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Research article

Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-space

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Abstract: In this paper, we study the singularities on a non-developable ruled surface according to Blaschke's frame in the Euclidean 3-space. Additionally, we prove that singular points occur on this kind of ruled surface and use the singularity theory technique to examine these singularities. Finally, we construct an example to confirm and demonstrate our primary finding.

Keywords: Blaschke frame; geodesic curvature; singularity theory; spherical indicatrix; ruled surfaces
Mathematics Subject Classification: 53A04, 53A05, 53A17

1. Introduction

A ruled surface is a surface produced by a one-parameter family of straight lines. Despite the fact that the ruled surfaces are indeed an old subject in classical differential geometry, many mathematicians currently find them to be attractive topics. Therefore, a large number of papers have been published in the literature which deal with ruled surfaces and their singularities. The basic goal of singularity investigation is to identify real-valued functions, such as the height function determined on a curve or surface, and the squared-distance function. The height function and the squared-distance function can indeed be considered singularities, which correspond to the classical invariants of extrinsic differential geometry. There is a substantial body of literature on the subject, including, for instance, numerous monographs; see [1–13]. For recent research related to these topics, we refer to these articles: [14–21].

On the other hand, involutes are curves on the tangent developable of a space curve that intersect the generating tangents at right angles. The converse statement yields the following definition: Evolutes are curves that admit a given curve as an involute. The concept of evolutes has been used by many investigators to study the intrinsic geometric properties of plane curves, spherical curves and ruled surfaces. In this paper, we use standard singularity theory methods for the height functions and discuss the connections between these functions' singularities and differential geometric invariants of the ruled surfaces.

The following is part of the current work: We provide a concise explanation of the essential definitions and findings on the curve and ruled surface of differential geometry in Euclidean 3-space $\mathbb{E}^3$ in Section 2. In In Section 3, we introduce the height function on the spherical curve of a ruled surface, and then the major conclusions are supported by some general findings on the singular theory applied to families of functions. Lastly, a detailed explanation of an application example is provided.

2. Basic concepts

The ideas, equations, and conclusions for ruled surfaces in Euclidean 3-space that can be encountered in differential geometry textbooks (see, for instance, [2–8]) are listed in this section. A ruled surface in Euclidean 3-space is a surface generated by a straight line L moving along a space curve $\mathbf{c}(s)$. The surface rulings are the different spots of the producing lines. As a result, such a surface has parametrization in the ruled form and references [2–6, 9, 11–17]:

$$\mathbb{M} : \mathbf{y}(s, v) = \mathbf{c}(s) + v\mathbf{x}(s) =, \ s \in I, \ v \in \mathbb{R}, \tag{1}$$

such that

$$\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}', \mathbf{x}' \rangle = 1, \ \langle \mathbf{c}', \mathbf{x}' \rangle = 0, \quad \left(' = \frac{d}{ds} \right). \tag{2}$$

The striction curve is represented by the base curve $\mathbf{c}(s)$, and the parameter s is the arc length of the spherical image $\mathbf{x} = \mathbf{x}(s)$ on the unit sphere $\mathbb{S}^2$. This parameterization allows for the investigation of kinematic geometry and associated geometric properties. We can get $\mathbf{t}(s) = \mathbf{x}'$, $\mathbf{g}(s) = \mathbf{x} \times \mathbf{t}$, and since s is a natural parameter of $\mathbf{x}(s)$, it follows that $\|\mathbf{t}\| = 1$. The frame $\{\mathbf{x} = \mathbf{x}(s), \ \mathbf{t}(s), \ \mathbf{g}(s)\}$ forms a moving orthonormal frame attached to each point of the spherical curve $\mathbf{x}(s)$. This frame is called the Blaschke frame relative to $\mathbf{x}(s)$. By construction, the Blaschke formula is defined as follows [10]:

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{t}' \\ \mathbf{g}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \\ \mathbf{g} \end{pmatrix} = \omega \times \begin{pmatrix} \mathbf{x} \\ \mathbf{t} \\ \mathbf{g} \end{pmatrix}, \tag{3}$$

where $\omega(s)$=$\gamma\mathbf{x} + \mathbf{g}$ is the Darboux vector along $\mathbf{x}(s)$, and $\gamma(s) = \det(\mathbf{x}, \mathbf{x}', \mathbf{x}'')$ is the geodesic curvature. Then, the curvature axis of $\mathbf{x}(s)$ is given by

$$\mathbf{b}(s) := \frac{\omega}{\|\omega\|} = \cos\vartheta\mathbf{x} + \sin\vartheta\mathbf{g}, \text{ with } \gamma(s) = \cot\vartheta, \tag{4}$$

where ϑ is the radius of curvature. It is evident that the spherical curve's $\mathbf{x}(s)$ evolution is $\mathbf{b}$. Therefore, we may write for the spherical curve $\mathbf{x} = \mathbf{x}(s)$ the following relationships:

$$\left.\begin{array}{l} \kappa(s) = \sqrt{1+\gamma^2} = \frac{1}{\sin\vartheta} = \frac{1}{\rho}, \\ \tau(s) = \frac{1}{\sigma} = -\vartheta' = \frac{1}{\kappa^2}\gamma', \end{array}\right\} \tag{5}$$

where $\kappa(s)$, $\rho(s)$ are the curvature and radius of curvature, and $\tau(s)$, $\sigma(s)$ are the torsion and radius of torsion of $\mathbf{x} = \mathbf{x}(s)$, respectively. The tangent of the striction curve $\mathbf{c}(s)$ is defined by

$$\mathbf{c}^{'}(s) = \Gamma\mathbf{x} + \mu\mathbf{g}. \tag{6}$$

The curvature functions or construction parameters of the ruled surface are functions $\gamma(s)$, $\Gamma(s)$ and $\mu(s)$. These invariants' geometrical interpretations are described as follows: γ is the spherical image curve's ($\mathbf{x} = \mathbf{x}(s)$) geodesic curvature. Γ represents the angle formed by the tangent of the striction curve and the surface ruling. μ is the distribution parameter of the ruled surface at the ruling.

3. Main results

As a well-known, family of planes with a single parameter's envelope is a developable ruled surface. The singular points of the tangent developable ruled surfaces are made up of the regression edge. The following three equations are satisfied by the edge of the envelope's regression by central planes [8].

$$\langle \mathbf{c} - \mathbf{z}, \mathbf{t}\rangle = 0,\ \frac{d}{ds}\langle \mathbf{c} - \mathbf{z}, \mathbf{t}\rangle = 0,\ \frac{d^2}{ds^2}\langle \mathbf{c} - \mathbf{z}, \mathbf{t}\rangle = 0, \tag{7}$$

where $\mathbf{z}$ is an arbitrary point of the edge of regression. The 2^{nd} and 3^{rd} equations in Eq (7) can be easily implemented by using the curvature functions Γ, μ and γ as

$$\langle \mathbf{c} - \mathbf{z}, \mathbf{x}\rangle = \frac{\gamma(\Gamma - \mu\gamma)}{\gamma'},\ \langle \mathbf{c} - \mathbf{z}, \mathbf{g}\rangle = \frac{\Gamma - \mu\gamma}{\gamma'}, \tag{8}$$

where $\gamma' \neq 0$ considers the scenario when the constant $\mathbf{g}$ has been dropped from the central tangent vector. It implies this developable ruled surface's border of regression:

$$\mathbf{z} = \mathbf{c} - \left(\frac{\Gamma - \gamma\mu}{\gamma'}\right)\omega. \tag{9}$$

The differentiation of $\mathbf{z}(s)$ can be obtained as:

$$\mathbf{z}' = \left[\mu - \frac{d}{ds}\left(\frac{\Gamma - \gamma\mu}{\gamma'}\right)\right]\omega. \tag{10}$$

Hence, the envelope of central planes can be parameterized as

$$\begin{aligned} \mathbf{q}(s, v) &= \mathbf{z} + v\frac{\mathbf{z}'}{\|\mathbf{z}'\|} \\ &= \mathbf{c} + \left(\frac{\mu\gamma - \Gamma}{\gamma'} + \frac{v}{\sqrt{1+\gamma^2}}\right)\omega,\ v \in \mathbb{R}. \end{aligned} \tag{11}$$

We call the surface $\mathbf{q}(s, v)$ as the central developable of $\mathbf{c}(s)$.

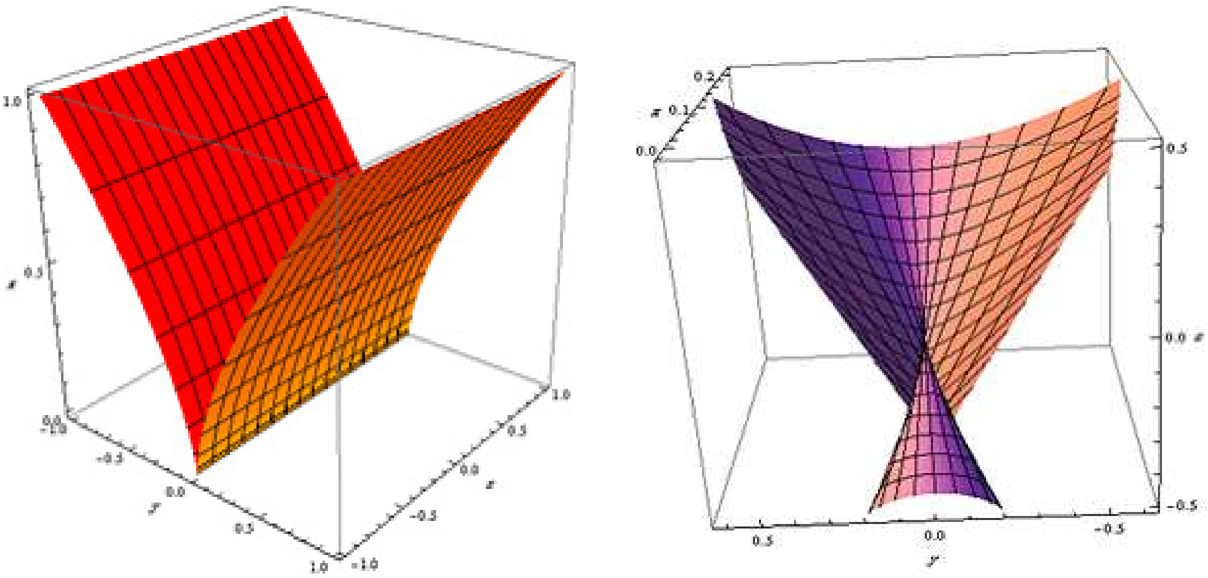
Theorem 3.1. *Let $\mathbf{x} = \mathbf{x}(s)$ be the spherical indicatrix of the ruled surface expressed by Eq (1) with $\gamma(s) \neq 0$. Then, we have the following.*
(1)-(i) The spherical evolute $\mathbf{b}(s)$ of $\mathbf{x}(s)$ is diffeomorphic to a fixed line at $s_0 \in \mathbb{R}$ if and only if

$\gamma'(s_0) = 0$.
(ii) The spherical evolute $\mathbf{b}(s)$ *of* $\mathbf{x}(s)$ *is diffeomorphic to the ordinary cusp* $\mathbf{C}$ *at* $s_0 \in \mathbb{R}$ *if and only if* $\gamma'(s_0) = 0$, *and* $\gamma''(s_0) \neq 0$.
(2)-(i) The ruled surface is locally diffeomorphic to the cuspidal edge $\mathbf{C} \times \mathbb{R}$ *at* $s_0 \in \mathbb{R}$ *if and only if* $\gamma'(s_0) = 0$, *and* $\gamma''(s_0) \neq 0$.
(ii) The central developable surface is locally diffeomorphic to the swallowtail (SW) *at* $s_0 \in \mathbb{R}$ *if and only if* $\gamma'(s_0) = \gamma''(s_0) = 0$, *and* $\gamma'''(s_0) \neq 0$.

The cuspidal edge $\mathbf{C} \times \mathbb{R} = \left\{(x_1, x_2) | x_1^2 - x_2^3 = 0\right\} \times \mathbb{R}$, and the swallowtail

$$SW = \left\{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v,\ x_2 = 4u^3 + 2uv,\ x_3 = v\right\}, \tag{12}$$

are shown in Figure 1.



(a) **(b)**

Figure 1. (a) The cuspidal edge; (b) the swallowtail.

The main aim of this work is to prove Theorem 3.1. To do this, we will use some general results on the singularity theory for families of functions and generic properties of regular curves in $\mathbb{E}^3$. There are detailed descriptions in [2–5], and we shall eliminate the variable s because some equations are quite long.

3.1. Height functions

In this part, we establish the families of functions on a spherical curve that can be used to investigate singularities of $\mathbf{b}(s)$ and $\mathbf{q}(s, v)$. Let $\mathbf{x} = \mathbf{x}(s)$ be the spherical indicatrix of the ruled surface expressed by Eq (1) with $\gamma(s) \neq 0$. As follows, we define a smooth function H:

$$H : I \times \mathbb{S}^2 \to \mathbb{R} \text{ by } H(s, \mathbf{b}_0) = \langle \mathbf{x}, \mathbf{b}_0 \rangle. \tag{13}$$

We call H the height function on $\mathbf{x}(s) \in \mathbb{S}^2$ and use the notation $h(s) = H(s, \mathbf{b}_0)$ for any fixed $\mathbf{b}_0 \in \mathbb{S}^2$.

Definition 3.1. *The fixed axis* $\mathbf{b}_0$ *of* $\mathbb{S}^2$ *will be said to be a* $\mathbf{b}_k$ *evolute of the curve* $\mathbf{x}(s) \in \mathbb{S}^2$ *at* $s \in \mathbb{R}$ *if for all i such that* $1 \leq i \leq k$, $\langle \mathbf{b}_0, \mathbf{x}^i(s)\rangle = 0$, *but* $\langle \mathbf{b}_0, \mathbf{x}^{k+1}(s)\rangle \neq 0$.
Here, $\mathbf{x}^i$ *denotes the i-th derivatives of* $\mathbf{x}$ *with respect to the arc-length of* $\mathbf{x}(s) \in \mathbb{S}^2$.

Let

$$\mathbb{S}(\mathbf{b}_0, \theta) = \{\mathbf{x} \in \mathbb{S}^2 | \ \langle \mathbf{x}, \mathbf{b_0}\rangle = \cos\theta\}, \tag{14}$$

be a circle on the unit sphere $\mathbb{S}^2$ whose center is a point on the fixed axis $\mathbf{b}_0 \in \mathbb{S}^2$.

Note that $\theta = \cos^{-1}(h(s))$ defines the radius of curvature between the unit vectors $\mathbf{x}$ and $\mathbf{b_0}$. Thus, we have the following proposition.

Proposition 3.1. *Let* $\mathbf{x} = \mathbf{x}(s)$ *be the spherical indicatrix of the ruled surface expressed by Eq* (1) *with* $\gamma(s) \neq 0$. *Then, by direct calculation, one has the following:*
(1) The angle θ *of the unit vectors* $\mathbf{x}$ *and* $\mathbf{b}_0$ *will be invariant in the first approximation if and only if* $\mathbf{b}_0 \in Sp\{\mathbf{x}, \mathbf{g}\}$, *that is,*

$$\theta' = 0 \Leftrightarrow \langle \mathbf{x}', \mathbf{b}_0\rangle = 0 \Leftrightarrow \mathbf{b}_0 = a_1\mathbf{x} + a_2\mathbf{g} \tag{15}$$

for some real numbers $a_1, a_2 \in \mathbb{R}$, *and* $a_1^2 + a_2^2 = 1$.
(2) The angle θ *of the unit vectors* $\mathbf{x}$ *and* $\mathbf{b}_0$ *will be invariant in the second approximation if and only if* $\mathbf{b}_0$ *is* $\mathbf{b}_2$ *evolute of* $\mathbf{x}(s) \in \mathbb{S}^2$, *that is,*

$$\theta' = \theta'' = 0 \Leftrightarrow \mathbf{b}_0 = \pm\,\mathbf{b}. \tag{16}$$

(3) The angle θ *of the unit vectors* $\mathbf{x}$ *and* $\mathbf{b}_0$ *will be invariant in the third approximation if and only if* $\mathbf{b}_0$ *is* $\mathbf{b}_3$ *evolute of* $\mathbf{x}(s) \in \mathbb{S}^2$, *that is,*

$$\theta' = \theta'' = \theta''' = 0 \Leftrightarrow \mathbf{b}_0 = \pm\mathbf{b}, \text{ and } \gamma' = 0. \tag{17}$$

(4) The angle θ *of the unit vectors* $\mathbf{x}$ *and* $\mathbf{b}_0$ *will be invariant in the fourth approximation if and only if* $\mathbf{b}_0$ *is* $\mathbf{b}_4$ *evolute of* $\mathbf{x}(s) \in \mathbb{S}^2$, *that is,*

$$\theta' = \theta'' = \theta''' = \overset{(4)}{\theta} = 0 \Leftrightarrow \mathbf{b}_0 = \pm\mathbf{b}, \text{ and } \gamma' = \gamma'' = 0. \tag{18}$$

Proof. (1) During the motion of the Blaschke frame along the striction curve $\mathbf{C}(s)$, the angle θ in Eq (14) naturally changes. For the first differential of θ, we get

$$\theta' = \frac{-\gamma'}{\sqrt{1+\gamma^2}} = \frac{-\langle \mathbf{x}', \mathbf{b}_0\rangle}{\sqrt{1+\langle \mathbf{x}, \mathbf{b}_0\rangle^2}}, \tag{19}$$

which leads to

$$\theta' = 0 \Leftrightarrow \gamma' = 0 \Leftrightarrow \langle \mathbf{x}', \mathbf{b}_0\rangle = 0 \Leftrightarrow \mathbf{b}_0 = a_1\mathbf{x} + a_2\mathbf{g}, \tag{20}$$

for some real numbers $a_1, a_2 \in \mathbb{R}$ and $a_1^2 + a_2^2 = 1$, So the result is clear.
(2) Differentiation of Eq (19) leads to

$$\theta'' = \frac{-\gamma''\sqrt{1+\gamma^2} + \gamma'\left(\sqrt{1+\gamma^2}\right)'}{1+\gamma^2}. \tag{21}$$

From Eqs (20) and (21), we have

$$\begin{aligned}\theta' &= \theta'' = 0 \Leftrightarrow \gamma' = \gamma'' = 0 \Leftrightarrow \langle \mathbf{x}', \mathbf{b}_0\rangle = \langle \mathbf{x}'', \mathbf{b}_0\rangle = 0\\ &\Leftrightarrow \mathbf{b}_0 = \pm\frac{\mathbf{x}'\times\mathbf{x}''}{\|\mathbf{x}'\times\mathbf{x}''\|} = \pm\mathbf{b}. \end{aligned}\tag{22}$$

(3) Differentiation of Eq (21) leads to

$$\begin{aligned}\theta''' &= \frac{-\gamma''' \sqrt{1+\gamma^2} + \gamma'\left(\sqrt{1+\gamma^2}\right)''}{1-(\langle \mathbf{x},\mathbf{b}_0\rangle)^2}\\ &\quad + \left\{-\gamma''' \sqrt{1+\gamma^2} + \gamma'\left(\sqrt{1+\gamma^2}\right)'\right\}\left(\frac{1}{1+\gamma^2}\right)'. \end{aligned}\tag{23}$$

Similarly, from Eqs (20) and (22), we get

$$\begin{aligned}\theta' &= \theta'' = \theta''' = 0 \Leftrightarrow \gamma' = \gamma'' = \gamma''' = 0\\ &\Leftrightarrow \langle \mathbf{x}', \mathbf{b}_0\rangle = \langle \mathbf{x}'', \mathbf{b}_0\rangle = \langle \mathbf{x}''', \mathbf{b}_0\rangle = 0\\ &\Leftrightarrow \mathbf{b}_0 = \pm\mathbf{b}, \text{ and } \gamma' = 0. \end{aligned}\tag{24}$$

(4) Differentiation of Eq (23) leads to

$$\begin{aligned}\theta^{(4)} &= \frac{-\gamma^{(4)} \sqrt{1+\gamma^2} - \gamma'''\left(\sqrt{1+\gamma^2}\right)' + \gamma''\left(\sqrt{1+\gamma^2}\right)'' + \gamma'\left(\sqrt{1+\gamma^2}\right)'''}{1+\gamma^2}\\ &\quad + \left\{-\gamma''' \sqrt{1+\gamma^2} + \gamma'\left(\sqrt{1+\gamma^2}\right)''\right\}\left(\frac{1}{1+\gamma^2}\right)'\\ &\quad + \left\{-\gamma'' \sqrt{1+\gamma^2} + \gamma'\left(\sqrt{1+\gamma^2}\right)'\right\}\left(\frac{1}{1+\gamma^2}\right)''. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\theta' = \theta'' = \theta''' = \theta^{(4)} = 0 &\Leftrightarrow \gamma' = \gamma'' = \gamma''' = \gamma^{(4)} = 0\\ &\Leftrightarrow \langle \mathbf{x}', \mathbf{b}_0\rangle = \langle \mathbf{x}'', \mathbf{b}_0\rangle = \langle \mathbf{x}''', \mathbf{b}_0\rangle = \langle \mathbf{x}^{(4)}, \mathbf{b}_0\rangle = 0, \end{aligned}$$

or equivalently,

$$\theta' = \theta'' = \theta''' = \theta^{(4)} = 0 \Leftrightarrow \mathbf{b}_0 = \pm\mathbf{b}, \text{ and } \gamma' = \gamma'' = 0.\tag{25}$$

Thus, the proof is completed. □

Now, various deductions can be made from the above statements as follows: If $\gamma' = 0$, then the spherical curve $\mathbf{x} = \mathbf{x}(s)$ has been classically known as a small circle on the unit sphere $\mathbb{S}^2$. Moreover, we have the following proposition:

Proposition 3.2. *Under the above notations, the spherical curve* $\mathbf{x} = \mathbf{x}(s)$ *is a small circle on the unit sphere* $\mathbb{S}^2$ *if and only if the Darboux vector* $\omega(s)$ *is a constant vector. In this case, we have the following assertions:*
(1) The direction of the center of the circle is given by the constant vector $\mathbf{b}_0$.
(2) The central developable $\mathbf{q}(s, v)$ *is a cylindrical surface.*

Proof. By the Blaschke formulae, we can show that

$$\mathbf{b}'(s)=\rho'\left(\frac{\mathbf{x}-\gamma\mathbf{g}}{\sqrt{\gamma^2+1}}\right). \tag{26}$$

Therefore, $\mathbf{b}$ is a constant vector if and only if $\rho' = 0\,(\kappa(s) \geq 1, \text{and } \tau(s) = 0)$. This condition is analogous to the condition that $\mathbf{x} = \mathbf{x}(s)$ is a small circle on the unit sphere $\mathbb{S}^2$. Assertion (2) is clear by definition, and then the proof is completed. □

According to Proposition 3.1, the osculating circle of $\mathbf{x} = \mathbf{x}(s)$ is determined by the equations:

$$h(s) = \langle \mathbf{x}, \mathbf{b}_0 \rangle,\ \langle \mathbf{b}_0, \mathbf{x}' \rangle = \langle \mathbf{b}_0, \mathbf{x}'' \rangle = 0 \Leftrightarrow \gamma' = 0, \tag{27}$$

which are obtained from the condition that the osculating circle should have contact of at least three orders with the curve. Moreover, the osculating circle has at least 4-points contact if and only if $\gamma' = \gamma'' = 0$. This is a necessary and sufficient condition for a higher vertex. An ordinary vertex occurs precisely at a simple maximum or minimum of geodesic curvature $\gamma(s)$ of $\mathbf{x}(s) \in \mathbb{S}^2$ [3, 5, 6]. So, the evolute is the locus of the center of geodesic curvature of $\mathbf{x} = \mathbf{x}(s)$. As a result, the singularities of the spherical evolute describe how the curve's shape is identical to a helix. On the contrary, the central developable's singularities explain how the curve's shape differs from a helix.

3.2. Unfolding of functions of one variable

We will apply the same method to families of smooth functions in this subsection. The books [5, 7] contain comprehensive descriptions. Let F: $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be a smooth function and $\mathfrak{F}(s) = F_{x_0}$, $F_{x_0}(S) = F(s, \mathbf{x}_0)$. Then, F is called an $r-$parameter unfolding of $\mathfrak{F}(s)$. We say that $\mathfrak{F}(s)$ has A_k singularity at s_0 if $\mathfrak{F}^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $\mathfrak{F}^{(p+1)}(s_0) \neq 0$. We also say that $\mathfrak{F}(s)$ has $A_{\geq k}$ singularity at s_0 if $\mathfrak{F}^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 be $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)\right) = \Sigma_{j=1}^{k-1} a_{ji} s^j$ (without the constant term) for $i = 1, ..., r$. Then, $F(s, \mathbf{x})$ is called a (p) versal unfolding if and only if the $(k-1) \times r$ matrix of coefficients $\left(a_{ji}\right)$ has rank $(k-1)$. (This certainly requires $k - 1 \leq r$, so the smallest value of r is $k - 1$).

We now provide some crucial information regarding how the preceding notations are evolving. The discriminant set of $F(s, \mathbf{x})$ is the set:

$$\mathbb{D}_F = \left\{\mathbf{x} \in \mathbb{S}^2 | \text{ there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, \mathbf{x})\right\}. \tag{28}$$

The bifurcation set $\mathbb{B}_F$ of F is the set:

$$\mathbb{B}_F = \left\{\mathbf{x} \in \mathbb{S}^2 | \text{ there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, \mathbf{x})\right\}. \tag{29}$$

The fundamental outcome of the developing theory [11, 13] is as follows:

Theorem 3.2. *Let F : $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $\mathfrak{F}(s)$, which has the A_k singularity $(k \geq 1)$ at s_0. Suppose that F is a (p) versal unfolding. then we have the following.*
(1) If $k = 1$, then $\mathbb{D}_F$ is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-1}$.

(2) If $k = 2$, *then* $\mathbb{D}_F$ *is locally diffeomorphic to* $\mathbf{C}\times\mathbb{R}^{r-2}$.
(3) If $k = 3$, *then* $\mathbb{D}_F$ *is locally diffeomorphic to* $SW\times\mathbb{R}^{r-3}$.
(4) If $k = 2$, *then* $\mathbb{B}_F$ *is locally diffeomorphic to* $\{\mathbf{0}\}\times\mathbb{R}^{r-1}$.
(5) If $k = 3$, *then* $\mathbb{B}_F$ *is locally diffeomorphic to* $\mathbf{C}\times\mathbb{R}^{r-2}$.

By Proposition 3.1, the discriminant set of $H(s, \mathbf{b}_0)$ is given as follows:

$$\mathbb{D}_H = \left\{(s, \mathbf{b}_0) \in I \times \mathbb{S}^2 : \text{by } H(s, \mathbf{b}_0) = \langle \mathbf{x}, \mathbf{b}_0 \rangle,\ \mathbf{b}_0 = a_1\mathbf{x} + a_2\mathbf{g}\right\}, \tag{30}$$

where $a_1, a_2 \in \mathbb{R}$, and $a_1^2 + a_2^2 = 1$. The bifurcation set of $H(s, \mathbf{b}_0)$ is

$$\mathbb{B}_H = \left\{\pm\mathbf{b} = \frac{\gamma\mathbf{x} + \mathbf{g}}{\sqrt{\gamma^2 + 1}}\right\}. \tag{31}$$

The following is our key pillar for proving Theorem 3.1:

Proposition 3.3. *For the unit speed curve* $\mathbf{x}(s) = (x_1(s), x_2(s), x_3(s))$ *on* $\mathbb{S}^2$ *with* $\gamma(s_0) \neq 0$, *if* $h(s) = H(s, \mathbf{b})$ *has* A_k *singularity* $(k = 2,\ 3)$ *at* $s_0 \in \mathbb{R}$, *then* $H(s, \mathbf{b})$ *is a* (p) *versal unfolding of* $h(s_0)$.

Proof. Since $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{S}^2$, $b_1^2 + b_2^2 + b_3^2 = 1$, b_1, b_2, and b_3 cannot all be all zero. Without loss of generality, suppose $b_3 \neq 0$. Then, $b_3 = \sqrt{1 - b_1^2 - b_2^2}$, and we have

$$H(s, \mathbf{b}) = b_1x_1(s) + b_2x_2(s) + \sqrt{1 - b_1^2 - b_2^2}\, x_3(s).$$

So, we get

$$\left.\begin{aligned} \frac{\partial H}{\partial b_1} &= \left(x_1(s) - \frac{b_1x_3(s)}{\sqrt{1-b_1^2-b_2^2}}\right), \\ \frac{\partial H}{\partial b_2} &= \left(x_2(s) - \frac{b_2x_3(s)}{\sqrt{1-b_1^2-b_2^2}}\right). \end{aligned}\right\}$$

Also, we have

$$\left.\begin{aligned} \frac{\partial}{\partial s}\frac{\partial H}{\partial b_1} &= \left(x_1'(s) - \frac{b_1x_3'(s)}{\sqrt{1-b_1^2-b_2^2}}\right), \\ \frac{\partial}{\partial s}\frac{\partial H}{\partial b_2} &= \left(x_2'(s) - \frac{b_2x_3'(s)}{\sqrt{1-b_1^2-b_2^2}}\right), \\ \frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial b_1} &= \left(x_1''(s) - \frac{b_1x_3''(s)}{\sqrt{1-b_1^2-b_2^2}}\right), \\ \frac{\partial^2}{\partial s^2}\frac{\partial H}{\partial b_2} &= \left(x_2''(s) - \frac{b_2x_3''(s)}{\sqrt{1-b_1^2-b_2^2}}\right). \end{aligned}\right\}$$

Let $\mathbf{b} = \mathbf{b}_0 = (b_{10}, b_{20}, b_{30}) \in \mathbb{S}^2$, and assume $b_{30} \neq 0$. Then:

$$\left.\begin{aligned} j^1\left(\frac{\partial H}{\partial b_1}(s, \mathbf{b}_0)\right) &= \left(x_1'(s_0) - \frac{b_{10}x_3'(s_0)}{b_{30}}\right)s, \\ j^1\left(\frac{\partial H}{\partial b_2}(s, \mathbf{b}_0)\right) &= \left(x_2'(s_0) - \frac{b_{20}x_3'(s_0)}{b_{30}}\right)s, \end{aligned}\right\}$$

and

$$\left.\begin{aligned} j^2\left(\frac{\partial H}{\partial b_1}(s, \mathbf{b}_0)\right) &= \left(x_1'(s_0) - \frac{b_{10}x_3'(s_0)}{b_{30}}\right)s + \frac{1}{2}\left(x_1''(s_0) - \frac{b_{10}x_3''(s_0)}{b_{30}}\right)s^2, \\ j^2\left(\frac{\partial H}{\partial b_2}(s, \mathbf{b}_0)\right) &= \left(x_2'(s_0) - \frac{b_{20}x_3'(s_0)}{b_{30}}\right)s + \frac{1}{2}\left(x_2''(s_0) - \frac{b_{20}x_3''(s_0)}{b_{30}}\right)s^2. \end{aligned}\right\}$$

Now, we have the following:
(i) If $h(s_0)$ has an A_2-singularity at $s_0 \in \mathbb{R}$, then $\gamma'(s_0) = 0$. So, the $(2-1)\times 2$ matrix of coefficients $\left(a_{ji}\right)$ is

$$A = \begin{pmatrix} x_1'(s_0) - \frac{b_{10}x_3'(s_0)}{b_{30}} & x_2'(s_0) - \frac{b_{20}x_3'(s_0)}{b_{30}} \end{pmatrix}.$$

Assume matrix A has rank zero and in this case, we have

$$x_1'(s_0) = \frac{b_{10}x_3'(s_0)}{b_{30}},\ x_2'(s_0) = \frac{b_{20}x_3'(s_0)}{b_{30}}.$$

Since $\|\mathbf{x}'(s_0)\| = \|\mathbf{t}(s_0)\| = 1$, we have $x_3'(s_0) \neq 0$, and it leads to the contradictions

$$\begin{aligned} 0 &= \langle (x_1'(s_0), x_2'(s_0), x_3'(s_0)), (b_{10}, b_{20}, b_{30})\rangle \\ &= x_1'(s_0)b_{10} + x_2'(s_0)b_{20} + x_3'(s_0)b_{30} \\ &= \frac{b_{10}^2 x_3'(s_0)}{b_{30}} + \frac{b_{20}^2 x_3'(s_0)}{b_{30}} + x_3'(s_0)b_{30} \\ &= \frac{x_3'(s_0)}{b_{30}} \neq 0. \end{aligned}$$

Therefore, rank $A = 1$, and H is the (p) versal unfolding of h at s_0.
(ii) If $\gamma(s_0)$ has an A_3-singularity at $s_0 \in \mathbb{R}$, then by Proposition 3.1, we have

$$\mathbf{b}_0 = \pm\left(\frac{\gamma\mathbf{x} + \mathbf{g}}{\sqrt{\gamma^2 + 1}}\right)(s_0),$$

where $\gamma'(s_0) = 0$ and $\gamma''(s_0) \neq 0$. So, the $(3-1)\times 2$ matrix of the coefficients $\left(a_{ji}\right)$ is

$$B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} x_1' - \frac{b_1 x_3'}{\sqrt{1-b_1^2-b_2^2}} & x_2' - \frac{b_2 x_3'}{\sqrt{1-b_1^2-b_2^2}} \\ x_1'' - \frac{b_1 x_3''}{\sqrt{1-b_1^2-b_2^2}} & x_2'' - \frac{b_2 x_3''}{\sqrt{1-b_1^2-b_2^2}} \end{pmatrix}.$$

We also need the 2×2 matrix B, which is always non-singular, to serve this purpose. In essence, this matrix's determinant at s_0 is

$$\begin{aligned} \det(B) &= \frac{1}{b_{30}}\begin{vmatrix} x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \\ b_{10} & b_{20} & b_{30} \end{vmatrix} \\ &= \frac{1}{b_{30}}\langle \mathbf{x}'\times\mathbf{x}'', \mathbf{b}_0\rangle \\ &= \frac{1}{b_{30}}\langle \mathbf{x}'\times\mathbf{x}'', \left(\frac{\gamma\mathbf{x}+\mathbf{g}}{\sqrt{\gamma^2+1}}\right)(s_0)\rangle. \end{aligned}$$

Since $\mathbf{x}' = \mathbf{t}$, we have $\mathbf{x}'' = -\mathbf{x}+\gamma\mathbf{g}$. By substituting these relations in the preceding equality, we get

$$\det(B) = \frac{\sqrt{\gamma^2(s_0)+1}}{b_{30}} \neq 0.$$

This means that rank $B = 2$, and hence, this completes the proof. □

3.3. Example

Let us demonstrate the above considerations in a simple example. So, consider the ruled surface is defined by

$$M : \mathbf{y}(\psi, v) = \mathbf{c}(\psi) + v\mathbf{x}(\psi); v \in \mathbb{R},$$

where

$$\begin{aligned}
\mathbf{c}(\psi) &= \left(\frac{-2}{1+\cos^2\psi}, \frac{2\cos^3\psi}{1+\cos^2\psi}, \frac{-\sin\psi - 2\ \sin\psi\cos^2\psi}{1+\cos^2\psi}\right), \\
\mathbf{x}(\psi) &= \left(\sin\psi, \sin\psi\cos\psi, \cos^2\psi\right).
\end{aligned}$$

It is easy to show that $\langle \mathbf{x}, \mathbf{x} \rangle = 1$, and $\langle \mathbf{c}', \mathbf{x}' \rangle = 0$, i.e., $\mathbf{c}(\psi)$ is the striction curve of the ruled surface M. Here we use $\prime = \frac{d}{d\psi}$. Consequently, by the standard arguments, we get

$$\mathbf{t}(\psi) = \left(\frac{\sqrt{2}\cos\psi}{\sqrt{3+\cos 2\psi}}, \frac{\sqrt{2}\cos 2\psi}{\sqrt{3+\cos 2\psi}}, -\frac{\sqrt{2}\sin 2\psi}{\sqrt{3+\cos 2\psi}}\right),$$

$$\mathbf{g}(\psi) = -\left(\frac{\sqrt{2}\cos\psi^2}{\sqrt{3+\cos 2\psi}}, \frac{-5\cos\psi + \cos 3\psi}{2\sqrt{2}\sqrt{3+\cos 2\psi}}, \frac{\sqrt{2}\sin\psi^3}{\sqrt{3+\cos 2\psi}}\right).$$

We can calculate the geodesic curvature as follows:

$$\gamma(\psi) = \frac{\det(\mathbf{x}, \mathbf{x}', \mathbf{x}'')}{\|\mathbf{x}'\|^3} = -\frac{9\sin\psi + \sin 3\psi}{\sqrt{2}(3+\cos 2\psi)^{3/2}}.$$

Also, we get

$$\Gamma(\psi) = \frac{\det(\mathbf{c}', \mathbf{g}, \mathbf{g}')}{\|\mathbf{g}'\|^2} = \frac{(3+\cos 2\psi)^2}{9\sin\psi + \sin 3\psi},$$

$$\mu(\psi) = \frac{\det(\mathbf{c}', \mathbf{x}, \mathbf{x}')}{\|\mathbf{x}'\|^2} = -\frac{\sin 2\psi^2}{(3+\cos 2\psi)^2}.$$

Using Eq (4) and some algebraic manipulations, we get

$$\mathbf{b}(\psi) = (b_1, b_2, b_3),$$

where

$$\begin{aligned}
b_1 &= -\frac{4}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}}, \\
b_2 &= \frac{4\cos\psi^3}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}}, \\
b_3 &= -\frac{2(2+\cos 2\psi)\sin\psi}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}}.
\end{aligned}$$

Also, from Eq (11) the associated central developable surface of $\mathbf{c}(\psi)$ is given by

$$\mathbf{q}(\psi, v) = (\eta_1, \eta_2, \eta_3)\,,$$

where

$$\begin{aligned}\eta_1 \;=\; & -\frac{4}{3+\cos 2\psi} - \frac{4v}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}} \\ & +\frac{1}{6}(3+\cos 2\psi)\sec v\left(\frac{4\sqrt{2}\cos\psi^2(5+\cos 2\psi)\sin\psi^3}{(3+\cos 2\psi)^{7/2}} - \frac{(3+\cos 2\psi)^2}{9\sin\psi+\sin 3\psi}\right),\end{aligned}$$

$$\begin{aligned}\eta_2 \;=\; & \frac{4\cos\psi^3}{3+\cos 2\psi} + \frac{4v\cos\psi^3}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}} \\ & -\frac{1}{6}\cos\psi^2(3+\cos 2\psi)\left(\frac{4\sqrt{2}\cos\psi^2(5+\cos 2\psi)\sin\psi^3}{(3+\cos 2\psi)^{7/2}} - \frac{(3+\cos 2\psi)^2}{9\sin\psi+\sin 3\psi}\right),\end{aligned}$$

$$\begin{aligned}\eta_3 \;=\; & -\frac{2v(2+\cos 2\psi)\sin\psi}{(3+\cos 2\psi)^{3/2}\sqrt{\frac{26+6\cos 2\psi}{(3+\cos 2\psi)^3}}} + 2\left(-1+\frac{1}{3+\cos 2\psi}\right)\sin\psi \\ & +\frac{1}{12}(2+\cos 2\psi)(3+\cos 2\psi)\left(\frac{4\sqrt{2}\cos\psi^2(5+\cos 2\psi)\sin\psi^3}{(3+\cos 2\psi)^{7/2}} - \frac{(3+\cos 2\psi)^2}{9\sin\psi+\sin 3\psi}\right)\tan v.\end{aligned}$$

By a straightforward calculation, we have $\gamma'(\frac{3}{2}\pi) = 0$, and $\gamma''(\frac{3}{2}\pi) = -6 \neq 0$. Hence, according to Theorem 3.2, the spherical evolute $\mathbf{b} = \mathbf{b}(\psi)$ is locally diffeomorphic to the ordinary cusp at $\psi = \frac{3}{2}\pi$ (see Figure 2).

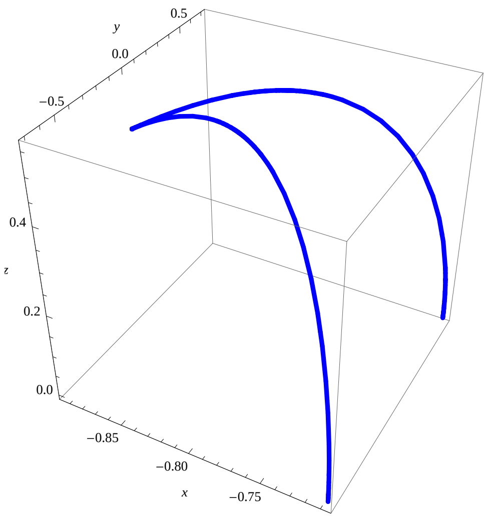


Figure 2. The spherical evolute $\mathbf{b} = \mathbf{b}(\psi)$ has an ordinary cusp.

The red curve is the spherical curve $\mathbf{x} = \mathbf{x}(\psi)$, and the blue curve is its evolute (see Figure 3). For $\pi \leq \psi \leq 2\pi$, and $-3 \leq v \leq 3$, the ruled surface M and the central developable surface $\mathbf{q}(\psi, v)$ are shown in Figure 4, respectively. All calculations and figures in this paper were computed and plotted by using Wolfram Mathematica 7.0.

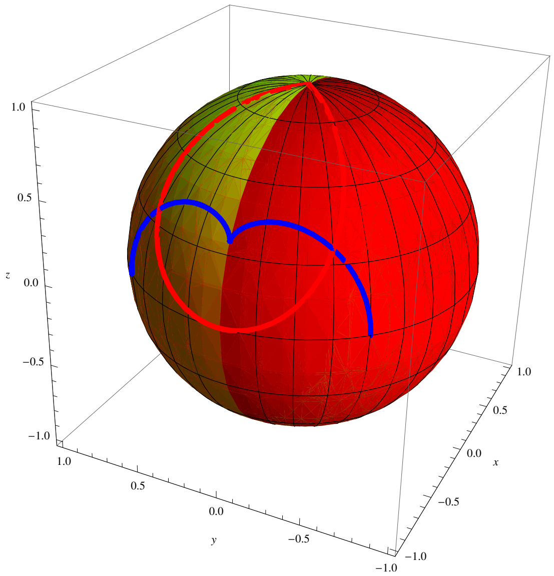
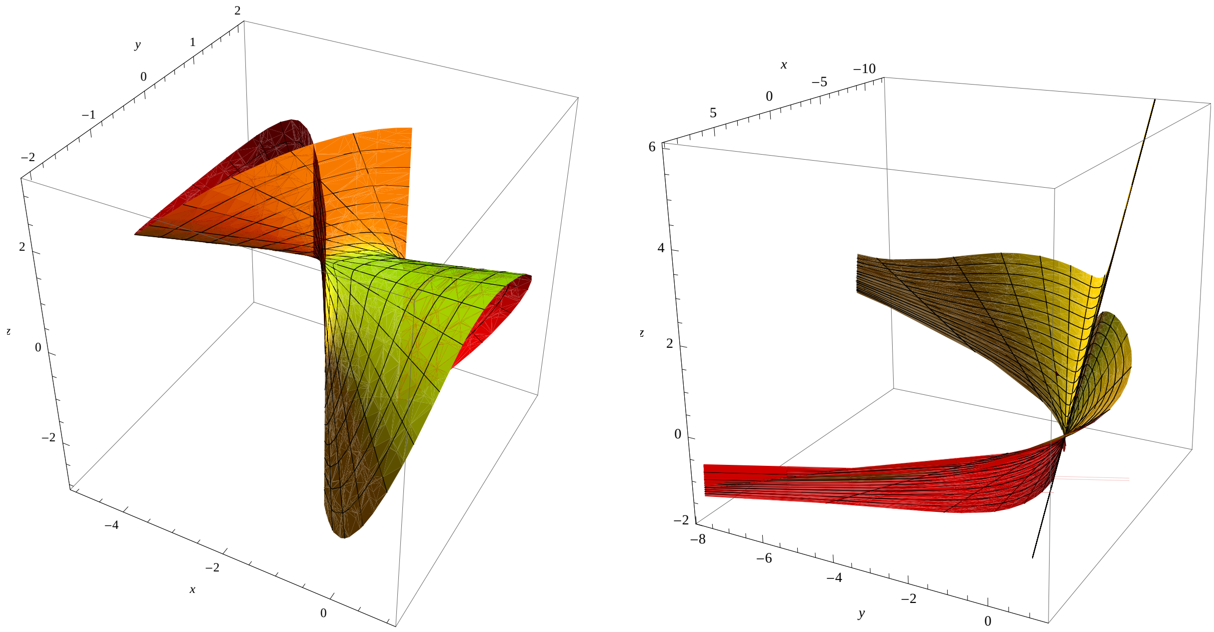


Figure 3. The spherical curve $\mathbf{x} = \mathbf{x}(\psi)$ (*red*) and its evolute $\mathbf{b}(\psi)$ (*blue*).



(a) **(b)**

Figure 4. (a) The ruled surface $\mathbf{y}(\psi, v)$; (b) the swallowtail surface $\mathbf{q}(\psi, v)$.

4. Conclusions

The singularities of height function intrinsically related to the Blaschke frame along a spherical curve of the ruled surface have been studied. We have introduced the generic properties of the evolute of a spherical curve of a ruled surface as the application of the singularity theory of space curves. An example, to confirm the main results, which are meaningful, is given and plotted.

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Conflict of interest

The authors declare no conflict of interest.

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