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FUZZY PREUNIFORM STRUCTURE BASED ON WAY BELOW RELATION

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In this paper, a preuniform structure based on way below relation (or L-fuzzifying preuniform structure) was defined and their properties were studied. Also, the concept of interior and closure operators in the L-fuzzifying setting were established. Furthermore, the relation between L-fuzzifying preuniform and L-fuzzifying topological spaces were explained. Finally, the continuity of L-fuzzifying preuniform spaces was studied.

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1. INTRODUCTION

There is no doubt that scientists worked hard to introduce a new concept of uniform space among of them in [8] kelley who said that a uniformity for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ satisfies some condition. He also said that there may be different uniformities for a set X. The largest of these is the family of all those subsets of $X \times X$ which contain Δ the diagonal and the smallest is the family whose only member is $X \times X$. Also, in [2] Badard defined a fuzzy preuniform structure and studied some of its properties. Furthermore, in [13, 14] the authors defined L-fuzzy preuniform space and L-fuzzy quasi-uniform spaces by using $(L, \leq, \odot, \longrightarrow, \oplus, *)$ is a strictly two-sided commutative quantale with an order reversing involution $*$. In [6, 12] the authors defined and studied uniformites on fuzzy topological spaces. In this paper was organized as follow: In section 2, the notion of fuzzy preuniform space based on way below relation was established and some of its properties were studied. Furthermore, the concepts of interior and closure operators relative to L-fuzzifying preuniform space were investigated. In section 3, the L-fuzzifying preuniform topology was studied. In section 4, the L-fuzzifying preuniform continuity was introduced.

Throughout this work $L = (L, \leq, \wedge, \vee, ')$ is a completely distributive complete lattice with an order reversing involution $'$, i.e,$(L, \leq, \wedge, \vee, ')$ is a complete lattice, for every $i \in I$ and for $A_i \subseteq L$,

$\bigwedge_{i\in I} \vee A_i = \bigvee_{\Psi \in \prod_{i\in I} A_i} \bigwedge_{i\in I} \Psi(i)$. and $': L \longrightarrow L$ is a function such that for every $\alpha, \beta \in L$, $(\alpha')' = \alpha$ and if $\alpha \leq \beta$, then $\alpha' \geq \beta'$. The upper(resp.lower) universal element of L will denoted by $\top$ (resp.$\bot$).

2. PRELIMINARIES

Definition 2.1.[3] Let L be a complete lattice. We say that x is way below y, in symbols $x \ll y$, if for any directed subset $D \subseteq L$ the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

Proposition 2.2.[3] In a complete lattice L one has the following statements for all $u, x, y, z \in L$:

(i) $x \ll y$ implies $x \leq y$;

(ii) $u \leq x \ll y \leq z$ implies $u \ll z$;

(iii) $x \ll z$ and $y \ll z$ together imply $x \vee y \ll z$;

(iv) $\bot \ll x$.

(v) $x \ll y$ and $y \leq z$ implies $x \ll z$.

(vi) If $\top \not\ll \top$, then $\bigvee_{\alpha < \top} \alpha = \top$.

Definition 2.3.[5] Let X be a nonempty set, L be a complete lattice and $\tau: 2^X \longrightarrow L$ be a function that satisfies the following conditions:

(O1) $\tau(X) = \tau(\phi) = 1$;

(O2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, for all $A, B \subseteq 2^X$;

(O3) for each $\{A_j : j \in J\} \subseteq 2^X$, $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$.

Then τ is called an L-fuzzifying topology on X and the pair (X, τ) is called an L-fuzzifying topological space.

Definition 2.4.[5] Let (X, τ_1) and (Y, τ_2) be two L-fuzzifying topological spaces.

A function $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is called an L-fuzzifying continuous if for all $B \in 2^Y$, $\tau_2(B) \leq \tau_1(f^{-1}(B))$.

3. *L*-FUZZIFYING PREUNIFORM STRUCTURE

Definition 3.1. A function $\mathcal{U}: 2^{X\times X} \longrightarrow L$ is called an L-fuzzifying preuniform structure on X if it satisfies the following axioms:

PU1 : For any $u \in 2^{X\times X}$, if $\mathcal{U}(u) \neq \bot$, then $\Delta \subseteq u$.

PU2 : If $\mathcal{U}(u) \ll r$ and $u \subseteq v$, then $\mathcal{U}(v) \ll r$, where $r \in L - \{\perp\}$.

The pair $(X, \mathcal{U})$ is called an L-fuzzifying preuniform space.

An L-fuzzifying preuniform is called of type **D** for any $u_1, u_2 \in 2^{X \times X}$.

PU3 : If $(\mathcal{U}(u_1) \wedge \mathcal{U}(u_2)) \ll r$, then $\mathcal{U}(u_1 \cap u_2) \ll r$.

An L-fuzzifying preuniform is called symmetrical for any $u \in 2^{X \times X}$.

PU4 : If $\mathcal{U}(u) \ll r$, then $\mathcal{U}(u^{-1}) \ll r$.

Proposition 3.2. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ two L-fuzzifying preuniform structure. Then satisfies the following:-

(1) $\mathcal{U}_1 \wedge \mathcal{U}_2$ and $\mathcal{U}_1 \vee \mathcal{U}_2$ are L-fuzzifying preuniform structures.

(2) When $\mathcal{U}_1$, $\mathcal{U}_2$ are symmetrical, then so are $\mathcal{U}_1 \wedge \mathcal{U}_2$ and $\mathcal{U}_1 \vee \mathcal{U}_2$.

(3) When $\mathcal{U}_1$, $\mathcal{U}_2$ are of type **D**, then so is $\mathcal{U}_1 \wedge \mathcal{U}_2$.

Proof.

(1) First, we prove that $\mathcal{U}_1 \wedge \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

PU1 : If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \neq \perp$, then $\mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \neq \perp$. So, $\mathcal{U}_1(u) \neq \perp$ and $\mathcal{U}_2(u) \neq \perp$. Hence $\Delta \subseteq u$ for any $u \in 2^{X \times X}$.

PU2: Suppose $u \subseteq v$. If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \ll r$, then $\mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \ll r$.

So, $\mathcal{U}_1(u) \ll r$ or $\mathcal{U}_2(u) \ll r$. Hence $\mathcal{U}_1(v) \ll r$ or $\mathcal{U}_2(v) \ll r$. Thus $(\mathcal{U}_1 \wedge \mathcal{U}_2)(v) = \mathcal{U}_1(v) \wedge \mathcal{U}_2(v) \ll r$. Therefore $\mathcal{U}_1 \wedge \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

Second, we prove that $\mathcal{U}_1 \vee \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

PU1 : If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) \neq \perp$, then $\mathcal{U}_1(u) \vee \mathcal{U}_2(u) \neq \perp$. So, $\mathcal{U}_1(u) \neq \perp$ or $\mathcal{U}_2(u) \neq \perp$. Hence $\Delta \subseteq u$ for any $u \in 2^{X \times X}$

PU2: Suppose $u \subseteq v$. If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) \ll r$, then $\mathcal{U}_1(u) \vee \mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u) \ll r$ and $\mathcal{U}_2(u) \ll r$. Hence $\mathcal{U}_1(v) \ll r$ and $\mathcal{U}_2(v) \ll r$. Thus $(\mathcal{U}_1 \vee \mathcal{U}_2)(v) = \mathcal{U}_1(v) \vee \mathcal{U}_2(v) \ll r$. Therefore $\mathcal{U}_1 \vee \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

(2) Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical.

If $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u) = \mathcal{U}_1(u) \wedge \mathcal{U}_2(u) \ll r$, then $\mathcal{U}_1(u) \ll r$ or $\mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u^{-1}) \ll r$ or $\mathcal{U}_2(u^{-1}) \ll r$ which implies $(\mathcal{U}_1(u^{-1}) \wedge \mathcal{U}_2(u^{-1})) = (\mathcal{U}_1 \wedge \mathcal{U}_2)(u^{-1}) \ll r$. Hence $\mathcal{U}_1 \wedge \mathcal{U}_2$ is symmetrical.

If $(\mathcal{U}_1 \vee \mathcal{U}_2)(u) = \mathcal{U}_1(u) \vee \mathcal{U}_2(u) \ll r$, then $\mathcal{U}_1(u) \ll r$ and

$\mathcal{U}_2(u) \ll r$. So, $\mathcal{U}_1(u^{-1}) \ll r$ and $\mathcal{U}_2(u^{-1}) \ll r$ which implies $(\mathcal{U}_1(u^{-1}) \vee \mathcal{U}_2(u^{-1})) = (\mathcal{U}_1 \vee \mathcal{U}_2)(u^{-1}) \ll r$. Hence $\mathcal{U}_1 \vee \mathcal{U}_2$ is symmetrical.

(3) Suppose $\mathcal{U}_1$, $\mathcal{U}_2$ are of type **D**.

If $((\mathcal{U}_1 \wedge \mathcal{U}_2)(u) \wedge (\mathcal{U}_1 \wedge \mathcal{U}_2)(v)) \ll r$, then $((\mathcal{U}_1(u) \wedge \mathcal{U}_1(v)) \wedge (\mathcal{U}_2(u) \wedge \mathcal{U}_2(v))) \ll r$. So, $(\mathcal{U}_1(u) \wedge \mathcal{U}_1(v)) \ll r$ or $(\mathcal{U}_2(u) \wedge \mathcal{U}_2(v)) \ll r$. Hence $(\mathcal{U}_1(u \cap v) \wedge \mathcal{U}_2(u \cap v)) \ll r$. Then $(\mathcal{U}_1 \wedge \mathcal{U}_2)(u \cap v) \ll r$. Therefore, $\mathcal{U}_1 \wedge \mathcal{U}_2$ is of type **D**.

Definition 3.3. Let $\mathcal{U}_1$, $\mathcal{U}_2$ are two L-fuzzifying preuniform structures on X . We denote $\mathcal{U}_1 \odot \mathcal{U}_2: 2^{X\times X} \longrightarrow L$ defined by $(\mathcal{U}_1 \odot \mathcal{U}_2)(u) = \bigvee\{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) | v \cap w \subseteq u\}$

Proposition 3.4. Let $\mathcal{U}_1$, $\mathcal{U}_2$ are two L-fuzzifying preuniform structures on X. Then

(i) $\mathcal{U}_1 \odot \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

(ii) When $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical, then so is $\mathcal{U}_1 \odot \mathcal{U}_2$.

(iii) When $\mathcal{U}_1$ and $\mathcal{U}_2$ are of type **D**, then so is $\mathcal{U}_1 \odot \mathcal{U}_2$.

Proof.

(i) Suppose $\mathcal{U}_1$, $\mathcal{U}_2$ are two L-fuzzifying preuniform structures on X.

PU1 : If $\mathcal{U}_1 \odot \mathcal{U}_2(u) = \bigvee\{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) | v \cap w \subseteq u\} \neq \perp$, then there exist $v, w \in 2^{X\times X}$ such that $\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) \neq \perp$ and $v \cap w \subseteq u$. So, $\mathcal{U}_1(v) \neq \perp$ and $\mathcal{U}_2(w) \neq \perp$. Hence $\Delta \subseteq v \cap w \subseteq u$.

PU2 : Suppose $\mathcal{U}_1 \odot \mathcal{U}_2(u) \ll r$. Then $\bigvee\{\mathcal{U}_1(x) \wedge \mathcal{U}_2(y) | x \cap y \subseteq u\} \ll r$. Since $u \subseteq v$, then

$\bigvee\{\mathcal{U}_1(x) \wedge \mathcal{U}_2(y) | x \cap y \subseteq v\} \ll r$. So, $\mathcal{U}_1 \odot \mathcal{U}_2(v) \ll r$. Therefore $\mathcal{U}_1 \odot \mathcal{U}_2$ is an L-fuzzifying preuniform structure.

(ii) Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are symmetrical.

If $(\mathcal{U}_1 \odot \mathcal{U}_2)(u) = \bigvee\{\mathcal{U}_1(v) \wedge \mathcal{U}_2(w) | v \cap w \subseteq u\} \ll r$, then $(\mathcal{U}_1(v) \wedge \mathcal{U}_2(w)) \ll r$ for all $v \cap w \subseteq u$. So, $\mathcal{U}_1(v) \ll r$ or $\mathcal{U}(w) \ll r$ which implies $\mathcal{U}_1(v^{-1}) \ll r$ or $\mathcal{U}(w^{-1}) \ll r$. Thus $(\mathcal{U}_1(v^{-1}) \wedge \mathcal{U}_2(w^{-1})) \ll r$ for all $v^{-1} \cap w^{-1} \subseteq u^{-1}$.

Then $\bigvee\{\mathcal{U}_1(v^{-1}) \wedge \mathcal{U}_2(w^{-1}) | v^{-1} \cap w^{-1} \subseteq u^{-1}\} = (\mathcal{U}_1 \odot \mathcal{U}_2)(u^{-1}) \ll r$. Therefore $\mathcal{U}_1 \odot \mathcal{U}_2$ is symmetrical.

(iii) Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are **D** and

$((\mathcal{U}_1 \odot \mathcal{U}_2)(u) \wedge (\mathcal{U}_1 \odot \mathcal{U}_2)(v)) = \big((\bigvee\{\mathcal{U}_1(x_1) \wedge \mathcal{U}_2(y_1) | x_1 \cap$

$y_1 \subseteq u\})\wedge(\vee\{\mathcal{U}_1(x_2)\wedge\mathcal{U}_2(y_2)|x_2\cap y_2\subseteq v\})\big) \ll$. Then $(\vee\{(\mathcal{U}_1(x_1)\wedge\mathcal{U}_1(x_2))\wedge(\mathcal{U}_2(y_1)\wedge\mathcal{U}_2(y_2))|(x_1\cap x_2)\cap(y_1\cap y_2)\subseteq u\cap v\}) \ll r$. So, for some $x_1, x_2, y_1, y_2 \in 2^{X\times X}$ such that $(\mathcal{U}_1(x_1)\wedge\mathcal{U}_1(x_2)) \ll r$ or $(\mathcal{U}_2(y_1)\wedge\mathcal{U}_2(y_2)) \ll r$. Hence $\mathcal{U}_1(x_1\cap x_2) \ll r$ or $\mathcal{U}_2(y_1\cap y_2) \ll r$. Then $(\vee\{\mathcal{U}_1(x_1\cap x_2)\wedge\mathcal{U}_2(y_1\cap y_2)|(x_1\cap x_2)\cap(y_1\cap y_2)\subseteq u\cap v\}) = (\vee\{\mathcal{U}_1(x)\wedge\mathcal{U}_2(y)|x\cap y\subseteq u\cap v\}) = \mathcal{U}_1\odot\mathcal{U}_2(u\cap v) \ll r$, where $x = x_1\cap x_2$ and $y = y_1\cap y_2$. Therefore, $(\mathcal{U}_1\odot\mathcal{U}_2)$ is of type **D**.

Theorem 3.5. Let $(X,\mathcal{U})$ be an L-fuzzifying preuniform space. Define the function $\mathcal{I}_{\mathcal{U}}: 2^X\times(L-\{\top\})\longrightarrow 2^X$ as follows:-
$\mathcal{I}_{\mathcal{U}}(A,r) = \cup\left\{D\in 2^X\,|\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right)\ll r'\right\}$.

Then $\mathcal{I}_{\mathcal{U}}$ satisfies the following axioms:

(1) $\mathcal{I}_{\mathcal{U}}(X,r) = X$; $\mathcal{I}_{\mathcal{U}}(\phi,r) = \phi$.

(2) $\mathcal{I}_{\mathcal{U}}(A,r)\subseteq A$.

(3) If $A\subseteq B$, then $\mathcal{I}_{\mathcal{U}}(A,r)\subseteq\mathcal{I}_{\mathcal{U}}(B,r)$.

(4) $\mathcal{I}_{\mathcal{U}}(A\cap B,r)\subseteq\mathcal{I}_{\mathcal{U}}(A,r)\cap\mathcal{I}_{\mathcal{U}}(B,r)$, but if $\mathcal{U}$ is of type **D** the equality holds.

(5) If $r_1\leq r_2$, then $\mathcal{I}_{\mathcal{U}}(A,r_1)\supseteq\mathcal{I}_{\mathcal{U}}(A,r_2)$.

The function $\mathcal{I}_{\mathcal{U}}$ is called an L-fuzzifying preuniform interior operator.

Proof.

(1) Since $w[D]\subseteq X$, then $\mathcal{I}_{\mathcal{U}}(X,r) = \cup\left\{D\in 2^X\,|\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq X}(\mathcal{U}(w))'\right)\ll r'\right\} = X$. So, $\mathcal{I}_{\mathcal{U}}(X,r) = X$.

Since $w[D]\subseteq\phi$. Then $w[D] = D = \phi$ for all r. So,

$$\mathcal{I}_{\mathcal{U}}(\phi,r) = \cup\left\{D\in 2^X\,|\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq\phi}(\mathcal{U}(w))'\right)\ll r'\right\} = \cup\phi = \phi$$

(2) Suppose $x\in\mathcal{I}_{\mathcal{U}}(A,r)$. Then there exist $D\in 2^X$ such that $x\in D$, $\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right)\ll r'$, where $x\in D\subseteq w[D]\subseteq A$. Hence $x\in A$. So $\mathcal{I}_{\mathcal{U}}(A,r)\subseteq A$.

(3) Suppose $A\subseteq B$. Then $\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right)\geq\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq B}(\mathcal{U}(w))'\right)$.

When $\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right) \ll r'$, we have $\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq B}(\mathcal{U}(w))'\right) \ll r'$. Hence $\mathcal{I}_{\mathcal{U}}(A,r) \subseteq \mathcal{I}_{\mathcal{U}}(B,r)$.

(4) It is clear from (3) that $A\cap B\subseteq A$ implies $\mathcal{I}_{\mathcal{U}}(A\cap B,r)\subseteq \mathcal{I}_{\mathcal{U}}(A,r)$ and $A\cap B\subseteq B$ implies $\mathcal{I}_{\mathcal{U}}(A\cap B,r)\subseteq \mathcal{I}_{\mathcal{U}}(B,r)$. Hence $\mathcal{I}_{\mathcal{U}}(A\cap B,r)\subseteq \mathcal{I}_{\mathcal{U}}(A,r)\cap \mathcal{I}_{\mathcal{U}}(B,r)$.

Now, suppose that $x\in \mathcal{I}_{\mathcal{U}}(A,r)\cap \mathcal{I}_{\mathcal{U}}(B,r)$. Then $x\in \mathcal{I}_{\mathcal{U}}(A,r)$ and $x\in \mathcal{I}_{\mathcal{U}}(B,r)$. Hence there exist $D_1, D_2\in 2^X$ such that

$x\in D_1$, $\left(\bigwedge_{w_1\in 2^{X\times X}, w_1[D_1]\subseteq A}(\mathcal{U}(w_1))'\right) \ll r'$ and $x\in D_2$, $\left(\bigwedge_{w_2\in 2^{X\times X}, w_2[D_2]\subseteq B}(\mathcal{U}(w_2))'\right) \ll r'$. So, $x\in D_1\cap D_2$ and $\left(\left(\bigwedge_{w_1\in 2^{X\times X}, w_1[D_1]\subseteq A}(\mathcal{U}(w_1))'\right) \ll r'\right) \wedge \left(\left(\bigwedge_{w_2\in 2^{X\times X}, w_2[D_2]\subseteq B}(\mathcal{U}(w_2))'\right) \ll r'\right)$. Then $x\in D_1\cap D_2$ such that $\left(\bigwedge_{w_1\in 2^{X\times X}, w_1[D_1]\subseteq A}(\mathcal{U}(w_1))'\right) \vee \left(\bigwedge_{w_2\in 2^{X\times X}, w_2[D_2]\subseteq B}(\mathcal{U}(w_2))'\right) \ll r'$. Thus

$\left(\bigwedge_{w_1\in 2^{X\times X}, w_1[D_1]\subseteq A, w_2\in 2^{X\times X}, w_2[D_2]\subseteq B}(\mathcal{U}(w_1)\wedge \mathcal{U}(w_2)) \ll r'\right)$. Since $\mathcal{U}$ is of type **D**, then

$\left(\bigwedge_{w_1\cap w_2\in 2^{X\times X}, (w_1\cap w_2)[D_1\cap D_2]\subseteq A\cap B}(\mathcal{U}(w_1\cap w_2))' \ll r'\right)$. Therefore $x\in D$ such that $\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A\cap B}(\mathcal{U}(w))' \ll r'\right)$ which implies $x\in \mathcal{I}_{\mathcal{U}}(A\cap B,r)$. Hence $\mathcal{I}_{\mathcal{U}}(A\cap B,r) = \mathcal{I}_{\mathcal{U}}(A,r)\cap \mathcal{I}_{\mathcal{U}}(B,r)$.

(5) Suppose $r_1 \le r_2$. Then $\mathcal{I}_{\mathcal{U}}(A,r_2) = \cup\left\{D\in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right) \ll r_2'\right\}$. Since $r_2' \le r_1'$, then

$\left(\bigwedge_{w\in 2^{X\times X}, w[D]\subseteq A}(\mathcal{U}(w))'\right) \ll r_1'$. So, $\mathcal{I}_{\mathcal{U}}(A,r_1) \supseteq \mathcal{I}_{\mathcal{U}}(A,r_2)$.

Theorem 3.6. Let $(X,\mathcal{U})$ be an L-fuzzifying preuniform space. Define the function $\mathcal{C}_{\mathcal{U}}: 2^X \times (L-\{\top\}) \longrightarrow 2^X$ as follows:-

$$\mathcal{C}_{\mathcal{U}}(A,r) = \bigcap\left\{D \in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\}.$$

Then $\mathcal{C}_{\mathcal{U}}$ satisfies the following statements:

(1) $\mathcal{C}_{\mathcal{U}}(\phi,r) = \phi$; $\mathcal{C}_{\mathcal{U}}(X,r) = X$.

(2) $\mathcal{C}_{\mathcal{U}}(A,r) \supseteq A$.

(3) If $A \subseteq B$, then $\mathcal{C}_{\mathcal{U}}(A,r) \subseteq \mathcal{C}_{\mathcal{U}}(B,r)$.

(4) $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r)$, but if $\mathcal{U}$ is of type **D** the equality holds.

(5) If $r_1 \le r_2$, then $\mathcal{C}_{\mathcal{U}}(A,r_1) \subseteq \mathcal{C}_{\mathcal{U}}(A,r_2)$.

The function $\mathcal{C}_{\mathcal{U}}$ is called an L-fuzzifying preuniform closure operator.

Proof.

(1) Since $\phi \subseteq w[D]$, then $\mathcal{C}_{\mathcal{U}}(\phi,r) = \bigcap\left\{D \in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, \phi\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\} = \phi$. So, $\mathcal{C}_{\mathcal{U}}(\phi,r) = \phi$. Again, since $X \subseteq w[D]$, then $D = X$ for all r. So, $\mathcal{C}_{\mathcal{U}}(X,r) = \bigcap\left\{D \in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, X\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\} = \bigcap X = X$

(2) Suppose $x \in \mathcal{C}_{\mathcal{U}}(A,r)$. Then for all $D \in 2^X$ such that $x \in D$ we have $\left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'$. So, $\mathcal{C}_{\mathcal{U}}(A,r) \supseteq A$.

(3) Suppose $A \subseteq B$. Then $\left(\bigwedge_{w\in 2^{X\times X}, B\subseteq w[D]} (\mathcal{U}(w))'\right) \ge \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right)$. When $\left(\bigwedge_{w\in 2^{X\times X}, B\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'$, then $\left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'$. Hence $\mathcal{C}_{\mathcal{U}}(A,r) \subseteq \mathcal{C}_{\mathcal{U}}(B,r)$.

(4) It is clear from (3) when $A \cup B \supseteq A$, then $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r)$ and when $A \cup B \supseteq B$, then $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(B,r)$. Hence $\mathcal{C}_{\mathcal{U}}(A \cup B,r) \supseteq \mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r)$. Conversely, $\mathcal{C}_{\mathcal{U}}(A,r) \cup \mathcal{C}_{\mathcal{U}}(B,r) = \left(\bigcap\left\{D_1 \in 2^X \mid \left(\bigwedge_{w_1\in 2^{X\times X}, A\subseteq w_1[D_1]} (\mathcal{U}(w_1))'\right) \ll r'\right\}\right) \cup \left(\bigcap\left\{D_2 \in 2^X \mid \left(\bigwedge_{w_2\in 2^{X\times X}, B\subseteq w_2[D_2]} (\mathcal{U}(w_2))'\right) \ll r'\right\}\right)$.

$= \cap\left\{D_1 \cup D_2 \in 2^X \,\middle|\, \left(\left(\bigwedge_{w_1\in 2^{X\times X}, A\subseteq w_1[D_1]} (\mathcal{U}(w_1))'\right) \ll r'\right) or \left(\left(\bigwedge_{w_2\in 2^{X\times X}, B\subseteq w_2[D_2]} (\mathcal{U}(w_2))'\right) \ll r'\right)\right\} = \cap\left\{D_1 \cup D_2 \in 2^X \,\middle|\, \left(\left(\bigwedge_{w_1\in 2^{X\times X}, A\subseteq w_1[D_1]} (\mathcal{U}(w_1))'\right) \wedge \left(\bigwedge_{w_2\in 2^{X\times X}, B\subseteq w_2[D_2]} (\mathcal{U}(w_2))'\right)\right) \ll r'\right\} = \cap\left\{D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1\in 2^{X\times X}, A\subseteq w_1[D_1], w_2\in 2^{X\times X}, B\subseteq w_2[D_2]} (\mathcal{U}(w_1) \vee \mathcal{U}(w_2)) \ll r'\right)\right\} \supseteq$
$\cap\left\{D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1\in 2^{X\times X}, A\subseteq w_1[D_1], w_2\in 2^{X\times X}, B\subseteq w_2[D_2]} (\mathcal{U}(w_1) \wedge \mathcal{U}(w_2)) \ll r'\right)\right\} \supseteq \cap\left\{D_1 \cup D_2 \in 2^X \,\middle|\, \left(\bigwedge_{w_1\cap w_2\in 2^{X\times X}, A\cup B\subseteq (w_1\cap w_2)[D_1\cup D_2]} (\mathcal{U}(w_1 \cap w_2))' \ll r'\right)\right\} = \cap\{D \in 2^X | \left(\bigwedge_{w\in 2^{X\times X}, A\cup B\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\} = \mathcal{C}_{\mathcal{U}}(A \cup B, r)$, where $(w_1 \cap w_2)[D_1 \cup D_2] \subseteq w_1[D_1] \cap w_2[D_2]$. Hence $\mathcal{C}_{\mathcal{U}}(A \cup B, r) = \mathcal{C}_{\mathcal{U}}(A, r) \cup \mathcal{C}_{\mathcal{U}}(B, r)$.

(5) Suppose $r_1 \leq r_2$. Then $\mathcal{C}_{\mathcal{U}}(A, r_2) = \cap\left\{D \in 2^{X\times X} | \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r_2'\right\}$. Since $r_2' \leq r_1'$, then
$\left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r_1'$. So, $\mathcal{C}_{\mathcal{U}}(A, r_1) \subseteq \mathcal{C}_{\mathcal{U}}(A, r_2)$.

4. *L*-FUZZIFYING PREUNIFORM TOPOLOGY

Theorem 4.1. Let $(X, \mathcal{U})$ be an L-fuzzifying preuniform space of type **D** and $\top \not\ll \top$. Define a map $\tau_{\mathcal{U}}: 2^X \longrightarrow L$ by
$\tau_{\mathcal{U}}(A) = \sup\{r \in (L - \{\top\}) | \mathcal{I}_{\mathcal{U}}(A, r) = A\}$
Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Proof.

(O1) Since $\mathcal{I}_{\mathcal{U}}(X, r) = X$ and $\mathcal{I}_{\mathcal{U}}(\phi, r) = \phi$, for all $r \in (L - \{\top\})$. Then $\tau_{\mathcal{U}}(X) = \tau_{\mathcal{U}}(\phi) = \top$.

(O2) Suppose there exist $A, B \in 2^X$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) > t > \tau_{\mathcal{U}}(A \cap B)$. Then $\tau_{\mathcal{U}}(A) > t$ and $\tau_{\mathcal{U}}(B) > t$. So there exist $r_1, r_2 > t$ such that $\mathcal{I}_{\mathcal{U}}(A, r_1) = A$ and $\mathcal{I}_{\mathcal{U}}(B, r_2) = B$. Put $r = r_1 \wedge r_2$ and from Theorem 3.5 (4) and (5), we have $\mathcal{I}_{\mathcal{U}}(A \cap B, r) =$

$\mathcal{I}_{\mathcal{U}}(A,r) \cap \mathcal{I}_{\mathcal{U}}(B,r) \supseteq \mathcal{I}_{\mathcal{U}}(A,r_1) \cap \mathcal{I}_{\mathcal{U}}(B,r_2) = A \cap B$. So, $\mathcal{I}_{\mathcal{U}}(A \cap B,r) = A \cap B$. Thus, $\tau_{\mathcal{U}}(A \cap B) \geq r$ and $r > t$ and this is a contradiction. Hence $\tau_{\mathcal{U}}(A \cap B) \geq \tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B)$.

(O3) Suppose there exists a family $\{A_i \in 2^X | i \in \Gamma\}$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) < t < \bigwedge_{i\in\Gamma} \tau_\eta(A_i)$. Since $\bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i) > t$ for each $i \in \Gamma$, then there exist $r_i > t$ such that $\mathcal{I}_{\mathcal{U}}(A_i,r_i) = A_i$. Put $r = \bigwedge_{i\in\Gamma} r_i$. We have $\mathcal{I}_{\mathcal{U}}(\bigcup_{i\in\Gamma} A_i,r) \supseteq (\bigcup_{i\in\Gamma} \mathcal{I}_{\mathcal{U}}(A_i,r)) \supseteq (\bigcup_{i\in\Gamma} \mathcal{I}_{\mathcal{U}}(A_i,r_i)) = \bigcup_{i\in\Gamma} A_i$. So, $\mathcal{I}_{\mathcal{U}}(\bigcup_{i\in\Gamma} A_i,r) = \bigcup_{i\in\Gamma} A_i$. Thus, $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) \geq r$ and $r > t$ and this is a contradiction. Hence $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i)$. Thus, $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Theorem 4.2. Let $(X,\mathcal{U})$ is an L-fuzzifying preuniform space of type **D** and $\top \ll \top$. Define a map $\tau_{\mathcal{U}}: 2^X \longrightarrow L$ by
$\tau_{\mathcal{U}}(A) = \sup\{r \in (L - \{\top\}) | \mathcal{C}_{\mathcal{U}}(A^c,r) = A^c\}$

Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

proof

(O1) Since $\mathcal{C}_{\mathcal{U}}(X,r) = X$ and $\mathcal{C}_{\mathcal{U}}(\phi,r) = \phi$, for all $r \in (L - \{\top\})$, then $\tau_{\mathcal{U}}(X) = \tau_{\mathcal{U}}(\phi) = \top$.

(O2) Suppose there exist $A,B \in 2^X$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) > t > \tau_{\mathcal{U}}(A \cap B)$. Then $\tau_{\mathcal{U}}(A) > t$ and $\tau_{\mathcal{U}}(B) > t$. So there exist $r_1, r_2 > t$ such that $\mathcal{C}_{\mathcal{U}}(A^c,r_1) = A^c$ and $\mathcal{C}_{\mathcal{U}}(B^c,r_2) = B^c$. Put $r = r_1 \wedge r_2$. From theorem 3.6 (4) and (5), we have $\mathcal{C}_{\mathcal{U}}(A^c \cup B^c,r) = \mathcal{C}_{\mathcal{U}}(A^c,r) \cup \mathcal{C}_{\mathcal{U}}(B^c,r) \subseteq \mathcal{C}_{\mathcal{U}}(A^c,r_1) \cup \mathcal{C}_{\mathcal{U}}(B^c,r_2) = A^c \cup B^c$. So, $\mathcal{C}_{\mathcal{U}}((A \cap B)^c,r) = (A \cap B)^c$. Thus, $\tau_{\mathcal{U}}(A \cap B) \geq r > t$ and this is a contradiction. Hence $\tau_{\mathcal{U}}(A \cap B) \geq \tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B)$.

(O3) Suppose there exists a family $\{A_i \in 2^X | i \in \Gamma\}$ and $t \in L - \{\top\}$ such that $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) < t < \bigwedge_{i\in\Gamma} \tau_\eta(A_i)$. Since $\bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i) > t$ for each $i \in \Gamma$, then there exists $r_i > t$ such that $\mathcal{C}_{\mathcal{U}}((A_i)^c,r_i) = (A_i)^c$. Put $r = \bigwedge_{i\in\Gamma} r_i$. We have $\mathcal{C}_{\mathcal{U}}(\bigcap_{i\in\Gamma} (A_i)^c,r) \subseteq (\bigcap_{i\in\Gamma} \mathcal{C}_{\mathcal{U}}((A_i)^c,r)) \subseteq (\bigcap_{i\in\Gamma} \mathcal{C}_{\mathcal{U}}((A_i)^c,r_i)) = \bigcap_{i\in\Gamma} (A_i)^c$.

So, $\mathcal{C}_{\mathcal{U}}((\bigcup_{i\in\Gamma} A_i)^c,r) = (\bigcup_{i\in\Gamma} A_i)^c$. Thus, $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) \geq r > t$ and this is a contradiction.

Hence $\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i)$. Thus, $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Theorem 4.3. Let $(X,\mathcal{U})$ is an L-fuzzifying preuniform space of type **D**. Define a map $\tau_{\mathcal{U}}: 2^X \longrightarrow L$ by

$$\tau_{\mathcal{U}}(A) = \bigwedge_{x\in A} \bigvee_{u[x]\subseteq A} \mathcal{U}(u)$$

Then $\tau_{\mathcal{U}}$ is an L-fuzzifying topology on X.

Proof.

(O1) It is clear $\tau_{\mathcal{U}}(X) = \top$.

(O2)

$$\tau_{\mathcal{U}}(A) \wedge \tau_{\mathcal{U}}(B) = \left(\bigwedge_{x\in A} \bigvee_{u_1[x]\subseteq A} \mathcal{U}(u_1)\right) \wedge \left(\bigwedge_{x\in B} \bigvee_{u_2[x]\subseteq B} \mathcal{U}(u_2)\right) \leq \bigwedge_{x\in A, x\in B} \bigvee_{u_1[x]\subseteq A, u_2[x]\subseteq B} (\mathcal{U}(u_1) \wedge \mathcal{U}(u_2)) \leq \bigwedge_{x\in A\cap B} \bigvee_{u_1\cap u_2[x]\subseteq A\cap B} \mathcal{U}(u_1 \cap u_2) = \bigwedge_{x\in A\cap B} \bigvee_{u[x]\subseteq A\cap B} \mathcal{U}(u) = \tau_{\mathcal{U}}(A\cap B)$$

(O3)

$$\tau_{\mathcal{U}}\left(\bigcup_{i\in\Gamma} A_i\right) = \bigwedge_{x\in \bigcup_{i\in\Gamma} A_i} \bigvee_{u[x]\subseteq \bigcup_{i\in\Gamma} A_i} \mathcal{U}(u) = \bigwedge_{i\in\Gamma}\left(\bigwedge_{x\in A_i} \bigvee_{u[x]\subseteq \bigcup_{i\in\Gamma} A_i} \mathcal{U}(u)\right) \geq \bigwedge_{i\in\Gamma}\left(\bigwedge_{x\in A_i} \bigvee_{u[x]\subseteq A_i} \mathcal{U}(u)\right) = \bigwedge_{i\in\Gamma} \tau_{\mathcal{U}}(A_i).$$

5. THE CONTINUITY OF L-FUZZIFYING PREUNIFORM SPACES

Definition 5.1. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two L-fuzzifying preuniform spaces. Then $f: (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is called an L-fuzzifying preuniform continuous function if $\mathcal{V}(v) \ll r$, then $\mathcal{U}((f\times f)^{-1}(v)) \ll r \quad \forall v \in 2^{Y\times Y}$.

Lemma 5.2. Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ be three L-fuzzifying preuniform spaces. If $f: (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ and $g: (Y,\mathcal{V}) \longrightarrow (Z,\mathcal{W})$ are L-fuzzifying preuniform continuous functions, then $g\circ f: (X,\mathcal{U}) \longrightarrow (Z,\mathcal{W})$ is an L-fuzzifying preuniform continuous function.

Proof

If $\mathcal{W}(w) \ll r$, for all $w \in 2^{Z\times Z}$, then $\mathcal{V}(((g\times g)^{-1})(w)) \ll r$ for all $(g\times g)^{-1})(w) \in 2^{Y\times Y}$. So,

$\mathcal{U}((f\times f)^{-1}(g\times g)^{-1})(w) = \mathcal{U}(((g\times g)(f\times f))^{-1}(w)) = \mathcal{U}((g\circ f\times g\circ f)^{-1}(w)) \ll r$. Hence $g\circ f$ is an L-fuzzifying preuniform continuous function.

Lemma 5.3. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two L-fuzzifying preuniform spaces. f is L-fuzzifying preuniform continuous function if and only if $\mathcal{V}(v) \ll r$ for each $v \in 2^{Y\times Y}$, there exist $u \in 2^{X\times X}$ such that $\mathcal{U}(u) \ll r$ and $(f\times f)(u) \subseteq v$.

Proof.

Suppose f is an L-fuzzifying preuniform continuous function and $\mathcal{V}(v) \ll r$ for each $v \in 2^{Y\times Y}$. Then

$\mathcal{U}((f\times f)^{-1}(v)) \ll r$. Put $(f\times f)^{-1}(v) = u$. Then $\mathcal{U}(u) \ll r$ and $(f\times f)(u) \subseteq v$. If for each $v \in 2^{Y\times Y}$, there exist $u \in 2^{X\times X}$ such that $\mathcal{U}(u) \ll r$ and $(f\times f)(u) \subseteq v$, then $u \subseteq (f\times f)^{-1}(v)$. From $(\mathcal{U}5)$, we have $\mathcal{U}((f\times f)^{-1}(v)) \ll r$. Hence f is an L-fuzzifying preuniform continuous.

Theorem 5.4. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two L-fuzzifying preuniform spaces and $f:(X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ be an L-fuzzifying preuniform continuous function. Then:

(1) $f(\mathcal{C}_{\mathcal{U}}(A,r)) \subseteq \mathcal{C}_{\mathcal{V}}(f(A),r) \quad \forall A \in 2^X$;

(2) $\mathcal{C}_{\mathcal{U}}(f^{-1}(B),r) \subseteq f^{-1}(\mathcal{C}_{\mathcal{V}}(B,r)) \quad \forall B \in 2^Y$;

(3) $f:(X,\tau_{\mathcal{U}}) \longrightarrow (Y,\tau_{\mathcal{V}})$ is an L-fuzzifying continuous function.

Proof.

(1) $f(\mathcal{C}_{\mathcal{U}}(A,r)) = f\left(\cap\left\{D \in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\}\right) = \cap f\left(\left\{D \in 2^X \mid \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\}\right) = \cap\left\{f(D) \in 2^Y \mid \left(\bigwedge_{w\in 2^{X\times X}, A\subseteq w[D]} (\mathcal{U}(w))'\right) \ll r'\right\}$. So, there exist $w \in 2^{X\times X}$ such that $w = (f\times f)^{-1}(v)$ where $v \in 2^{Y\times Y}$ and $(\mathcal{U}((f\times f)^{-1}(v)))' \ll r'$. When f is continuous, then $(\mathcal{V}(v))' \ll r'$. Hence

$\cap\left\{f(D) \in 2^Y \mid \left(\bigwedge_{(f\times f)^{-1}(v)\in 2^{X\times X}, A\subseteq((f\times f)^{-1}(v))[D]} (\mathcal{U}(f\times f)^{-1}(v))'\right) \ll r'\right\} \subseteq \cap\left\{f(D) \in 2^Y \mid \left(\bigwedge_{v\in 2^{Y\times Y}, f(A)\subseteq v[f(D)]} (\mathcal{V}(v))'\right) \ll r'\right\} = \mathcal{C}_{\mathcal{V}}(f(A),r)$.

(2) From (1) we have $\mathcal{C}_{\mathcal{U}}(f^{-1}(B),r) \subseteq f^{-1}(f(\mathcal{C}_{\mathcal{U}}(f^{-1}(B),r))) \subseteq f^{-1}(\mathcal{C}_{\mathcal{V}}(f(f^{-1}(B)),r)) \subseteq f^{-1}(\mathcal{C}_{\mathcal{V}}(B,r))$.

(3) Suppose $\tau_{\mathcal{V}}(B) > t > \tau_{\mathcal{U}}(f^{-1}(B))$. Then $\tau_{\mathcal{V}}(B) > t$. So,

there exist $r > t$ such that $\mathcal{V}(B,r) = B$. Then $f^{-1}(\mathcal{C}_{\mathcal{V}}(B,r)) = f^{-1}(B)$. From (2) $\mathcal{C}_{\mathcal{U}}(f^{-1}(B),r) \subseteq f^{-1}(\mathcal{C}_{\mathcal{V}}(B,r)) = f^{-1}(B)$, then $\mathcal{C}_{\mathcal{U}}(f^{-1}(B),r) = f^{-1}(B)$ for each $B \in 2^Y$. Hence $\tau_{\mathcal{U}}(f^{-1}(B)) \geq r$ and it is contradiction. So, $\tau_{\mathcal{V}}(B) \leq \tau_{\mathcal{U}}(f^{-1}(B))$. Therefore f is an L-fuzzifying continuous function.

6. CONCLUSION

In the present paper, a new concept of preuniform structure based on way below relation was introduced and some of its properties were studied. Also, the relation between L-fuzzifying preuniform and L-fuzzifying topology was established. Furthermore, the notion of interior and closure operators were investigated.

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الخلاصة :في هذه الورقة البحثية عرف التركيب القبل منتظم القائم علي العلاقة التحتية(أوالتركيب القبل منتظم الفازى من النوع ل)ودراسة بعض خواصه .أيضا أنشئ مفهومالمعاملات الداخلية والاغلاقية في الاطار الفازى من النوع ل .علاوة علي ذلك وضحت
العلاقة بين الفضاءات القبل منتظمة والتوبولوجيه الفازية من النوع ل .أخيرا تم دراسة اتصال الفضاءات القبل منتظمة الفازية من النوع ل.