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Effects of a phase-damping cavity on entanglement and purity loss in two-qubit system

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Abstract We analyze two identical qubits interacting with a single-mode quantized radiation field, taking into account the influence of phase damping. The qubits are assumed to be initially in a superposition of the excited and the ground states, and the field is in a coherent state. The effects of the damping on the purity loss of the system and different bipartite partitions of the system [field-two qubits, qubit–(field + qubit)] through the tangles are considered. The effect of the damping on the entanglement of field qubits state is evaluated by the negativity. It is noted that the phenomenon of death and rebirth of the entanglement appears. With the increase in the phase parameter, this phenomenon disappears.

Keywords Entanglement-purity loss-phase · Damping negativity

1 Introduction

Quantum entanglement plays a crucial role in quantum information processing [1]. Quantum entangled states have become the key ingredient in the rapidly expanding field of quantum information science, with remarkable prospective applications such as quantum teleportation, quantum cryptography, quantum dense coding and parallel computing [1,2]. However, it has been shown that not all of the quantum entangled

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states are useful in quantum information processing. There exist bound entangled states from which no pure entangled states can be distilled under local operation and classical communication (LOCC) [2]. With bound entangled states as the entanglement resource, teleportation cannot be performed better than with a classical channel, even if conclusive teleportation is allowed [3]. It has been shown that bound entangled states can enhance the fidelity of teleportation for non-bound entangled states [4].

A common theme of the examples given above is that measurements are made on single copies of the quantum system of interest. In many situations, however, one does not have access to an individually addressable system. In a gas, for example, preparing and addressing individual qubits is extremely difficult. On the other hand, one may think of the entire ensemble as a single many-body system. Indeed, recent experiments [5,6] and theoretical proposals [7] have explored the control of such ensembles from the point of view of the model [8] where a collection of N two-level atoms is treated as a pseudo-spin with $J = \frac{N}{2}$. Measures of entanglement associated with spin-squeezed states have been studied in reference [8] under the assumption that all of the atoms in the ensemble are symmetrically coupled to the bus. However, completely quantifying entanglement in the most general cases is extremely difficult and as yet, an unsolved problem [9].

In this article, we consider the simplest possible ensemble consisting of two two-level qubits, and the simplest realization of the bus is a single-mode quantized electromagnetic field. The resulting physical system then corresponds to the two-atom Tavis–Cummings model (TCM) [10]. Thorough understanding of the dynamical evolution of TCM has obvious implications for the performance of quantum information processing, [1,11,12] as well as for our understanding of fundamental quantum mechanics [1,13]. Entanglement in tripartite systems has been studied in [14] for the case of three qubits. That study found that such quantum correlations cannot be arbitrarily distributed among the subsystems; the existence of three-body correlations that constrains the distribution of the bipartite entanglement which remains after tracing over any one of the qubits. This phenomenon of entanglement sharing was analyzed by using an entanglement monotone known as the tangle [14–18] that is a simple generalization of the more familiar concurrence. But the entanglement of two spatially separated qubits has been investigated by using the negativity with phase damping in [19,20]. Entanglement also compared with the total correlation that measured by the mutual entropy [19–23]. For two qubits, this comparison also has been investigated by different measures in [24–26]. But the entanglement of a qubit interacting with a field recently has been investigated in [27,28].

In this paper, the effects of the phase damping on the purity loss of the system and the tangles are considered. In particular, the effect of the phase damping on the amount of entanglement between qubits and field is evaluated by the negativity. The remainder of this article is organized as follows: Sect. 2 is devoted to the model and its solution. In Sect. 3, we employ the analytical results obtained in Sect. 2 to discuss the tangles and negativity. Finally in Sect. 4, we present our conclusion.

2 The model and its solution

Here we consider a two-qubit + field system in the dispersive regime with a reservoir. Two initially interacting identical qubits, labeled by A_1 and A_2, are chosen as the model. Each is a two-level atom with an excited state $|1\rangle$ and a ground state $|0\rangle$. The standard formalism for the calculations of the time evolution and correlation properties of a collective system of atoms is the master equation method. Suppose we have two qubits interacting with a quantized field inside a phase-damped cavity. The master equation that governs the dynamics of the whole system can be given as:

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} = i[\rho, \hat{H}] + \zeta_1\Big[2|0\rangle_{11}\langle 1|\rho|1\rangle_{11}\langle 0| - |1\rangle_{11}\langle 1|\rho - \rho|1\rangle_{11}\langle 1|\Big] + \zeta_2\Big[2|0\rangle_{22}\langle 1|\rho|1\rangle_{22}\langle 0| - |1\rangle_{22}\langle 1|\rho - \rho|1\rangle_{22}\langle 1|\Big]. \quad (1)$$

Here, the interaction Hamiltonian H of the two-qubit system interacting with the field in the dispersive regime is given by [29]:

$$\hat{H} = \sum_{i=A_1,A_2}\Big\{|1\rangle_{ii}\langle 1|\lambda\hat{a}\hat{a}^{\dagger} - |0\rangle_{ii}\langle 0|\lambda\hat{a}^{\dagger}\hat{a}\Big\} + \lambda\Big(|1\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 1| + |0\rangle_{11}\langle 1| \otimes |1\rangle_{22}\langle 0|\Big), \quad (2)$$

where $\hat{a}^{\dagger}$ ($\hat{a}$) is the creation (annihilation) operator, and λ is the effective interaction constant. Parameters ζ_1 and ζ_2 are the phase-damping constants to the environment.

The master equation Eq. 1 can be solved to obtain $\rho_{ij}(\tau)$, $(i,j = 1, 2, 3, 4)$ (for simplicity, we will take $\zeta_1 = \zeta_2 = \zeta$) . To do this, we suppose the qubits and field are initially in the form:

$$\rho(0) = \sum_{m,n=0} q_n q_m |m\rangle\langle n| \otimes |\nu\rangle\langle\nu|, \quad (3)$$

where $q_n = \mathrm{e}^{\frac{-|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$ and α is a complex number, and the field is initially in a coherent state $|\alpha\rangle\langle\alpha|$. While the qubits are initially in a superposition of the excited and the ground state as follows: $|\nu\rangle = (\sin\theta_1|1\rangle_1 + \cos\theta_1|0\rangle_1)(\sin\theta_2|1\rangle_2 + \cos\theta_2|0\rangle_2)$. Then, we can write the density matrix for the combined qubits–field system $\rho(\tau)$ in the flowing form:

$$\rho(\tau) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \quad (4)$$

We use the bases $|1\rangle = |1\rangle_1|1\rangle_2$, $|2\rangle = |1\rangle_1|0\rangle_2$, $|3\rangle = |0\rangle_1|1\rangle_2$ and $|4\rangle = |0\rangle_1|0\rangle_2$. Solving Eq. 1 under the initial conditions Eq. 3 gives the elements of $\rho_{ij}(\tau)$ in the following form:

$$\rho_{11}(\tau) = \mathrm{e}^{-4\gamma\tau}(\sin\theta_1 \sin\theta_2)^2 \sum_{m,n=0}^{\infty} q_m q_n^* \mathrm{e}^{-2i\tau(m-n)}|m\rangle\langle n|$$

$$\rho_{12}(\tau) = (\rho_{21}(\tau))^\dagger = \frac{\mathrm{e}^{-3\gamma\tau}}{2} \sin\theta_1 \sin\theta_2 \sum_{m,n=0}^{\infty} q_m q_n^* [\sin(\theta_1 - \theta_2)\mathrm{e}^{-2i\tau(m+1)} + \sin(\theta_1 + \theta_2)\mathrm{e}^{-2i\tau m}]|m\rangle\langle n|$$

$$\rho_{13}(\tau) = (\rho_{31}(\tau))^\dagger = \frac{\mathrm{e}^{-3\gamma\tau}}{2} \sin\theta_1 \sin\theta_2 \sum_{m,n=0}^{\infty} q_m q_n^* [\sin(\theta_2 - \theta_1)\mathrm{e}^{-2i\tau(m+1)} + \sin(\theta_1 + \theta_2)\mathrm{e}^{-2i\tau m}]|m\rangle\langle n|$$

$$\rho_{14}(\tau) = (\rho_{41}(\tau))^\dagger = \frac{\mathrm{e}^{-2\gamma\tau}}{4} \sin 2\theta_1 \sin 2\theta_2 \sum_{m,n=0}^{\infty} q_m q_n^* \mathrm{e}^{-2i\tau(1+m+n)}|m\rangle\langle n|$$

$$\rho_{22}(\tau) = \frac{1}{2} \sum_{m,n=0}^{\infty} q_m q_n^* \mathrm{e}^{-2\gamma\tau} [\sin(\theta_1 - \theta_2)\sin(\theta_1 + \theta_2)\cos 2\tau - \frac{2\gamma(\sin\theta_1 \sin\theta_2)^2}{\gamma + i(m-n)}(\mathrm{e}^{-2i\tau(m-n)-2\gamma\tau} - 1) + \sin^2\theta_1 \cos^2\theta_2 + \cos^2\theta_1 \sin^2\theta_2]|m\rangle\langle n|$$

$$\rho_{23}(\tau) = (\rho_{32}(\tau))^\dagger = \frac{\mathrm{e}^{-2\gamma\tau}}{2} \sum_{m,n=0}^{\infty} q_m q_n^* [i \sin(\theta_1 - \theta_2)\sin(\theta_1 + \theta_2)\sin 2\tau + \frac{1}{2}\sin 2\theta_1 \sin 2\theta_2]|m\rangle\langle n|$$

$$\rho_{24}(\tau) = (\rho_{42}(\tau))^\dagger = \frac{\mathrm{e}^{-\gamma\tau}}{2} \sum_{m,n=0}^{\infty} q_m q_n^* \left\{ \frac{\gamma \sin\theta_1 \sin\theta_2 \sin(\theta_1 - \theta_2)}{i(n-m-1) - \gamma}(\mathrm{e}^{-2\tau(\gamma + i(m+1))} - \mathrm{e}^{-2in\tau}) + \cos\theta_1 \cos\theta_2 \sin(\theta_2 - \theta_1)e^{-2in\tau} + \frac{\gamma \sin\theta_1 \sin\theta_2 \sin(\theta_1 + \theta_2)}{i(1+n-m) - \gamma} \times(\mathrm{e}^{-2\tau(im-\gamma)} - \mathrm{e}^{-2i\tau(1+n)}) + \frac{\mathrm{e}^{-2i\tau n}}{2}\sin(2\theta_1)\cos\theta_2 \right\}|m\rangle\langle n|$$

$$\rho_{33}(\tau) = \frac{1}{2} \sum_{m,n=0}^{\infty} q_m q_n^* \mathrm{e}^{-2\gamma\tau} [\sin(\theta_2 - \theta_1)\sin(\theta_1 + \theta_2)\cos 2\tau - \frac{2\gamma(\sin\theta_1 \sin\theta_2)^2}{\gamma + i(m-n)}(\mathrm{e}^{-2i\tau(m-n)-2\gamma\tau} - 1) + \sin^2\theta_1 \cos^2\theta_2 + \cos^2\theta_1 \sin^2\theta_2]|m\rangle\langle n|$$

$$\rho_{34}(\tau) = (\rho_{43}(t))^\dagger = \frac{\mathrm{e}^{-\gamma\tau}}{2} \sum_{m,n=0}^{\infty} q_m q_n^* \left\{ \frac{\gamma \sin\theta_1 \sin\theta_2 \sin(\theta_1 - \theta_2)}{i(n-m-1) - \gamma}(\mathrm{e}^{-2\tau(\gamma + i(m+1))} - \mathrm{e}^{-2in\tau}) + \cos\theta_1 \cos\theta_2 \sin(\theta_2 - \theta_1)\mathrm{e}^{-2in\tau} - \frac{\gamma \sin\theta_1 \sin\theta_2 \sin(\theta_1 + \theta_2)}{i(1+n-m) - \gamma}\right.$$

$$\times(\mathrm{e}^{-2\tau(im-\gamma)}-\mathrm{e}^{-2i\tau(1+n)})-\frac{\mathrm{e}^{-2i\tau n}}{2}\sin(2\theta_1)\cos\theta_2\Bigg\}|m\rangle\langle n|$$

$$\rho_{44}(\tau)=\sum_{m,n=0}^{\infty}q_m q_n^*\Bigg\{\frac{\gamma^2\sin^2\theta_1\sin^2\theta_2}{(\gamma+i(m-n))^2}[\mathrm{e}^{-2i\tau(m-n)-4\gamma\tau}+\mathrm{e}^{2i\tau(m-n)}-2\mathrm{e}^{-2\gamma\tau}]$$
$$-\frac{\gamma(\sin^2\theta_1\cos^2\theta_2+\cos^2\theta_1\sin^2\theta_2)}{\gamma+i(m-n)}[\mathrm{e}^{-2\gamma\tau}-\mathrm{e}^{2i\tau(m-n)}]$$
$$+(\cos\theta_1\cos\theta_2)^2[\mathrm{e}^{-2\gamma\tau}-\mathrm{e}^{-2i\tau(m-n)}]\Bigg\}|m\rangle\langle n|,$$

with $\tau=\lambda t$ and $\gamma=\frac{\zeta}{\lambda}$. The dynamic of purity loss and entanglement for two qubits interacting with a single-mode field are discussed in the next section using the above analytic description.

3 Purity loss and entanglement

Quantum information processing often requires a state with high purity and a large amount of entanglement. In this section, we will explain the basic notions and physical implication of bipartite tangle. As many experiments nowadays aim at the generation of multiparticle entangled states. However, the study of bipartite entangled states will already enable us to introduce the central concepts of entanglement detection.

3.1 Bipartite tangles in the two qubits

Let the two identical qubits in the ensemble be denoted by A_1 and A_2, respectively, and the field, or quantum bus, by F. Because of the assumed exchange symmetry, there are two nonequivalent partitions of the two qubits model into tensor products of bipartite subsystems: (i) the two qubits ensemble times the field, $A_1A_2\otimes(F)$; (ii) one qubit times the remaining qubit and the field, $A_1\otimes(A_2F)$. We calculate the tangle of the partition (i), (ii) to discuss the purity for the system under consideration. The tangle between two qubits in an arbitrary state is defined in terms of the concurrence [30,31]. An analytic form for the tangle τ of a bipartite system AB is given by [32]:

$$\tau_{A(B)}=2[1-Tr(\rho_B^2)] \tag{5}$$

Field-ensemble and one-atom-remainder tangles Under the assumption that the system is in an overall pure state, we may easily calculate the tangles in partition (i) and (ii) above by applying Eq. 5 as:

$$\tau_{A_1A_2(F)}=2[1-Tr(\rho_F^2)],\qquad \tau_{A_1(A_2F)}=2[1-Tr(\rho_{A_2F}^2)]. \tag{6}$$

If the density operator ρ of any state can be written as the outer product of a ket and its associated bra, then the state is said to be pure. The density operator of all pure states satisfies $Tr\rho^2=1$. Otherwise, it is called a mixed state, $Tr\rho^2<1$. The last

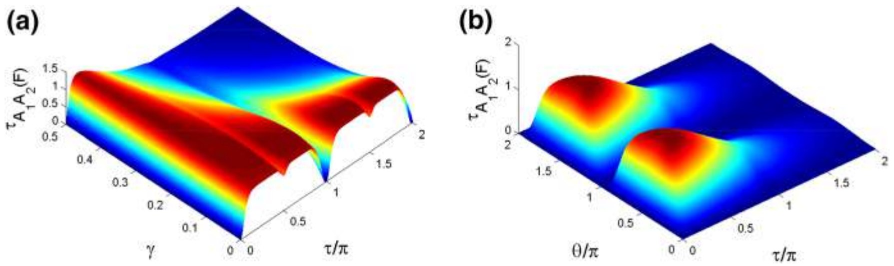

Fig. 1 $\tau_{A_1A_2(F)}$ when the qubits are initially in a superposition of the excited and the ground state and the field is in a coherent state with initial mean photon number $|\alpha|^2 = 4$ and $\theta_1 = \theta_2 = \theta$: **a** $\tau_{F(A_1A_2)}$ against τ and γ with $\theta = \frac{\pi}{4}$. **b**$\tau_{F(A_1A_2)}$ against τ and θ with $\gamma = 0.4$

inequality is a signature of a mixed state. There are two sources of purity loss in the present system. One of them is due to the unitary interaction. This process is usually called entanglement, and from the point of view of one of the subsystems, a purity loss (or coherence loss) will take place. On the other hand, the interaction of any subsystem with the environment also induces purity loss, and this process is usually called coherence loss induced by the environment. The purity of the state represented by a density operator ρ is measured by tangles $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$. In the following, we use the terminologies $\tau_{A_1A_2(F)} < 1$ for weak tangle and $\tau_{A_1A_2(F)} > 1$ for strong tangle.

In Figs. 1 and 3a, the tangles $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$ are plotted against scaled time and γ for the two qubits with the initial mean photon number $|\alpha|^2 = 4$ and $\theta = \frac{\pi}{4}$. At first, we can note that the initial values of $\tau_{A_1A_2(F)}$, for the initial product states, are zero. But, from Fig. 1a, b, we can find that the increase in $\tau_{A_1A_2(F)}$ exceeds its initial value, and it growths to the maximum value. Then after a certain time, the tangles evolve to zero and the qubits are in a pure state with a period π. So, we can observe the qubits which are initially separated can generate entanglement ($\gamma = 0$). In absence damping, we find that the unitary qubits–cavity interaction can generate entanglement as shown in Fig. 1a,b. Also, when $\tau_{A_1A_2(F)}$ is plotted against τ and θ, this generation (birth) of the entanglement from the initial product states (which have zero entanglement) is clearly observed in Fig. 1b, where the peak of the tangle appears, and this means that the large entanglement can be prepared by choosing the initial phase angle.

We can see that the influence of the phase damping leading to the amplitudes of the local maxima and minima of the tangles $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$ decreases with increasing the parameter γ. The tangles quite vanish, and the states (Eq. 4) finally go into a pure state and the mixedness is completely lost. So, after a short time, the damping destroys the tangles and we can determine a particular region in which there is no coherence in the qubits and the field due to the phase damping. We can see that the phenomenon of the generation of mixedness is a very strong sensitivity for the phase-damping parameter γ.

To see clearly the influence of the damping on the evolution of the mixedness, the tangles for different values of the damping parameter ($\gamma = 0, 0.2, 0.5$) for $0 \le \tau \le \pi$, are plotted in Figs. 2 and 4a. When $\gamma = 0$, (i.e., in the absence of the damping), we can observe that the tangles evolve with a period π. At $\tau = n\pi$ ($n = 0, 1, ...$),

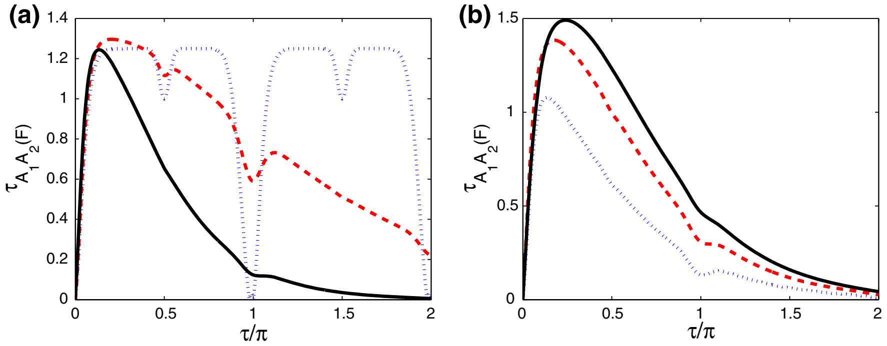

Fig. 2 $\tau_{A_1A_2(F)}$ as a function of τ, when the qubits are initially in a superposition of the excited and the ground state and the field is in a coherent state with initial mean photon number $|\alpha|^2 = 4$ and $\theta_1 = \theta_2 = \theta$: **a** $\theta = \frac{\pi}{4}$ for various values of the damping parameter $\gamma = 0$ (*dot curve*), $\gamma = 0.2$ (*dashed curve*) and $\gamma = 0.5$ (*solid curve*). **b** $\gamma = 0.4$ where $\theta = 0.2\pi$ (*dot curve*), $\theta = 0.3\pi$ (*dashed curve*) and $\theta = 0.4\pi$ (*solid curve*)

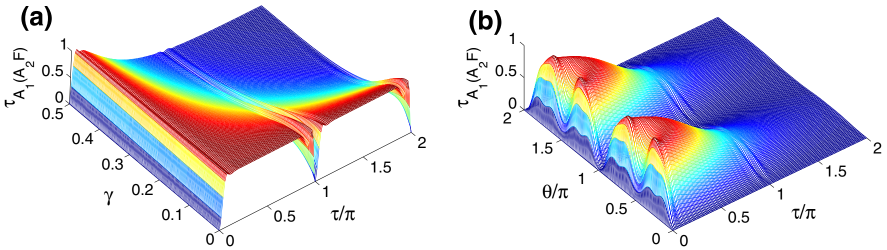

Fig. 3 $\tau_{A_1(A_2F)}$ when the qubits are initially in a superposition of the excited and the ground state and the field in a coherent state with initial mean photon number $|\alpha|^2 = 4$ and $\theta_1 = \theta_2 = \theta$: **a** $\tau_{A_1(A_2F)}$ against τ and γ with $\theta = \frac{\pi}{4}$. **b** $\tau_{A_1(A_2F)}$ against τ and θ with $\gamma = 0.4$

the tangles drop to zero, and this measure has observable periodic behavior. This periodical dynamic behavior of the tangles depend on the parameter γ. Increasing the damping parameter not only disturbs the evolution period of the tangles, but also affects their amplitudes. Physically, all these features can be attributed to the change in the qubit–field interaction time due to damping parameter (see Figs. 2 and 4a).

In Figs. 2 and 4a, the damping parameter reflects the presence of the reservoir and shows that it affects the coherence properties. The states of qubits and qubit–fields lose their mixedness more than gain due to the phase damping ($\gamma = 0.2, 0.5$). To see the asymptotic behavior of $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$, we put ($\gamma \to \infty$). Therefore, $\tau_{A_1A_2(F)}(\gamma \to \infty) \approx 0$ and , $\tau_{A_1(A_2F)}(\gamma \to \infty) \approx 0$ i.e., the asymptotic behavior of the qubits closely follows the qubit–field which are in pure states.

To see the effect of the distribution angel of the initial qubits states, the tangles $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$ are plotted against scaled time and θ with $|\alpha|^2 = 4$ and $\gamma = 0.4$ in Figs. 1 and 3b. The time evolution of the tangles is very regular, and it is symmetric about $\theta = \frac{\pi}{2}$ and is periodic in time with period π. This is due to the periodic nature of the interaction and the symmetry of the two qubits. We can note that the

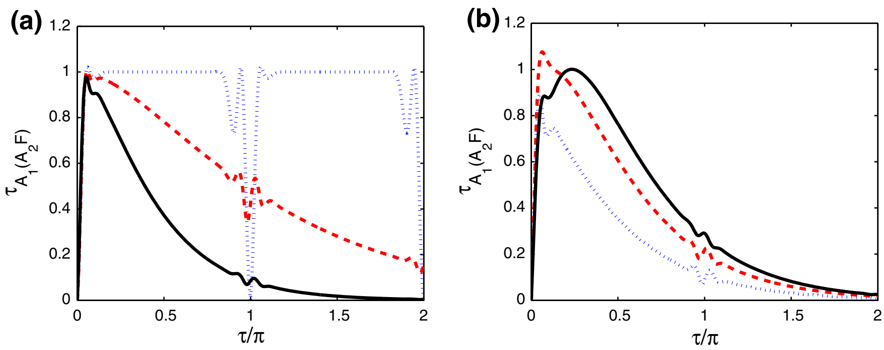

Fig. 4 $\tau_{A_1(A_2F)}$ as a function of τ, when the qubits are initially in a superposition of the excited and the ground state and the field in a coherent state with initial mean photon number $|\alpha|^2 = 4$ and $\theta_1 = \theta_2 = \theta$: **a** $\theta = \frac{\pi}{4}$ for various values of the damping parameter $\gamma = 0$ (*dot curve*), $\gamma = 0.2$ (*dashed curve*) and $\gamma = 0.5$ (*solid curve*). **b** $\gamma = 0.4$ where $\theta = 0.2\pi$ (*dot curve*), $\theta = 0.3\pi$ (*dashed curve*) and $\theta = 0.4\pi$ (*solid curve*)

phase damping decreases the amplitude of $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$ and changes in its oscillations; therefore, the local extreme (maxima or minima) values of $\tau_{A_1A_2(F)}$ and $\tau_{A_1(A_2F)}$ decrease. These effects become more pronounced at higher values of the damping parameter, the tangles completely vanish, and the peak centered at $\theta = \frac{\pi}{2}$ disappears.

The obvious remark from Figs. 1,2, 3 and 4 is that $\tau_{A_1(A_2F)} \leq \tau_{A_1A_2(F)}$, i.e., the tangle for the one qubit-remainder tangle is smaller than that for the field qubit. Also, this shows that quantum tangle cannot be equally distributed among many different objects in the system. The physical implication for two definitions of the tangles shows clearly the effects by the damping parameter γ. Despite differences in the temporal evolution of the curve in both definitions, the maximum and minimum values of them occur at the same time. This means that the tangle may be used to measure the coherence loss of the bipartite partitions.

3.2 Negativity

The entanglement creation by the spontaneous emission is illustrated more clearly if one assumes that a system of two qubits decays spontaneously from initially unentangled (uncorrelated) states. Several different measures have been proposed to identify entanglement between two qubits, and we chose the Peres-Horodecki (negativity) measure for entanglement [33,34]. The negativity criterion is given by the quantity

$$E = \max\left(0, -2\sum_i \mu_i\right), \tag{7}$$

where the sum is taken over the negative eigenvalues μ_i of the partial transposition of the density matrix $\rho_{A_1A_2}(\tau)$ for two qubits. The value $E = 1$ corresponds to maximum entanglement between the qubits, while $E = 0$ describes completely separated qubits.