gx, gy) \in \curlyvee_r(F, g)$. Then we have

$$gx = g(gx) \in [F(gx, gy)]_r$$

and

$$gy = g(gy) \in [F(gy, gx)]_r.$$

Hence (gx, gy) is a fuzzy coupled common fixed point of F and g.

Finally, suppose that (d) holds. That is, if we have two fuzzy coupled coincidence points for F and g (say, (x, y) and (u, v)) then

$$d(gx, gu) \leq H([F(x, y)]_r, [F(u, v)]_r) \tag{2.20}$$

and

$$d(gy, gv) \leq H([F(y, x)]_r, [F(v, u)]_r).$$

Now we aim to proof that the set $g\Big(\curlyvee_r(F, g)\Big)$ is singleton, i.e., if we have $(x, y), (u, v), \ldots \in \curlyvee_r(F, g)$ then $gx = gu = \ldots$ and $gy = gv = \ldots$. By (2.1), (2.20) and Lemma 1.13

$$\begin{aligned} d(gx, gu) &\leq H\Big([F(x, y)]_r, [F(u, v)]_r\Big) \\ &\leq \frac{\alpha d(gu, [F(u, v)]_r)\Big[1 + d(gx, [F(x, y)]_r)\Big]}{1 + d(gx, gu)} + \beta d(gx, gu) \\ &+ \gamma\Big[d(gx, [F(x, y)]_r) + d(gu, [F(u, v)]_r)\Big] + \delta\Big[d(gx, [F(u, v)]_r) + d(gu, [F(x, y)]_r)\Big] \\ &\leq (\beta + 2\delta) d(gx, gu) \\ [1 - (\beta + 2\delta)] d(gx, gu) &\leq 0. \end{aligned}$$

Thus, $gx = gu$. By a similar way we can conclude that $gy = gv$. Therefore, $g\Big(\curlyvee_r(F, g)\Big) = \{(gx, gy)\}$. We have to discuss two cases

- Case 1. Firstly, if $g\Big(\curlyvee_r(F, g)\Big) \subseteq \curlyvee_r(F, g)$ then $x = gx =\in [F(x, y)]_r$ and $y = gy =\in [F(y, x)]_r$.
- Case 2. Secondly, if F and g are $r - w-$compatible. Therefore, from (2.12), (2.13) we get

  $$g(gx) \in [F(gx, gy)]_r$$

  and

  $$g(gy) \in [F(gy, gx)]_r.$$

  Thus,

  $$(gx, gy) \in \curlyvee_r(F, g).$$

  Hence, we have (x, y) and (gx, gy) are in $\curlyvee_r(F, g)$. From above we conclude that

  $$gx = g(gx) \in [F(gx, gy)]_r$$

  and

  $$gy = g(gy) \in [F(gy, gx)]_r.$$

Hence (gx, gy) is a fuzzy coupled common fixed point of F and g.

At the end of the proof we shall claim that the existed fuzzy coupled common fixed point for hybrid pair F and g is unique under the conditions mentioned in (d). Let (x, y) and $(\acute{x}, \acute{y})$ are two fuzzy coupled common fixed point for F and g, then we have $x = gx \in F(x, y)$, $y = gy \in F(y, x)$, $\acute{x} = g\acute{x} \in F(\acute{x}, \acute{y})$ and $\acute{y} = g\acute{y} \in F(\acute{y}, \acute{x})$. Therefore, by (2.1) we obtain

$$\begin{aligned}
d(x,\acute{x}) = d(gx, g\acute{x}) &\leq H\Big([F(x,y)]_r, [F(\acute{x},\acute{y})]_r\Big) \\
&\leq \alpha\Big(\frac{d(g\acute{x},[F(\acute{x},\acute{y})]_r)\big[1+d(gx,[F(x,y)]_r)\big]}{1+d(gx,g\acute{x})}\Big) + \beta d(gx,g\acute{x}) \\
&+ \gamma\Big[d(gx,[F(x,y)]_r) + d(g\acute{x},[F(\acute{x},\acute{y})]_r)\Big] + \delta\Big[d(gx,[F(\acute{x},\acute{y})]_r) + d(g\acute{x},[F(x,y)]_r)\Big] \\
&\leq \alpha\Big(\frac{d(g\acute{x},g\acute{x})\big[1+d(gx,gx)\big]}{1+d(gx,g\acute{x})}\Big) + \beta d(gx,g\acute{x}) + \gamma\Big[d(gx,gx) + d(g\acute{x},g\acute{x})\Big] \\
&+ \delta\Big[d(gx,g\acute{x}) + d(g\acute{x},gx)\Big] \\
&\leq \beta d(gx,g\acute{x}) + 2\delta d(gx,g\acute{x}) \\
&\leq (\beta+2\delta) d(gx,g\acute{x}),
\end{aligned}$$

which implies that

$$[1-(\beta+2\delta)]d(x,\acute{x}) \leq 0.$$

Thus, $x = \acute{x}$, by a similar way we can prove that $y = \acute{y}$. That is, F and g have a unique fuzzy common fixed point.

3. Coupled coincidence and common fixed point theorems for single-valued and multi-valued mappings

Under special choice for the mapping $F(x, y)$, for any $x, y \in X$, we obtain the corresponding multi-valued and single-valued results as follows:
We know that

$$\begin{aligned}
F :& X \times X \to W(X), \\
&(x,y) \mapsto F(x,y)
\end{aligned}$$

and

$$\begin{aligned}
F(x,y) :& X \to I = [0,1], \\
&u \mapsto r \in I.
\end{aligned}$$

Now we consider that $F(x, y)$, for all $(x, y) \in X \times X$ is the constant mapping defined by

$$F(x,y)(u) = \begin{cases} 1, & u \in T(x,y), \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $T : X \times X \to CB(X)$ is multi-valued mapping from $X \times X$ into $CB(X)$.
Then for $r \in (0, 1]$, we have

$$\begin{aligned}
[F(x,y)]_r &= \{u \in X : F(x,y)(u) \geq r\} \\
&= \{u \in X : F(x,y)(u) = 1 \geq r\} \\
&= \{u \in X : u \in T(x,y) \geq r\} \\
&= T(x,y).
\end{aligned}$$

Therefore, we can get the specific form of inequality (2.1) as

$$\begin{aligned} H\Big([F(x,y)]_r,[F(u,v)]_r\Big) &\le \alpha\Big(\frac{d(gu,[F(u,v)]_r)\Big[1+d(gx,[F(x,y)]_r)\Big]}{1+d(gx,gu)}\Big) \\ &+\beta d(gx,gu)+\gamma\Big[d(gx,[F(x,y)]_r)+d(gu,[F(u,v)]_r)\Big] \\ &+\delta\Big[d(gx,[F(u,v)]_r)+d(gu,[F(x,y)]_r)\Big] \\ H\Big(T(x,y),T(u,v)\Big) &\le \alpha\Big(\frac{d(gu,T(u,v))\Big[1+d(gx,T(x,y))\Big]}{1+d(gx,gu)}\Big) \\ &+\beta d(gx,gu)+\gamma\Big[d(gx,T(x,y))+d(gu,T(u,v))\Big] \\ &+\delta\Big[d(gx,T(u,v))+d(gu,T(x,y))\Big]. \end{aligned}$$

Consider $F(x,y)$ as in (3.1) we obtain the following corollary.

Corollary 3.1. Let (X,d) be a $b-$metric space with constant $s\ge 1$, $T: X\times X\to CB(X)$ be a multi-valued mapping and $g: X\to X$ be a single-valued mapping on X. Suppose that there exist non-negative real numbers $\alpha,\beta,\gamma,\delta\in[0,1)$ with $(\alpha+\gamma+\delta s)+(\beta+\gamma+\delta s)s<1$ and for all $x,y,u,v\in X$ we have

$$\begin{aligned} H\Big(T(x,y),T(u,v)\Big) &\le \alpha\Big(\frac{d(gu,T(u,v))\Big[1+d(gx,T(x,y))\Big]}{1+d(gx,gu)}\Big)+\beta d(gx,gu) \\ &+\gamma\Big[d(gx,T(x,y))+d(gu,T(u,v))\Big]+\delta\Big[d(gx,T(u,v))+d(gu,T(x,y))\Big]. \end{aligned} \tag{3.2}$$

Suppose also that $T(x,y)\subseteq g(X)$ for all $x,y\in X$ and $g(X)$ is complete subspace of X.
Then T and g have at least one coupled coincidence point in X. Moreover, if T and g are $w-$compatible and

$$d(gx,gu)\le H(T(x,y),T(u,v)) \text{ and } d(gy,gv)\le H(T(y,x),T(v,u)),$$

for any two coupled coincidence points (x,y) and (u,v) for F and g. Then F and g have unique coupled common fixed point.

4. Coupled coincidence and common fixed point theorems for single-valued mappings defined on a space with partial order

Theorem 4.1. Let $(X,\preceq)$ be a partially ordered set and $d: X\times X\to R^+$ be a $b-$metric with constant $s\ge 1$. Suppose that $F: X\times X\to X$ and $g: X\to X$ be two single-valued mappings on X, F has the $g-$mixed monotone property, $F(x,y)\subseteq g(X)$ for all $x,y\in X$ and $g(X)$ is complete subspace of X. Suppose that there exist non-negative real numbers $\alpha,\beta,\gamma,\delta\in[0,1)$ with $(\alpha+\gamma+\delta s)+(\beta+\gamma+\delta s)s<1$ such that

$$\begin{aligned} d\Big(F(x,y),F(u,v)\Big) &\le \frac{\alpha d(gx,F(x,y))d(gu,F(u,v))}{1+d(gx,gu)}+\beta d(gx,gu) \\ &+\gamma\Big[d(gx,F(x,y))+d(gu,F(u,v))\Big]+\delta\Big[d(gx,F(u,v))+d(gu,F(x,y))\Big], \end{aligned} \tag{4.1}$$

for all $x,y,u,v\in X$ with $gx\preceq gu$ and $gy\succeq gv$.
Also suppose that X has the following properties:

(i) if a sequence $\{x_n\}\subset X$ is a non-decreasing sequence with $x_n\to x\in X$, then $x=\sup_{\forall n}\{x_n\}$,
(ii) if a sequence $\{y_n\}\subset X$ is a non-increasing sequence with $y_n\to y\in X$, then $y=\inf_{\forall n}\{y_n\}$.

Furthermore, if there exists two elements $x_0, y_0 \in X$ with $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then there exists $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Using $F(X \times X) \subseteq g(X)$ and beginning with these points $x_0, y_0 \in X$, we can find $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Also for $x_1, y_1 \in X$ there exist $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. By the $g-$mixed monotone property of F, $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, we have

$$gx_0 \preceq gx_1 \text{ and } gy_0 \succeq gy_1 \Rightarrow$$
$$gx_1 = F(x_0, y_0) \preceq F(x_1, y_1) = gx_2 \text{ and } gy_1 = F(y_0, x_0) \succeq F(y_1, x_1) = gy_2.$$

Continuing in this way, we construct two sequences $\{gx_n\}_{n\geq 0}$ and $\{gy_n\}_{n\geq 0}$ in X such that

$$\begin{cases} gx_{n+1} &= F(x_n, y_n), \\ gy_{n+1} &= F(y_n, x_n). \end{cases}$$

By mathematical induction we obtain

$$gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1} \ \forall n \geq 0.$$

If $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$ for some $n \geq 1$ then $gx_n = F(x_n, y_n)$ and $gy_n = F(gy_n, gx_n)$, i.e., (x_n, y_n) is a coupled coincidence point of F and g and this completes the proof. So from now on, we assume that either $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n. Since $gx_{n-1} \preceq gx_n$ and $gy_{n-1} \succeq gy_n$, then from (4.1), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d\Big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\Big) \\ &\leq \frac{\alpha d(gx_{n-1}, F(x_{n-1}, y_{n-1}))d(gx_n, F(x_n, y_n))}{1 + d(gx_{n-1}, gx_n)}\Big) \\ &+ \beta d(gx_{n-1}, gx_n) + \gamma\Big[d(gx_{n-1}, F(x_{n-1}, y_{n-1})) + d(gx_n, F(x_n, y_n))\Big] \\ &+ \delta\Big[d(gx_{n-1}, F(x_n, y_n)) + d(gx_n, F(x_{n-1}, y_{n-1}))\Big] \\ &\leq \frac{\alpha d(gx_{n-1}, gx_n)d(gx_n, gx_{n+1})}{1 + d(gx_{n-1}, gx_n)}\Big) \\ &+ \beta d(gx_{n-1}, gx_n) + \gamma\Big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\Big] \\ &+ \delta\Big[d(gx_{n-1}, gx_{n+1}) + d(gx_n, gx_n)\Big] \\ &\leq \alpha d(gx_n, gx_{n+1}) + \beta d(gx_{n-1}, gx_n) + \gamma\Big[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\Big] \\ &+ \delta\Big[sd(gx_{n-1}, gx_n) + sd(gx_n, gx_{n+1})\Big] \\ &\leq (\alpha + \gamma + \delta s)d(gx_n, gx_{n+1}) + (\beta + \gamma + \delta s)d(gx_{n-1}, gx_n). \end{aligned}$$

So we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)} d(gx_{n-1}, gx_n) \\ &\ \vdots \\ &\leq k^n d(gx_0, gx_1). \end{aligned} \tag{4.2}$$

Similarly,

$$\begin{aligned}
d(gy_{n+1}, gy_n) &= d\Big(F(y_n, x_n), F(y_{n-1}, x_{n-1})\Big) \\
&\leq \frac{\alpha d(gy_n, F(y_n, x_n))d(gy_{n-1}, F(y_{n-1}, x_{n-1}))}{1 + d(gy_n, gy_{n-1})}\Big) \\
&+ \beta d(gy_{n-1}, gy_n) + \gamma\Big[d(gy_n, F(y_n, x_n)) + d(gy_{n-1}, F(y_{n-1}, x_{n-1}))\Big] \\
&+ \delta\Big[d(gy_n, F(y_{n-1}, x_{n-1})) + d(gy_{n-1}, F(y_n, x_n))\Big] \\
&\leq (\alpha + \gamma + \delta s)d(gy_n, gy_{n+1}) + (\beta + \gamma + \delta s)d(gy_{n-1}, gy_n) \\
&\leq \frac{\beta + \gamma + \delta s}{1 - (\alpha + \gamma + \delta s)} d(gy_{n-1}, gy_n) \\
&\vdots \\
&\leq k^n d(gy_0, gy_1).
\end{aligned}$$

Let $n \in N$ and $p \geq 1$. Now we will prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X.

$$\begin{aligned}
d(gx_n, gx_{n+p}) &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots + s^p d(gx_{n+p-1}, gx_{n+p}) \\
&\leq [sk^n + s^2 k^{n+1} + \cdots + s^p k^{n+p-1}]d(gx_0, gx_1) \\
&\leq sk^n[1 + sk + \cdots + (sk)^{p-1}]d(gx_0, gx_1) \\
&\leq sk^n \frac{1 - (sk)^p}{1 - sk} d(gx_0, gx_1) \to 0 \ \text{as} \ n \to \infty.
\end{aligned}$$

It follows that $\{gx_n\}$ is Cauchy sequence. Similarly, we can show that $\{gy_n\}$ is also Cauchy sequence. Since $g(X)$ is complete, there exist $x, y \in X$ such that

$$gx_n \to gx \ \text{and} \ gy_n \to gy. \tag{4.3}$$

Now we show that (x, y) is a coupled coincidence point for F and g. For this purpose we shall use (4.1) with $x = x_n$, $y = y_n$, $u = x$ and $v = y$, then take limit on both sides as n tends to infinity and use Eq. (4.3), Lemma 1.2 and properties on X.

$$\begin{aligned}
d(gx, F(x, y)) &\leq sd(gx, gx_{n+1}) + sd\Big(gx_{n+1}, F(x, y)\Big) \\
d(gx, F(x, y)) - sd(gx, gx_{n+1}) &\leq sd\Big(F(x_n, y_n), F(x, y)\Big) \\
&\leq s\Big(\frac{\alpha d(gx_n, F(x_n, y_n))d(gx, F(x, y))}{1 + d(gx_n, gx)} + \beta d(gx, gx_n) \\
&+ \gamma\Big[d(gx_n, F(x_n, y_n)) + d(gx, F(x, y))\Big] + \delta\Big[d(gx_n, F(x, y)) + d(gx, F(x_n, y_n))\Big]\Big) \\
&\leq s\Big(\frac{\alpha d(gx_n, gx_{n+1})d(gx, F(x, y))}{1 + d(gx_n, gx)}\Big) + \beta d(gx, gx_n) \\
&+ \gamma\Big[d(gx_n, gx_{n+1}) + d(gx, F(x, y))\Big] + \delta\Big[d(gx_n, F(x, y)) + d(gx, gx_{n+1})\Big]\Big) \\
\Big[1 - s(\gamma + s\delta)\Big]&d(gx, F(x, y)) \leq 0.
\end{aligned}$$

This implies that $gx = F(x, y)$. A similar argument can be derived to show that $F(y, x) = gy$. This completes the proof and (x, y) is a coupled coincidence point of the mappings F and g.

Remark 4.2. If F is continuous and commutes with g, we can get the same result without using properties (i) and (ii) on X. $F(gx, gy) = F(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n) = \lim_{n\to\infty} F(gx_n, gy_n) = \lim_{n\to\infty} gF(x_n, y_n) =$

$\lim_{n\to\infty} g(gx_{n+1}) = gx$ and
$F(gy, gx) = F(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gx_n) = \lim_{n\to\infty} F(gy_n, gx_n) = \lim_{n\to\infty} gF(y_n, x_n)$
$= \lim_{n\to\infty} g(gy_{n+1}) = gy.$

Now we proof the uniqueness of the coupled point of coincidence under additional condition and the existence of the coupled common fixed point by using the notion of weak compatibility.

Theorem 4.3. By adding to the hypotheses of Theorem 4.1 the condition: for every two coupled coincidence points of F and g, (x, y) and (x^*, y^*), there exists $(u, v) \in X^2$ such that (gu, gv) is comparable, at the same time, to (gx, gy) and (gx^*, gy^*). Then F and g have a unique point of coincidence. Furthermore, if F and g are $w-$compatible then they have a unique coupled common fixed point.

Proof. Suppose that (x, y) and (x^*, y^*) are two coupled coincidence points of F and g, that is, $gx = F(x, y)$, $gy = F(y, x)$, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$. We shall prove that $gx = gx^*$ and $gy = gy^*$. Consider the following two cases:

(Case 1) If (gx, gy) and (gx^*, gy^*) are comparable, say $(gx, gy) \preceq_p (gx^*, gy^*)$, then we have

$$\begin{aligned} d(gx, gx^*) = d\Big(F(x, y), F(x^*, y^*)\Big) &\le \frac{\alpha d(gx, F(x, y))d(gx^*, F(x^*, y^*))}{1 + d(gx, gx^*)} + \beta d(gx, gx^*) \\ &+ \gamma\Big[d(gx, F(x, y)) + d(gx^*, F(x^*, y^*))\Big] + \delta\Big[d(gx, F(x^*, y^*)) + d(gx^*, F(x, y))\Big] \\ &\le \Big(\beta + 2\delta\Big)d(gx, gx^*), \end{aligned}$$

which gives $d(gx, gx^*) = 0$. Thus, $gx = gx^*$. Also, we have

$$d(gy, gy^*) = d\Big(F(y, x), F(y^*, x^*)\Big) = 0.$$

Hence, $gy = gy^*$.

(Case 2) If (gx, gy) and (gx^*, gy^*) are not comparable. By assumption there exists $(u, v) \in X\times X$ such that (gu, gv) is comparable to (gx, gy) and to (gx^*, gy^*).
Put $u_0 = u$, $v_0 = v$, $x_0 = x$, $y_0 = y$, $x^*_0 = x^*$ and $y^*_0 = x^*$. Using that $F(X \times X) \subseteq g(X)$, for $u_0, v_0 \in X$, there exist $u_1, v_1 \in X$ with $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. For $n \ge 1$, continuing this process we can construct the sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = F(gu_n, gv_n) \text{ and } gv_{n+1} = F(gv_n, gu_n).$$

On the same way, for (x, y) and $(x^*, y^*) \in X \times X$, define the sequences $\{gx_n\}, \{gy_n\}, \{gx^*_n\}$ and $\{gy^*_n\}$ as

$$gx_{n+1} = F(gx_n, gy_n), \quad gy_{n+1} = F(gy_n, gx_n)$$

and

$$gx^*_{n+1} = F(gx^*_n, gy^*_n), \quad gy^*_{n+1} = F(gy^*_n, gx^*_n).$$

Since F has $g-$mixed monotone property and $gx_1 = F(x_0, y_0) = gx_0$ (that is, $gx_1 \preceq gx_0$ and $gx_1 \succeq gx_0$), then we have

$$gx_{n+1} \preceq gx_n \text{ and } gx_n \succeq gx_{n+1} \ \forall n \ge 0.$$

Hence

$$gx_{n+1} = gx_n \text{ or } gx_n = gx = F(x, y).$$

By a similar way, we get

$$gy_n = F(y, x), \quad gx^*_n = F(x^*, y^*) \text{ and } gy^*_n = F(y^*, x^*).$$

Since (gu_0, gv_0) is comparable with (gx, gy), say $gu_0 \leq gx_0$ and $gv_0 \geq gy_0$, then we have

$$gu_n \leq gx_n = gx \text{ and } gv_n \geq gy_n = gy \ \forall n \geq 0. \tag{4.4}$$

Using that (gu_0, gv_0) is also comparable to (gx^*, gy^*), we obtain that (gu_n, gv_n) is comparable to (gx^*, gy^*). By (4.1) and (4.4), we get

$$\begin{aligned}
d(gu_{n+1}, gx) = d\Big(F(u_n, v_n), F(x, y)\Big) &\leq \frac{\alpha d(gu_n, F(u_n, v_n))d(gx, F(x, y))}{1 + d(gu_n, gx)} \\
&+ \beta d(gu_n, gx) + \gamma\Big[d(gu_n, F(u_n, v_n)) + d(gx, F(x, y))\Big] \\
&+ \delta\Big[d(gu_n, F(x, y)) + d(gx, F(u_n, v_n))\Big] \\
&\leq \beta d(gu_n, gx) + \gamma d(gu_n, gu_{n+1}) + \delta\Big[d(gu_n, gx) + d(gx, gu_{n+1})\Big] \\
\Big[1 - (\gamma s + \delta)\Big]d(gu_{n+1}, gx) &\leq (\beta + \gamma s + \delta)d(gu_n, gx) \\
d(gu_{n+1}, gx) &\leq \frac{\beta + \gamma s + \delta}{1 - (\gamma s + \delta)} d(gu_n, gx) \\
&\vdots \\
&\leq \left(\frac{\beta + \gamma s + \delta}{1 - (\gamma s + \delta)}\right)^n d(gu, gx) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Also, we have

$$\begin{aligned}
d(gy, gv_{n+1}) = d\Big(F(y, x), F(v_n, u_n)\Big) &\leq \frac{\alpha d(gy, F(y, x))d(gv_n, F(v_n, u_n))}{1 + d(gy, gv_n)} \\
&+ \beta d(gy, gv_n) + \gamma\Big[d(gy, F(y, x)) + d(gv_n, F(v_n, u_n))\Big] \\
&+ \delta\Big[d(gy, F(v_n, u_n)) + d(gv_n, F(y, x))\Big] \\
&\leq \beta d(gy, gv_n) + \gamma d(gv_n, gv_{n+1}) + \delta\Big[d(gy, gv_{n+1}) + d(gv_n, gy)\Big] \\
&\leq \frac{\beta + \gamma s + \delta}{1 - (\gamma s + \delta)} d(gy, gv_n) \\
&\vdots \\
&\leq \left(\frac{\beta + \gamma s + \delta}{1 - (\gamma s + \delta)}\right)^n d(gy, gv) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(gu_n, gx) = \lim_{n\to\infty} d(gv_n, gy) = 0.$$

By a similar way we can prove that

$$\lim_{n\to\infty} d(gu_n, gx^*) = \lim_{n\to\infty} d(gv_n, gy^*) = 0.$$

By the uniqueness of the limit, we obtain $gx = gx^*$ and $gy = gy^*$.

Therefore, F and g have a unique point of coincidence (gx, gy), that is $g(\curlyvee(F, g)) = \{(gx, gy)\}$. Furthermore, if F and g are w−compatible then we have

$$g^2x = g(F(x, y)) = F(gx, gy) \text{ and } g^2y = g(F(y, x)) = F(gy, gx).$$

That is (gx, gy) is another coupled coincidence point for F and g, i.e., $(g^2x, g^2y) \in g(\curlyvee(F, g))$. By the uniqueness of the point of coincidence, we obtain

$$gx = g^2x = F(gx, gy) \text{ and } gy = g^2y = F(gy, gx). \tag{4.5}$$

Hence, $(w_1, w_2) = (gx, gy)$ is a coupled common fixed point for F and g.
Now we will claim the uniqueness of this coupled common fixed point. Suppose that (z_1, z_2) is another coupled common fixed point of F and g. Thus,

$$z_1 = gz_1 = F(z_1, z_2) \text{ and } z_2 = gz_2 = F(z_2, z_1). \tag{4.6}$$

Therefore, (z_1, z_2) is point of coincidence of F and g. Since (w_1, w_2) is the unique point of coincidence, then we have $z_1 = w_1$ and $z_2 = w_2$. Hence (w_1, w_2) is a unique coupled common fixed point of F and g.

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