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Abstract

Motivated by recent investigations, the main object of this paper is to construct the new (p, q)-analogy definitions of the various families of (p, q)-Bessel functions using the generating function method as a starting point. We derive the explicit representations, especially differential recurrence relations and these classes results of the (p, q)-Bessel functions.

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1. Introduction

The theory of post quantum calculus, or (p, q)-calculus has recently been applied in many branches of pure and applied mathematics and engineering, such as biology, physics, electrochemistry, economics,

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engineering, probability theory, statistics, statistical sciences, quantum theory, number theory and statistical mechanics, etc. (see [1, 13]) The (*p, q*)-special functions have important roles in many areas of mathematical physics and mathematics, see, for example, [2, 3, 4, 7, 10, 11, 12, 14, 18, 22, 25, 26]. In a recent paper, the authors discuss a *q*-analogue of the *q*-Bessel functions in [5, 6, 9, 8, 15, 16, 19, 20, 21].

In the present work might suggest that the various families of (*p, q*)-Bessel functions are more suitable for (*p, q*)-calculus analysis, both within and without the context of quantum groups. Future research will help to clarify the merits of the various types of (*p, q*)-Bessel functions. The study is organized as follows. More precisely, we define the new (*p, q*)-Bessel functions and derive some significant properties such as the explicit representations, recurrence relations and some new generating functions in Section 2. Especially recurrence relations and some interesting differential recurrence relations for the (*p, q*)-Bessel functions are discussed in Section 3.

1.1 *Basic definitions and miscellaneous relations*

Here, we provide some basic definitions of (*p, q*)-calculus, *p* and *q* are complex numbers, $0 < |q| < |p| \leq 1$, operations and mathematical notations that we need to be used in this work.

For *n* non-negative integer, the (*p, q*)-number (or basic number) $[n]_{p,q}$ is defined by [23]

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, 0 < |q| < |p| \leq 1 \tag{1.1}$$

and

$$\lim_{p \to 1} [n]_{p,q} = [n]_q.$$

A (*p, q*)-number factorial $[n]_{p,q}!$ is defined as

$$[n]_{p,q}! = \begin{cases} \prod_{k=1}^{n} [k]_{p,q}, & n \geq 1; \\ 1, & n = 0. \end{cases} \tag{1.2}$$

Two (*p, q*)-exponential functions are defined as[6]

$$e_{p,q}(x) = \sum_{k=0}^{\infty} \frac{p^{\frac{k(k-1)}{2}} x^k}{[k]_{p,q}!} \tag{1.3}$$

and

$$E_{p,q}(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{[k]_{p,q}!}. \tag{1.4}$$

Let $f(x)$ be a function defined on a subset of real or complex plane, then the (p, q)-derivative operator of the function $f(x)$ defined as follows [1, 12]

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0 \tag{1.5}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that $f(x)$ is differentiable at 0, which satisfies the following relations

$$\begin{aligned} D_{p,q}e_{p,q}(ax) &= ae_{p,q}(apx), \\ D_{p,q}E_{p,q}(ax) &= aE_{p,q}(aqx), \end{aligned} \tag{1.6}$$

where a is a complex number.

The product rule of (p, q)-derivative of functions is given as:

$$D_{p,q}[f_1(x)f_2(x)] = f_2(px)D_{p,q}\{f_1(x)\} + f_1(qx)D_{p,q}\{f_2(x)\}, \tag{1.7}$$

provided $f(x)$ is differentiable at the origin.

Our purpose is to generalize the class of Bessel functions, by using the same approach exposed above, to define our main problem of the generalized (p, q)-Bessel functions. Our aim is introducing and investigating, in a rather systematic manner, some particular cases of functions belonging to the family of (p, q)-Bessel functions introduced.

2. Definitions of new (p, q)-analogue of the (p, q)-Bessel functions and basic properties

Here, we define of (p, q)-analogue of the generating function to give explicit formulas for (p, q)-Bessel functions and derive some interesting significant properties for these functions, the generalizations of the above mentioned identities.

Definition 2.1 : Let us define the product of symmetric (p, q)-exponential functions as the generating function for (p, q)-Bessel functions

$$F_1(x;t\,|\,p,q) = e_{p,q}\left(\frac{xt}{2}\right)e_{p,q}\left(-\frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n^{(1)}(x\,|\,p,q)t^n. \qquad (2.1)$$

From (2.1) and using (1.3), we have

$$F_1(x;t\,|\,p,q) = e_{p,q}\left(\frac{xt}{2}\right)e_{p,q}\left(-\frac{x}{2t}\right) = \sum_{i=0}^{\infty}\frac{p^{\frac{i(i-1)}{2}}}{[i]_{p,q}!}\left(\frac{xt}{2}\right)^i \sum_{k=0}^{\infty}\frac{p^{\frac{k(k-1)}{2}}}{[k]_{p,q}!}\left(-\frac{x}{2t}\right)^k$$

$$= \sum_{i,k=0}^{\infty}\frac{(-1)^k p^{\frac{i(i-1)}{2}+\frac{k(k-1)}{2}}}{[k]_{p,q}![i]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r} t^{i-k}.$$

Replace n by $i-k$, we get

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\frac{(n+k)(n+k-1)+k(k-1)}{2}}}{[k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k} t^n = \sum_{n=-\infty}^{\infty} J_n^{(1)}(x\,|\,p,q)t^n.$$

Explicitly, we obtain the explicit expression of (p, q)-Bessel functions $J_n^{(1)}(x\,|\,p, q)$ as

$$J_n^{(1)}(x\,|\,p,q) = \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}!![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k}. \qquad (2.2)$$

From the above equations (2.2), the new (p, q)-analogy of the explicit representations for (p, q)-Bessel functions $J_n^{(1)}(x\,|\,p, q)$ are defined.

Remark 2.1 : Nota that in eq. (2.2), if we put $p = 1$, then (p, q)-Bessel functions reduces to the q-Bessel functions defined by [27].

Lemma 2.1 : *The $J_n^{(1)}(x\,|\,p, q)$ satisfies the relation*

$$J_{-n}^{(1)}(x\,|\,p,q) = (-1)^n J_n^{(1)}(x\,|\,p,q), \qquad (2.3)$$

where n is integer.

Proof : From the definition of (p, q)-Bessel functions $J_n^{(1)}(x\,|\,p, q)$, we have

$$J_{-n}^{(1)}(x\,|\,p,q) = \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{-n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-n+k]_{p,q}!}\left(\frac{x}{2}\right)^{-n+2k} = \sum_{k=n}^{\infty}\frac{(-1)^k p^{\binom{-n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-n+k]_{p,q}!}\left(\frac{x}{2}\right)^{-n+2k}.$$

Replacing $s = k - n$, we obtain (2.3). □

Now, we prove that the generating function leads to the second kind of (p, q)-Bessel functions $J_n^{(2)}(x \mid p, q)$.

Definition 2.2 : The generating function $F_2(x; t \mid p, q)$ of the $J_n^{(2)}(x \mid p, q)$ is defined by

$$F_2(x;t\mid p,q) = E_{p,q}\left(\frac{xt}{2}\right)E_{p,q}\left(-\frac{qx}{2t}\right) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} J_n^{(2)}(x\mid p,q)t^n. \quad (2.4)$$

From the generating function for the $J_n^{(2)}(x \mid p, q)$, we have

$$\begin{aligned} F_2(x;t\mid p,q) &= E_{p,q}\left(\frac{xt}{2}\right)E_{p,q}\left(-\frac{qx}{2t}\right) \\ &= \sum_{i=0}^{\infty}\frac{q^{\frac{i(i-1)}{2}}}{[i]_{p,q}!}\left(\frac{xt}{2}\right)^i \sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}}{[k]_{p,q}!}\left(-\frac{qx}{2t}\right)^k \\ &= \sum_{k,i=0}^{\infty}\frac{(-1)^k q^{\binom{i}{2}+\binom{k+1}{2}}}{[k]_{p,q}![i]_{p,q}!}\left(\frac{x}{2}\right)^{k+i} t^{i-k}. \end{aligned}$$

Now, replace n by $i - k$ to get

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{n+k}{2}+\binom{k+1}{2}}}{[k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k} t^n = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} J_n^{(2)}(x\mid p,q)t^n.$$

Explicitly, we get the explicit expression of (p, q)-Bessel functions $J_n^{(2)}(x \mid p, q)$ by

$$J_n^{(2)}(x\mid p,q) = \sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![n+k]_{p,q}!} q^{k(n+k)}\left(\frac{x}{2}\right)^{n+2k}. \quad (2.5)$$

Lemma 2.2 : *The connection between generating functions of* $J_n^{(1)}(x \mid p, q)$ *and* $J_n^{(2)}(x \mid p, q)$ *is given by*

$$J_n^{(1)}\left(\sqrt{q}x\mid \frac{1}{p},\frac{1}{q}\right) = q^{\frac{n^2}{2}} J_n^{(2)}(x\mid p,q). \quad (2.6)$$

Proof : If we set that

$$x = \sqrt{q}x, t = q^{-\frac{1}{2}}t,$$

in (2.1), and using $e_{\frac{1}{p},\frac{1}{q}}(x) = E_{p,q}(x)$, we obtain

$$F_1\left(\sqrt{q}x; q^{-\frac{1}{2}}t \mid \frac{1}{p}, \frac{1}{q}\right) = E_{p,q}\left(\frac{xt}{2}\right)E_{p,q}\left(-\frac{qx}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n^{(1)}\left(\sqrt{q}x \mid \frac{1}{p}, \frac{1}{q}\right)q^{-\frac{n}{2}}t^n$$

and using (2.4), we obtain

$$F_2(x;t \mid p,q) = E_{p,q}\left(\frac{xt}{2}\right)E_{p,q}\left(-\frac{qx}{2t}\right) = \sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} J_n^{(2)}(x \mid p,q)t^n.$$

□

Other proofs may be performed in the above way. Therefore we have obtained representations (2.1) in the following.

Definition 2.3 : The $F_3(x; t \mid p, q)$ of the function $J_n^{(3)}(x \mid p, q)$ is defined by

$$F_3(x;t \mid p,q) = e_{p,q}\left(\frac{xt}{2}\right)E_{p,q}\left(-\frac{qx}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n^{(3)}(x \mid p,q)t^n \tag{2.7}$$

and

$$J_n^{(3)}(x \mid p,q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{p,q}![n+k]_{p,q}!} p^{\binom{n+k}{2}} q^{\binom{k+1}{2}} \left(\frac{x}{2}\right)^{n+2k}. \tag{2.8}$$

Definition 2.4 : The $F_4(x; t \mid p, q)$ of the function $J_n^{(4)}(x \mid p, q)$ is given as

$$F_4(x;t \mid p,q) = E_{p,q}\left(\frac{qxt}{2}\right)e_{p,q}\left(-\frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n^{(4)}(x \mid p,q)t^n \tag{2.9}$$

and

$$J_n^{(4)}(x \mid p,q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{p,q}!![n+k]_{p,q}!} p^{\binom{k}{2}} q^{\frac{1}{2}[(n+k)(n+k+1)]} \left(\frac{x}{2}\right)^{n+2k}. \tag{2.10}$$

Furthermore, we show the relations between generating functions for the (p, q)-Bessel functions.

Theorem 2.1 : *The connection between generating functions of* (*p*, *q*)*-Bessel functions is given by*

$$F_4(x;t\,|\,p,q) = F_3\left(qx;t\,|\,\frac{1}{p},\frac{1}{q}\right). \tag{2.11}$$

Further examples can be considered, but are omitted for the sake of conciseness. Before deriving the multiplication theorems, we state a straightforward but important identity [23]

$$e_{p,q}(x)E_{p,q}(-x) = 1. \tag{2.12}$$

Theorem 2.2 : *The link between generating functions for* (*p*, *q*)*-Bessel functions* (*Multiplication theorems*)

$$\begin{aligned} F_1(x;t\,|\,p,q) &= F_3(x;t\,|\,p,q)e_{p,q}\left(\frac{qx}{2t}\right)e_{p,q}\left(-\frac{x}{2t}\right), \\ F_3(x;t\,|\,p,q) &= F_1(x;t\,|\,p,q)E_{p,q}\left(-\frac{qx}{2t}\right)E_{p,q}\left(\frac{x}{2t}\right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} F_2(x;t\,|\,p,q) &= F_3(x;t\,|\,p,q)E_{p,q}\left(-\frac{xt}{2}\right)E_{p,q}\left(\frac{xt}{2}\right), \\ F_3(x;t\,|\,p,q) &= F_2(x;t\,|\,p,q)e_{p,q}\left(\frac{xt}{2}\right)e_{p,q}\left(-\frac{xt}{2}\right) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} F_1(x;t\,|\,p,q) &= F_4(x;t\,|\,p,q)e_{p,q}\left(-\frac{qxt}{2}\right)e_{p,q}\left(\frac{xt}{2}\right), \\ F_4(x;t\,|\,p,q) &= F_1(x;t\,|\,p,q)E_{p,q}\left(-\frac{xt}{2}\right)E_{p,q}\left(\frac{qxt}{2}\right). \end{aligned} \tag{2.15}$$

3. Differential and recurrence relations for (*p*, *q*)-Bessel functions $J_n^{(1)}(x\,|\,p,q)$

Here, we derive the significant interesting recurrence relations of the (*p*, *q*)-Bessel functions so far introduced can be established with respect to *x* on their generating functions in different ways.

Theorem 3.1 : *The* $(p,\ q)$*-Bessel functions* $J_n^{(1)}(x\,|\,p,\ q)$ *satisfy the differential recurrence relations*

$$J_{n-1}^{(1)}(px\,|\,p,q) - p^{-\frac{n+1}{2}} q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{pq}x\,|\,p,q) = 2D_{p,q}\{J_n^{(1)}(x\,|\,p,q)\} \tag{3.1}$$

and

$$p^{\frac{n-1}{2}} q^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{pq}x\,|\,p,q) - J_{n+1}^{(1)}(px\,|\,p,q) = 2D_{p,q} J_n^{(1)}(x\,|\,p,q). \tag{3.2}$$

Proof : By using (1.7) and applying the $D_{p,q}$ derivative of both sides of the first of eq. (2.1), we get

$$\frac{1}{2}\left[te_{p,q}\left(\frac{pxt}{2}\right) e_{p,q}\left(-\frac{px}{2t}\right) - \frac{1}{t} e_{p,q}\left(\frac{qxt}{2}\right) e_{p,q}\left(-\frac{px}{2t}\right)\right] = \sum_{n=-\infty}^{\infty} D_{p,q} J_n^{(1)}(x\,|\,p,q) t^n . (3.2)$$

If we substitute $x = px$ in the generating relation (2.1), then we get the result

$$te_{p,q}\left(\frac{pxt}{2}\right) e_{p,q}\left(-\frac{px}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n^{(1)}(px\,|\,p,q) t^{n+1}. \tag{3.4}$$

Replacing $x = \sqrt{pq}x$ and $t = p^{-\frac{1}{2}}\sqrt{q}t$ and using (2.1), we have

$$\frac{1}{p^{-\frac{1}{2}}\sqrt{qt}} e_{p,q}\left(\frac{qxt}{2}\right) e_{p,q}\left(-\frac{px}{2t}\right) = \sum_{n=-\infty}^{\infty} p^{-\frac{n}{2}} q^{\frac{n}{2}} J_{n+1}^{(1)}(\sqrt{pq}x\,|\,p,q) t^n. \tag{3.5}$$

Using equations (3.3), (3.4) and (3.5), we give the relation

$$\frac{1}{2}\left[\sum_{n=-\infty}^{\infty} J_{n-1}^{(1)}(px\,|\,p,q) t^n - \sum_{n=-\infty}^{\infty} p^{-\frac{n+1}{2}} q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{pq}x\,|\,p,q) t^n \right] = \sum_{n=-\infty}^{\infty} D_{p,q} J_n^{(1)}(x\,|\,p,q) t^n.$$

Thus, we obtain the recurrence relation (3.1). Similarly, the other equations of this theorem can be proved. □

Similarly, we can derive the next result.

Theorem 3.2 : *The* $J_n^{(1)}(x\,|\,p,\ q)$ *have the pure recurrence relation*

$$J_{n-1}^{(1)}(px\,|\,p,q) + J_{n+1}^{(1)}(px\,|\,p,q) = p^{-\frac{n+1}{2}} q^{\frac{n+1}{2}} J_{n+1}^{(1)}(\sqrt{pq}x\,|\,p,q) + p^{\frac{n-1}{2}} q^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{pq}x\,|\,p,q). \tag{3.6}$$

Theorem 3.3 : *The* (*p*, *q*)*-Bessel functions* $J_n^{(1)}(x\,|\,p,q)$ *satisfy the relations*

$$2\frac{[n]_{p,q}}{x}J_n^{(1)}(x\,|\,p,q)=p^{\frac{n-1}{2}}J_{n-1}^{(1)}(p^{-\frac{1}{2}}x\,|\,p,q)+p^{\frac{n}{2}}q^nJ_{n+1}^{(1)}(p^{-\frac{1}{2}}x\,|\,p,q) \tag{3.7}$$

and

$$2\frac{[n]_{p,q}}{x}J_n^{(1)}(x\,|\,p,q)=q^{\frac{n-1}{2}}J_{n-1}^{(1)}(q^{-\frac{1}{2}}x\,|\,p,q)+q^{\frac{n}{2}}p^nJ_{n+1}^{(1)}(q^{-\frac{1}{2}}x\,|\,p,q). \tag{3.8}$$

Proof : Multiplying both sides of the (2.2) by $[n]_{p,q}$ and using

$$[n]_{p,q}\ =\ p^{-k}[n+k]_{p,q}-p^{-k}q^n[k]_{p,q} \tag{3.9}$$

and

$$\begin{aligned}
[n]_{p,q}J_n^{(1)}(x\,|\,p,q)\ &=\ \sum_{k=0}^{\infty}\frac{(-1)^k[n]_{p,q}}{[k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k}\\
&=\ \sum_{k=0}^{\infty}\frac{(-1)^kp^{-k}[n+k]_{p,q}}{[k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k}-\sum_{k=0}^{\infty}\frac{(-1)^kp^{-k}q^n[k]_{p,q}}{[k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{2}\right)^{n+2k}\\
&=\ p^{\frac{n-1}{2}}\frac{x}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{p^{-\frac{1}{2}}x}{2}\right)^{n-1+2k}\\
&\quad+p^{\frac{n}{2}}q^n\frac{x}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![n+k+1]_{p,q}!}\left(\frac{p^{-\frac{1}{2}}x}{2}\right)^{n+2k+2}.
\end{aligned} \tag{3.10}$$

Using (3.10) and (2.2), we obtain (3.7). Similarly, we can prove (3.8). □

Theorem 3.4 : *The* (*p*, *q*)*-Bessel functions have the relations*

$$q^{-\frac{n-1}{2}}J_{n-1}^{(1)}(\sqrt{q}x\,|\,p,q)=p^{-\frac{n-1}{2}}J_{n-1}^{(1)}(\sqrt{p}x\,|\,p,q)+(p-q)\frac{x}{2}J_n^{(1)}(x\,|\,p,q), \tag{3.11}$$

$$p^{-\frac{n-1}{2}}J_{n-1}^{(1)}(\sqrt{p}x\,|\,p,q)=q^{-\frac{n-1}{2}}J_{n-1}^{(1)}(\sqrt{q}x\,|\,p,q)-(p-q)\frac{x}{2}J_n^{(1)}(x\,|\,p,q), \tag{3.12}$$

$$(pq)^{-\frac{n-1}{2}}J_{n-1}^{(1)}(\sqrt{pq}x\,|\,p,q)=p^{-n+1}J_{n-1}^{(1)}(px\,|\,p,q)+(p-q)p^{-\frac{n-2}{2}}\frac{x}{2}J_n^{(1)}(\sqrt{p}x\,|\,p,q) \tag{3.13}$$

and

$$(pq)^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{pq}x \mid p,q) = q^{-n+1} J_{n-1}^{(1)}(qx \mid p,q) - (p-q) q^{-\frac{n-2}{2}} \frac{x}{2} J_n^{(1)}(\sqrt{q}x \mid p,q). \tag{3.14}$$

Proof : We consider

$$q^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{q}x \mid p,q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{p,q}![n+k-1]_{p,q}!} \left(\frac{x}{2}\right)^{n-1+2k} q^k.$$

Using the identity

$$q^k = p^k - (p-q)[k]_{p,q},$$

we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k p^k}{[k]_{p,q}![n+k-1]_{p,q}!} \left(\frac{x}{2}\right)^{n-1+2k} - (p-q) \sum_{k=0}^{\infty} \frac{(-1)^k [k]_{p,q}}{[k]_{p,q}![n+k-1]_{p,q}!} \left(\frac{x}{2}\right)^{n-1+2k}$$
$$= p^{-\frac{n-1}{2}} J_{n-1}^{(1)}(\sqrt{p}x \mid p,q) + (p-q) \frac{x}{2} J_n^{(1)}(x \mid p,q).$$

Thus, the (3.11) is proved. In the same way, Eqs. (3.12), (3.13) and (3.14) can be proved. □

Similar pure recurrence relations can be achieved by using the generating function; in fact, by differentiating with respect to t, we have:

Theorem 3.5 : *The* (p, q)*-Bessel functions satisfy the properties:*

$$\frac{x}{2}[p J_n^{(1)}(px \mid p,q) + p^{-\frac{n+1}{2}} q^{\frac{n-1}{2}} J_{n+2}^{(1)}(\sqrt{pq}x \mid p,q)] = [n+1]_{p,q} J_{n+1}^{(1)}(x \mid p,q) \tag{3.15}$$

and

$$\frac{x}{2}[p J_{n+2}^{(1)}(px \mid p,q) + p^{\frac{n-1}{2}} q^{-\frac{n+1}{2}} J_n^{(1)}(\sqrt{pq}x \mid p,q)] = [n+1]_{p,q} J_{n+1}^{(1)}(x \mid p,q). \tag{3.16}$$

Proof : Differentiating with respect to t in (2.1) and using (1.7), we have

$$\frac{x}{2}\left[e_{p,q}\left(\frac{pxt}{2}\right)e_{p,q}\left(-\frac{px}{2t}\right)+\frac{1}{t^2}e_{p,q}\left(\frac{qxt}{2}\right)e_{p,q}\left(-\frac{px}{2t}\right)\right]=\sum_{n=-\infty}^{\infty}[n]_{p,q}J_n^{(1)}(x\,|\,p,q)t^{n-1}. \tag{3.17}$$

Replacing $x = px$, we get

$$\frac{px}{2}e_{p,q}\left(\frac{pxt}{2}\right)e_{p,q}\left(-\frac{px}{2t}\right)=\frac{px}{2}\sum_{n=-\infty}^{\infty}J_n^{(1)}(px\,|\,p,q)t^n. \tag{3.18}$$

Putting $x=\sqrt{pq}x$ and $t=p^{-\frac{1}{2}}\sqrt{q}t$ in (2.1), we have

$$\frac{\sqrt{pq}x}{2(p^{-\frac{1}{2}}\sqrt{q}t)^2}e_{p,q}\left(\frac{qxt}{2}\right)e_{p,q}\left(-\frac{px}{2t}\right)=\frac{\sqrt{pq}x}{2}\sum_{n=-\infty}^{\infty}p^{-\frac{n}{2}}q^{\frac{n}{2}}J_{n+2}^{(1)}(\sqrt{pq}x\,|\,p,q)t^n. \tag{3.19}$$

Adding the result (3.16), (3.17) and using (3.15) to give

$$\frac{x}{2}\left[\sum_{n=-\infty}^{\infty}pJ_n^{(1)}(px\,|\,p,q)t^n+\sum_{n=-\infty}^{\infty}p^{-\frac{n+1}{2}}q^{\frac{n-1}{2}}J_{n+2}^{(1)}(\sqrt{pq}x\,|\,p,q)t^n\right]$$
$$=\sum_{n=-\infty}^{\infty}[n+1]_{p,q}J_n^{(1)}(x\,|\,p,q)t^n.$$

Thus, we obtain the result (3.15). In similarly way, we prove (3.16). □

We have seen that, within such a context, the (*p, q*)-Bessel functions is, indeed, rich enough to require a separate treatment.

4. Concluding remarks

In this work, we have mentioned the (*p, q*)-Bessel functions, using the family of generating function method as a starting point. In a forthcoming works will be carried out in proposing modified forms (*p, q*)-Bessel functions in other fields of mathematical physics and engineering sciences and so on.

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