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On some generating relations involving generalized Humbert functions *

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Abstract The main objective of this paper is to construct new analogous definitions of the families of Humbert functions using the generating function method as the starting point. We study a class of various results in the family of Humbert functions with the help of the families of generating functions, explicit representations, especially differential recurrence relations and study some of the significant properties of this family of functions.

Key words Humbert functions, generating functions, recurrences relations.

2020 Mathematics Subject Classification 33C10, 33C45, 33C50, 33C99, 44A20.

1 Introduction

In [3, 4, 7, 9], the Humbert functions are defined by the generating function

$$\exp\left[\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right]=\sum_{m,n=-\infty}^{\infty}J_{m,n}(x)u^{m}t^{n}.$$

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It is a well-known fact that the Humbert function is defined as

$$J_{m,n}(x) = \left(\frac{x}{3}\right)^{m+n} \frac{1}{\Gamma(m+1)\Gamma(n+1)}\, {}_0F_2\left(-; m+1, n+1; -\frac{x^3}{27}\right)$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)\Gamma(n+k+1)} \left(\frac{x}{3}\right)^{m+n+3k}.$$

We now make use of the Mellin-Barnes contour integral for the Gamma function (see, for details, ([5, p. 17, equation 2.7(4)],] and [8, p. 219, Equation 4.1(5)]):

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi\, i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^t t^{-a} dt.$$

where $Re(a) > 0$ and $\sigma > 0$, the contour in the complex s-plane is of the familiar Mellin-Barnes type. The subject of generating relations plays an important role in the development and study of special functions. We will further generalize the class of Bessel functions, by using the same approach as exposed above, to define the family of generalized Humbert functions of different types.

2 Definitions of new Humbert functions and their properties

In this section, we apply the generating functions to get explicit formulas for the family of generalized Humbert functions of different types and discuss some interesting significant properties for these functions as the generalizations of the above mentioned identities.

Definition 2.1. The product of symmetric exponential functions is defined by the generating function

$$F_1(x; u, t; p, q, l) = \exp\left[\frac{x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] = \sum_{m,n=-\infty}^{\infty} J_{m,n}^{p,q,l}(x) u^m t^n. \tag{2.1}$$

From (2.1), we have

$$F_1(x; u, t; p, q, l) = \exp\left(\frac{xu^p}{p+q+l}\right) \exp\left(\frac{xt^q}{p+q+l}\right) \exp\left(-\frac{x}{(p+q+l)(ut)^l}\right)$$
$$= \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{xu^p}{p+q+l}\right)^r \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{xt^q}{p+q+l}\right)^i \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{x}{(p+q+l)(ut)^l}\right)^k$$
$$= \sum_{i,r,k=0}^{\infty} \frac{(-1)^k}{k!\; i!\; r!}\left(\frac{x}{p+q+l}\right)^{k+i+r} u^{pr-lk} t^{qi-lk}.$$

Now set $pr - lk = m$ and $qi - lk = n$ to get

$$\sum_{k,i,r=0}^{\infty} \frac{(-1)^k}{k!\; i!\; r!}\left(\frac{x}{p+q+l}\right)^{k+i+r} u^{pr-lk} t^{qi-lk}$$
$$= \sum_{m,n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p+q+l}\right)^{\frac{m+lk}{p}+\frac{n+lk}{q}+k} u^m t^n$$
$$= \sum_{m,n=-\infty}^{\infty} J_{m,n}^{p,q,l}(x) u^m t^n.$$

Explicitly, we obtain the explicit expression of Humbert function $J_{m,n}^{p,q,l}(x)$ as

$$J_{m,n}^{p,q,l}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p+q+l}\right)^{\frac{m+lk}{p}+\frac{n+lk}{q}+k}. \tag{2.2}$$

Theorem 2.2. *The Humbert function $J^{p,q,l}_{m,n}(x)$ satisfy the relations*

$$(p+q+l)\frac{d}{dx}J^{p,q,l}_{m,n}(x) = J^{p,q,l}_{m,n-q}(x) + J^{p,q,l}_{m-p,n}(x) - J^{p,q,l}_{m+l,n+l}(x), \tag{2.3}$$

$$\frac{(m+1)(p+q+l)}{x}J^{p,q,l}_{m+1,n}(x) = pJ^{p,q,l}_{m-p+1,n}(x) + lJ^{p,q,l}_{m+l+1,n+l}(x), \tag{2.4}$$

and

$$\frac{(n+1)(p+q+l)}{x}J^{p,q,l}_{m,n+1}(x) = qJ^{p,q,l}_{m,n-q+1}(x) + lJ^{p,q,l}_{m+l,n+l+1}(x). \tag{2.5}$$

Proof. In (2.1), we start to derive with respect to x in the generating functions

$$\frac{1}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\exp\left[\frac{x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] = \sum_{m,n=-\infty}^{\infty}\frac{d}{dx}J^{p,q,l}_{m,n}(x)u^m t^n.$$

after explicating the l.h.s., we obtain (2.3).

By differentiating with respect to u and t in (2.1), separately, we have:

$$\frac{x}{p+q+l}\left(pu^{p-1} + \frac{l}{u(ut)^l}\right)\exp\left[\frac{x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] = \sum_{m,n=-\infty}^{\infty}J^{p,q,l}_{m,n}(x)mu^{m-1}t^n.$$

and

$$\frac{x}{p+q+l}\left(qt^{q-1} + \frac{l}{t(ut)^l}\right)\exp\left[\frac{x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] = \sum_{m,n=-\infty}^{\infty}J^{p,q,l}_{m,n}(x)u^m nt^{n-1}.$$

and by equating the same power of the indexes, we obtain the recurrence relations (2.4) and (2.5). □

Theorem 2.3. *The Humbert function $J^{p,q,l}_{m,n}(x)$ satisfies the multiplication formula:*

$$J^{p,q,l}_{m,n}(\mu\, x) = \mu^{m+n}\sum_{k=0}^{\infty}\left(\frac{x(1-\mu^{3l})}{p+q+l}\right)^k J^{p,q,l}_{m+lk,n+lk}(x). \tag{2.6}$$

Proof. Substituting, in fact, μx with x in equation (2.1) and by expanding the exponential function, we get:

$$\begin{aligned}
F_1(\mu\, x; u, t; p, q, l) &= \exp\left[\frac{\mu\, x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] \\
&= \exp\left[\frac{x}{p+q+l}\left(\mu\, u^p + \mu\, t^q - \frac{1}{(\mu^2\, ut)^l}\right)\right]\exp\left[\frac{x}{p+q+l}\left(\frac{1}{(\mu^2\, ut)^l} - \frac{\mu}{(ut)^l}\right)\right] \\
&= \exp\left[\frac{x}{p+q+l}\left(\mu\, u^p + \mu\, t^q - \frac{1}{(\mu^2\, ut)^l}\right)\right]\exp\left[\frac{x}{p+q+l}\left(\frac{\mu^{-2l}-\mu}{(ut)^l}\right)\right] \\
&= \exp\left[\frac{x(\mu^{-2l}-\mu)}{(p+q+l)(ut)^l}\right]\sum_{m,n=-\infty}^{\infty}J^{p,q,l}_{m,n}(x)(\mu\, u)^m(\mu\, t)^n \\
&= \sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\left(\frac{x(\mu^{-2l}-\mu)}{p+q+l}\right)^k J^{p,q,l}_{m,n}(x)\mu^{m+n}u^{m-lk}t^{n-lk} \\
&= \sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\left(\frac{x(\mu^{-2l}-\mu)}{p+q+l}\right)^k J^{p,q,l}_{m+lk,n+lk}(x)\mu^{m+n+2lk}u^m t^n.
\end{aligned}$$

Setting $h-s=n$ and $k-s=m$, after equating the same power of n and m, we get the multiplication formula:

$$J^{p,q,l}_{m,n}(\mu\, x) = \sum_{k=0}^{\infty}\left(\frac{x(\mu^{-2l}-\mu)}{p+q+l}\right)^k \mu^{m+n+2lk}J^{p,q,l}_{m+lk,n+lk}(x) = \mu^{m+n}\sum_{k=0}^{\infty}\left(\frac{x(1-\mu^{3l})}{p+q+l}\right)^k J^{p,q,l}_{m+lk,n+lk}(x).$$

□

Note that for $\mu = -1$, we get the relation:

$$J^{p,q,l}_{m,n}(-x) = \sum_{m,n=-\infty}^{\infty} J^{p,q,l}_{m,n}(x)(\mu\, u)^m(\mu\, t)^n . J^{p,q,l}_{m,n}(-x) = \sum_{k=0}^{\infty}\left(\frac{2x}{p+q+l}\right)^k (-1)^{m+n+2lk} J^{p,q,l}_{m+lk,n+lk}(x).$$

Theorem 2.4. *The Humbert function $J^{p,q,l}_{m,n}(x)$ satisfies the addition formula:*

$$J^{p,q,l}_{m,n}(x+y) = \sum_{r,s=-\infty}^{\infty} J^{p,q,l}_{m-r,n-s}(x) J^{p,q,l}_{r,s}(y). \tag{2.7}$$

Proof. From (2.1), we have

$$\sum_{m,n=-\infty}^{\infty} J^{p,q,l}_{m,n}(x+y)u^m t^n = \exp\left[\frac{x+y}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right]$$

$$= \exp\left[\frac{x}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right] \exp\left[\frac{y}{p+q+l}\left(u^p + t^q - \frac{1}{(ut)^l}\right)\right]$$

$$= \sum_{m,n=-\infty}^{\infty} J^{p,q,l}_{m,n}(x)u^m t^n \sum_{r,s=-\infty}^{\infty} J^{p,q,l}_{r,s}(y)u^r t^s = \sum_{m,n,r,s=-\infty}^{\infty} J^{p,q,l}_{m,n}(x) J^{p,q,l}_{r,s}(y) u^{m+r} t^{n+s}$$

$$= \sum_{m,n,r,s=-\infty}^{\infty} J^{p,q,l}_{m-r,n-s}(x) J^{p,q,l}_{r,s}(y) u^m t^n.$$

after explicating the l.h.s., we obtain the relation (2.7). □

Theorem 2.5. *The Humbert function $J^{p,q,l}_{m,n}(x)$ satisfies the integral*

$$J^{p,q,l}_{m,n}(x) = \frac{1}{(2\pi\, i)^2}\left(\frac{x}{p+q+l}\right)^{\frac{m}{p}+\frac{n}{q}} \int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}$$
$$e^{t+s} t^{-\frac{m}{p}-1} s^{-\frac{n}{q}-1} \exp\left[-\frac{1}{t^{\frac{i}{p}} s^{\frac{l}{q}}}\left(\frac{x}{p+q+l}\right)^{\frac{l}{p}+\frac{l}{q}+1}\right] dtds. \tag{2.8}$$

Proof. Starting from the formula

$$\frac{1}{\Gamma(\frac{m+kl}{p}+1)} = \frac{1}{2\pi\, i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^t t^{-\frac{m+kl}{p}-1} dt,$$

and

$$\frac{1}{\Gamma(\frac{n+kl}{q}+1)} = \frac{1}{2\pi\, i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^s s^{-\frac{n+kl}{q}-1} ds,$$

and substituting the above expression into the series expression of the Humbert function given in (2.1), it follows that

$$J^{p,q,l}_{m,n}(x) = \frac{1}{(2\pi\, i)^2}\left(\frac{x}{p+q+l}\right)^{\frac{m}{p}+\frac{n}{q}} \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}$$
$$\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{t+s} t^{-\frac{m+kl}{p}-1} s^{-\frac{n+kl}{q}-1}\left(\frac{x}{p+q+l}\right)^{\frac{lk}{p}+\frac{lk}{q}+k} dtds$$

□

Definition 2.6. Let us define the product of symmetric exponential functions as the generating function

$$F_2(x;u,t;p,q,l) = \exp\left[\left(\frac{xu}{p+q+l}\right)^p + \left(\frac{xt}{p+q+l}\right)^q - \left(\frac{x}{(p+q+l)ut}\right)^l\right] = \sum_{m,n=-\infty}^{\infty} \mathbf{J}^{p,q,l}_{m,n}(x)u^m t^n. \tag{2.9}$$

From (2.9), we have

$$F_2(x;u,t;p,q,l) = \exp\left[\left(\frac{xu}{p+q+l}\right)^p\right]\exp\left[\left(\frac{xt}{p+q+l}\right)^q\right]\exp\left[-\left(\frac{x}{(p+q+l)ut}\right)^l\right]$$
$$= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xu}{p+q+l}\right)^{pr}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{xt}{p+q+l}\right)^{qi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{(p+q+l)ut}\right)^{lk}$$
$$= \sum_{i,r,k=0}^{\infty}\frac{(-1)^k}{k!\ i!\ r!}\left(\frac{x}{p+q+l}\right)^{lk+qi+pr} u^{pr-lk}t^{qi-lk}.$$

Now, replace m by $pr - lk$ and n by $qi - lk$ to get

$$\sum_{k,i,r=0}^{\infty}\frac{(-1)^k}{k!\ i!\ r!}\left(\frac{x}{p+q+l}\right)^{lk+qi+pr} u^{pr-lk}t^{qi-lk}$$
$$= \sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p+q+l}\right)^{m+n+3lk} u^m t^n$$
$$= \sum_{m,n=-\infty}^{\infty}\mathbf{J}_{m,n}^{p,q,l}(x)u^m t^n.$$

Explicitly, we obtain the explicit form of Humbert function $\mathbf{J}_{m,n}^{p,q,l}(x)$ as the series

$$\mathbf{J}_{m,n}^{p,q,l}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p+q+l}\right)^{m+n+3lk}. \tag{2.10}$$

Theorem 2.7. *The following integral holds true for the Humbert function* $\mathbf{J}_{m,n}^{p,q,l}(x)$*:*

$$\mathbf{J}_{m,n}^{p,q,l}(x) = \frac{1}{(2\pi\ i)^2}\left(\frac{x}{p+q+l}\right)^{m+n}$$
$$\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{t+s}t^{-\frac{m}{p}-1}s^{-\frac{n}{q}-1}\exp\left[-\frac{1}{t^{\frac{i}{p}}s^{\frac{l}{q}}}\left(\frac{x}{p+q+l}\right)^{3l}\right]dtds. \tag{2.11}$$

Definition 2.8. Let us consider the product of symmetric exponential functions by the generating function

$$F_3(x;u,t;p,q,l) = \exp\left(\left(\frac{xu}{p}\right)^p + \left(\frac{xt}{q}\right)^q - \left(\frac{x}{lut}\right)^l\right) = \sum_{m,n=-\infty}^{\infty}\mathbb{J}_{m,n}^{p,q,l}(x)u^m t^n. \tag{2.12}$$

From (2.12), we have

$$F_3(x;u,t;p,q,l) = \exp\left[\left(\frac{xu}{p}\right)^p\right]\exp\left[\left(\frac{xt}{q}\right)^q\right]\exp\left[-\left(\frac{x}{lut}\right)^l\right]$$
$$= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xu}{p}\right)^{pr}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{xt}{q}\right)^{qi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{lut}\right)^{lk}$$
$$= \sum_{i,r,k=0}^{\infty}\frac{(-1)^k}{k!\ i!\ r!}\left(\frac{x}{p}\right)^{pr}\left(\frac{x}{q}\right)^{qi}\left(\frac{x}{l}\right)^{lk} u^{pr-lk}t^{qi-lk}.$$

Now, replace m by $pr-lk$ and n by $qi-lk$ to get

$$\sum_{k,i,r=0}^{\infty}\frac{(-1)^k}{k!\,i!\,r!}\left(\frac{x}{p}\right)^{pr}\left(\frac{x}{q}\right)^{qi}\left(\frac{x}{l}\right)^{lk}u^{pr-lk}t^{qi-lk}$$
$$=\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p}\right)^{m+lk}\left(\frac{x}{q}\right)^{n+lk}\left(\frac{x}{l}\right)^{lk}u^m t^n$$
$$=\sum_{m,n=-\infty}^{\infty}\mathbb{J}_{m,n}^{p,q,l}(x)u^m t^n.$$

Explicitly, we state the explicit form of Humbert function as the power series

$$\mathbb{J}_{m,n}^{p,q,l}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x}{p}\right)^{m+lk}\left(\frac{x}{q}\right)^{n+lk}\left(\frac{x}{l}\right)^{lk}. \tag{2.13}$$

Theorem 2.9. *The integral representation of the Humbert function $J_{m,n}^{p,q,l}(x)$ is given by*

$$\mathbb{J}_{m,n}^{p,q,l}(x)=\frac{1}{(2\pi\, i)^2}\left(\frac{x}{p}\right)^m\left(\frac{x}{q}\right)^n$$
$$\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{t+s}t^{-\frac{m}{p}-1}s^{-\frac{n}{q}-1}\exp\left[-\frac{1}{t^{\frac{i}{p}}s^{\frac{l}{q}}}\left(\frac{x}{p}\right)^l\left(\frac{x}{q}\right)^l\left(\frac{x}{l}\right)^l\right]dtds. \tag{2.14}$$

3 Generalized Humbert functions

Here, we discuss and derive explicit formulas and some interesting relations linking the various families of Humbert functions of three variables by using the techniques of generating functions.

Definition 3.1. Let us define the product of exponential functions as the generating function

$$F_4(x,y,z;u,t;p,q,l)=\exp\left[\frac{1}{p+q+l}\left((xu)^p+(yt)^q-(\frac{z}{ut})^l\right)\right]=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{p,q,l}(x,y,z)u^m t^n. \tag{3.1}$$

From (3.1), we have

$$F_4(x,y,z;u,t;p,q,l)=\exp\left(\frac{(xu)^p}{p+q+l}\right)\exp\left(\frac{(yt)^q}{p+q+l}\right)\exp\left(-\frac{z^l}{(p+q+l)(ut)^l}\right)$$
$$=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{(xu)^p}{p+q+l}\right)^r\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{(yt)^q}{p+q+l}\right)^i\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{z^l}{(p+q+l)(ut)^l}\right)^k$$
$$=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k}{k!\,i!\,r!}\left(\frac{x^p}{p+q+l}\right)^r\left(\frac{y^q}{p+q+l}\right)^i\left(\frac{z^l}{p+q+l}\right)^k u^{pr-lk}t^{qi-lk}.$$

Now set $pr-lk=m$ and $qi-lk=n$ to get

$$\sum_{k,i,r=0}^{\infty}\frac{(-1)^k}{k!\,i!\,r!}\left(\frac{x^p}{p+q+l}\right)^r\left(\frac{y^q}{p+q+l}\right)^i\left(\frac{z^l}{p+q+l}\right)^k u^{pr-lk}t^{qi-lk}$$
$$=\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)}\left(\frac{x^p}{p+q+l}\right)^{\frac{m+kl}{p}}\left(\frac{y^q}{p+q+l}\right)^{\frac{n+kl}{q}}\left(\frac{z^l}{p+q+l}\right)^k u^m t^n$$
$$=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{p,q,l}(x,y,z)u^m t^n.$$

Explicitly, we get the explicit formula for Humbert function $J^{p,q,l}_{m,n}(x,y,z)$ as the power series

$$J^{p,q,l}_{m,n}(x,y,z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\Big(\frac{m+kl}{p}+1\Big)\Gamma\Big(\frac{n+kl}{q}+1\Big)}\frac{x^{m+kl}y^{n+kl}z^{kl}}{(p+q+l)^{\frac{m+lk}{p}+\frac{n+lk}{q}+k}}. \tag{3.2}$$

Theorem 3.2. *The Humbert function $J^{p,q,l}_{m,n}(x,y,z)$ satisfies the relations*

$$\frac{\partial}{\partial x}J^{p,q,l}_{m,n}(x,y,z)=\frac{px^{p-1}}{p+q+l}J^{p,q,l}_{m-p,n}(x,y,z), \tag{3.3}$$

$$\frac{\partial}{\partial y}J^{p,q,l}_{m,n}(x,y,z)=\frac{qy^{q-1}}{p+q+l}J^{p,q,l}_{m,n-q}(x,y,z), \tag{3.4}$$

and

$$\frac{\partial}{\partial z}J^{p,q,l}_{m,n}(x,y,z)=-\frac{l\,z^{l-1}}{p+q+l}J^{p,q,l}_{m+l,n+l}(x,y,z). \tag{3.5}$$

Proof. By deriving in equation (3.1) separately with respect to x, we have

$$\sum_{m,n=-\infty}^{\infty}\frac{\partial}{\partial x}J^{p,q,l}_{m,n}(x,y,z)u^m t^n=\frac{pu(xu)^{p-1}}{p+q+l}\exp\Big[\frac{1}{p+q+l}\Big((xu)^p+(yt)^q-(\frac{z}{ut})^l\Big)\Big]$$
$$=\frac{pu(xu)^{p-1}}{p+q+l}\sum_{m,n=-\infty}^{\infty}J^{p,q,l}_{m,n}(x,y,z)u^m t^n=\frac{px^{p-1}}{p+q+l}\sum_{m,n=-\infty}^{\infty}J^{p,q,l}_{m,n}(x,y,z)u^{m+p}t^n.$$

By following the same procedure, we can obtain the recurrence relations (3.4) and (3.5). □

Theorem 3.3. *The Humbert function $J^{p,q,l}_{m,n}(x,y,z)$ satisfy the properties*

$$qy^{q-1}\frac{\partial}{\partial x}J^{p,q,l}_{m+p,n}(x,y,z)=px^{p-1}\frac{\partial}{\partial y}J^{p,q,l}_{m,n+q}(x,y,z), \tag{3.6}$$

$$lz^{l-1}\frac{\partial}{\partial x}J^{p,q,l}_{m+p,n}(x,y,z)+px^{p-1}\frac{\partial}{\partial z}J^{p,q,l}_{m-l,n-l}(x,y,z)=0, \tag{3.7}$$

and

$$lz^{l-1}\frac{\partial}{\partial y}J^{p,q,l}_{m,n+q}(x,y,z)+qy^{q-1}\frac{\partial}{\partial z}J^{p,q,l}_{m-l,n-l}(x,y,z)=0. \tag{3.8}$$

We now define another kind of generalized Humbert functions $\mathbb{J}^{p,q,l}_{m,n}(x,y,z)$ of three variables by the approach of generating functions in the next definition:

Definition 3.4. The generating function of the Humbert functions $\mathbb{J}^{p,q,l}_{m,n}(x,y,z)$ is defined by the relation

$$F_5(x,y,z;u,t;p,q,l)=\exp\Bigg(\Big(\frac{xu}{p}\Big)^p+\Big(\frac{yt}{q}\Big)^q-\Big(\frac{z}{lut}\Big)^l\Bigg)=\sum_{m,n=-\infty}^{\infty}\mathbb{J}^{p,q,l}_{m,n}(x,y,z)u^m t^n. \tag{3.9}$$

From (3.9), we have

$$F_5(x,y,z;u,t;p,q,l)=\exp\Bigg(\Big(\frac{xu}{p}\Big)^p\Bigg)\exp\Bigg(\Big(\frac{yt}{q}\Big)^q\Bigg)\exp\Bigg(-\Big(\frac{z}{lut}\Big)^l\Bigg)$$
$$=\sum_{r=0}^{\infty}\frac{1}{r!}\Big(\frac{xu}{p}\Big)^{pr}\sum_{i=0}^{\infty}\frac{1}{i!}\Big(\frac{yt}{q}\Big)^{qi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Big(\frac{z}{lut}\Big)^{kl}$$
$$=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k}{k!\,i!\,r!}\Big(\frac{x}{p}\Big)^{pr}\Big(\frac{y}{q}\Big)^{qi}\Big(\frac{z}{l}\Big)^{kl}u^{pr-lk}t^{qi-lk}.$$

Now set $pr - lk = m$ and $qi - lk = n$ to get

$$\sum_{k,i,r=0}^{\infty} \frac{(-1)^k}{k!\, i!\, r!} \left(\frac{x}{p}\right)^{pr} \left(\frac{y}{q}\right)^{qi} \left(\frac{z}{l}\right)^{kl} u^{pr-lk} t^{qi-lk}$$
$$= \sum_{m,n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)} \left(\frac{x}{p}\right)^{m+kl} \left(\frac{y}{q}\right)^{n+kl} \left(\frac{z}{l}\right)^{kl} u^m t^n$$
$$= \sum_{m,n=-\infty}^{\infty} \mathbb{J}_{m,n}^{p,q,l}(x,y,z) u^m t^n.$$

Explicitly, we get the explicit expression of Humbert function $\mathbb{J}_{m,n}^{p,q,l}(x,y,z)$ as the power series

$$\mathbb{J}_{m,n}^{p,q,l}(x,y,z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{m+kl}{p}+1\right)\Gamma\left(\frac{n+kl}{q}+1\right)} \left(\frac{x}{p}\right)^{m+kl} \left(\frac{y}{q}\right)^{n+kl} \left(\frac{z}{l}\right)^{kl}. \tag{3.10}$$

Theorem 3.5. *For the Humbert function $\mathbb{J}_{m,n}^{p,q,l}(x,y,z)$, we have*

$$\frac{\partial}{\partial x}\mathbb{J}_{m,n}^{p,q,l}(x,y,z) = \left(\frac{x}{p}\right)^{p-1} \mathbb{J}_{m-p,n}^{p,q,l}(x,y,z), \tag{3.11}$$

$$\frac{\partial}{\partial y}\mathbb{J}_{m,n}^{p,q,l}(x,y,z) = \left(\frac{y}{q}\right)^{q-1} \mathbb{J}_{m,n-q}^{p,q,l}(x,y,z) \tag{3.12}$$

and

$$\frac{\partial}{\partial z}\mathbb{J}_{m,n}^{p,q,l}(x,y,z) = -\left(\frac{z}{l}\right)^{l-1} \mathbb{J}_{m+l,n+l}^{p,q,l}(x,y,z). \tag{3.13}$$

Proof. We start by partially differentiating with respect to x both sides of (3.9):

$$\sum_{m,n=-\infty}^{\infty} \frac{\partial}{\partial x}\mathbb{J}_{m,n}^{p,q,l}(x,y,z) u^m t^n = p\frac{u}{p}\left(\frac{xu}{p}\right)^{p-1} \exp\left(\left(\frac{xu}{p}\right)^p + \left(\frac{yt}{q}\right)^q - \left(\frac{z}{lut}\right)^l\right)$$
$$= \frac{\partial}{\partial x}F_5(x,y,z;u,t;p,q,l) = p\frac{u}{p}\left(\frac{xu}{p}\right)^{p-1} F_5(x,y,z;u,t;p,q,l)$$
$$= \left(\frac{x}{p}\right)^{p-1} \sum_{m,n=-\infty}^{\infty} \mathbb{J}_{m,n}^{p,q,l}(x,y,z) u^{m+p} t^n$$
$$= \left(\frac{x}{p}\right)^{p-1} \sum_{m,n=-\infty}^{\infty} \mathbb{J}_{m-p,n}^{p,q,l}(x,y,z) u^m t^n$$

and by equating the same power of the indexes, we get the recurrence relation (3.11). The relations (3.12) and (3.13) follow in the same manner. □

Theorem 3.6. *The Humbert function $\mathbb{J}_{m,n}^{p,q,l}(x,y,z)$ satisfies the derivative relations*

$$\left(\frac{y}{q}\right)^{q-1} \frac{\partial}{\partial x}\mathbb{J}_{m+p,n}^{p,q,l}(x,y,z) = \left(\frac{x}{p}\right)^{p-1} \frac{\partial}{\partial y}\mathbb{J}_{m,n+q}^{p,q,l}(x,y,z), \tag{3.14}$$

$$\left(\frac{z}{l}\right)^{l-1} \frac{\partial}{\partial x}\mathbb{J}_{m+p,n}^{p,q,l}(x,y,z) + \left(\frac{x}{p}\right)^{p-1} \frac{\partial}{\partial z}\mathbb{J}_{m-l,n-l}^{p,q,l}(x,y,z) = 0 \tag{3.15}$$

and

$$\left(\frac{z}{l}\right)^{l-1} \frac{\partial}{\partial y}\mathbb{J}_{m,n+q}^{p,q,l}(x,y,z) + \left(\frac{y}{q}\right)^{q-1} \frac{\partial}{\partial z}\mathbb{J}_{m-l,n-l}^{p,q,l}(x,y,z) = 0. \tag{3.16}$$

4 Concluding remarks

We have seen some interesting particular cases of functions that can be considered belonging to the many families of Humbert functions and consequences of our results have been discussed. Further investigations will be carried out in the future in other fields of interest and this will be another open problem for further studies and applications in mathematics and physics.

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