Available online at http://vonneumann-publishing.com/cna
Commun. Nonlinear Anal. 4 (2017), 167–182
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.vonneumann-publishing.com/cna

Common and coupled random fixed point theorems in $S-$metric spaces

Rashwan Ahmed Rashwan*, Shimaa Ibraheem Moustafa

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.

Abstract

The aim of this paper is to prove some common and coupled random fixed point theorems for a pair of weakly monotone random operators satisfying some rational type contraction in the setting of partially ordered $S-$ metric space. Our results extend and generalize many existing results in the literature. Moreover, an example is given to support our results. Finally, the results are used to prove the existence and uniqueness of solution of some random functional equations.

Keywords: Measurable mapping, random operator, coupled random coincidence point, S-Metric space, partially ordered set, mixed weakly monotone property.
2010 MSC: 47H10, 54H25, 60H25.

1. Introduction

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization and approximation theory. There were many authors introduced the generalizations of a metric space such as Gahler [6] (called 2-metric space) and Dhage [4] (called $D-$metric spaces). In 2003, Mustafa and Sims [16] presented some remarks and examples which show that many of the claims concerning the topological structure of $D-$metric space are incorrect. Consequently, they introduced a more generalized metric space (called $G-$metric spaces). In 2012, Sedghi et al. [26] introduced a more general space namely $S-$metric space which generalizes $D-$metric and $G-$metric spaces and proved fixed point theorems in this space.

The existence of fixed points for monotone operators in partially ordered metric spaces has been considered in [24] with some applications to matrix equations, then Nieto and López [21, 22] extended these results and applied them to study a problem of ordinary differential equations. In [1] Gnana-Bhaskar and Lakshmikantham introduced the concept of coupled fixed point for mixed monotone operator and established

*Corresponding author
Email addresses: rr_rashwan54@yahoo.com (Rashwan Ahmed Rashwan), shimaa1362011@yahoo.com (Shimaa Ibraheem Moustafa)

Received 2017-10-20

fixed point theorems in partially ordered metric spaces, then they discussed the existence and uniqueness of solution for a periodic boundary value problem as an application. Moreover, Lakshmikantham and Ćirić [14] studied coupled coincidence and common fixed point theorems for nonlinear contractions in partially ordered metric spaces. The same authors Ćirić and Lakshmikantham [3] also studied some coupled random coincidence and coupled random fixed point theorems for a pair of random mappings under certain contractive conditions in partially ordered complete separable metric spaces. Following that, many authors worked on these topics (see e. g. [9, 13, 15, 28, 29]).

Random fixed point theorems are stochastic generalizations of classical fixed point theorems and play an important role in the theory of random integral equations and random differential equations. Because of their importance, random fixed point theorems for contractive mapping on complete separable metric space have been proved by several authors (see [2, 11, 12, 20, 23, 27, 30, 31]).

2. Preliminaries

First we begin with the following definitions and results in the framework of $S-$metric space.

Definition 2.1 ([26]). Let X be a non-empty set. An $S-$metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for each $x, y, z, a \in X$,

(S_1) $S(x, y, z) \geq 0$,

(S_2) $S(x, y, z) = 0 \Leftrightarrow x = y = z$,

(S_3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an $S-$metric space.

Example 2.2. Let $X = R$ and the distance function $S : X^3 \to [0, \infty)$ be defined as

$$S(x, y, z) = |x - z| + |y - z|, \;\; \forall\, x, y, z \in X.$$

Then (X, S) is a complete $S-$ metric space.

Lemma 2.3 ([26]). *In an $S-$metric space, we have $S(x, x, y) = S(y, y, x)$.*

Lemma 2.4 ([5]). *Let (X, S) be an $S-$ metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ for all $x, y, z \in X$.*

Definition 2.5 ([26]). Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and the closed ball $\overline{B}_S(x, r)$ with center x and radius r as follows

$$B_S(x, r) = \{y \in X : S(x, x, y) < r\}, \; \overline{B}_S(x, r) = \{y \in X : S(x, x, y) \leq r\}.$$

Definition 2.6 ([26]). Let (X, S) be an S-metric space and $A \subseteq X$:

(1) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

(3) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.7 ([26]). *Let (X, S) be an $S-$metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

Definition 2.8. The $S-$ metric space (X, S) is called separable if it has a countable dense subset $A \subset X$. That is there are $x_1, x_2, ... \in X$ such that $\overline{\{x_1, x_2, ...\}} = X$ ($\overline{A}$ denotes the closure of A).

Lemma 2.9. *A set $A \subset X$ is dense in X if and only if for any $x \in X$ and $r > 0$ we can find $x_i \in A$ such that $x_i \in B_S(x, r)$.*

Definition 2.10. A triple $(X, S, \preceq)$ is called a partially ordered $S-$ metric space if the pair $(X, \preceq)$ is a partially ordered set endowed with an $S-$ metric on X.

Definition 2.11 ([1])**.** Let $(X, \preceq)$ be an ordered set and $F : X \times X \to X$ be a mapping. Then F is said to has the mixed monotone property if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \;\; x_1 \preceq x_2 \;\text{ implies }\; F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, \;\; y_1 \preceq y_2 \;\text{ implies }\; F(x, y_1) \succeq F(x, y_2).$$

Definition 2.12 ([1])**.** An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$, if

$$F(x, y) = x, \;\; F(y, x) = y.$$

In 2012, Gordii et al. [7] introduced the concept of the mixed weakly increasing property of mappings and proved a coupled common fixed point result for two single-valued mappings. Then Gupta and Deep [10] used altering distance function to generalize these results in $S-$metric space.

Definition 2.13 ([7])**.** Let $(X, \preceq)$ be a partially ordered set and $F, G : X \times X \to X$ be mappings. We say that a pair F, G has the mixed weakly monotone property on X if for any $x, y \in X$

$$\begin{gathered} x \preceq F(x, y), \;\; y \succeq F(y, x), \\ \Rightarrow F(x, y) \preceq G(F(x, y), F(y, x)), \;\; F(y, x) \succeq G(F(y, x), F(x, y)), \end{gathered}$$

and

$$\begin{gathered} x \preceq G(x, y), \;\; y \succeq G(y, x), \\ \Rightarrow G(x, y) \preceq F(G(x, y), G(y, x)), \;\; G(y, x) \succeq F(G(y, x), G(x, y)). \end{gathered}$$

Throughout the paper, we follow the notions of Ćirić and Lakshmikantham [3]: Let (Ω, Σ) be a measurable space with sigma algebra Σ generated by all measurable subsets of Ω and (X, S) be an $S-$ metric space with Borel $\sigma-$ algebra $\mathcal{B} = \mathcal{B}(X)$ (which is the smallest $\sigma-$ algebra that contains all open subsets of X). A mapping $\xi : \Omega \to X$ is called $\sigma-$ measurable if for any open subset U of X, the set $\xi^{-1}(U) = \{\omega \in \Omega : \xi(\omega) \in U\}$ is measurable. Notice that when we say that a set A is "measurable" we mean that A is $\sigma-$ measurable. A mapping $f : \Omega \times X \to X$ is called a random operator if $f(., x)$ is measurable for any $x \in X$. A measurable mapping $\xi : \Omega \to X$ is called a random fixed of $f : \Omega \times X \to X$ if $\xi(\omega) = f(\omega, \xi(\omega))$ for all $\omega \in \Omega$. A mapping $T : \Omega \times X \to X$ is called a random operator if $T(., x)$ is measurable for any $x \in X$. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of a random function $T : \Omega \times X \to X$ if $\xi(\omega) = T(\omega, \xi(\omega))$ for every $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a random coincidence of $T : \Omega \times X \to X$ and $g : \Omega \times X \to X$ if $g(\omega, \xi(\omega)) = T(\omega, \xi(\omega))$ for every $\omega \in \Omega$.

In this paper, we consider the following class of pairs of functions $\Im$ (for more discussion and examples on these functions, one can see [25]).

Definition 2.14 ([25])**.** A pair of functions (φ, ϕ) is said to belong to the class $\Im$ if they satisfy the following conditions:

(a1) $\varphi, \phi : [0, \infty) \to [0, \infty)$;

(a2) $\varphi(t) \le \phi(s)$ implies $t \le s$, for $t, s \in [0, \infty)$;

(a3) For $\{t_n\}$ and $\{s_n\}$ sequences in $[0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$, if $\varphi(t_n) \leq \phi(s_n)$ for any $n \in N$, then $a = 0$.

Remark 2.15 ([25]). If $(\varphi,\phi) \in \Im$ and $\varphi(t) \leq \phi(t)$, then $t = 0$ (we can apply $(a3)$ with $t_n = s_n = t$).

Example 2.16. The conditions $(a1) - (a3)$ of the above definition are fulfilled for the functions $\varphi, \phi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = t$ and $\phi(t) = \lambda t$, for any $\lambda < 1$ and $t \in [0,\infty)$.

Example 2.17 ([25])**.** Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function such that $\varphi(t) = 0$ if and only if $t = 0$ (these functions are known in the literature as altering distance functions). Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi \leq \varphi$. Then the pair $(\varphi, \varphi - \phi) \in \Im$.
An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$ and $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi(t) \leq t$ for any $t \in [0,\infty)$.

Example 2.18 ([25])**.** Let S be the class of functions defined by $S = \{\alpha : [0,\infty) \to [0,\infty) : \alpha(t_n) \to 1 \Rightarrow t_n \to 0\}$. Let us consider the pairs of functions $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$, where $\alpha \in S$ and $\alpha 1_{[0,\infty)}$ is defined by $\alpha 1_{[0,\infty)}(t) = \alpha(t)t$, for $t \in [0,\infty)$. Then $(1_{[0,\infty)}, \alpha 1_{[0,\infty)}) \in \Im$.

Using some of these pairs of functions satisfying certain assumptions, Singh et al. [18] proved the following fixed point theorem.

Theorem 2.19. *Let $(X,\preceq)$ be a partially ordered set. Suppose that there exists an $S-$metric on X such that (X,S) be a complete $S-$metric space. Let $f : X \to X$ be a non-decreasing mapping such that there exists a pair of functions $(\varphi,\phi) \in \Im$ satisfying*

$$\varphi\big(S(fx,fx,fy)\big) \leq \max\big\{\phi(S(x,x,y)), \phi(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)})\big\}, \tag{2.1}$$

for all comparable elements $x,y \in X$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in N$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

The aim of this work is to prove some common random fixed point and coupled random fixed point theorems for a pair of weakly monotone random operators satisfying some contractive conditions of rational type in the setting of partially ordered $S-$ metric space. These results are some random versions and extensions of results of Singh et al. [18]. Our results generalize the results of [17] and many of the well-known results in the current literature. An example is also established to support the usability of our results. The results are used to prove the existence and uniqueness of solution of some random functional equations.

3. Random fixed point theorems

First we introduce the notion of random weakly monotone property.

Definition 3.1. Let $(X,\preceq)$ be a partially ordered set and $F,G : \Omega \times X \times X \to X$ be random mappings. We say that a pair (F,G) has the mixed weakly monotone property on X if for any $x,y \in X$

$$\begin{gathered} x \preceq F(\omega,x,y),\;\; y \succeq F(\omega,y,x), \\ \Rightarrow F(\omega,x,y) \preceq G(\omega,F(\omega,x,y),F(\omega,y,x)),\;\; F(\omega,y,x) \succeq G(\omega,F(\omega,y,x),F(\omega,x,y)), \end{gathered}$$

and

$$\begin{gathered} x \preceq G(\omega,x,y),\;\; y \succeq G(\omega,y,x), \\ \Rightarrow G(\omega,x,y) \preceq F(\omega,G(\omega,x,y),G(\omega,y,x)),\;\; G(\omega,y,x) \succeq F(\omega,G(\omega,y,x),G(\omega,x,y)). \end{gathered}$$

Note that, may all the arguments like x,y depend also on ω, i.e., $x = x(\omega)$.

Now we state our main results as follows:

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set, (X, S) be a complete separable $S-$metric space, (Ω, Σ) be a measurable space and $f, g : \Omega \times X \to X$ be two mappings such that*

(i) *$f(\omega, .)$, $g(\omega, .)$ are continuous, for all $\omega \in \Omega$,*

(ii) *$f(., x)$, $g(., x)$ are measurable, for all $x \in X$,*

(iii) *The pair (f, g) satisfy the weakly monotone property,*

(iv) *There exists a pair of functions $(\varphi, \phi) \in \Im$, such that*

$$\varphi\big(S(f(w,x), f(w,x), g(w,u))\big) \leq \max\big\{\phi(S(x,x,u)), \phi(\frac{S(u,u,g(w,u))[1+S(x,x,f(w,x))]}{1+S(f(w,x),f(w,x),g(w,u))})\big\}, \quad (3.1)$$

for comparable elements x and $u \in X$.

If there exists a measurable mapping $\xi_0 : \Omega \to X$ with $\xi_0(\omega) \preceq f(\omega, \xi_0(\omega))$ or $\xi_0(\omega) \preceq g(\omega, \xi_0(\omega))$. Then f and g have a common random fixed point in X.

Proof. Let $\theta = \{\xi : \Omega \to X\}$ be a family of measurable mappings. Since $f(\omega, \xi_0(\omega))$ is measurable in its first argument ω, then the mapping defined by $\xi_1(\omega) = f(\omega, \xi_0(\omega))$ is measurable, that is, $\xi_1 \in \theta$. Similarly, as $g(\omega, \xi_1(\omega))$ is measurable, then there is $\xi_2 \in \theta$ such that $\xi_2(\omega) = g(\omega, \xi_1(\omega))$. Continuing this process, we can construct sequence $\{\xi_n(\omega)\}$ in X with

$$\begin{aligned} &\xi_{2n+1}(\omega) = f(\omega, \xi_{2n}(\omega)), \\ &\xi_{2n+2}(\omega) = g(\omega, \xi_{2n+1}(\omega)), \quad \forall n \in N \cup \{0\}. \end{aligned} \quad (3.2)$$

Step 1. First we use mathematical induction to prove that

$$\xi_n(\omega) \preceq \xi_{n+1}(\omega), \quad (3.3)$$

for all $n \in N \cup \{0\}$. Let $n = 0$. By assumption we have $\xi_0(\omega) \preceq f(\omega, \xi_0(\omega)) = \xi_1(\omega)$. Therefore, (3.3) holds for $n = 0$. Suppose it is true for some fixed $n \geq 0$. Thus, we have to discuss two cases

* If n is even, say $n = 2m$. That is,

$$\xi_{2m}(\omega) \preceq \xi_{2m+1}(\omega) = f(\omega, \xi_{2m}(\omega)).$$

Since f and g satisfy the weakly monotone property, then

$$\xi_{n+1}(\omega) = \xi_{2m+1}(\omega) = f(\omega, \xi_{2m}(\omega)) \preceq g(\omega, f(\omega, \xi_{2m}(\omega))) = \xi_{2m+2}(\omega) = \xi_{n+2}(\omega).$$

* If n is odd, say $n = 2m + 1$. That is,

$$\xi_{2m+1}(\omega) \preceq \xi_{2m+2}(\omega) = g(\omega, \xi_{2m+1}(\omega)).$$

Again by weak monotonicity for f and g, we get

$$\xi_{n+1}(\omega) = \xi_{2m+2}(\omega) = g(\omega, \xi_{2m+1}(\omega)) \preceq f(\omega, g(\omega, \xi_{2m+1}(\omega))) = \xi_{2m+3}(\omega) = \xi_{n+2}(\omega).$$

Thus, (3.3) holds for all n.

Step 2. In this step we prove $\lim_{n\to\infty} S(\xi_n(\omega), \xi_{n+1}(\omega), \xi_{n+1}(\omega)) = 0$. By equations (3.1) - (3.3) we obtain

$$\begin{aligned}\varphi\big(S(f(w,\xi_{2n}(\omega)),f(w,\xi_{2n}(\omega)), g(w,\xi_{2n+1}(\omega)))\big) \le \max\Big\{&\phi\big(S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))\big),\\ &\phi(\frac{S(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega),g(w,\xi_{2n+1}(\omega)))[1+S(\xi_{2n}(\omega),\xi_{2n}(\omega),f(w,\xi_{2n}(\omega)))]}{1+S(f(w,\xi_{2n}(\omega)),f(w,\xi_{2n}(\omega)),g(w,\xi_{2n+1}(\omega)))})\Big\}\\ \varphi\big(S(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega),\xi_{2n+2}(\omega))\big) \le \max\Big\{&\phi\big(S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))\big),\\ &\phi(\frac{S(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega),\xi_{2n+2}(\omega))[1+S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))]}{1+S(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega),\xi_{2n+2}(\omega))})\Big\}.\end{aligned}$$

For simplicity, we denote $S_n = S(\xi_n(\omega), \xi_n(\omega), \xi_{n+1}(\omega))$.

$$\varphi(S_{2n+1}) \le \max\{\phi(S_{2n}), \phi(\frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}})\}. \tag{3.4}$$

Now, we can distinguish two cases.

* If $\max\{\phi(S_{2n}), \phi(\frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}})\} = \phi(\frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}})$, then Eq. (3.4) and property (c_2) of (φ,ϕ) yield

$$\begin{aligned}\varphi(S_{2n+1}) &\le \phi(\frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}})\\ \Rightarrow S_{2n+1} &\le \frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}}\\ \Rightarrow S_{2n+1} &\le S_{2n}.\end{aligned}$$

* Otherwise, $\max\{\phi(S_{2n}), \phi(\frac{S_{2n+1}[1+S_{2n}]}{1+S_{2n+1}})\} = \phi(S_{2n})$, then we have $S_{2n+1} \le S_{2n}$.

Hence, in both cases we obtain that

$$S_{2n+1} \le S_{2n}. \tag{3.5}$$

Interchanging the role of mappings f and g and using (3.1) - (3.3), we get

$$\begin{aligned}\varphi\big(S(g(w,\xi_{2n-1}(\omega)),g(w,\xi_{2n-1}(\omega)), f(w,\xi_{2n}(\omega)))\big) \le \max\Big\{&\phi\big(S(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega),\xi_{2n}(\omega))\big),\\ &\phi(\frac{S(\xi_{2n}(\omega),\xi_{2n}(\omega),f(w,\xi_{2n}(\omega)))[1+S(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega),g(w,\xi_{2n-1}(\omega)))]}{1+S(g(w,\xi_{2n-1}(\omega)),g(w,\xi_{2n-1}(\omega)),f(w,\xi_{2n}(\omega)))})\Big\}\\ \varphi\big(S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))\big) \le \max\Big\{&\phi\big(S(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega),\xi_{2n}(\omega))\big),\\ &\phi(\frac{S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))[1+S(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega),\xi_{2n}(\omega))]}{1+S(\xi_{2n}(\omega),\xi_{2n}(\omega),\xi_{2n+1}(\omega))})\Big\}\\ \varphi(S_{2n}) \le \max\{&\phi(S_{2n-1}), \phi(\frac{S_{2n}[1+S_{2n-1}]}{1+S_{2n}})\},\end{aligned}$$

which implies that

$$S_{2n} \le S_{2n-1}. \tag{3.6}$$

From (3.5)and (3.6) we conclude that $\{S_n\}$ is a decreasing sequence of non-negative real numbers and is bounded below, then there is $r(\omega) \ge 0$ such that

$$\lim_{n\to\infty} S_n = r(\omega).$$

Letting $n \to \infty$ in Eq. (3.4) and using property (c_3) of (φ,ϕ) imply that $r(\omega) = 0, \forall\omega$. Therefore,

$$\lim_{n\to\infty} S_n = 0. \tag{3.7}$$

Step 3. Now we show that $\{\xi_n(\omega)\}$ is a Cauchy sequence in X for every $\omega \in \Omega$, it is sufficient to prove that $\{\xi_{2n}(\omega)\}$ is a Cauchy sequence. Suppose the contrary, then there exist $\epsilon(\omega) > 0$ for which we can find two sub-sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that m_k is the smallest integer with

$$m_k > n_k > k, S(\xi_{2n_k}, \xi_{2n_k}, \xi_{2m_k}) \geq \epsilon(\omega), \tag{3.8}$$

and

$$S(\xi_{2n_k}, \xi_{2n_k}, \xi_{2m_k-2}) < \epsilon(\omega). \tag{3.9}$$

By Lemmas 2.3, 2.4 and using equations (3.8), (3.9), we have

$$\begin{aligned}\epsilon(\omega) &\leq S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k}(\omega)) = S(\xi_{2m_k}(\omega), \xi_{2m_k}(\omega), \xi_{2n_k}(\omega))\\ &\leq 2S(\xi_{2m_k}(\omega), \xi_{2m_k}(\omega), \xi_{2m_k-2}(\omega)) + S(\xi_{2m_k-2}(\omega), \xi_{2m_k-2}(\omega), \xi_{2n_k}(\omega))\\ &\leq 2[2S(\xi_{2m_k}(\omega), \xi_{2m_k}(\omega), \xi_{2m_k-1}(\omega)) + S(\xi_{2m_k-1}(\omega), \xi_{2m_k-1}(\omega), \xi_{2m_k-2}(\omega))]\\ &+ S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k-2}(\omega))\\ &< 2(2S_{2m_k-1} + S_{2m_k-2}) + \epsilon(\omega).\end{aligned}$$

On Letting the limit as $k \to \infty$ in the above inequality and using (3.7), we get

$$\lim_{k\to\infty} S(\xi_{2n_k}, \xi_{2n_k}, \xi_{2m_k}) = \epsilon(\omega). \tag{3.10}$$

Also we have,

$$\begin{aligned}S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega)) &\leq 2S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2n_k}(\omega)) + S(\xi_{2m_k+2}(\omega), \xi_{2m_k+2}(\omega), \xi_{2n_k}(\omega))\\ &\leq 2S_{2n_k} + 2S(\xi_{2m_k+2}(\omega), \xi_{2m_k+2}(\omega), \xi_{2m_k+1}(\omega)) + S(\xi_{2m_k+1}(\omega), \xi_{2m_k+1}(\omega), \xi_{2n_k}(\omega))\\ &\leq 2S_{2n_k} + 2S_{2m_k+1} + 2S_{2m_k} + S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k}(\omega)),\end{aligned}$$

and

$$S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k}(\omega)) \leq 2S_{2n_k} + 2S_{2m_k} + 2S_{2m_k+1} + S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega)).$$

That is,

$$| S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega)) - S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k}(\omega)) | \leq 2S_{2n_k} + 2S_{2m_k} + 2S_{2m_k+1}.$$

On making $k \to \infty$ above, we obtain that

$$\lim_{k\to\infty} S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega)) = \epsilon(\omega). \tag{3.11}$$

By a similar way, we obtain

$$| S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k+1}(\omega)) - S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k}(\omega)) | \leq 2S_{2m_k}.$$

Thus,

$$\lim_{k\to\infty} S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k+1}(\omega)) = \epsilon(\omega). \tag{3.12}$$

Since $n_k < m_k$ and $\xi_{2n_k} \leq \xi_{2m_k+1}$, then by contractive condition (3.1), we get

$$\begin{aligned}\varphi\big(S(f(w, \xi_{2n_k}(\omega)), f(w, \xi_{2n_k}(\omega)), g(w, \xi_{2m_k+1}(\omega)))\big) &= \varphi\big(S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega))\big)\\ &\leq \max\Big\{\phi\big(S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2m_k+1}(\omega))\big),\\ &\phi\big(\frac{S(\xi_{2m_k+1}(\omega), \xi_{2m_k+1}(\omega), \xi_{2m_k+2}(\omega))[1 + S(\xi_{2n_k}(\omega), \xi_{2n_k}(\omega), \xi_{2n_k+1}(\omega))]}{1 + S(\xi_{2n_k+1}(\omega), \xi_{2n_k+1}(\omega), \xi_{2m_k+2}(\omega))}\big)\Big\}.\end{aligned}$$

Now we have to discuss two cases

* If

$$\max\left\{\phi\big(S(\xi_{2n_k}(\omega),\xi_{2n_k}(\omega),\xi_{2m_k+1}(\omega))\big),\phi\big(\frac{S_{2m_k+1}[1+S_{2n_k}]}{1+S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))}\big)\right\}$$
$$=\phi\big(S(\xi_{2n_k}(\omega),\xi_{2n_k}(\omega),\xi_{2m_k+1}(\omega))\big),$$

then

$$\varphi\big(S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))\big)\leq\phi\big(S(\xi_{2n_k}(\omega),\xi_{2n_k}(\omega),\xi_{2m_k+1}(\omega))\big).$$

But since $(\varphi,\phi)\in\xi$ and by (3.11), (3.12), we get $\epsilon(\omega)=0$ which is a contradiction.

* If

$$\max\left\{\phi\big(S(\xi_{2n_k}(\omega),\xi_{2n_k}(\omega),\xi_{2m_k+1}(\omega))\big),\phi\big(\frac{S_{2m_k+1}[1+S_{2n_k}]}{1+S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))}\big)\right\}$$
$$=\phi\big(\frac{S_{2m_k+1}[1+S_{2n_k}]}{1+S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))}\big),$$

then

$$\varphi\big(S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))\big)\leq\phi\big(\frac{S_{2m_k+1}[1+S_{2n_k}]}{1+S(\xi_{2n_k+1}(\omega),\xi_{2n_k+1}(\omega),\xi_{2m_k+2}(\omega))}\big).$$

Letting $k\to\infty$ in the last inequality and using (3.7) we get $\epsilon(\omega)\leq 0$ which is a contradiction. Then our claim follows.

Since X is complete, then there exists $\xi\in\theta$ such that

$$\lim_{k\to\infty}\xi_n(\omega)=\xi(\omega).$$

Step 4. Here we show the existence of the fixed point. Since $f(\omega,.)$ and $g(\omega,.)$ are continuous for all $\omega\in\Omega$, then

$$\xi(\omega)=\lim_{k\to\infty}\xi_{2n+1}(\omega)=f(\omega,\lim_{k\to\infty}\xi_{2n}(\omega))=f(\omega,\xi(\omega)),$$

and

$$\xi(\omega)=\lim_{k\to\infty}\xi_{2n+2}(\omega)=g(\omega,\lim_{k\to\infty}\xi_{2n+1}(\omega))=g(\omega,\xi(\omega)).$$

So we have

$$\xi(\omega)=f(\omega,\xi(\omega))=g(\omega,\xi(\omega)).\tag{3.13}$$

That is $\xi(\omega)\in X$ is a common fixed point for f and g.

□

From Theorem 3.2 we obtain the following corollaries.

Corollary 3.3. *Let $(X,\preceq)$ be a partially ordered set, (X,S) be a complete separable $S-$metric space, (Ω,Σ) be a measurable space and $f,g:\Omega\times X\to X$ be two mappings such that for comparable elements x and $u\in X$, we have*

(i) *$f(\omega,.)$, $g(\omega,.)$ are continuous, for all $\omega\in\Omega$,*

(ii) *$f(.,x)$, $g(.,x)$ are measurable, for all $x\in X$,*

(iii) *The pair (f,g) satisfy the weakly monotone property,*

(iv) *There exist $\alpha, \beta > 0$ with $\alpha + \beta < 1$ such that*

$$S(f(w,x), f(w,x), g(w,u)) \leq \alpha \frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))} + \beta S(x,x,u). \tag{3.14}$$

If there exists a measurable mapping $\xi_0 : \Omega \to X$ with $\xi_0(\omega) \preceq f(\omega, \xi_0(\omega))$ or $\xi_0(\omega) \preceq g(\omega, \xi_0(\omega))$. Then f and g have a common random fixed point in X.

Proof. Since

$$\begin{aligned} S(f(w,x), f(w,x), g(w,u)) &\leq \alpha \frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))} + \beta S(x,x,u) \\ &\leq (\alpha+\beta) \max\big\{S(x,x,u), \frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\big\} \\ &\leq \max\big\{(\alpha+\beta) S(x,x,u), (\alpha+\beta)\frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\big\}. \end{aligned}$$

This condition is a particular case of the contractive condition appearing in Theorem 3.2 with the pair of functions$(\varphi, \phi) \in \Im$, given by $\varphi(t) = t$ and $\phi(t) = (\alpha+\beta)t$, (see Example 2.16). □

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set, (X, S) be a complete separable $S-$metric space, (Ω, Σ) be a measurable space and $f, g : \Omega \times X \to X$ be two mappings such that for comparable elements x and $u \in X$, we have*

(i) *$f(\omega, .)$, $g(\omega, .)$ are continuous, for all $\omega \in \Omega$,*

(ii) *$f(., x)$, $g(., x)$ are measurable, for all $x \in X$,*

(iii) *The pair (f, g) satisfy the weakly monotone property,*

(iv) *There exist two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ with the same conditions in Example 2.17 and*

$$\begin{aligned} \varphi\big(S(f(w,x), f(w,x), g(w,u))\big) \leq \max\big\{&\varphi(S(x,x,u)) - \phi(S(x,x,u)), \\ &\varphi\Big(\frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\Big) - \phi\Big(\frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\Big)\big\}, \end{aligned} \tag{3.15}$$

If there exists a measurable mapping $\xi_0 : \Omega \to X$ with $\xi_0(\omega) \preceq f(\omega, \xi_0(\omega))$ or $\xi_0(\omega) \preceq g(\omega, \xi_0(\omega))$. Then f and g have a common random fixed point in X.

Note that, Condition (3.15) is a particular case of the contractive condition appearing in Theorem 3.2 with the pair of functions$(\varphi, \varphi - \phi) \in \Im$, (see Example 2.17).

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set, (X, S) be a complete separable $S-$metric space, (Ω, Σ) be a measurable space and $f, g : \Omega \times X \to X$ be two mappings such that for comparable elements x and $u \in X$, we have*

(i) *$f(\omega, .)$, $g(\omega, .)$ are continuous, for all $\omega \in \Omega$,*

(ii) *$f(., x)$, $g(., x)$ are measurable, for all $x \in X$,*

(iii) *The pair (f, g) satisfy the weakly monotone property,*

(iv) *There exists $\alpha \in S$, such that*

$$\begin{aligned} S(f(w,x), f(w,x), g(w,u)) \leq \max\big\{&\alpha(S(x,x,u)) S(x,x,u), \\ &\alpha\Big(\frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\Big)\frac{S(u,u,g(w,u))[1 + S(x,x,f(w,x))]}{1 + S(f(w,x), f(w,x), g(w,u))}\big\}, \end{aligned} \tag{3.16}$$

If there exists a measurable mapping $\xi_0 : \Omega \to X$ with $\xi_0(\omega) \preceq f(\omega, \xi_0(\omega))$ or $\xi_0(\omega) \preceq g(\omega, \xi_0(\omega))$. Then f and g have a common random fixed point in X.

Note that, Condition (3.16) is a particular case of the contractive condition appearing in Theorem 3.2 with the pair of functions$(\varphi, \phi) \in \Im$, given by $\varphi = 1_{[0,\infty)}$ and $\phi = \alpha_{[0,\infty)}$, (see Example 2.18).

4. Coupled random fixed point theorems

Lemma 4.1. *If (X, S) is a separable $S-$ metric space, then $\mathcal{B}$ equals the $\sigma-$ algebra generated by the open balls of X, where $\mathcal{B}$ is the Borel $\sigma-$ algebra that contains all open (closed) subsets of X.*

Proof. Denote $\mathcal{A} := \sigma-$ algebra generated by the open balls of X. Clearly, $\mathcal{A} \subset \mathcal{B}$. Let D and U be countable dense subset and open subset of X, respectively. Since U is open, then for any $x \in U$ there exist $r > 0$ such that $B(x, r) \subset U$. Also since D is dense in X, then there exist $y_x \in D$ such that $y_x \in B(x, \frac{r}{3}) \Rightarrow x \in B(y_x, \frac{r}{2}) \subset U$. Set $r_x = \frac{r}{2}$, we have

$$U = \cup\{B(y_x, r_x) : x \in U\},$$

which is a countable union of elements in $\mathcal{A}$. Therefore $U \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{B}$. □

Lemma 4.2. *Consider that $\overline{S} : Z \times Z \to R^+$, where $Z = X^2$, defined by*

$$\overline{S}((x_1, y_1), (x_1, y_1), (x_2, y_2)) = S(x_1, x_1, x_2) + S(y_1, y_1, y_2).$$

Then $\overline{S}$ is an $S-$ metric on Z and if (X, S) is separable then $(Z, \overline{S})$ is separable. Moreover, if (X, S) is complete then $(Z, \overline{S})$ is complete, too.

Proof. Clearly $\overline{S}(z_1, z_2, z_3) \geq 0$ for all $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3) \in Z$ and

$$\begin{aligned} \overline{S}(z_1, z_2, z_3) = 0 &\Leftrightarrow S(x_1, x_2, x_3) = S(y_1, y_2, y_3) = 0 \\ &\Leftrightarrow x_1 = x_2 = x_3 \texttt{ and } y_1 = y_2 = y_3 \Leftrightarrow z_1 = z_2 = z_3. \end{aligned}$$

Also, for $z_1, z_2, z_3, (a, b) \in Z$ we have

$$\begin{aligned} \overline{S}(z_1, z_2, z_3) &= S(x_1, x_2, x_3) + S(y_1, y_2, y_3) \\ &\leq S(a, x_2, x_3) + S(a, a, x_1) + S(b, y_2, y_3) + S(b, b, y_1) \\ &\leq \overline{S}((a, b), (x_2, y_2), (x_3, y_3)) + \overline{S}((a, b), (a, b), (x_1, y_1)) \\ &\leq \overline{S}((a, b), z_2, z_3) + \overline{S}((a, b), (a, b), z_1). \end{aligned}$$

Thus $\overline{S}$ is an $S-$ metric on Z. Furthermore, if (X, S) is separable, then X has a countable dense subset A. Now we show that A^2 is countable dense subset in Z. Since A is dense in X then for $x \in X$ and $r/2 > 0$ there is $a \in A$ such that $S(x, x, a) < \frac{r}{2}$ and for $y \in X$ and $r/2 > 0$ there is $b \in A$ such that $S(y, y, b) < \frac{r}{2}$. Hence, for any $z = (x, y) \in Z$ and $r > 0$ there exist $(a, b) \in A^2$ such that

$$\overline{S}(z, z, (a, b)) = S(x, x, a) + S(y, y, b) < r.$$

Then A^2 is dense in Z and $(Z, \overline{S})$ is separable $S-$metric.
Finally, we prove that $(Z, \overline{S})$ is complete if (X, S) is complete. Let $\{z_n\}$ be Cauchy sequence in $(Z, \overline{S})$, then we have for all $\epsilon > 0$ there exists $n_0 \in N$ such that $\overline{S}(z_n, z_n, z_m) < \epsilon$ for all $n > m > n_0$. Say $z_n = (x_n, y_n)\ \forall n$, then $S(x_n, x_n, x_m) < \epsilon$ and $S(y_n, y_n, y_m) < \epsilon$, that is, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete then there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then we have $z_n \to (x, y) \in Z$. Hence, $(Z, \overline{S})$ is complete. □

Lemma 4.3. *Let (X,S) be separable $S-$ metric space. The mapping $\zeta:\Omega\to Z$ defined by $\zeta(\omega)=\big(\xi(\omega),\eta(\omega)\big)$ is measurable, whenever $\xi,\eta:\Omega\to X$ are measurable mappings.*

Proof. Since ζ must be measurable iff $\zeta^{-1}(U)$ is measurable for each open set U in Z. By Lemmas 4.1 and 4.2 we can find $z=(x,y)\in X^2$ and $r>0$ such that $U=B(z,r)$.

$$\begin{aligned}\zeta^{-1}(U)&=\{\omega:\zeta(\omega)\in B(z,r)\}\\&=\{\omega:\overline{S}(z,z,\zeta(\omega))<r\}\\&=\{\omega:\overline{S}\big((x,y),(x,y),(\xi(\omega),\eta(\omega))\big)<r\}\\&=\{\omega:S(x,x,\xi(\omega))+S(y,y,\eta(\omega))<r\}\\&=\{\omega:S(x,x,\xi(\omega))<\frac{r}{2}\}\cap\{\omega:S(y,y,\eta(\omega))<\frac{r}{2}\}\in\Sigma.\end{aligned}$$

Hence ζ is measurable mapping. □

Lemma 4.4. *Let (X,S) be separable $S-$ metric space and $F:\Omega\times X\times X\to X$ be Carathéodory function (that is, it is measurable in ω for each $x,y\in X$ and continuous in x and y for each $\omega\in\Omega$). Then the function $T:\Omega\times Z\to Z$ defined by $T(\omega,(x,y))=\big(F(\omega,x,y),F(\omega,y,x)\big)$ is also Carathéodory function.*

Proof. We shall show that $T(\omega,z)$ is measurable in $\omega\in\Omega$ and continuous in $z\in Z$. By Lemmas 4.1 and 4.2 we can find $z_0=(x_0,y_0)\in Z$ and $r>0$ such that $U=B(z_0,r)$ for any open set $U\subset Z$.

$$\begin{aligned}T_z^{-1}(U)&=\{\omega:T_z(\omega)=T(\omega,z)\in B(z_0,r)\}\\&=\{\omega:\overline{S}(z_0,z_0,T_z(\omega))<r\}\\&=\{\omega:\overline{S}\big((x_0,y_0),(x_0,y_0),(F(\omega,x,y),F(\omega,y,x))\big)<r\}\\&=\{\omega:S(x_0,x_0,F(\omega,x,y))+S(y_0,y_0,F(\omega,y,x))<r\}\\&=\{\omega:S(x_0,x_0,F(\omega,x,y))<\frac{r}{2}\}\cap\{\omega:S(y_0,y_0,F(\omega,y,x))<\frac{r}{2}\}\\&=\{\omega:F(\omega,x,y)\in B(x_0,\frac{r}{2})\}\cap\{\omega:F(\omega,y,x)\in B(y_0,\frac{r}{2})\}\in\Sigma.\end{aligned}$$

Hence T is measurable in ω for all $z\in Z$. To prove that T is continuous we have to prove that $T_\omega(z_n)\to T_\omega(z)$ for any convergent sequence $z_n\to z$ in Z. Consider $z_n=(x_n,y_n)$ for all $n\in N$. Since $z_n\to z$, say there exist $x,y\in X$ with $z=(x,y)$, $x_n\to x$ and $y_n\to y$. From the continuity of F in $x,y\in X$ we have

$$\lim_{n\to\infty}T_\omega(z_n)=\big(\lim_{n\to\infty}F_\omega(x_n,y_n),\lim_{n\to\infty}F_\omega(y_n,x_n)\big)=\big(F_\omega(x,y),F_\omega(y,x)\big)=T_\omega(z).$$

This finishes our proof. □

Remark 4.5. Let $(X,\preceq)$ be a partially ordered set. We can endow the product space $X\times X$ with the partial order $\preceq_p$ given by

$$(x,y)\preceq_p(u,v)\Leftrightarrow x\preceq u,y\succeq v.$$

Theorem 4.6. *Let $(X,\preceq)$ be a partially ordered set, (X,S) be a complete separable $S-$metric space, (Ω,Σ) be a measurable space and $F,G:\Omega\times X\times X\to X$ be two mappings such that*

(i) *$F(\omega,.)$, $G(\omega,.)$ are continuous, for all $\omega\in\Omega$,*

(ii) *$F(.,v)$, $G(.,v)$ are measurable, for all $v\in X\times X$,*

(iii) *The pair (F,G) satisfy the weakly monotone property,*

(iv) *There exists a pair of functions* $(\varphi, \phi) \in \Im$, *such that*

$$\varphi\big(S(F(w,x,y),F(w,x,y),G(w,u,v)) + S(F(w,y,x),F(w,y,x),G(w,v,u))\big) \leq \max\Big\{\phi(S(x,x,u)+S(y,y,v)),$$
$$\phi\Big(\frac{[S(u,u,G(w,u,v))+S(v,v,G(w,v,u))][1+S(x,x,F(w,x,y))+S(y,y,F(w,y,x))]}{1+S(F(w,x,y),F(w,x,y),G(w,u,v))+S(F(w,y,x),F(w,y,x),G(w,v,u))}\Big)\Big\}, \tag{4.1}$$

for comparable elements (x, y) *and* $(u, v) \in X \times X$.

If there exist measurable mappings $\xi_0, \eta_0 : \Omega \to X$ *with* $\xi_0(\omega) \preceq F(or\ G)(\omega, \xi_0(\omega), \eta_0(\omega))$ *and* $\eta_0(\omega) \preceq F(or\ G)(\omega, \eta_0(\omega), \xi_0(\omega))$. *Then* F *and* G *have a coupled common random fixed point in* X.

Proof. Consider the functional $\overline{S} : X^2 \times X^2 \to R_+$, defined by

$$\overline{S}(z_1, z_2, z_3) = S(x_1, x_2, x_3) + S(y_1, y_2, y_3),\ \forall z_1 = (x_1, y_1), z_2 = (x_2, y_2) and z_3 = (x_3, y_3) \in X^2.$$

By Lemma 4.2, $(X^2, \overline{S})$ is a complete separable $S-$ metric space.
Define $T_1, T_2 : \Omega \times X^2 \to X^2$ as

$$T_1(\omega,(x,y)) = \big(F(\omega,x,y),F(\omega,y,x)\big),$$
$$T_2(\omega,(x,y)) = \big(G(\omega,x,y),G(\omega,y,x)\big).$$

By Lemma 4.4, T_1 and T_2 are measurable in $\omega \in \Omega$ and continuous in $(x, y) \in X^2$. Moreover, since F and G have mixed weakly monotone property with respect to "$\preceq$", then T_1 and T_2 have weakly monotone property with respect to "$\preceq_p$", i.e.,

$$(x,y) \in X^2, (x,y) \preceq_p T_1(\omega,(x,y)) \Rightarrow T_1(\omega,(x,y)) \preceq_p T_2(\omega, T_1(\omega,(x,y))).$$

Also, we define a sequence of measurable mappings $\zeta_n : \Omega \to X^2$ by $\zeta_n(\omega) = (\xi_n(\omega), \eta_n(\omega))$ for all $n \in N \cup \{0\}$. Since $\xi_0(\omega) \preceq F(\omega, \xi_0(\omega), \eta_0(\omega))$ and $\eta_0(\omega) \succeq F(\omega, \eta_0(\omega), \xi_0(\omega))$, then we have that

$$(\xi_0(\omega),\eta_0(\omega)) \preceq_p \big(F(\omega,\xi_0(\omega),\eta_0(\omega)),F(\omega,\eta_0(\omega),\xi_0(\omega))\big)$$
$$\zeta_0(\omega) \preceq_p T_1(\omega,(\xi_0(\omega),\eta_0(\omega))) = T_1(\omega,\zeta_0(\omega)).$$

Finally the contraction condition implies that

$$\varphi\big(\overline{S}(T_1(w,(x,y)),T_1(w,(x,y)),T_2(w,(u,v)))\big) \leq \max\Big\{\phi\big(\overline{S}((x,y),(x,y),(u,v))\big),$$
$$\phi\Big(\frac{\overline{S}((u,v),(u,v),T_2(w,(u,v)))[1+\overline{S}((x,y),(x,y),T_1(w,(x,y)))]}{1+\overline{S}(T_1(w,(x,y)),T_1(w,(x,y)),T_2(w,(u,v)))}\Big)\Big\}.$$

Therefore, now we can apply the conclusion of Theorem 2.19 and get that T_1 and T_2 have at least one random fixed point in X^2. That is, there exists a measurable mapping $\zeta : \Omega \to X^2$ such that

$$\zeta(\omega) = T_1(\omega,\zeta(\omega)) = T_2(\omega,\zeta(\omega)).$$

This implies

$$\xi(\omega) = F(\omega,\xi(\omega),\eta(\omega)) = G(\omega,\xi(\omega),\eta(\omega)), \tag{4.2}$$

and

$$\eta(\omega) = F(\omega,\eta(\omega),\xi(\omega)) = G(\omega,\eta(\omega),\xi(\omega)). \tag{4.3}$$

That is, F and G have a coupled common random fixed point in X. □

Now we construct an example to support our results.

Example 4.7. Let $X = [0, \frac{\pi}{4}]$ be equipped with the usual order " $\leq$ " and $S-$metric defined by

$$S(x, y, z) = \frac{1}{2}\big[|x - z| + |y - z|\big], \;\; \forall\, x, y, z \in X.$$

Let $\Omega = [0, 1]$ and Σ be the sigma algebra of all Lebesgue measurable subsets of Ω. Define the following partial order on X^2:

$$(x, y) \preceq (u, v) \Leftrightarrow x \leq u \;\; \texttt{and} \;\; y \geq v.$$

Consider the mappings $F, G : \Omega \times X \times X \to X$ defined by

$$F(\omega, x, y) = G(\omega, x, y) = \frac{1-\omega^2}{10}[\sin x + \sin y], \;\; \forall \;\; x, y \in X \;\; \texttt{and} \quad \omega \in X.$$

Define the functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ as follows

$$\varphi(t) = \sqrt{\frac{1}{12} + \frac{5t}{12}} \;\; \texttt{and} \;\; \phi(t) = \sqrt{\frac{1}{12} + \frac{3t}{12}}, \;\; \forall \;\; t \in [0, \infty).$$

Now we will check the contraction condition of Theorem 3.2 is satisfied for comparable elements (x, y) and $(u, v) \in X^2$.

$$\begin{aligned} &S(F(w, x, y), F(w, x, y), G(w, u, v)) + S(F(w, y, x), F(w, y, x), G(w, v, u)) \\ &\qquad = \big|F(w, x, y) - G(w, u, v)\big| + \big|F(w, y, x) - G(w, v, u)\big| \\ &\qquad = \frac{1-\omega^2}{5}\big|(\sin x - \sin u) + (\sin y - \sin v)\big|, \end{aligned}$$

$$\begin{aligned} &\varphi\big(S(F(w, x, y), F(w, x, y), G(w, u, v)) + S(F(w, y, x), F(w, y, x), G(w, v, u))\big) \\ &= \sqrt{\frac{1}{12} + \frac{1-\omega^2}{12}|(\sin x - \sin u) + (\sin y - \sin v)|} \\ &\leq \sqrt{\frac{1}{12} + \frac{1}{12}\big[\big|(\sin x - \sin u)\big| + \big|(\sin y - \sin v)\big|\big]} \\ &\leq \sqrt{\frac{1}{12} + \frac{3}{12}\big[\big|(\sin x - \sin u)\big| + \big|(\sin y - \sin v)\big|\big]} \\ &\leq \phi(S(x, x, u) + S(y, y, v)) \\ &\leq \max\Big\{\phi(S(x, x, u) + S(y, y, v)), \\ &\phi\Big(\frac{[S(u, u, G(w, u, v)) + S(v, v, G(w, v, u))][1 + S(x, x, F(w, x, y)) + S(y, y, F(w, y, x))]}{1 + S(F(w, x, y), F(w, x, y), G(w, u, v)) + S(F(w, y, x), F(w, y, x), G(w, v, u))}\Big)\Big\}. \end{aligned}$$

It is obvious that other hypothesis of Theorem 3.2 are satisfied. We deduce that F and G have a coupled random fixed point $\xi, \eta : \Omega \to X$ defined as

$$\xi(\omega) = \eta(\omega) = 0, \;\; \forall \;\; \omega \in \Omega.$$

5. Application

Now, we introduce the existence and uniqueness solution for a class of random functional equations by using Theorem 3.2.

Let A, B be Banach spaces, $D \subset A$, $E \subset B$ and R be the field of real numbers.
Let $X = \mathbf{B}(D)$ be the set of all bounded real valued functions on D with $S-$ metric defined by

$$\begin{aligned} S\big(f(\omega, x(\omega)), h(\omega, x(\omega)), t(\omega, x(\omega))\big) = &\sup_{(\omega,x)\in\Omega\times D} \big|f(\omega, x(\omega)) - h(\omega, x(\omega))\big| \\ &+ \sup_{(\omega,x)\in\Omega\times D} \big|f(\omega, x(\omega)) + t(\omega, x(\omega) - 2h(\omega, x(\omega))\big|, \end{aligned} \tag{5.1}$$

where, $f, h, t : \Omega \times A \to R$ be bounded i.e., $f, h, t \in X$. Then (X, S) is complete $S-$ metric space. Let (Ω, Σ) be a given probability space. we will study the existence and uniqueness of a solution of the following functional equation:-

$$P\big(x(\omega)\big) = \sup_{y\in E} \big[g(\omega, x(\omega), y(\omega)) + G(x(\omega), y(\omega), P(\tau(x(\omega), y(\omega))))\big], \tag{5.2}$$

where, $x : \Omega \to D$ and $y : \Omega \to E$ represent the random state and decision vector, respectively, $P : D \to R$ is bounded, i.e., $P \in X$, $\tau : D \times E \to D$ represents the optimal return function with initial state.
Also, $g : \Omega \times D \times E \to R$, $G : D \times E \times R \to R$ are bonded functions.
Let $T : \Omega \times X \to X$ be mapping defined as

$$T(\omega, h(x(\omega))) = \sup_{y\in E} \big[g(\omega, x(\omega), y(\omega)) + G(x(\omega), y(\omega), h(\tau(x(\omega), y(\omega))))\big], \tag{5.3}$$

where, $h : D \to R$, is in X.

Theorem 5.1. *Assume that for every $(x, y) \in D \times E$, $h, k : D \to R$ and $\omega \in \Omega$ such that*

$$\begin{aligned} &\big|G(x(\omega), y(\omega), h(\tau(x(\omega), y(\omega)))) - G(x(\omega), y(\omega), k(\tau(x(\omega), y(\omega))))\big| \leq \\ &\quad \alpha(w) \frac{S(k(x(\omega)), k(x(\omega)), T(\omega, k(x(\omega)))[1 + S(h(x(\omega)), h(x(\omega)), T(\omega, h(x(\omega))))]}{1 + S(T(\omega, h(x(\omega))), T(\omega, h(x(\omega))), T(\omega, k(x(\omega))))} \\ &\quad + \beta(\omega) S(h(x(\omega)), h(x(\omega)), k(x(\omega))), \end{aligned} \tag{5.4}$$

where $\alpha, \beta : \Omega \to [0, \infty)$ with $\alpha(\omega) + \beta(\omega) < 1$.
Then the functional equation (5.2) has a unique solution in X.

Proof. Let μ be positive real number and for any $x(\omega) \in D$, $y_1(\omega), y_2(\omega) \in E$, from Equation (5.3) we have:

$$\begin{aligned} T(\omega, h(x(\omega))) &\leq g(\omega, x(\omega), y_1(\omega)) + G(x(\omega), y_1(\omega), h(\tau(x(\omega), y_1(\omega)))) + \mu \\ T(\omega, k(x(\omega))) &\leq g(\omega, x(\omega), y_2(\omega)) + G(x(\omega), y_2(\omega), k(\tau(x(\omega), y_2(\omega)))) + \mu. \end{aligned} \tag{5.5}$$

On the other hand from the definition of T and equation (5.3)

$$\begin{aligned} T(\omega, h(x(\omega))) &\geq g(\omega, x(\omega), y_2(\omega)) + G(x(\omega), y_2(\omega), h(\tau(x(\omega), y_2(\omega)))) \\ T(\omega, k(x(\omega))) &\geq g(\omega, x(\omega), y_1(\omega)) + G(x(\omega), y_1(\omega), k(\tau(x(\omega), y_1(\omega)))). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we have

$$\begin{aligned} T(\omega, h(x(\omega))) - T(\omega, k(x(\omega))) < &\ G(x(\omega), y_1(\omega), h(\tau(x(\omega), y_1(\omega)))) \\ &- G(x(\omega), y_1(\omega), k(\tau(x(\omega), y_1(\omega)))) + \mu \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} T(\omega, k(x(\omega))) - T(\omega, h(x(\omega))) < &\ G(x(\omega), y_2(\omega), k(\tau(x(\omega), y_2(\omega)))) \\ &- G(x(\omega), y_2(\omega), h(\tau(x(\omega), y_2(\omega)))) + \mu. \end{aligned} \tag{5.8}$$

From (5.1), (5.4), (5.7) and (5.8) we have

$$\begin{aligned} S(T(\omega, h(x(\omega))), T(\omega, h(x(\omega))), T(\omega, k(x(\omega)))) &= \sup_{\omega\in\Omega} \left|T(\omega, k(x(\omega))) - T(\omega, h(x(\omega)))\right| \leq \\ &\alpha(w)\frac{S(k(x(\omega)), k(x(\omega)), T(\omega, k(x(\omega)))[1 + S(h(x(\omega)), h(x(\omega)), T(\omega, h(x(\omega))))]}{1 + S(T(\omega, h(x(\omega))), T(\omega, h(x(\omega))), T(\omega, k(x(\omega))))} \\ &+ \beta(\omega)S(h(x(\omega)), h(x(\omega)), k(x(\omega)))). \end{aligned} \tag{5.9}$$

Therefore all conditions of Theorem 3.2 are satisfied. Hence the random functional equation (5.2) has a solution. □

References

[1] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393. 1, 2.11, 2.12

[2] Lj. B. Ćirić, S. N. Ješić, J. S. Ume, *On random coincidence for a pair of measurable mappings,*J. Inequal. Appl., **2006** (2006), 1–12. Article ID 81045. 1

[3] Lj. B. Ćirić, V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces,* Stochastic Anal. Appl.,, **27** (2009), 1246–1259. 1, 2

[4] B. C. Dhage, *Generalized metric space and mapping with fixed point,* Bulletin of the Calcutta Mathematical Society, **84** (1992), 329–336. 1

[5] N. V. Dung, *On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces,* Fixed Point Theory Appl., **48** (2013), 1–17. 2.4

[6] S. Gahler, *2-metrische Räume und ihre topologische Struktur,* Math. Nachr., **26** (1963), 115–148. 1

[7] M. E. Gordji, E. Akbartabar, Y. J. Cho, and M. Remezani, *Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces,* Fixed Point Theory Appl.,, **2012** (2012), 12 pages. 2, 2.13

[8] A. Gupta, *A common unique random fixed point theorems in S metric spaces,* Mathematical Theory and Modeling, **5(9)** (2015), 139–150.

[9] A. Gupta, E. Karapinar, *On random coincidence point and random coupled fixed point theorems,* Palest. J. Math., **4(2)** (2015), 348–359. 1

[10] V. Gupta, R. Deep, *Some coupled fixed point theorems in partially ordered S−metric spaces,* Miskolc Math. Notes, **16(1)** (2015), 181–194. 2

[11] C. J. Himmelberg, *Measurable relations,*Fund. Math., **87** (1975), 53–72. 1

[12] S. Itoh, *A random fixed point theorem for a multi-valued contraction mapping,*Pacific J. Math., **68** (1977), 85–90. 1

[13] B. Jiang, S. Xu, L. Shi, *Coupled coincidence points for mixed monotone random operators in partially ordered metric spaces,* Abstr. Appl. Anal.,**2014** (2014), 9 pages. 1

[14] V. Lakshmikantham, Lj. B. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces,* Nonlinear Anal., **70** (2009), 4341–4349. 1

[15] S. Mehta, A. D. Singh, *Coupled random fixed point theorems in partially ordered metric spaces,* Advances in Fixed Point Theory, **2(2)** (2012), 176–196. 1

[16] Z. Mustafa, B. Sims, *Some remarks concerning D−metric spaces.* Proceedings of the International Conferences on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, (2003), 189–198. 1

[17] S. R. Singh, R. D. Daheriya and M. Ughade, *Common random fixed point theorems for contractions of rational type in ordered metric spaces,* International Journal of Advanced Mathematical Sciences ,**4(2)** (2016), 37–43. 2

[18] S. R. Singh, M. Ughade and R. D. Daheriya, *Some fixed point results in ordered s-metric spaces,* Asian Research Journal of Mathematics ,**1(3)** (2016), 1–19. 2, 2

[19] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear and Convex Anal., **7(2)** (2006), 289–297.

[20] J. J. Nieto, A. Ouahab and R. Rodríguez-López, *Random fixed point theorems in partially ordered metric spaces,* Fixed Point Theory Appl., **2016** (2016), 19 pages. 1

[21] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order, **22** (2005), 223–239. 1

[22] J. J. Nieto, R. Rodríguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,* Acta Math. Sin. Engl. Ser., **23** (2007), 2205–2212. 1

[23] N. S. Papageorgiou, *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proc. Amer. Math. Soc., **97** (1986), 507–514. 1

[24] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., **132(5)** (2004), 1435–1443. 1

[25] J. Rocha, B. Rzepka and K. Sadarangani, *Fixed point theorems for contraction of rational type with PPF dependence in Banach spaces,* J. Funct. Spaces, **2014** (2014), 8 pages. 2, 2.14, 2.15, 2.17, 2.18

[26] S. Sedghi, N. Shobe and A. Aliouche, *A generalization of fixed point theorems in S-metric spaces,* Mat. Bech., **64(3)** (2012), 258–266. 1, 2.1, 2.3, 2.5, 2.6, 2.7

[27] V. M. Sehgal, S. P. Singh, *On random approximations and a random fixed point theorem for set valued mappings,* Proc. Amer. Math. Soc., **95** (1985), 91–94. 1

[28] W. Shatanawi, Z. Mustafa, *On coupled random fixed point results in partially ordered metric spaces,* Matemat. Bech., **64(2)** (2012), 139–146. 1

[29] R. Shrivastava, R. Bhardwaj, M. Sharma, *On quadruple random fixed point theorems in partially ordered metric spaces,* Journal of Information Engineering and Applications, **4(11)** (2014), 42–52. 1

[30] D. H. Wagner, *Survey of measurable selection theorems,* SIAM J. Control Optim., **15** (1977), 859–903. 1

[31] X.-H. Zhu and J.-Z. Xiao, *Random periodic point and fixed point results for random monotone mappings in ordered polish spaces,* Fixed Point Theory Appl., **2010** (2010), 13 pages. 1